Ricerche di Matematica (2023) 72:803–813
https://doi.org/10.1007/s11587-021-00588-y

An existence result for strongly pseudomonotone quasi-variational inequalities

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Received: 31 January 2021 / Accepted: 26 April 2021 / Published online: 4 May 2021

Abstract
By applying the Brouwer fixed point theorem, we prove an existence result of solutions for strongly pseudomonotone quasi-variational inequalities which extends an analogous result in Kocvara and Outrata (Optim Methods Softw 5:275–295, 1995). The result is based on a new result concerning continuity property of solutions to a parametric variational inequality. Examples are given to illustrate our results.

Keywords Parametric variational inequalities · Strongly pseudomonotone · Quasi-variational inequalities

Mathematics Subject Classification 47J20 · 49J40

1 Introduction and preliminaries

Let F be a single-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ and let K be a set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $K(x)$ is a closed convex set in $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$. We consider the quasi-variational inequality (in short, QVI): Find $x^* \in K(x^*)$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in K(x^*). \tag{1.1}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$. When $K(x) = K$, a nonempty closed convex subset of $\mathbb{R}^n$, for all $x \in \mathbb{R}^n$, the quasi-variational inequality (1.1) reduces to the classical variational inequality (in short, VIP): Find $x^* \in K$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \quad \text{for all } x \in K. \tag{1.2}$$

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Quasi-variational inequalities were introduced by Bensoussan and Lions in [2–4], where some impulse control problems are studied. They have become a powerful mathematical tool for modelling of various complicated equilibria which appeared in many areas such as mechanics [9,16,17], traffic [5], statistics [13], biology [10], economics and finance [18,23]. For more analytical results of quasi-variational inequalities, we refer the reader to, for examples, [6,19].

It is well-known that the problem (1.1) can be written as the following fixed point problem

$$x = P_{K(x)}[x - \lambda F(x)] \tag{1.3}$$

where λ is a positive real number and $P_C[\cdot]$ is the projection operator on a closed convex set C in $\mathbb{R}^n$. Thus, we can get sufficient conditions for the existence of solutions of a quasi-variational inequality by applying a suitable fixed point theorem. Some existence results in finite dimensional spaces can be found in [7]. Applying the Brouwer fixed point theorem, the authors in [7] proved the following existence result.

Theorem 1.1 [7] *Suppose that there exists a nonempty compact convex set* $C \subset \operatorname{dom} K$ *such that*

(H1) $K(C) \subset C$,
(H2) *F is continuous on C,*
(H3) *K is continuous on C.*

Then the quasi-variational inequality (1.1) *has at least one solution.*

However, in some applications, e.g., the discretized Coulomb friction model (see, e.g., [16]), there is no compact set C for which condition (H1) is satisfied. In [16], to obtain the existence of solutions of the quasi-variational inequality (1.1), the authors replaced the compactness of C by a weaker condition, namely the closedness and required, in addition, that the function F is strongly monotone. More precisely, by using the Brouwer fixed point theorem, the authors in [16] proved the following existence result.

Theorem 1.2 [16] *Let F be continuously differentiable on $\mathbb{R}^n$ and be strongly monotone on $\mathbb{R}^n$, i.e., there exists $\alpha > 0$ such that*

$$\langle F(x) - F(y), x - y\rangle \geq \alpha ||x - y||^2 \quad \forall x, y \in \mathbb{R}^n.$$

Here, $||\cdot||$ denotes the norm induced by the inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^n$. Suppose that there exists a nonempty convex closed set $C \subset \operatorname{dom} K$ such that

(i) $K(C) \subset C$,
(ii) $\bigcap_{x\in C} K(x) \neq \emptyset$,
(iii) *for each $y \in \mathbb{R}^n$, the mapping $x \mapsto P_{K(x)}[y]$ is continuous on an open set containing C.*

Then the quasi-variational inequality (1.1) *has at least one solution.*

For other existence results of of solutions of the quasi-variational inequality concerning strong monotonicity, we refer the reader to [22] and [20]. Both results need some contraction property of the projection operator on the set $K(\cdot)$, i.e., they require the existence of a positive constant $\ell < 1$ such that

$$||P_{K(x)}[z] - P_{K(y)}[z]|| \leq \ell||x - y|| \quad \forall x, y, z \in \mathbb{R}^n. \tag{1.4}$$

An example for $K(\cdot)$ satisfying (1.4) is a moving set where $K(x) = c(x) + K$ with K being a nonempty closed convex subset of $\mathbb{R}^n$ and $c : \mathbb{R}^n \to \mathbb{R}^n$ a Lipschitz continuous mapping with the same modulus ℓ. In [21], the authors extended the existence results in [20,22] to the case when F is strongly pseudomonotone on $\mathbb{R}^n$, i.e., there exists a modulus $\gamma > 0$ such that for all $x, y \in \mathbb{R}^n$

$$\langle F(x), y - x\rangle \geq 0 \quad \Rightarrow \quad \langle F(y), y - x\rangle \geq \gamma||y - x||^2.$$

Since strongly monotone mappings are also strongly pseudomonotone, but not vice versa, there are examples that we can apply the existence result in [21] but we cannot apply any existence results in [16,20,22]. Note that, in general setting, (1.4) is hard to prove and it is not often satisfied. The following example shows that there are situations in which we cannot apply above mentioned results.

Example 1.1 We consider the quasi-variational inequality problem (1.1) with $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$F(x) = \left(5^{-||x||^2} + \frac{1}{6}\right)(x_1, 2x_2) \quad \forall x = (x_1, x_2) \in \mathbb{R}^2,$$

and $K : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$defined by:

$$K(x) = [-2|x_1|, 2|x_1|] \times [0, \infty) \quad \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Now, assume that $x^* = (x_1^*, x_2^*) \in \mathbb{R}^2$ is a solution of the quasi-variational inequality (1.1). Then, $(x_1^*, x_2^*) \in [-2|x_1^*|, 2|x_1^*|] \times [0, \infty)$ and

$$\left\langle \left(5^{-||x^*||^2} + \frac{1}{6}\right)(x_1^*, 2x_2^*), (y_1 - x_1^*, y_2 - x_2^*)\right\rangle \geq 0$$

for all $(y_1, y_2) \in [-2|x_1^*|, 2|x_1^*|] \times [0, \infty)$. This is equivalent to $(x_1^*, x_2^*) \in [-2|x_1^*|, 2|x_1^*|] \times [0, \infty)$ and

$$x_1^*(y_1 - x_1^*) + 2x_2^*(y_2 - x_2^*) \geq 0, \quad \forall (y_1, y_2) \in [-2|x_1^*|, 2|x_1^*|] \times [0, \infty).$$

This implies that $x_1^* = x_2^* = 0$. Thus, $(0, 0)$ is a solution of the quasi-variational inequality (1.1).

We cannot apply Theorem 1.1 to this example since there is no compact set $C \subset \mathbb{R}^2$ such that $K(C) \subset C$. We also cannot apply Theorem 1.2 to this example since the mapping F is not strongly monotone. Indeed, taking $x = (1, 0)$ and $y = (2, 0)$, we have

$$\langle F(x) - F(y), x - y\rangle = \left\langle \left(5^{-1} + \frac{1}{6}\right)(1, 0) - \left(5^{-4} + \frac{1}{6}\right)(2, 0), (1, 0) - (2, 0)\right\rangle$$
$$= -\frac{113}{3570} < 0.$$

One can see that F is strongly pseudomonotone on $\mathbb{R}^n$ with modulus $\frac{1}{6}$. Indeed, for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ with $\langle F(x), y - x\rangle \geq 0$, i.e., $\langle (x_1, 2x_2), (y_1 - x_1, 2y_2 - 2x_2\rangle \geq 0$, we have

$$\begin{aligned}
\langle F(y), y - x\rangle &= \left(5^{-||y||^2} + \frac{1}{6}\right)\langle (y_1, 2y_2), (y_1 - x_1, 2y_2 - x_2)\rangle \\
&\geq \left(5^{-||y||^2} + \frac{1}{6}\right)(\langle (y_1, 2y_2), (y_1 - x_1, 2y_2 - x_2)\rangle \\
&\quad -\langle (x_1, 2x_2), (y_1 - x_1, 2y_2 - 2x_2\rangle) \\
&= \left(5^{-||y||^2} + \frac{1}{6}\right)\langle (y_1 - x_1, 2y_2 - x_2), (y_1 - x_1, 2y_2 - x_2)\rangle \\
&= \left(5^{-||y||^2} + \frac{1}{6}\right)[(y_1 - x_1)^2 + 2(y_2 - x_2)^2] \\
&\geq \frac{1}{6}\left[(y_1 - x_1)^2 + (y_2 - x_2)^2\right] = \frac{1}{6}||x - y||^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left\| P_{K((0,1))}(3, 1) - P_{K((1,1))}(3, 1)\right\| &= \left\| P_{\{0\}\times[0,\infty)}(3, 1) - P_{[-2,2]\times[0,\infty)}(3, 1)\right\| \\
&= \|(0, 1) - (2, 1)\| = 2 > 1 = \|(0, 1) - (1, 1)\|.
\end{aligned}$$

That is, the projection map is not contractive. So, we cannot apply [21, Theorem 3.1] to this example.

Our aim is to prove an existence result for strongly pseudomonotone quasi-variational inequalities without requiring (1.4). For this aim, we will extend Theorem 1.2 by replacing the strong monotonicity of F by the strong pseudomonotonicity. Our result bases on a new result about continuity property of solutions to parametric variational inequalities.

To conclude this section, we recall the following results which will be used in the sequel.

Lemma 1.1 (see, e.g., [15]) *Let Ω be a nonempty closed convex subset of $\mathbb{R}^n$. Then for a given $z \in \mathbb{R}^n$, $u \in \Omega$ satisfies $u = P_\Omega[z]$ if and only if*

$$\langle u - z, v - u\rangle \geq 0, \qquad \forall v \in \Omega.$$

Theorem 1.3 [15, Theorem 4.2, page 13] *Let $K \subset \mathbb{R}^n$ be closed and convex and $F : K \to \mathbb{R}^n$ be continuous. A necessary and sufficient condition that there exists a solution to VIP (1.2) is that there exists an $R > 0$ such that $||x_R|| < R$, where $x_R \in K \cap \overline{B}(\boldsymbol{0}, R)$ satisfying*

$$\langle F(x_R), y - x_R \rangle \geq 0 \quad \forall y \in K \cap \overline{B}(\boldsymbol{0}, R).$$

Here, $\overline{B}(\boldsymbol{0}, R)$ denotes the closed ball of radius R and center $\boldsymbol{0}$, the zero vector of $\mathbb{R}^n$.

2 Main results

In this section, by applying the Brouwer fixed point theorem, we derive an existence result for quasi-variational inequality (1.1) with strongly pseudomonotone mapping F. With this aim, we first prove a new result concerning continuity property of solution map to a parametric variational inequality.

2.1 Parametric variational inequalities

Let $M \subset \mathbb{R}^m$ and $\Lambda \subset \mathbb{R}^d$ be two sets of parameters. Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a vector-valued function and $K : \Lambda \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. We consider the following parametric variational inequality problem with parameters $(\mu, \lambda) \in M \times \Lambda$:

Find $x \in K(\lambda)$ such that

$$\langle f(x, \mu), y - x \rangle \geq 0 \ \text{ for all } \quad y \in K(\lambda). \tag{2.1}$$

Throughout this subsection, we always require the following assumptions:

(A1) $K(\lambda)$ is nonempty, closed and convex for each $\lambda \in \Lambda$.
(A2) There exists $\ell > 0$ such that, for all $x_1, x_2 \in K(\Lambda)$ and $\mu_1, \mu_2 \in M$,

$$||f(x_1, \mu_1) - f(x_2, \mu_2)|| \leq \ell(||x_1 - x_2|| + ||\mu_1 - \mu_2||). \tag{2.2}$$

(A3) There exists $\alpha > 0$ such that, for all $\mu \in M$ and for all $x, y \in K(\Lambda)$,

$$\langle f(x, \mu), y - x \rangle \geq 0 \quad \Rightarrow \quad \langle f(y, \mu), y - x \rangle \geq \alpha ||x - y||^2.$$

Under our assumptions, for each $(\lambda, \mu) \in M \times \Lambda$, the parametric variational inequality problem (2.1) has a unique solution (see [14, Theorem 2.1]). Hence, we can define a mapping $u : M \times \Lambda \to \mathbb{R}^n$ assigning to each $(\lambda, \mu) \in M \times \Lambda$ the corresponding unique solution of (2.1).

The main result of this subsection is stated as follows.

Theorem 2.1 *Let $(\lambda_0, \mu_0) \in M \times \Lambda$ and $x_0 = u(\lambda_0, \mu_0)$. If the mapping $\lambda \mapsto P_{K(\lambda)}[x_0 - \rho f(x_0, \mu_0)]$ is continuous at λ_0 for some $\rho > 1/\alpha$, then u is continuous at (λ_0, μ_0).*

Proof Fix $\rho > 1/\alpha$ and let $(\lambda, \mu) \in M \times \Lambda$. We have

$$u(\lambda_0, \mu_0) = P_{K(\lambda_0)}[u(\lambda_0, \mu_0) - \rho f(u(\lambda_0, \mu_0), \mu_0)],$$

and

$$u(\lambda, \mu) = P_{K(\lambda)}[u(\lambda, \mu) - \rho f(u(\lambda, \mu), \mu).$$

Set

$$y = P_{K(\lambda)}[u(\lambda_0, \mu_0) - \rho f(u(\lambda_0, \mu_0), \mu_0)].$$

By Lemma 1.1, one has

$$\langle y - u(\lambda_0, \mu_0) + \rho f(u(\lambda_0, \mu_0), \mu_0), u(\lambda, \mu) - y\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle u(\lambda_0, \mu_0) - y, u(\lambda, \mu) - y\rangle &\leq \rho f(u(\lambda_0, \mu_0), \mu_0), u(\lambda, \mu) - y\rangle \\ &= \rho f(u(\lambda_0, \mu_0), \mu_0) - f(y, \mu), u(\lambda, \mu) - y\rangle \\ &\quad -\rho\langle f(y, \mu), y - u(\lambda, \mu)\rangle \end{aligned} \tag{2.3}$$

Since $y \in K(\lambda)$, we have

$$f(u(\lambda, \mu), \mu), y - u(\lambda, \mu)\rangle \geq 0.$$

Thus, by (A3), one has

$$\langle f(y, \mu), y - u(\lambda, \mu)\rangle \geq \alpha||y - u(\lambda, \mu)||^2. \tag{2.4}$$

Using (A2), we have from (2.3) and (2.4) that

$$\begin{aligned}&2\langle u(\lambda_0, \mu_0) - y, u(\lambda, \mu) - y\rangle \\ &\quad \leq 2\rho\ell(||u(\lambda_0, \mu_0) - y|| + ||\mu - \mu_0||).||u(\lambda, \mu) - y|| \\ &\qquad -2\rho\alpha||u(\lambda, \mu) - y||^2 \\ &\quad \leq \rho^2\ell^2(||u(\lambda_0, \mu_0) - y|| + ||\mu - \mu_0||)^2 + ||u(\lambda, \mu) - y||^2 \\ &\qquad -2\rho\alpha||u(\lambda, \mu) - y||^2 \\ &\quad \leq 2\rho^2\ell^2||u(\lambda_0, \mu_0) - y||^2 + 2\rho^2\ell^2||\mu - \mu_0||^2 + ||u(\lambda, \mu) - u(\lambda_0, \mu_0)||^2 \\ &\qquad +||u(\lambda_0, \mu_0) - y||^2 + 2\langle u(\lambda, \mu) - u(\lambda_0, \mu_0), u(\lambda_0, \mu_0) - y\rangle \\ &\qquad -2\rho\alpha||(u(\lambda, \mu) - u(\lambda_0, \mu_0)) - (y - u(\lambda_0, \mu_0))||^2 \\ &\quad \leq (2\rho^2\ell^2 + 1)||u(\lambda_0, \mu_0) - y||^2 + 2\rho^2\ell^2||\mu - \mu_0||^2 \\ &\qquad +||u(\lambda, \mu) - u(\lambda_0, \mu_0)||^2 + 2\langle u(\lambda, \mu) - u(\lambda_0, \mu_0), u(\lambda_0, \mu_0) - y\rangle \\ &\qquad -\rho\alpha||u(\lambda, \mu) - u(\lambda_0, \mu_0)||^2 + 2\rho\alpha||u(\lambda_0, \mu_0) - y||^2.\end{aligned}$$

Here, we obtain the last inequality by using the inequality $-2||a-b||^2 \leq -||a||^2 + 2||b||^2$. Then,

$$2\langle u(\lambda_0,\mu_0)-y, u(\lambda_0,\mu_0)-y\rangle \leq (2\rho^2\ell^2+2\rho\alpha+1)||u(\lambda_0,\mu_0)-y||^2 + 2\rho^2\ell^2||\mu-\mu_0||^2 \\ +(1-\rho\alpha)||u(\lambda,\mu)-u(\lambda_0,\mu_0)||^2.$$

Thus,

$$||u(\lambda,\mu)-u(\lambda_0,\mu_0)||^2 \leq \frac{2\rho^2\ell^2}{\rho\alpha-1}||\mu-\mu_0||^2 \\ +\frac{2\rho^2\ell^2+2\rho\alpha-1}{\rho\alpha-1}||u(\lambda_0,\mu_0)-y||^2. \tag{2.5}$$

Since $\rho\alpha > 1$, we have $2\rho^2\ell^2+2\rho\alpha-1 > \max\{1, 2\rho^2\ell^2\}$. It follows from (2.5) that

$$||u(\lambda,\mu)-u(\lambda_0,\mu_0)|| \leq \frac{2\rho^2\ell^2+2\rho\alpha-1}{\rho\alpha-1}[||\mu-\mu_0|| + ||u(\lambda_0,\mu_0)-y||].$$

Hence,

$$||u(\lambda,\mu)-u(\lambda_0,\mu_0)|| \leq \frac{2\rho^2\ell^2+2\rho\alpha-1}{\rho\alpha-1}||P_{K(\lambda)}[x_0-\rho f(x_0,\mu_0)] \\ -P_{K(\lambda_0)}[x_0-\rho f(x_0,\mu_0)]|| \\ +\frac{2\rho^2\ell^2+2\rho\alpha-1}{\rho\alpha-1}||\mu-\mu_0||. \tag{2.6}$$

Since $\lambda \mapsto P_{K(\lambda)}[x_0 - \rho f(x_0,\mu_0)]$ is continuous at λ_0, it follows from (2.6) that $u(\lambda,\mu)$ is continuous at (λ_0,μ_0). □

Remark 2.1 It follows from (2.6) that properties of the solution mapping u depend on the properties of the mapping $\lambda \mapsto P_{K(\lambda)}[x_0 - \rho f(x_0,\mu_0)]$. In fact, if the mapping $\lambda \mapsto P_{K(\lambda)}[x_0 - \rho f(x_0,\mu_0)]$ is Hölder continuous (respectively, Lipschitz continuous), then the solution mapping u is also Hölder continuous (respectively, Lipschitz continuous). Examples of set-valued mappings K in which $\lambda \mapsto P_{K(\lambda)}[x_0 - \rho f(x_0,\mu_0)]$ is Hölder continuous or Lipschitz continuous can be seen, for instance, in [25,26]. For examples of set-valued mappings K in which $\lambda \mapsto P_{K(\lambda)}[x_0 - \rho f(x_0,\mu_0)]$ is continuous, we refer the reader, e.g., to [16]. For more results concerning regularity property of solution maps to parametric variational inequalities, we refer the reader to, e.g., [1,8,12,24–26].

The next result concerns the boundedness of the images of the solution mapping u.

Proposition 2.1 *Assume that there exists a nonempty set $\Omega \subset \mathrm{dom}K \cap \Lambda$ such that*

(A4) $$\bigcap_{\lambda\in\Omega} K(\lambda) \neq \emptyset.$$

Then there exists $R > 0$ *such that* $||u(\mu, \lambda)|| < R$ *for all* $\lambda \in \Omega$ *and* $\mu \in M$.

Proof Let $\mu_0 \in M$ and $\lambda_0 \in \Omega$. Let $v_0 \in \bigcap_{\lambda \in \Omega} K(\lambda)$ and fix $H > f(v_0, \mu_0)$. Set

$$R = ||v_0|| + \frac{H}{\alpha}.$$

Let z^* be the unique solution of the variational inequality: Find $z \in K(\lambda_0) \cap \overline{B}(\mathbf{0}, R)$ such that

$$\langle f(z, \mu_0), y - z\rangle \geq 0 \quad \text{for all } y \in K(\lambda_0) \cap \overline{B}(\mathbf{0}, R). \tag{2.7}$$

Assume that $||z^*|| \geq R$. Since $v_0 \in K(\lambda_0)$ and

$$||v_0|| = R - \frac{H}{\alpha} < R,$$

$v_0 \in K(\lambda_0) \cap \overline{B}(\mathbf{0}, R)$. Hence, we have

$$\langle f(z^*, \mu_0), v_0 - z^*\rangle \geq 0.$$

By (A3), one has

$$\begin{aligned} \alpha||v_0 - z^*||^2 &\leq \langle f(v_0, \mu_0), v_0 - z^*\rangle \\ &\leq ||f(v_0, \mu_0)||.||v_0 - z^*|| \\ &< H.||v_0 - z^*||. \end{aligned}$$

This implies that

$$||v_0 - z^*|| < \frac{H}{\alpha}.$$

We have

$$||z^*|| \leq ||v_0|| + ||v_0 - z^*|| < ||v_0|| + \frac{H}{\alpha} = R.$$

This is a contradiction. Thus, the solution z^* of (2.7) satisfies $||z^*|| < R$ for some $R > 0$. It follows from Theorem 1.3 that z^* is also a solution of the the parametric variational inequality problem (2.1) with the pair of parameters (μ_0, λ_0). Therefore, $||u(\mu_0, \lambda_0)|| < R$. This ends the proof. □

2.2 Existence result

We are now in a position to state the main result of this paper.

Theorem 2.2 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping and let $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping with closed convex values. Suppose that there exists a nonempty convex closed set $C \subset \mathrm{dom} K$ such that*

(i) $K(C) \subset C$,
(ii) *F is Lipschitz continuous on C and f is strongly pseudomonotone on C,*
(iii) $\bigcap_{x \in C} K(x) \neq \emptyset$,
(iv) *for each $y \in \mathbb{R}^n$, the mapping $x \mapsto P_{K(x)}[y]$ is continuous on an open set containing C.*

Then the quasi-variational inequality (1.1) has at least one solution.

Proof We define a mapping $S : \mathrm{dom} K \to \mathbb{R}^n$ by assigning to each $y \in \mathrm{dom} K$ the corresponding unique solution of the following parameter variational inequality: Find $z \in K(y)$ such that

$$\langle F(z), w - z \rangle \geq 0 \qquad \text{for all } \ w \in K(y).$$

By Theorem 2.1, S is continuous on C. Moreover, by Proposition 2.1, there exists $R > 0$ such that $||S(y)|| < R$ for all $y \in C$. That is, S maps $C \cap \{x \in \mathbb{R}^n : ||x|| \leq R\}$ into itself. Hence, by the Brouwer fixed point theorem, S has a fixed point in $C \cap \{x \in \mathbb{R}^n : ||x|| \leq R\}$, which is a solution of (1.1). This ends the proof. □

Example 2.1 We consider F and K as in Example 1.1. As shown in Example 1.1, the mapping F is strongly pseudomonotone. One can also see that $C = \{0\} \times [0, \infty)$ is closed convex such that condition (i) and (iii) of Theorem 2.2 are satisfied. It is easy to see that F is Lipschitz continuous on C. For $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$, one can compute that

$$P_{K(x)}[y] = \begin{cases} (-2|x_1|, y_2) & \text{if} \qquad y_1 \leq -2|x_1|, \\ \quad (y_1, y_2) & \text{if } -2|x_1| \leq y_1 \leq 2|x_1|, \\ (2|x_1|, y_1) & \text{if} \qquad 2|x_1| \leq y_1. \end{cases}$$

Then the map $x \mapsto P_{K(x)}[y]$ is continuous for each $y \in \mathbb{R}^2$ and so the condition (iv) of Theorem 2.2 is satisfied. Therefore, we can apply Theorem 2.2 to this example.

Example 2.2 (see, e.g. [16]) Let m be an even integer and consider $K : \mathbb{R}^m \to \mathbb{R}^m$ which is defined by: for $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$,

$$K(x) = \{v = (v_1, \ldots, v_m) \in \mathbb{R}^m : |v_{2i}| \leq \alpha |x_{2i-1}|, v_{2i-1} \leq 0, i = 1, 2, \ldots, m/2\},$$

where $\alpha > 0$ is a constant. This type of set-valued mappings appears in the discretized Coulomb friction model (see, [11]).

If we set $C = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_{2i-1} \leq 0, i = 1, 2, \ldots, m/2\}$, then conditions (i), (iii) and (iv) of Theorem 2.2 are satisfied.

To conclude this paper, we present a general class of strongly pseudomonotone mappings which are not strongly monotone in general.

Example 2.3 (see, e.g., [14]) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$F(x) = h(x)(Qx + q) \quad \forall x \in \mathbb{R}^n, \tag{2.8}$$

where $Q : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping satisfying

$$\langle Qx, x \rangle \geq \gamma ||x||^2, \quad \forall x \in \mathbb{R}^n$$

for some constant $\gamma > 0$, $h : \mathbb{R}^n \to [a, \infty)$ for some constant $\alpha > 0$ and $q \in \mathbb{R}^n$.

Let $x, y \in \mathbb{R}^n$ be such that $\langle F(x), y - x \rangle \geq 0$, i.e., $\langle h(x)(Qx + q), y - x \rangle \geq 0$. This implies

$$\langle Qx + q, y - x \rangle \geq 0.$$

Thus,

$$\begin{aligned} \langle F(y), y - x \rangle &= \langle h(y)(Qy + q), y - x \rangle \\ &\geq h(y)[\langle Qy + q, y - x \rangle - \langle Qx + q, y - x \rangle] \\ &= h(y)\langle Q(y - x), y - x \rangle \\ &\geq \alpha\gamma ||y - x||^2. \end{aligned}$$

Hence, F is strongly pseudomonotone with modulus $\alpha\gamma$. Note that the mapping F considered in Example 1.1 is a particular case of the form (2.8). So, in general, a mapping of the form (2.8) is not strongly monotone.

Declarations

Conflict of interest The author declares that he has no conflict of interest.

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