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Existence and general decay of solution for nonlinear viscoelastic two-dimensional beam with a nonlinear delay

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Abstract
We investigate the longitudinal and transversal vibrations of the viscoelastic beam with nonlinear tension and nonlinear delay term under the general decay rate for relaxation function. The existence theorem is proved by the Faedo–Galerkin method and using suitable Lyapunov functional to establish the general decay result.

Keywords Nonlinear vibrations · General decay · Kelvin Voigt damping · Faedo–Galerkin method · Nonlinear delay term

Mathematics Subject Classification 35B40 · 35L05 · 35G61

1 Introduction

Consider the following viscoelastic nonlinear beam in two-dimensional space:

$$\begin{cases} \rho w_{tt} + D w_{xxxxt} + E_I \{w - h * w\}_{xxxx} - \left\{\left[T_0 + E_A\left(v_x + \frac{1}{2}w_x^2\right)\right] w_x\right\}_x = 0, \\ \rho v_{tt} - E_A\left(v_x + \frac{1}{2}w_x^2\right)_x + \mu_1 g_1(v_t) + \mu_2 g_2(v_t(t-\tau)) = 0, \\ \forall (x,t) \in (0,L) \times (0,+\infty[, \end{cases} \tag{1}$$

with homogeneous boundary conditions

$$\begin{cases} w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0, \\ v(0,t) = v(L,t) = 0, \qquad \forall t > 0, \end{cases} \tag{2}$$

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and initial conditions

$$\begin{cases} w(x,0) = w_0(x), w_t(x,0) = w_1(x), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \qquad \forall x \in (0,L), \\ v_t(x,-t) = f(x,t), \forall\,(x,t) \in (0,L) \times (0,\tau), \end{cases} \tag{3}$$

where x and t denotes the space variable along the beam of length L and the time variable, respectively, $w(x,t)$ and $v(x,t)$ are the displacements in the transverse and longitudinal directions of the beam at the position x for time t, subscripts mean partial derivatives. ρ, D, E_I, T_0 and E_A are the uniform mass per unit length of the beam, the Kelvin–Voigt damping coefficient, bending stiffness, the tension and the stiffness of the beam, respectively. μ_1 and μ_2 are two positive real numbers, g_1 and g_2 are two functions, τ is a time delay, and f is history function, $h * w_{xx}$ is the viscoelastic damping term defined by

$$(h * w_{xx})(t) = \int_0^t h(t-s) w_{xx}(x,s) \mathrm{d}s, \quad \forall t \geq 0,$$

which describes the relationship between the stress and the strain from the Boltzmann Principle [6,8]. The relaxation function h represents the kernel of the memory term.

The term $g_2(v_t(t-\tau))$ represents distributed delay term. Time delay is the property of a physical system by which the response to an applied force is delayed in its effect [31]. The presence of the delay can become a source of instability. For example, the authors in [27,28,30] proved that the system is unstable under the condition $\mu_1 < \mu_2$, but otherwise the system is stable.

In recent years, energy decay in viscoelastic systems has become an important research title, while the behaviour of the relaxation function influences the energy decay rate. These behaviour of the relaxation function are generalized by the following extended class of kernels, namely,

$$h'(t) \leq -H(h(t)), \qquad t \geq 0,$$

where H satisfying some additional conditions imposed. We refer to previous studies [9,12–14,16–18,32] that proved a general energy decay rate.

For a system with delay term and viscoelastic damping, It is important to mention that the authors in [1,15,19,22,24] established the existence of solutions and general decay rates under the assumption $\mu_1 > \mu_2$.

In the absence of a delay, damping term ($\mu_1 = \mu_2 = 0$) and the relaxation function satisfies

$$h'(t) \leq -\gamma h(t),$$

where γ is a positive constant. Lekdim et al. [20] investigated the problem (1)–(3),with the following boundary conditions

$$\begin{cases} w(0,t) = w(L,t) = w_x(0,t) = w_x(L,t) = 0, \\ v(0,t) = 0, \qquad EAv_x(L,t) = U(t), \quad \forall t > 0. \end{cases} \tag{4}$$

They established an exponential stability under a suitable boundary control $U(t)$.

Motivated by the previous works, in this work we consider (1)–(3) in which we generalize the results obtained in [20], without boundary control. By expanding the class of relaxation functions into which the existence and unconditional stability are established.

The rest of our paper is organized as follows. In Sect. 2, we present some notations, assumptions and technical lemmas which will be needed later. In Sect. 3, we establish the existence and uniqueness results. The general decay rate is provided in Sect. 4.

2 Hypothesis and preliminary results

In this section, we give some notations, hypotheses and lemmas necessary to prove our results.

Notation. Let $L^2(0, L)$ be the Hilbert space with the inner product $(\cdot, \cdot)$ and norms $\|\cdot\|$.

We introduce the Hilbert spaces

$$V{=}H_0^1(0, L) \cap H^2(0, L),\ \ W{=}H_0^2(0, L) \cap H^4(0, L) \text{ and } Z{=}L^2\left(0, L; H^1\,(0, 1)\right).$$

As in [29], we introduce the new dependent variable

$$z(x, p, t) = v_t(x, t - \tau p), \qquad (x, p) \in (0, L) \times (0, 1)\,, \quad t \geq 0, \tag{5}$$

which satisfies

$$\tau z_t(x, p, t) + z_p(x, p, t) = 0, \quad (x, p) \in (0, L) \times (0, 1)\,. \tag{6}$$

The problem (1)–(3) is equivalent to

$$\begin{cases} \rho w_{tt} + D w_{xxxxt} + E_I \{w - h * w\}_{xxxx} - \left\{\left[T_0 + E_A \left(v_x + \frac{1}{2} w_x^2\right)\right] w_x\right\}_x = 0, \\ \rho v_{tt} - E_A \left(v_x + \frac{1}{2} w_x^2\right)_x + \mu_1 g_1 (v_t) + \mu_2 g_2 (z(x, p, t)) = 0, \\ \tau z_t(x, p, t) + z_p(x, p, t) = 0, & p \in (0, 1)\,, \\ z(x, 0, t) = v_t(x, t), & \forall\, (x, t) \in (0, L) \times (0, +\infty\,[, \\ w(x, 0) = w_0(x), w_t(x, 0) = w_1(x), \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), & \forall x \in (0, L), \\ z(x, p, 0) = f(x, p\tau), & \forall\, (x, p) \in (0, L) \times (0, 1), \end{cases} \tag{7}$$

with boundary conditions (2).

Hypothesis on memory kernels, the damping and the delay functions

As in [1,25], we make the following hypotheses on the kernel functions :

(H1) $h \in \mathcal{C}^2(\mathbb{R}_+, \mathbb{R}_+)$ is differentiable function satisfying

$$h(0) > 0, \qquad 1 - \int_0^\infty h(s)ds = 1 - \bar{h} > 0. \tag{8}$$

(**H2**) There exists a $\mathcal{C}^1$ function $H : \mathbb{R}_+ \to \mathbb{R}_+$ witch is linear or strictly increasing and strictly convex $\mathcal{C}^2$ function on $(0, \epsilon]$, $\epsilon < 1$, with $H(0) = H'(0) = 0$, such that

$$h'(t) \le -H(h(t)), \qquad \forall t \ge 0. \tag{9}$$

For the weight of the delay, following [4,10], we assume that

(**H3**) $g_1 \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ is nondecreasing function, such that there exist ϵ, c_1, $c_2 > 0$, such that

$$\begin{cases} c_1 |s| \le |g_1(s)| \le c_2 |s| & \text{if } |s| \ge \epsilon, \\ s^2 + g_1^2(s) \le H^{-1}(s g_1(s)) & \text{if } |s| \le \epsilon. \end{cases} \tag{10}$$

(**H4**) Let $g_2 \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ odd nondecreasing function, such that there exist c_3, a_1, $a_2 > 0$ with

$$\left|g_2'(s)\right| \le c_3 \tag{11}$$

$$a_1 s g_2(s) \le G(s) \le a_2 s g_1(s), \tag{12}$$

where $G(s) = \int_0^s g_2(r)dr$ and

$$a_2\mu_2 < a_1\mu_1. \tag{13}$$

Remark 1 (see [26]) By (**H1**), we have $\lim_{s\to+\infty} h(s) = 0$ and assume that $\lim_{s\to+\infty} h'(s) = 0$. This implies that there exists $t_0 > 0$ large enough such that

$$\max\left\{h(t), -h'(t)\right\} < \min\{\epsilon, H(\epsilon), H_0(\epsilon)\}, \quad \forall t \ge t_0, \tag{14}$$

where $H_0(t) = H(D(t))$ provided that D is a positive $\mathcal{C}^1$ function, $D(0) = 0$, for which H_0 is strictly increasing and strictly convex $\mathcal{C}^2$ function on $(0, \epsilon]$ and

$$\int_0^{+\infty} \frac{h(s)}{H_0^{-1}(-h'(s))} ds < +\infty. \tag{15}$$

By the nonincreasing of h, we get

$$0 < h(t_0) \le h(t) \le h(0), \quad \forall t \in [0, t_0], \tag{16}$$

the continuity and positivity of H imply that

$$a \le H(h(t)) \le b, \quad \forall t \in [0, t_0], \tag{17}$$

for some positive constants a and b. Then there exists $\gamma > 0$ such that

$$h'(t) \le -\gamma h(t), \quad \forall t \in [0, t_0]. \tag{18}$$

We define the energy functional of problem (1)–(3) by

$$E(t) = \frac{\rho}{2}\left[\|w_t\|^2 + \|v_t\|^2\right] + \left(1 - \int_0^t h(s)ds\right)\frac{E_I}{2}\|w_{xx}\|^2 + \frac{T_0}{2}\|w_x\|^2 + \frac{E_A}{2}\left\|v_x + \frac{1}{2}w_x^2\right\|^2 + \frac{E_I}{2}(h \circ w_{xx}) + \xi\int_0^L\int_0^1 G(z(x,p,t))\,dpdx. \tag{19}$$

where ξ is a positive constant such that

$$\tau\frac{\mu_1(1-a_1)}{a_1} < \xi < \tau\frac{\mu_1 - a_2\mu_1}{a_2}$$

and

$$(h \circ u)(t) = \int_0^t h(t-s)\|u(t) - u(s)\|^2 ds.$$

Lemma 1 *Let (w, v, z) be a solution of the problem (7), then, for $t \geq 0$, the time derivative of $E(t)$ can be upper bounded by*

$$E'(t) \leq -D\|w_{xxt}\|^2 + \frac{E_I}{2}h' \circ w_{xx} - \frac{E_I}{2}h(t)\|w_{xx}\|^2 - \eta_1(v_t, g_1(v_t)) - \eta_2(z(x,1,t), g_2(z(x,1,t))) \leq 0. \tag{20}$$

where $\eta_1 = \mu_1 - a_2\frac{\xi}{\tau} - a_2\mu_2$ and $\eta_2 = a_1\frac{\xi}{\tau} - \mu_2(1-a_1)$.

Proof Taking the inner product in $L^2(0, L)$ of the first equation of (7) with w_t, the second equation with v_t, , then integrating by parts, we obtain

$$\frac{d}{dt}\left\{\frac{\rho}{2}\left[\|w_t\|^2 + \|v_t\|^2\right] + \frac{E_I}{2}\|w_{xx}\|^2 + \frac{T_0}{2}\|w_x\|^2 + \frac{E_A}{2}\left\|v_x + \frac{1}{2}w_x^2\right\|^2\right\} = -D\|w_{xxt}\|^2 + (h * w_{xx}, w_{xxt}) - \mu_1(v_t, g_1(v_t)) - \mu_2(v_t, g_2(z(x,1,t))). \tag{21}$$

The second term of the right hand side of the above equality gives

$$2(h * w_{xx}, w_{xxt}) = h' \circ w_{xx} - h(t)\|w_{xx}\|^2 - \frac{d}{dt}\left\{(h \circ w_{xx}) - \left(\int_0^t h(s)ds\right)\|w_{xx}\|^2\right\}. \tag{22}$$

By multiplying the third equation of (7) by $\xi g_2(z(x,p,t))$ then integrating over $(0,L)\times(0,1)$, we find

$$\xi\int_0^1\int_0^L z_t g_2(z(x,p,t))\,dpdx = -\frac{\xi}{\tau}\int_0^1\int_0^L \frac{\partial}{\partial p}G(z(x,p,t))\,dpdx = -\frac{\xi}{\tau}\int_0^L [G(z(x,1,t)) - G(z(x,0,t))]\,dx. \tag{23}$$

Combining (21)–(23), using the fact that $z(0,t)=v_t(t)$ and assumption (**H4**), we get

$$\frac{d}{dt}E(t) = -D\|w_{xxt}\|^2 + \frac{E_I}{2}h'\circ w_{xx} - \frac{E_I}{2}h(t)\|w_{xx}\|^2 - \frac{\xi}{\tau}\int_0^L G(z(x,1,t))\,dx - \left(\mu_1 - a_2\frac{\xi}{\tau}\right)(v_t, g_1(v_t)) - \mu_2(v_t, g_2(z(x,1,t))). \tag{24}$$

The conjugate function G^* of the differentiable convex function G (see [7], pp. 9), i.e.

$$G^*(s) = \sup_{t\geq 0}(st - G(t)).$$

On the other hand G^* is the Legendre transform of G. We refer to ([2], pp. 61-62), that is given by

$$G^*(s) = s\left(G'\right)^{-1}(s) - G\left[\left(G'\right)^{-1}(s)\right], \qquad \forall s\geq 0, \tag{25}$$

which satisfies the generalized Young inequality :

$$st \leq G^*(s) + G(t), \qquad \forall t,s\geq 0. \tag{26}$$

By the definition of G, we obtain

$$G^*(s) = sg_2^{-1}(s) - G\left[g_2^{-1}(s)\right]. \tag{27}$$

Applying (26) with $s = g_2(z(1,t))$ and $t = v_t(t)$, from the last term of (24), we obtain

$$\frac{d}{dt}E(t) \leq -D\|w_{xxt}\|^2 + \frac{E_I}{2}h'\circ w_{xx} - \frac{E_I}{2}h(t)\|w_{xx}\|^2 - \left(\mu_1 - a_2\frac{\xi}{\tau}\right)(v_t, g_1(v_t)) + \mu_2\int_0^L \left(G(v_t) + G^*(g_2(z(x,1,t)))\right)dx - \frac{\xi}{\tau}\int_0^L G(z(x,1,t))\,dx. \tag{28}$$

The equality (27) and assumption (**H4**) imply that

$$G^{*}(g_2(z(x,1,t))) = z(x,1,t)g_2(z(x,1,t)) - G(z(x,1,t)) \\ \le (1-a_1)\, z(x,1,t)g_2(z(x,1,t)). \tag{29}$$

Combining (28) and (29), we have (20). □

Remark 2 The Lemma 1 imply that $E(t)$ nonincreasing, moreover

$$E(t) \le E(0), \qquad \forall t \ge 0. \tag{30}$$

The following lemmas will be used frequently in the sequel

Lemma 2 ([11]) *Let $u \in C^1([0,L])$ satisfying $u(0,t) = 0$. Then the following inequality hold:*

$$\left\| u^2(t) \right\|_{\infty} \le 2\, \|u(t)\|\, \|u_x(t)\|, \qquad \forall t \ge 0,$$

where $\|.\|_\infty$ is the norm of $L^\infty([0,L])$.

Lemma 3 ([23]) *If w is a solution of problem (1)–(3), assuming that h satisfies **(H1)** and **(H2)**. then we have*

$$\int_0^L (h \diamond w)^2_{xx}\, dx \le \bar{h}\,(h \circ w_{xx}), \tag{31}$$

$$\int_0^L \left(h' \diamond w\right)^2_{xx}\, dx \le -h(0)\left(h' \circ w_{xx}\right), \tag{32}$$

where $(h \diamond u)(t) = \int_0^t h(t-s)\,(u(s)-u(t))\,ds$.

3 Well possedness

The main aim of this section is to prove the following existence and uniqueness theorem:

Theorem 1 *Let $(w_0, v_0, z_0) \in W \times V \times Z$ and $(w_1, v_1) \in H_0^2(0,L) \times H_0^1(0,L)$. Assumee that **(H1)**–**(H4)** hold and satisfy*

$$z(x,p,t) = v_t(x, t-\tau p), \quad (x,p) \in (0,L)\times(0,1),\, t \ge 0. \tag{33}$$

Then the system (1)–(3) has a unique solution (w, v, z) in the sense that

$$w \in L^\infty\left(0,T; H_0^2(0,L)\right), \quad v \in L^\infty\left(0,T; H_0^1(0,L)\right), \quad z \in L^\infty(0,T;Z).$$
$$w_t \in L^\infty\left(0,T; H_0^2(0,L)\right), \quad v_t \in L^\infty\left(0,T; H_0^1(0,L)\right), \quad z_t \in L^\infty\left(0,T; L^2((0,L)\times(0,1))\right).$$
$$w_{tt} \in L^\infty\left(0,T; L^2(0,L)\right) \cap L^2\left(0,T; H_0^2(0,L)\right), \quad v_{tt} \in L^\infty\left(0,T; L^2(0,L)\right).$$

Proof We employ the Faedo–Galerkin technique to construct a solution.

Approximate solutions: Let $(w_i, v_i, z_i)_{i\leq m}$ be a complete orthogonal system of $W \times V \times Z$. For each $m \in \mathbb{N}$, let $W_1^m = span\{w_1, w_2, \ldots, w_m\}$, $W_2^m = span\{v_1, v_2, \ldots, v_m\}$ and $W_3^m = span\{z_1, z_2, \ldots, z_m\}$, such as the sequence $z_i(x, p)$ defined by $z_i(x, 0) = v_i(x)$, we prolong $z_i(x, 0)$ in Z by $z_i(x, p)$. For $w(0), w_t(0) \in W_1^m$, $v(0), v_t(0) \in W_2^m$ and $z(p, 0, s) \in W_3^m$, searching for functions $w^m(x, t) = \sum_{i=1}^m k_i^1(t)w_i(x)$, $v^m(x, t) = \sum_{i=1}^m k_i^2(t)v_i(x)$ and $z^m(x, t, p) = \sum_{i=1}^m k_i^3(t)z_i(x, p)$, that satisfy the following equations

$$\begin{aligned}
&\rho\left(w_{tt}^m, \varphi\right) + D\left(w_{xxt}^m, \varphi_{xx}\right) + E_I\left(w_{xx}^m, \varphi_{xx}\right) + \left(T_0 w_x^m, \varphi_x\right)\\
&\quad + E_A\left(\left(v_x^m + \frac{1}{2}\left(w_x^m\right)^2\right) w_x^m, \varphi_x\right) - E_I(h * w_{xx}^m, \varphi_{xx}) = 0, \qquad (34)\\
&\rho\left(v_{tt}^m, \phi\right) + E_A\left(v_x^m, \phi_x\right) + \frac{E_A}{2}\left(\left(w_x^m\right)^2, \phi_x\right)\\
&\quad + \left(\mu_1 g_1\left(v_t^m\right) + \mu_2 g_2\left(z^m(x, 1, t)\right), \phi\right) = 0 \qquad (35)
\end{aligned}$$

and

$$\int_0^1 \left(\tau\left(z_t^m, \psi\right) + \left(z_p^m, \psi\right)\right) dp = 0, \qquad (36)$$

for all $(\varphi, \phi, \psi) \in H_0^2(0, L) \times H_0^1(0, L) \times L^2((0, L) \times (0, 1))$, with the initial conditions

$$\begin{aligned}
&(w^m(0), v^m(0), z^m(p, 0, s)) = \left(w_0^m, v_0^m, z_0^m\right) \to (w_0, v_0, z_0) \quad \text{in} \quad W \times V \times Z,\\
&\left(w_t^m(0), v_t^m(0)\right) = \left(w_1^m, v_1^m\right) \to (w_1, v_1) \quad \text{in} \quad H_0^2(0, L) \times H_0^1(0, L).
\end{aligned}$$

A Priori Estimates

Throughout this part, B_i, $i = 1, 2, \ldots$, denote positive constants independent of m and $t \in [0, T]$.

Estimate 1: Let E_m the energy defined by (19), for the solutions w^m, v^m and z^m. By utilizing the same steps used in the proof of Lemma 1, we find

$$\begin{aligned}
&E_m(t) + \int_0^t \left[D\left\|w_{xxt}^m\right\|^2 + \eta_1\left(v_t^m, g_1\left(v_t^m\right)\right)\right] ds\\
&\quad + \int_0^t \eta_2\left(z^m(1, s), g_2\left(z^m(1, s)\right)\right) ds \leq E_m(0) \leq B_1. \qquad (37)
\end{aligned}$$

where η_1, η_2 positive constants.

Estimate 2: Firstly, we estimate $\left\|w_{tt}^m(0)\right\|^2$, $\left\|v_{tt}^m(0)\right\|^2$ and $\left\|z_t^m(0)\right\|^2$.

Fixed $t = 0$ and taking $\varphi = w_{tt}^m(0)$, $\phi = v_{tt}^m(0)$ and $\psi = z_t^m(0)$ in (34), (35) and (36), respectively, then integrate them by parts and apply Young's inequality. From the assumptions **(H3)**, **(H4)** and the initial data are sufficiently smooth, we can be infer that

$$\left\|w_{tt}^m(0)\right\|^2, \left\|v_{tt}^m(0)\right\|^2, \int_0^1 \left\|z_t^m(0)\right\|^2 dp \leq B_2. \qquad (38)$$

Now, we estimate $\left\|w_{tt}^m\right\|^2$, $\left\|v_{tt}^m\right\|^2$ and $\left\|z_t^m\right\|^2$.

Let us fix $t,\ \zeta > 0$ such that $\zeta < T - t$. Taking the difference of (34), (35) and (36) with $t = t + \zeta$ and $t = t$, and simultaneously replacing φ, ϕ and ψ with $w_t^m(t+\zeta) - w_t^m(t)$, $v_t^m(t+\zeta) - v_t^m(t)$ and $(z^m(p, t+\zeta) - z^m(p, t))$, respectively. Then gather the two first relations and integrating the last relation over $(0, 1)$, we get

$$\begin{aligned}
&\frac{\rho}{2}\frac{d}{dt}\left\|w_t^m(t+\zeta) - w_t^m(t)\right\|^2 + D\left\|w_{xxt}^m(t+\zeta) - w_{xxt}^m(t)\right\|^2 \\
&\quad + \frac{E_I}{2}\frac{d}{dt}\left\|w_{xx}^m(t+\zeta) - w_{xx}^m(t)\right\|^2 + \frac{T_0}{2}\frac{d}{dt}\left\|w_x^m(t+\zeta) - w_x^m(t)\right\|^2 \\
&\quad + \frac{\rho}{2}\frac{d}{dt}\left\|v_t^m(t+\zeta) - v_t^m(t)\right\|^2 + \frac{E_A}{2}\frac{d}{dt}\left\|v_x^m(t+\zeta) - v_x^m(t)\right\|^2 \\
&\quad + \mu_2\left(v_t^m(t+\zeta) - v_t^m(t), g_2\left(z^m(1, t+\zeta)\right) - g_2\left(z^m(1, t)\right)\right) \\
&\quad + \mu_1\left(v_t^m(t+\zeta) - v_t^m(t), g_1\left(v_t^m(t+\zeta)\right) - g_1\left(v_t^m(t)\right)\right) = F_1 + F_2, \qquad (39)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\tau\int_0^1\left\|z^m(p, t+\zeta) - z^m(p, t)\right\|^2 dp &= -\frac{1}{2}\int_0^1\frac{d}{dp}\left\|z^m(p, t+\zeta) - z^m(p, t)\right\|^2 dp \\
&= \frac{1}{2}\left\|v_t^m(t+\zeta) - v_t^m(t)\right\|^2 \\
&\quad -\frac{1}{2}\left\|z^m(1, t+\zeta) - z^m(1, t)\right\|^2, \qquad (40)
\end{aligned}$$

where

$$\begin{aligned}
F_1 = &-\frac{E_A}{2}\int_0^L\left[\left(w_x^m(t+\zeta)\right)^3 - \left(w_x^m(t)\right)^3\right]\left[w_{xt}^m(t+\zeta) - w_{xt}^m(t)\right]dx \\
&-E_A\int_0^L\left[w_x^m v_x^m(t+\zeta) - w_x^m v_x^m(t)\right]\left[w_{xt}^m(t+\zeta) - w_{xt}^m(t)\right]dx \\
&-\frac{E_A}{2}\int_0^L\left[\left(w_x^m(t+\zeta)\right)^2 - \left(w_x^m(t)\right)^2\right]\left[v_{xt}^m(t+\zeta) - v_{xt}^m(t)\right]dx,
\end{aligned}$$

and

$$F_2 = E_I\left(\int_0^{t+\zeta} h(t+\zeta-s)w_{xx}(s)ds - \int_0^t h(t-s)w_{xx}(s)ds,\ w_{xxt}(t+\zeta) - w_{xxt}(t)\right).$$

Taking the first estimate, Young, Poincaré’s inequalities and Lemma 2 into account, we can estimate F_1 and F_2 as follows :

$$\begin{aligned}
|F_1| \le \frac{D}{4}&\left\|w_{xxt}^m(t+\zeta) - w_{xxt}^m(t)\right\|^2 + B_3\left\|w_x^m(t+\zeta) - w_x^m(t)\right\|^2 \\
&+B_3\left(\left\|w_{xx}^m(t+\zeta) - w_{xx}^m(t)\right\|^2 + \left\|v_t^m(t+\zeta) - v_t^m(t)\right\|^2\right), \qquad (41)
\end{aligned}$$

$$|F_2| \leq B_4 \int_0^L \left(\int_0^{t+\zeta} h(t+\zeta-s) w_{xx}(s) ds - \int_0^t h(t-s) w_{xx}(s) ds \right)^2 dx$$
$$+ \frac{D}{4} \|w_{xxt}(t+\zeta) - w_{xxt}(t)\|^2 . \tag{42}$$

Combining (39)–(42), then dividing both sides by ζ^2 and taking the limit as $\zeta \to 0$, we get

$$\frac{\rho}{2}\frac{d}{dt}\left\|w_{tt}^m(t)\right\|^2 + \frac{D}{2}\left\|w_{xxtt}^m(t)\right\|^2 + \frac{E_I}{2}\frac{d}{dt}\left\|w_{xxt}^m(t)\right\|^2 + \frac{T_0}{2}\frac{d}{dt}\left\|w_{xt}^m(t)\right\|^2$$
$$+\frac{\rho}{2}\frac{d}{dt}\left\|v_{tt}^m(t)\right\|^2 + \frac{E_A}{2}\frac{d}{dt}\left\|v_{xt}^m(t)\right\|^2 + \mu_2\left(v_{tt}^m(t) z_t^m(1,t), g_2'\left(z^m(1,t)\right)\right)$$
$$+\mu_1\left(\left(v_{tt}^m(t)\right)^2, g_1'\left(v_t^m(t)\right)\right) + \frac{1}{2}\frac{d}{dt}\tau\int_0^1 \left\|z_t^m(p,t)\right\|^2 dp + \frac{1}{2}\left\|z_t^m(1,t)\right\|^2$$
$$\leq B_5\left(\left\|w_{xt}^m(t)\right\|^2 + \left\|w_{xxt}^m(t)\right\|^2 + \left\|v_{tt}^m(t)\right\|^2\right)$$
$$+B_4\int_0^L\left(h(0)w_{xx}^m + \int_0^t h'(t-s) w_{xx}^m(s) ds\right)^2 dx. \tag{43}$$

On the other hand, we have

$$\mu_1\left(\left(v_{tt}^m(t)\right)^2, g_1'\left(v_t^m(t)\right)\right) \geq 0, \tag{44}$$
$$\mu_2\left(v_{tt}^m(t) z_t^m(1,t), g_2'\left(z^m(1,t)\right)\right) \leq \frac{\mu_2 c_3}{2}\left(\left\|z_t^m(1,t)\right\|^2 + \left\|v_{tt}^m(t)\right\|^2\right), \tag{45}$$

and

$$\int_0^L \int_0^t h'(t-s) w_{xx}^m(s) ds dx \leq \sup_{[0,T]} \left\|w_{xx}^m\right\| \left(\int_0^T \left|h'(s)\right| ds\right) \leq B_6. \tag{46}$$

Integrating (43) over $(0, t)$, taking (37), (44)–(46) under consideration and noting that the initial data are sufficiently smooth. By Gronwall's lemma, we conclude

$$\left\|w_{tt}^m\right\|^2 + \int_0^t \left\|w_{xxtt}^m\right\|^2 ds + \left\|w_{xxt}^m\right\|^2 + \left\|w_{xt}^m\right\|^2 + \left\|v_{tt}^m\right\|^2$$
$$+ \left\|v_{xt}^m\right\|^2 + \tau\int_0^1 \left\|z_t^m(p,t)\right\|^2 dp + \int_0^t \left\|z_t^m(1,s)\right\|^2 dpds \leq B_7 e^{B_8 T}. \tag{47}$$

Passage to the limit

The estimate (37) and (38) permits to deduce

$$\begin{cases} \left(w^m, v^m\right), \ \left(w_t^m, v_t^m\right) \ \text{are bounded in } L^\infty\left(0, T; H_0^2(0,L)\right) \times L^\infty\left(0, T; H_0^1(0,L)\right), \\ \left(w_{tt}^m, v_{tt}^m\right) \ \text{is bounded in } L^\infty\left(0, T; L^2(0,L)\right) \cap L^2\left(0, T; H_0^2(0,L)\right) \times L^\infty\left(0, T; L^2(0,L)\right), \\ \left(z^m, z_t^m\right) \ \text{is bounded in } L^\infty(0, T; Z) \times L^\infty\left(0, T; L^2((0,L)\times(0,1))\right), \\ v_t^m g_1\left(v_t^m\right), \ z^m(1,t) g_2\left(z^m(1,t)\right) \ \text{are bounded in } L^1((0,L)\times(0,T)), \\ G\left(z^m\right) \ \text{is bounded in } L^\infty\left(0, T; L^2((0,L)\times(0,1))\right) \\ \left(v_x + \frac{1}{2}\left(w_x^m\right)^2\right) \ \text{is bounded in } L^\infty\left(0, T; L^2(0,L)\right). \end{cases} \tag{48}$$

Therefore, there exists subsequences of (w^m), (v^m) and (z^m), still denoted by (w^m), (v^m) and (z^m), respectively, such that

$$\begin{cases} \left(w^m, w_t^m\right) \to (w, w_t) \ \text{weak star in } L^\infty\left(0, T; H_0^2(0,L)\right) \times L^\infty\left(0, T; H_0^2(0,L)\right), \\ \left(v^m, v_t^m\right) \to (v, v_t) \ \text{weak star in } L^\infty\left(0, T; H_0^1(0,L)\right) \times L^\infty\left(0, T; H_0^1(0,L)\right), \\ w_{tt}^m \to w_{tt} \text{ weak star in } L^\infty\left(0, T; L^2(0,L)\right) \cap L^2\left(0, T; H_0^2(0,L)\right), \\ v_{tt}^m \to v_{tt} \text{ weak star in } L^\infty\left(0, T; L^2(0,L)\right), \\ \left(z^m, z_t^m\right) \to (z, z_t) \ \text{weak star in } L^\infty(0, T; Z) \times L^\infty\left(0, T; L^2((0,L)\times(0,1))\right), \\ \left(g_1(v^m), g_2\left(z_t^m\right)\right) \to (G_1, G_2) \ \text{weak star in } \left[L^2((0,T)\times(0,L))\right]^2, \\ \left(v_x^m + \frac{1}{2}\left(w_x^m\right)^2\right) \to \Gamma \text{ weak star in } L^\infty\left(0, T; L^2(0,L)\right). \end{cases} \tag{49}$$

Analysis of the nonlinear terms

By Aubin-Lions compactness (see [3]), we conclude from (49), that

$$\begin{aligned} w^m &\to w \text{ strongly in } W^{1,\infty}\left(0, T; H_0^1(0,L)\right), \\ v^m &\to v \text{ strongly in } W^{1,\infty}\left(0, T; L^2(0,L)\right), \end{aligned} \tag{50}$$

therefore

$$z^m \to z \text{ strongly and a.e in } [0,T]\times[0,L]. \tag{51}$$

Lemma 4 ([5]) *We have the convergence* $g_1\left(v_t^m\right) \to g_1(v_t)$ *and* $g_2(z^m) \to g_2(z)$ *in* $L^1([0,T]\times[0,L])$. *Hence,*

$$\begin{aligned} g_1\left(v_t^m\right) &\to g_1(v_t) \ \text{weak in } L^2([0,T]\times[0,L]), \\ g_2(z^m) &\to g_2(z) \ \text{weak in } L^2([0,T]\times[0,L]). \end{aligned} \tag{52}$$

From (50) Lions Lemma ([21], pp. 12), we concluded that $\Gamma = \left(v_x + \frac{1}{2}(w_x)^2\right)$ and

$$\left(v_x^m + \frac{1}{2}\left(w_x^m\right)^2\right) w_x^m \to \left(v_x + \frac{1}{2}(w)^2\right) w_x \text{ weakly in } L^2([0,T]\times[0,L]). \tag{53}$$

Now, we can pass to the limit in the approximate problem (34)–(35) to get a weak solution of the problem (1)–(3), (see [21,33]).

Uniqueness The uniqueness can be proved by following the same procedures as in estimation 2. □

4 Asymptotic behavior

The prove of energy decay relies heavily on the construction of Lyapunov functional and exploitation of convex analysis. For this intent, we start by constructing a Lyapunov functional :

$$\mathcal{L}(t) = E(t) + \beta_1 \Phi(t) + \beta_2 \Psi(t) + \beta_3 \chi(t) \tag{54}$$

where $\beta_1,\ \beta_2$ and β_3 are positive constants, $E(t)$ is given by (19) and

$$\Phi(t) = \frac{\rho}{2}(w, w_t) + \rho(v, v_t) + \frac{D}{4}\|w_{xx}\|^2, \tag{55}$$

$$\Psi(t) = \rho((h \diamond w), w_t). \tag{56}$$

$$\chi(t) = \int_0^L \int_0^1 e^{-2\tau p} G(z(x, p, t))\, dp dx \tag{57}$$

It is our aim to prove that functional $\mathcal{L}(t)$ satisfies an estimates. To pass this estimate to $E(t)$, we will need the following proposition.

Proposition 1 *Let $\mathcal{L}(t)$ and $E(t)$ be the functional defined by (54) and (19), respectively. Then, for β_1 β_2 and β_3 small enough, we have*

$$\alpha_1 E(t) \le \mathcal{L}(t) \le \alpha_2 E(t), \qquad \forall t > 0. \tag{58}$$

where α_1 and α_2 are positive constants.

Proof By Young's inequality, we have

$$|\Phi(t)| \le \frac{D}{4}\|w_{xx}\|^2 + \frac{\rho}{4}\|w_t\|^2 + \frac{\rho L^2}{4}\|w_x\|^2 + \frac{\rho}{2}\|v_t\|^2 + \frac{\rho L^2}{2}\|v_x\|^2. \tag{59}$$

Obviously,

$$\|v_x\|^2 \le 2\left\|v_x + \frac{1}{2}w_x^2\right\|^2 + \frac{1}{2}\left\|w_x^2\right\|^2,$$

then using Holder's inequality, Lemma 2 and inequality (30), we have

$$\|v_x\|^2 \le 2\left\|v_x + \frac{1}{2}w_x^2\right\|^2 + \frac{A_1}{2}\|w_x\|^2, \tag{60}$$

where $A_1 = 4E(0)/\sqrt{T_0\left(E_I\left(1-\bar{h}\right)\right)}$.

Substituting (60) into (59), we obtain

$$|\Phi(t)| \le \frac{\rho}{4}\|w_t\|^2 + \frac{\rho}{2}\|v_t\|^2 + \frac{D}{4}\|w_{xx}\|^2 + \frac{(1+A_1)\rho L^2}{4}\|w_x\|^2 + \rho L^2\left\|v_x + \frac{1}{2}w_x^2\right\|^2. \tag{61}$$

For Ψ, by Young's inequality and Lemma 3, we get

$$|\Psi(t)| \leq \frac{\rho}{2}\|w_t\|_2^2 + \frac{L^4\rho\bar{h}}{2}(h \circ w_{xx}). \tag{62}$$

For χ, from the decay of the function $e^{-2\tau p}$, we have

$$|\chi(t)| \leq \frac{1}{\xi}E(t). \tag{63}$$

By combining (61), (62) and (63), we deduce that

$$-\left(\lambda + \frac{\beta_3}{\xi}\right)E(t) \leq \beta_1\Phi(t) + \beta_2\Psi(t) + \beta_3\chi(t) \leq \left(\lambda + \frac{\beta_3}{\xi}\right)E(t),$$

where $\lambda = \max\left(\frac{\beta_1}{2} + \beta_2, \beta_1, \frac{D}{2E_I(1-\bar{h})}\beta_1, \frac{(1+A_1)\rho L^2}{2T_0}\beta_1, \frac{2\rho L^2}{E_A}\beta_1, \frac{L^4\rho\bar{h}}{E_I}\beta_2\right)$.

We take β_1, β_2 and β_3 small, so that $\lambda + \frac{\beta_3}{\xi} < 1$, this complete proof. □

Lemma 5 *Let $\Phi(t)$ be the functional given by (55), then, for $t \geq 0$,*

$$\begin{aligned}\Phi'(t) \leq{}& \frac{\rho}{2}\|w_t\|^2 + \rho\|v_t\|^2 - \frac{E_I\left(1-\bar{h}\right)}{4}\|w_{xx}\|^2 + \frac{E_I\bar{h}}{4\left(1-\bar{h}\right)}(h \circ w_{xx}) \\ &- \left[\frac{T_0}{2} - \theta_1 A_\mu\frac{A_1}{2}\right]\|w_x\|^2 - \left[E_A - 2\theta_1 A_\mu\right]\left\|v_x + \frac{1}{2}w_x^2\right\|^2 \\ &+ \frac{\mu_1}{4\theta_1}\|g_1(v_t)\|^2 + \frac{\mu_2 c_3}{4\theta_1}(z(x,1,t), g_2(z(x,1,t))).\end{aligned} \tag{64}$$

Proof Differentiating $\Phi(t)$ and using the equations (1), we obtain

$$\Phi'(t) = \frac{\rho}{2}\|w_t\|^2 + \rho\|v_t\|^2 + J_1 + J_2 + J_3 + J_4, \tag{65}$$

where

$$\begin{cases} J_1 = -\frac{E_I}{2}\left(\{w - h * w\}_{xxxx}, w\right), \\ J_2 = \left(-\frac{D}{2}w_{xxxxt} + \frac{T_0}{2}w_{xx}, w\right) + \frac{D}{2}(w_{xxt}, w_{xx}), \\ J_3 = \frac{E_A}{2}\left(\left\{\left(v_x + \frac{1}{2}w_x^2\right)w_x\right\}_x, w\right) + E_A\left(\left(v_x + \frac{1}{2}w_x^2\right)_x, v\right), \\ J_4 = -\left(\mu_1 g_1(v_t) + \mu_2 g_2(z(x,1,t)), v\right). \end{cases}$$

Using integrating J_1 by parts twice and the boundary conditions (2), we get

$$J_1 = -\frac{E_I}{2}\|w_{xx}\|^2 + \frac{E_I}{2}(h * w_{xx}, w_{xx}),$$

add and subtract the term $\frac{E_I}{2}\left(\left(\int_0^t h(s)ds\right) w_{xx}, w_{xx}\right)$ in the above equality then using Young's inequality and Lemma 3, we have

$$J_1 \leq -\frac{E_I\left(1-\bar{h}-\theta\right)}{2}\|w_{xx}\|^2 + \frac{E_I\bar{h}}{8\theta}\left(h \circ w_{xx}\right). \tag{66}$$

Integrating by parts the terms J_2 and J_3, we find

$$J_2 = -\frac{T_0}{2}\|w_x\|^2, \tag{67}$$

$$J_3 = -E_A\left\|v_x + \frac{1}{2}w_x^2\right\|^2. \tag{68}$$

By Young's inequality, Poincaré's inequality, inequality (60) and assumption (**H4**), we infer for $\theta_1 > 0$

$$\begin{aligned} J_4 &\leq \theta_1 A_\mu \|v_x\|^2 + \frac{\mu_1}{4\theta_1}\|g_1(v_t)\|^2 + \frac{\mu_2}{4\theta_1}\|g_2(z(x,1,t))\|^2 \\ &\leq \theta_1 A_\mu\left(2\left\|v_x + \frac{1}{2}w_x^2\right\|^2 + \frac{A_1}{2}\|w_x\|^2\right) + \frac{\mu_1}{4\theta_1}\|g_1(v_t)\|^2 \\ &\quad + \frac{\mu_2 c_3}{4\theta_1}\left(z(x,1,t), g_2(z(x,1,t))\right), \end{aligned} \tag{69}$$

where $A_\mu = (\mu_1+\mu_2)L^2$.

Now, Substituting (66)–(69) into (65) and taking $\theta = \frac{(1-\bar{h})}{2}$, we have (64). □

Lemma 6 *Let $\Psi(t)$ be the functional given by (56), then, for $t \geq 0$,*

$$\begin{aligned} \Psi'(t) \leq &-\left[\left(\int_0^t h(s)ds\right) - \frac{\theta_2}{2}\right]\rho\|w_t\|^2 - \frac{\rho L^4 h(0)}{2\theta_2}\left(h' \circ w_{xx}\right) + A_2\bar{h}\left(h \circ w_{xx}\right) \\ &+ \theta_4 T_0\|w_x\|^2 + \theta_3 D\|w_{xxt}\|^2 + \theta_3\left(1+\bar{h}\right)E_I\|w_{xx}\|^2 \\ &+ \theta_4 E_A\left\|v_x + \frac{1}{2}w_x^2\right\|^2, \end{aligned} \tag{70}$$

where θ_i, $i = 2, 3, 4$. are positive constants and

$$A_2 = \left(\theta_3 + \frac{3}{4\theta_3}\right)E_I + \frac{D}{4\theta_3} + \frac{L^2 T_0 + L^2 E_A A_1}{4\theta_4}.$$

Proof Substituting the first equation in (1) into the derivative of $\Psi(t)$, we obtain

$$\Psi'(t) = -\rho\int_0^t h(s)ds\,\|w_t\|_2^2 + K_1 + K_2 + K_3 + K_4, \tag{71}$$

where

$$\begin{cases} K_1 = \rho\left(h' \diamond w, w_t\right), \\ K_2 = -\left(h \diamond w, Dw_{xxxxt} + E_I w_{xxxx}\right), \\ K_3 = E_I\left(h \diamond w, h * w_{xxxx}\right), \\ K_4 = E_A\left(h \diamond w, \left\{\left[T_0 + \left(v_x + \frac{1}{2}w_x^2\right)\right] w_x\right\}_x\right). \end{cases}$$

Let's estimate these terms, for K_1 similarly to (62), we get

$$K_1 \leq \frac{\rho\theta_2}{2}\|w_t\|_2^2 - \frac{\rho L^4 h(0)}{2\theta_2}\left(h' \circ w_{xx}\right). \tag{72}$$

For K_2 and K_3, using integrating by parts twice, Young's inequality and Lemma 3, we obtain

$$K_2 \leq \theta_3 D\|w_{xxt}\|^2 + \theta_3 E_I\|w_{xx}\|^2 + \frac{\bar{h}\left(E_I + D\right)}{4\theta_3}\left(h \circ w_{xx}\right), \tag{73}$$

$$K_3 \leq \theta_3 E_I \bar{h}\|w_{xx}\|^2 + \left(\theta_3 + \frac{1}{2\theta_3}\right) E_I \bar{h}\left(h \circ w_{xx}\right). \tag{74}$$

Integrating K_4 by parts, applying Young's inequality, Holder's inequality and Lemma 3,we get

$$K_4 \leq \theta_4 T_0\|w_x\|^2 + \theta_4 E_A\left\|v_x + \frac{1}{2}w_x^2\right\|^2 + \frac{L^2 T_0 + L^2 E_A A_1}{4\theta_4}\bar{h}\left(h \circ w_{xx}\right). \tag{75}$$

Combining (71)–(75), we obtain (70). □

Lemma 7 *Let $\chi(t)$ be the functional given by (55), then, for $t \geq 0$*

$$\chi'(t) \leq -2\chi(t) + \frac{a_2}{\tau}\left(v_t, g_1\left(v_t\right)\right) + \frac{a_1 e^{-2\tau}}{\tau}\left(z(x,1,t), g_2\left(z(x,1,t)\right)\right). \tag{76}$$

Proof Take derivative of $\chi(t)$ with respect to t, using the identity (62) and integrating by parts, we obtain

$$\chi'(t) = -\frac{1}{\tau}\int_0^L \left[e^{-2\tau} G\left(z\left(x,1,t\right)\right) - G\left(z\left(x,0,t\right)\right)\right] dx$$
$$-2\int_0^L \int_0^1 e^{-2\tau p} G\left(z\left(x,p,t\right)\right) dp dx.$$

In view of the hypotheses (H3) and the above equality, we have (70). □

Theorem 2 *For the system dynamics described by (1)–(3), under assumptions **(H1)–(H4)**, given that $(w_0, v_0, z_0) \in W \times V \times Z$ and $(w_1, v_1) \in H_0^2(0,L) \times H_0^1(0,L)$, where*

$$z(x,p,t,s) = v_t(x, t - p\tau), \quad p \in (0,1). \tag{77}$$

Then, there exist strictly positive constants ω_1, ω_2, ω_3 and ε such that

$$E(t) \le \omega_1 H_1^{-1}(\omega_2 t + \omega_3), \qquad \forall t > 0, \tag{78}$$

where

$$H_1(t) = \int_t^1 \frac{ds}{H_2(s)},$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear}, \\ tH'(\varepsilon t) & \text{if } H \text{ is nonlinear}. \end{cases} \tag{79}$$

Proof Using the results (20), (64), (70) and (76), for all $t \ge t_0 > 0$, we obtain

$$\begin{aligned}
\mathcal{L}'(t) \le & \left[\frac{E_I}{2} - \frac{\rho L^4 h(0)}{2\theta_2}\beta_2\right](h' \circ w_{xx}) + \left[\frac{E_I \bar{h}}{4\left(1-\bar{h}\right)}\beta_1 + A_2 \bar{h}\beta_2\right](h \circ w_{xx}) \\
& - \left[\frac{\left(1-\bar{h}\right)}{4}\beta_1 - \theta_3\left(1+\bar{h}\right)\beta_2\right] E_I \|w_{xx}\|^2 - [1-\theta_3\beta_2] D \|w_{xxt}\|^2 \\
& - \left[\left(h_0 - \frac{\theta_2}{2}\right)\beta_2 - \frac{\beta_1}{2}\right]\rho \|w_t\|^2 \\
& - \left[\left(\frac{T_0}{2} - \theta_1 A_\mu \frac{A_1}{2}\right)\beta_1 - \theta_4 T_0 \beta_2\right]\|w_x\|^2 \\
& - \left[\left(E_A - 2\theta_1 A_\mu\right)\beta_1 - \theta_4 E_A \beta_2\right]\left\| v_x + \frac{1}{2}w_x^2\right\|^2 \\
& - \left[\eta_1 - \frac{a_2}{\tau}\beta_3\right](v_t, g_1(v_t)) + \rho\beta_1\|v_t\|^2 + \frac{\mu_1}{4\theta_1}\beta_1 \|g_1(v_t)\|^2 \\
& - \left[\eta_2 - \frac{a_1 e^{-2\tau}}{\tau}\beta_3 - \frac{\mu_2 c_2}{4\theta_1}\beta_1\right](z(x,1,t), g_2(z(x,1,t))) - 2\beta_3\chi(t)
\end{aligned} \tag{80}$$

where $h_0 = \int_0^{t_0} h(t)dt$.

Right now, we do select our parameters very carefully. First pick $\theta_2 \le h_0$, after that take θ_i, $i = 1, 3, 4.$, β_1, β_2 and β_3 sufficiently small so that

$$\begin{cases}
\theta_1 \le \frac{1}{2}\min\left\{\frac{T_0}{A_\mu A_1}, \frac{E_A}{2A_\mu}\right\}, \\
\theta_3 < \min\left\{\frac{\beta_1}{\beta_2}\frac{(1-\bar{h})}{(1+\bar{h})}, \frac{1}{\beta_2}\right\}, \\
\theta_4 < \frac{\beta_1}{4\beta_2} < \frac{h_0}{4}, \\
\beta_2 \le \frac{E_I\theta_2}{\rho L^4 h(0)}, \\
\beta_3 < \frac{\tau}{a_2}\eta_1, \\
\frac{a_1 e^{-2\tau}}{\tau}\beta_3 + \frac{\mu_2 c_2}{4\theta_1}\beta_1 < \eta_2, \\
A_3 > 0.
\end{cases} \tag{81}$$

Thus, (80) becomes

$$\mathcal{L}'(t) \leq -A_3 E(t) + A_4 \left(\|v_t\|^2 + \|g_1(v_t)\|^2 + (h \circ w_{xx})(t) \right), \quad \forall t \geq t_0, \tag{82}$$

where A_3 and A_4 are two positive constants.

To complete the proof, we partition the interval $[0, L]$ into

$$L^{>} = \{0 \leq x \leq L : |v_t| > \epsilon\}, \qquad L^{<} = \{0 \leq x \leq L : |v_t| \leq \epsilon\}.$$

From (H3), we estimate that

$$\int_{L^{>}} \left(v_t^2 + g_1^2(v_t) \right) dx \leq (1/c_1 + c_2) \int_{L^{>}} v_t g_1(v_t)\, dx \leq -A_5 E'(t) \tag{83}$$

where $A_5 = (1/c_1 + c_2)/\eta_1$.

For the estimate of the last term in the right hand side of (82) on $L^{<}$, we distinguish two cases

Case 1 : H is linear. From the assumption(H3), we deduce that there exists c_4, such that $s^2 + g_1^2(s) \leq c_4 s g_1(s)$ in $[-\epsilon, \epsilon]$, and therefore

$$\int_{L^{<}} \left(v_t^2 + g_1^2(v_t) \right) dx \leq c_4 \int_{L^{<}} v_t g_1(v_t)\, dx \leq -A_6 E'(t) \tag{84}$$

and the assumption (H2) gives

$$\int_0^t h(s) \int_{\Omega} |w_{xx}(t) - w_{xx}u(t-s)|^2 dx ds \leq -A_6 E'(t). \tag{85}$$

Combining (82)–(84), we obtain

$$(\mathcal{L}(t) + \sigma E(t))' \leq -A_3 H_2(E(t)). \tag{86}$$

where $\sigma = A_4(A_5 + 2A_6)$.

Case 2: H is nonlinear. First, we deduce from (18) and (20) that

$$\begin{aligned} \int_0^{t_0} h(s) \|w_{xx}(t) - w_{xx}(t-s)\|^2 ds &\leq \frac{-1}{\gamma} \int_0^{t_0} h'(s) \|w_{xx}(t) - w_{xx}(t-s)\|^2 ds \\ &\leq -\frac{2}{\gamma E_I} E'(t). \end{aligned} \tag{87}$$

Next, we define the functions κ_p and κ by

$$\kappa_p(t) = p \int_{t_0}^t \frac{h(s)}{H_0^{-1}(-h'(s))} \|w_{xx}(t) - w_{xx}(t-s)\|^2 ds \tag{88}$$

$$\kappa(t) = -\int_{t_0}^{t} h'(s)\frac{h(s)}{H_0^{-1}\left(-h'(s)\right)}\|w_{xx}(t) - w_{xx}(t-s)\|^2 ds, \tag{89}$$

where $1/p > \frac{8E(0)}{E_I(1-\bar{h})}\int_0^{+\infty}\frac{h(s)}{H_0^{-1}(-h'(s))}ds$ and H_0 is defined in (14), we find that $\kappa_p(t) < 1$, for all $t \geq 0$.

The properties of the functions H_0, D and h gives

$$\frac{h(s)}{H_0^{-1}\left(-h'(s)\right)} \leq \frac{h(s)}{H_0^{-1}(H(h(s)))} = \frac{h(s)}{D^{-1}(h(s))} \leq \kappa_0, \tag{90}$$

for some positive constant κ_0.

We can easily verify that

$$\begin{aligned}
\kappa(t) &\leq -\kappa_0 \int_{t_0}^{t} h'(s)\|w_{xx}(t) - w_{xx}(t-s)\|^2 ds \\
&\leq -\frac{8\kappa_0 E(0)}{E_I(1-\bar{h})}\int_{t_0}^{t} h'(s)ds \\
&\leq \frac{8\kappa_0 E(0)}{E_I(1-\bar{h})}h(t_0) \leq \frac{1}{2}\min\{\epsilon, H(\epsilon), H_0(\epsilon)\}, \qquad \forall t \geq t_0.
\end{aligned}$$

We have, by the convexity property of H_0 and $H_0(0) = 0$ that

$$H_0(\nu x) \leq \nu H_0(x), \qquad x \in [0, \epsilon],\ \nu \in [0, 1]. \tag{91}$$

By assumption (H2), identity (91) and Jensen's inequality, we get

$$\begin{aligned}
\kappa(t) &\geq \frac{1}{\kappa_p(t)}\int_{t_0}^{t}\frac{H_0\left(\kappa_p(t)H_0^{-1}\left(-h'(s)\right)\right)}{H_0^{-1}\left(-h'(s)\right)}h(s)\|w_{xx}(t) - w_{xx}(t-s)\|^2 ds \\
&\geq H_0\left(\frac{1}{\kappa_p(t)}\int_{t_0}^{t}\frac{\kappa_p(t)H_0^{-1}\left(-h'(s)\right)}{H_0^{-1}\left(-h'(s)\right)}h(s)\|w_{xx}(t) - w_{xx}(t-s)\|^2 ds\right) \\
&= H_0\left(\int_{t_0}^{t} h(s)\|w_{xx}(t) - w_{xx}(t-s)\|^2 ds\right), \qquad \forall t \geq t_0,
\end{aligned}$$

this is equivalent to

$$\int_{t_0}^{t} h(s)\|w_{xx}(t) - w_{xx}(t-s)\|^2 ds \leq H_0^{-1}(\kappa(t)), \qquad \forall t \geq t_0. \tag{92}$$

We can assume that ϵ is small enough such that $sg_1(s) \leq \frac{1}{2}\min\{\epsilon, H(\epsilon), H_0(\epsilon)\}$ for all $|s| \leq \epsilon$. With (H3) and reversed Jensen's inequality for concave function and the concavity of H^{-1}, we obtain

$$\int_{L^<} \left(v_t^2 + g_1^2(v_t)\right) dx \le \int_{L^<} H^{-1}(v_t g_1(v_t))\, dx \le c H^{-1}(\vartheta(t)), \tag{93}$$

where $\vartheta(t) = \frac{1}{L}\int_{L^<} v_t g_1(v_t)\, dx$.

The inequalities (82), (83), (87), (92) and (93), gives

$$(\mathcal{L}(t) + A_7 E(t))' \le -A_3 E(t) + A_4 c H^{-1}(\vartheta(t)) + A_4 H_0^{-1}(\kappa(t)),\ \forall t \ge t_0.$$

Since $H_0^{-1}(t){=}D^{-1}(H^{-1}(t))$, $D^{-1}(H^{-1}(0) = D^{-1}(0)){=}0$ and $H^{-1}(\kappa(t)){\le}\epsilon$. Moreover, the function $D^{-1}(H^{-1}(t))$ is concave, so its graph is below its tangent, that $H_0^{-1}(\kappa(t)) \le c H^{-1}(\kappa(t))$. Therefore, for all $t{\ge}t_0$,

$$\begin{aligned}(\mathcal{L}(t) + A_7 E(t))' &\le -A_3 E(t) + A_4 c H^{-1}(\vartheta(t)) + A_4 c H^{-1}(\kappa(t)),\\ &\le -A_3 E(t) + A_4 c H^{-1}(\vartheta(t) + \kappa(t)).\end{aligned} \tag{94}$$

Take into account $E'(t) \le 0$, $H'(t) > 0$, $H''(t) > 0$, and using the inequality (94), for $\varepsilon < \epsilon E(0)$, we infer

$$\begin{aligned}&\left[H'(\varepsilon E(t))\{\mathcal{L}(t) + A_7 E(t)\} + A_8 E(t)\right]'\\ &\quad= \varepsilon E'(t) H''(\varepsilon E(t))\{\mathcal{L}(t){+}A_7 E(t)\} + H'(\varepsilon E(t))\left\{\mathcal{L}'(t){+}A_7 E'(t)\right\} + A_8 E'(t)\\ &\quad\le -A_3 E(t) H'(\varepsilon E(t)) {+} A_4 c H'(\varepsilon E(t)) H^{-1}(\vartheta(t){+}\kappa(t)) + A_8 E'(t), \quad \forall t{\ge}t_0.\end{aligned} \tag{95}$$

Let H^* by the convex conjugate of H, given by (25), then the increasing of the functions $(H')^{-1}$, H and the fact that $H(0) = 0$, yield

$$H^*(s) \le s\left(H'\right)^{-1}(s). \tag{96}$$

Applying inequalities (26) and (96) to the second term of the right hand side of (95), we obtain

$$\begin{aligned}H'(\varepsilon E(t)) H^{-1}(\vartheta(t) + \kappa(t)) &\le H^*(H'(\varepsilon E(t))) - (\vartheta(t) + \kappa(t))\\ &\le \varepsilon E(t) H'(\varepsilon E(t)) - A_8 E'(t),\end{aligned} \tag{97}$$

combining (95) and (97), we have

$$\begin{aligned}\left[H'(\varepsilon E(t))\{\mathcal{L}(t) + A_7 E(t)\} + A_8 E(t)\right]' &\le -(A_3 - A_4 c\varepsilon) E(t) H'(\varepsilon E(t))\\ &= -A_9 H_2(E(t)).\end{aligned} \tag{98}$$

Let us define

$$\tilde{\mathcal{L}}(t) = \begin{cases}\mathcal{L}(t) + \sigma E(t) & \text{if } H \text{ is linear},\\ H'(\varepsilon E(t))\{\mathcal{L}(t) + A_7 E(t)\} + A_8 E(t) & \text{if } H \text{ is nonlinear}.\end{cases} \tag{99}$$

From (86) and (98), we conclude

$$\tilde{\mathcal{L}}(t) \leq -\tilde{C} H_2(E(t)), \qquad t \geq t_0.$$

Since $\mathcal{L}(t)$ and $E(t)$ are equivalent and $0 \leq H'(\varepsilon E(t)) \leq H'(\varepsilon E(0))$. So, there exist $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ two positive constants such that

$$\tilde{\alpha}_1 E(t) \leq \tilde{\mathcal{L}}(t) \leq \tilde{\alpha}_2 E(t).$$

Now we put $\mathcal{L}_\alpha(t) = \alpha \tilde{\mathcal{L}}(t)$ for $\alpha \leq 1/\tilde{\alpha}_2$, using the fact that H_2 is increasing, we obtain

$$\begin{aligned} \mathcal{L}_\alpha(t) = \alpha \tilde{\mathcal{L}}(t) &\leq -\alpha \tilde{C} H_2(\frac{1}{\tilde{\alpha}_2} \tilde{\mathcal{L}}(t)) \\ &\leq -\alpha \tilde{C} H_2(\mathcal{L}_\alpha(t)), \qquad t \geq t_0. \end{aligned}$$

Taking into consideration that $H_1' = -1/H_2$, the above inequalities become

$$\mathcal{L}_\alpha(t) H_1'(\mathcal{L}_\alpha(t)) \geq \alpha \tilde{C}, \qquad t \geq t_0,$$

integrate this differential inequality over (t_0, t), we obtain

$$H_1\left(\mathcal{L}_\alpha(t)\right) \geq H_1\left(\mathcal{L}_\alpha\left(t_0\right)\right) + \alpha \tilde{C}\left(t - t_0\right).$$

Choosing α small enough such that $H_1(\mathcal{L}_\alpha(t_0)) - \alpha \tilde{C} t_0 > 0$. The decay of H_1^{-1}, yields

$$\mathcal{L}_\alpha(t) \leq H_1^{-1}\left(\alpha \tilde{C} t + \left(H_1(\mathcal{L}_\alpha(t_0)) - \alpha \tilde{C} t_0\right)\right).$$

finally, the equivalence of $\mathcal{L}(t)$, $\tilde{\mathcal{L}}(t)$, $\mathcal{L}_\alpha(t)$ and $E\left(t\right)$, result

$$E(t) \leq \omega_1 H_1^{-1}\left(\omega_2 t + \omega_3\right), \qquad t \geq t_0.$$

One can easily find a similar estimate over the interval $[0, t_0]$, by using decreasing of E and H_1^{-1}. This completes the proof. □

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