Ricerche di Matematica (2023) 72:709–721
https://doi.org/10.1007/s11587-021-00567-3

New results on periodic solutions for second order damped vibration systems

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Received: 25 October 2020 / Revised: 28 January 2021 / Accepted: 2 February 2021 /
Published online: 10 March 2021

Abstract
The purpose of this paper is to study the multiplicity of periodic solutions for a class of non-autonomous second-order damped vibration systems. New results are obtained by using Fountain theorem. These results improve the related ones in the literature.

Keywords Damped vibration systems · Periodic solutions · Fountain theorem · Asymptotically quadratic conditions · Critical point

Mathematics Subject Classification 34C25 · 37j45 · 49j35

1 Introduction and main results

Consider the second-order damped vibration system:

$$\begin{cases} \ddot{u}(t) + A\dot{u}(t) + \nabla_u V(t, u(t)) = 0, & \forall \text{ t } \in \mathbb{R}, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, & T > 0, \end{cases} \tag{1.1}$$

where A is a skew-symmetric matrix, $V(t, u) = -K(t, u) + W(t, u)$ and $K, W \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ with conditions

$$K(t + T, u) = K(t, u), \quad W(t + T, u) = W(t, u), \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^n.$$

When $A \equiv 0$, (1.1) is just the following second order non-autonomous Hamiltonian system:

$$\begin{cases} \ddot{u}(t) + \nabla_u V(t, u(t)) = 0, & \forall \text{ t } \in \mathbb{R}, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, & T > 0. \end{cases} \tag{1.2}$$

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During the last several decades, the existence and multiplicity of periodic solutions for second-order Hamiltonian systems have been extensively studied via critical point theory, such as [2,7,8,14,15,17,20,22,24–26] and the references therein. In those papers, $V(t,u)$ was required to satisfy some growth conditions as $|u| \to +\infty$, such as asymptotically linear, subquadratic, asymptotically quadratic or superquadratic growth. In [8,20] the authors considered the case that $V(t,u)$ satisfies subquadratic potential condition. In 2009, when $V(t,u) = -K(t,u) + W(t,u)$, Zhao [27] established the existence result of system (1.2) with conditions that $W(t,u)$ is asymptotically linear and $K(t,u)$ satisfies "pinched" condition:

$$a_1|u|^2 \leq K(t,u) \leq a_2|u|^2,$$

where constants $a_1, a_2 > 0$.

In 2011, when $V(t,u) = \frac{1}{2}(L(t)u \cdot u) + W(t,u)$, where $L(t)$ is a $n \times n$ symmetric matrix, Zhang and Liu [25] considered the multiplicity of periodic solutions for system 1.2 with condition that $W(t,u)$ is asymptotically quadratic or superquadratic. In 2013, Gu and An [7] investigated the multiplicity of periodic solutions for system (1.2) with subquadratic condition. In 2018, Wang and Zhang [22] studied the existence of periodic solutions of system (1.2) with locally asymptotically quadratic condition or locally superquadratic condition. Motivated by [25,27], in this paper, firstly we generalized the above results by replacing the 'pinching' condition by the following conditions:

(H_1) There exist constants $d > 0$ and $L_1 > 0$ such that $K(t,u) \geq -d|u|^2$ for all $t \in [0,T]$ and $|u| \geq L_1$,
(H_2) There exists a constant $L_2 > 0$ such that $(\nabla K(t,u) \cdot u) \leq 2K(t,u)$ for all $t \in [0,T]$ and $|u| \geq L_2$.

Secondly, if $A \not\equiv 0$ and $V(t,u) = -K(t,u) + W(t,u)$, we discuss the case that $W(t,u)$ satisfies an asymptotically quadratic condition. By using Fountain theorem, we study the existence of infinitely many nontrivial odd T-periodic solutions of (1.1). However, the space in Fountain theorem is not a regular Sobolev space. What's more, Long (see [10]) introduced the bi-even condition and one space, which is a closed subspace of the commonly used Sobolev space H_T^1 consisting of odd functions, denoted by E in this paper. We would like to remind the readers, if $V(t,u)$ only satisfies $V(t,-u) = V(t,u)$, then the multiplicity result of periodic solutions but not necessarily odd solutions for problem (1.1) can also be obtained via Fountain theorem.

Now, we will use the following assumptions to prove our results.

(H_3) $\lim_{|u|\to+\infty}[(\nabla W(t,u) \cdot u) - 2W(t,u)] = +\infty$ uniformly for $t \in [0,T]$.
(H_4) There exist a function $b \in L^1([0,T],(0,+\infty))$ and a constant $L_3 > 0$ such that

$$0 < W(t,u) \leq b(t)|u|^2, \quad \forall t \in [0,T], \quad |u| \geq L_3.$$

(H_5) $V(t,u)$ is bi-even, which means that it is even in t and u variables respectively.
(H_6) $\|A\| \leq 1$.

Now, we are ready to state the main results of this paper as follows.

Theorem 1.1 *Assume that* $(H_1)-(H_6)$ *hold. Then problem* (1.1) *possesses infinitely many odd T-periodic solutions* $\{u_k\}$ *satisfying* $\|u_k\|_\infty \to +\infty$ *as* $k \to +\infty$.

Remark 1.1 Conditions (H_3) and (H_4) imply that $W(t,u)$ satisfies the asymptotically quadratic condition, that is

$$0 < \liminf_{|u|\to+\infty} \frac{W(t,u)}{|u|^2} \leq \limsup_{|u|\to+\infty} \frac{W(t,u)}{|u|^2} < +\infty, \quad \text{a.e.t} \in [0,T].$$

Obviously, if $K(t,u)$ is a quadratic form, then K satisfies (H_1) and (H_2). So functions K in Theorem 1.1 not only can be all quadratic forms, but also may be subquadratic (for example, $K(t,u) = \frac{|u|^2}{\ln(10+|u|^2)}$, $\forall t \in \mathbb{R}, \forall u \in \mathbb{R}^n$ or asymptotically quadratic (for example, $K(t,u) = \frac{1}{3}|u|^2 + \ln(10+|u|^2)$, $\forall t \in \mathbb{R}, \forall u \in \mathbb{R}^n$). The variant fountain theorem requires that the partial functional is nonnegative, which needs $W(t,u) \geq 0$ for all $t \in [0,T]$ and $u \in \mathbb{R}^n$. However, the Fountain theorem used in this paper requires that the functional φ is infinitely large quantity in the subspace of E, which only needs $W(t,u) > 0$ with $|u|$ large enough. Function V satisfying our assumptions of Theorem 1.1 do really exist, but may not be covered by [25, Theorem 1.1].

Now, we give an example as an application of this result.

Example 1.1 Set $T = \pi$ and define $K, G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with

$$K(t,x) = \cos^2 t . \frac{x^2}{ln(e+x^2)}$$

and

$$G(t,x) = \frac{1+\cos^2 t}{2}\left[1 - \frac{1}{ln(100+x^2)}\right]x^2.$$

Choose a function $\lambda \in C^\infty(\mathbb{R}^+, [0,1])$ such that $\lambda(t) = 1$ for $t \leq 1$, $\lambda(t) = 0$ for $t \geq 2$, and $\lambda'(t) \leq 0$ for $t \in [1,2]$. Set

$$W(t,x) = -\frac{1}{2}\lambda(|x|)x^2 + (1-\lambda(|x|))G(t,x), \quad \forall t, x \in \mathbb{R}.$$

Then functions $K, W \in C^1(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ hold and are π-periodic with respect to the variable t. Obviously, both K and W satisfy condition (V). For $K(t,x)$, one has $-|x|^2 \leq 0 \leq K(t,x) \leq |x|^2$ and

$$\begin{aligned}\frac{\partial K(t,x)}{\partial x} &= \cos^2 t . \left[\frac{2x^2}{ln(e+x^2)} - \frac{2x^4}{(e+x^2)ln^2(e+x^2)}\right] \\ &= \cos^2 t . \frac{2x^2}{ln(e+x^2)} \\ &= 2K(t,x), \quad \forall t, x \in \mathbb{R}.\end{aligned}$$

Hence, $K(t,x)$ satisfies (H_1) with $d = 1$ and (H_2) condition. It is evident that W satisfies (H_4) with $b(t) \equiv 1$ and $L_3 = 2$. Furthermore,

$$\begin{aligned}\frac{\partial W(t,x)}{\partial x}.x - 2W(t,x) &= \frac{(1+\cos^2 t)x^4}{(100+x^2)ln^2(100+x^2)} \\ &\geq \frac{x^4}{(100+x^2)ln^2(100+x^2)} \to +\infty \quad \text{as} \quad |x| \to +\infty.\end{aligned}$$

So

$$\frac{\partial W(t,x)}{\partial x}.x - 2W(t,x) = +\infty \quad \text{uniformly for} \quad t \in \mathbb{R}.$$

Hence $W(t,x)$ satisfies condition (H_3). By Theorem 1.1, problem (1.1) possesses infinitely many odd π-periodic solutions for above K and W. Obviously, W in Example 1.1 cannot be covered by conditions of [25, Theorem 1.1] because not only $K(t,x)$ is not a quadratic form, but also $W(t,x) \leq 0$ with $|x| \leq 1$.

2 Variational setting and preliminaries

As usually, we denote

$$\begin{aligned}H_T^1 = \Big\{&u : [0,T] \to \mathbb{R}^n,\ u \textit{ is absolutely continuous}, \\ &u(0) = u(T) \textit{ and } \dot{u} \in L^2([0,T],\mathbb{R}^n)\Big\}.\end{aligned}$$

Then H_T^1 is a Hilbert space with the norm

$$\|u\|_{H_T^1} = \left[\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right]^{1/2}, \quad u \in H_T^1$$

and the associated inner product

$$(u \cdot v) = \int_0^T (u(t) \cdot v(t))dt + \int_0^T (\dot{u}(t) \cdot \dot{v}(t))dt, \quad u, v \in H_T^1.$$

Let E be the subspace of H_T^1 consisting of odd functions. The corresponding norm is

$$\|u\| = \left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}} \quad \forall u \in E, \tag{2.1}$$

which is equivalent to the norm $\|u\|_{H_T^1}$ on E. Note that E is a closed subspace of H_T^1, so it is a reflective Hilbert space.

Lemma 2.1 [12] *The space H_1^T is compactly embedded in $C([0,T],\mathbb{R}^n)$. In addition,*

$$\|u\|_\infty \le C_\infty \|u\|_{H_T^1}, \quad \forall u \in H_T^1,$$

where C_∞ is a positive constant.

We consider the functional φ: $H_T^1 \to \mathbb{R}$ defined by

$$\varphi(u) = \int_0^T \left[\frac{1}{2}|\dot{u}(t)|^2 + \frac{1}{2}(Au(t)\cdot \dot{u}(t)) + K(t,u(t)) - W(t,u(t))\right]dt. \quad (2.2)$$

It is well know that φ is continuously differentiable on H_T^1 and by using the skew-symmetry of A, we have

$$(\varphi'(u)\cdot v) = \int_0^T (\dot{u}(t).\dot{v}(t))dt - \int_0^T (A\dot{u}(t)\cdot v(t))dt + \int_0^T (\nabla K(t,u(t)).v(t))dt$$
$$- \int_0^T \nabla(W(t,u(t))\cdot v(t))dt, \quad \forall u, v \in H_T^1. \quad (2.3)$$

Furthermore, a point $u \in H_T^1$ is a T-periodic solution of system (1.1) if and only if u is a critical point to the functional φ.

The following lemmas will be needed in the proof of our results.

Lemma 2.2 [10] *For $T > 0$, suppose $V \in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$, $V(t+T,u) = V(t,u)$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^n$, and V is bi-even. Then $\varphi \in C^1(E,\mathbb{R})$ and that $u \in E$ is a critical point of φ restricted to E if and only if it is an odd $C^2([0,T],\mathbb{R}^n)$-solution of problem* (1.1).

So the odd solutions of problem (1.1) correspond to the critical points of the functional $\varphi_{|E}$ via Lemma 2.2. From now on, for simplicity, we use φ for $\varphi_{|E}$. By a simple calculation, we obtain that the differential system

$$\begin{cases} -\ddot{u}(t) = \lambda u(t), & \forall t \in [0,T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

has the eigenvalues $\lambda_j = \frac{4j^2\pi^2}{T^2}(j \in \mathbb{N})$ with $\lambda_j \to +\infty$ as $j \to +\infty$ and the corresponding normalized eigenfunctions $\{e_j\}$ $(-\ddot{e}_j = \lambda_j e_j$, and e_j has the form of $a_j\cos(\sqrt{\lambda}t) + b_j\sin(\sqrt{\lambda}t)$, $a_j,\ b_j \in \mathbb{R}^n)$ with $\int_0^T (e_j(t), e_k(t))dt = \delta_{jk}$ and $\int_0^T (\dot{e}_j(t), \dot{e}_k(t))dt = \lambda_k.\delta_{jk}$ for every $j, k \in \mathbb{N}^*$, we define subspaces

$$X_j = span\{e_j\} \cap E \neq \varnothing, \quad Y_k = \bigoplus_{j=1}^{k} X_j \quad and \quad Z_k = \overline{\bigoplus_{m\ge k} X_m}, \quad (2.4)$$

then $E = \overline{\bigoplus_{j\in\mathbb{N}^*} X_j} = Y_k \oplus Z_{k+1}$. For every $u_k \in Z_k$ and $c_j \in \mathbb{R}$, $u_k = \sum_{j=k}^{\infty} c_j e_j$, then one has

$$\|u_k\|_{L^2}^2 = \sum_{j=k}^{\infty} c_j^2 \quad \text{and} \quad \|\dot{u}_k\|_{L^2}^2 = \sum_{j=k}^{\infty} c_j^2 \lambda_j. \tag{2.5}$$

Lemma 2.3 [23, Theorem 3.6] *Assume that $\varphi \in C^1(E,\mathbb{R})$ satisfies $\varphi(u) = \varphi(-u)$ and the subspace X_j, Y_k, Z_k defined in (2.4). For every $k \in \mathbb{N}^*$, there exists $\rho_k > r_k > 0$ such that*

- (A_1) $\inf_{u\in Z_k, \|u\|=r_k} \varphi(u) \to +\infty$ *as* $k \to +\infty$.
- (A_2) $\max_{u\in Y_k, \|u\|=\rho_k} \varphi(u) \le 0$,
- (A_3) φ *satisfies the* $(PS)_c$ *condition for every* $c > 0$.

If k is large enough, and set $c_k = \inf_{h\in\Gamma_k} \max_{u\in B_k} \varphi(h(u))$, where $B_k := \{u \in Y_k : \|u\| \le \rho_k\}$, $\Gamma_k := \{h \in C(B_k, E) : h \text{ is odd and } h_{|\partial B_k} = id\}$ then $c_k \ge \inf_{u\in Z_k,\|u\|=r_k} \varphi(u)$, furthermore, c_k is an unbounded sequence of critical values of φ
.

Remark 2.1 As shown in [1], a deformation lemma can be proved with the weaker (C) condition which is due to Cerami (see [3]) replacing the usual $(PS)_c$ condition, and it turns out that Lemma 2.3 holds under (C) condition.

Lemma 2.4 *Set $\beta_k := \sup_{u\in Z_k, \|u\|=1} \|u\|_\infty$ $k \in \mathbb{N}^*$, one has*

$$\beta_k \to 0 \quad as \quad k \to +\infty.$$

Proof The main idea comes from [11]. The definition of β_k implies that $0 \le \beta_{k+1} \le \beta_k$, so $\lim_{k\to+\infty} \beta_k$ does really exist. For every $k \in \mathbb{N}^*$, there exists $u_k \in Z_k \subseteq E = Z_1$ such that $\|u_k\| = 1$ and

$$\|u_k\|_\infty > \frac{1}{2}\beta_k. \tag{2.6}$$

For the above $u_k(t) = (u_k^1(t), u_k^2(t), \dots, u_k^n(t)) \in Z_k \subseteq E$, one has that $\int_0^T u_k^i(t)dt = 0$ $(i = 1, 2, \dots, n)$. From the mean value theorem, there exists a $\xi_i \in (0, T)$ such that $u_k^i(\xi_i) = 0$ $(i = 1, 2, \dots, n)$. Then one has

$$|u_k^i(t)|^2 = 2\int_\xi^t \dot{u}_k^i(s).u_k^i(s)ds \le 2\|\dot{u}_k^i\|_{L^2}\|u_k^i\|_{L^2}, \quad i = 1, 2, \dots, n, \quad \forall\ t \in [0, T],$$

which implies that

$$
\begin{aligned}
|u_k(t)|^2 &= \sum_{i=1}^{n} |u_k^i(t)|^2 \\
&\le 2\sum_{i=1}^{n} \|\dot u_k^i\|_{L^2}\|u_k^i\|_{L^2} \\
&\le 2\left(\sum_{i=1}^{n} \|\dot u_k^i\|_{L^2}\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} \|u_k^i\|_{L^2}\right)^{\frac{1}{2}} \\
&= 2\|\dot u_k\|_{L^2}\|u_k\|_{L^2}.
\end{aligned}
\tag{2.7}
$$

By (2.5), there holds

$$1 = \|u_k\|^2 = \int_0^T |\dot u_k(t)|^2 dt \ge \lambda_k \|u_k\|_{L^2}^2, \quad \forall u_k \in Z_k,$$

which implies that $\|u_k\|_{L^2} \to 0$ as $k \to +\infty$. By (2.7), one has $\|u_k\|_\infty^2 \le 2\|\dot u_k\|_{L^2}\|u_k\|_{L^2} \le 2\|u_k\|_{L^2} \to 0$ as $k \to +\infty$. So (2.6) implies that $\beta_k \to 0$ as $k \to +\infty$. □

Lemma 2.5 [16, Lemma 1] *Suppose that Ω is a Lebesgue measurable subset of $\mathbb{R}$ with meas(Ω) $< +\infty$ ('meas' denotes the Lebesgue measure) and $\{f_n(t)\}$ is a sequence of Lebesgue measurable functions such that $f_n(t) \to +\infty$ as $n \to +\infty$ for a.e. $t \in \Omega$. Then for every $\delta > 0$, there exists a subset Ω_δ with meas($\Omega\backslash\Omega_\delta$) $< \delta$ such that $f_n(t) \to +\infty$ as $n \to \infty$ uniformly for all $t \in \Omega_\delta$.*

3 Proof of Theorem 1.1

Before the proof of Theorem 1.1, we need the following lemmas.

Lemma 3.1 [4, Lemma 1] *Assume that $W(t,x)$ satisfies (H_3), $K(t,x)$ satisfies (H_2), then there exists a constant $M > L_2$ large enough such that*

$$W(t,x) \ge \frac{|x|^2}{M^2}.\min_{|x|=M} W(t,x), \quad \text{if} \quad |x| \ge M \quad \text{and} \quad t \in [0,T],$$
$$K(t,x) \le \frac{|x|^2}{L_2^2}.\max_{|x|=L_2} K(t,x), \quad \text{if} \quad |x| \ge L_2 \quad \text{and} \quad t \in [0,T].$$

Remark 3.1 Lemma 3.1 implies that there exists a function $c(t) > 0$ such that

$$W(t,x) \ge c(t)|x|^2, \quad |x| \ge M. \tag{3.1}$$

Condition (H_4) and inequality (3.1) imply that

$$c(t) \leq \frac{W(t,x)}{|x|^2} \leq b(t).$$

Then $W(t,x)$ is an asymptotically quadratic function.

Lemma 3.2 *Assume that $W(t,x)$ satisfies (H_3) and (H_4), $K(t,x)$ satisfies (H_1) and (H_2), then the functional φ satisfies the (C) condition.*

Proof Let $\{u_m\} \subset E$ be a C-sequence, that is, $\sup_{m\in\mathbb{N}^*}\{|\varphi(u_m)|\} < +\infty$ and $(1 + \|u_m\|)\|\varphi'(u_m)\| \to 0$ as $m \to +\infty$. Then there exists a constant $L > 0$ such that

$$|\varphi(u_m)| \leq L, \quad (1 + \|u_m\|)\|\varphi'(u_m)\| \leq L \quad \forall\, m \in \mathbb{N}^*. \tag{3.2}$$

We claim that $\{u_m\}$ is bounded. Otherwise, there exists a subsequence of $\{u_{m_k}\}$ such that $\|u_{m_k}\| \to +\infty$ as $k \to +\infty$, and we still denote $\{u_{m_k}\}$ by $\{u_m\}$. Set $z_m(t) = \frac{u_m(t)}{\|u_m\|}$, then $\|z_m\| = 1$. So there exists a $z \in E$ with $\|z\| \leq 1$ such that $z_m \rightharpoonup z$ in E. By Lemma 2.1, one has $z_m \to z$ in $C([0,T], \mathbb{R}^n)$ as $m \to +\infty$. We consider the following cases $z(t) \not\equiv 0$ and $z(t) \equiv 0$ respectively.

Case 1: $z(t) \not\equiv 0$. Set $\Omega := \{t \in [0,T]/|z(t)| > 0\}$, then meas $(\Omega) > 0$, By lemma 2.5 and $\|u_m\| \to +\infty$ as $m \to +\infty$, there exists a subset $\Omega_{\delta_0} \subseteq \Omega$ with meas $(\Omega_{\delta_0}) > 0$ and meas $(\Omega \backslash \Omega_{\delta_0}) < \delta_0$ such that

$$|u_m(t)| = \|u_m\|.|z_m(t)| \to +\infty \text{ uniformly } \forall t \in \Omega_{\delta_0} \text{ as } m \to +\infty. \tag{3.3}$$

It follows from (H_3) that there exists a constant $M_1 > 0$ large enough such that

$$(\nabla W(t,x).x) - 2W(t,x) \geq -M_1, \ \forall (t,x) \in [0,T] \times \mathbb{R}^n. \tag{3.4}$$

By (H_2), there exists a constant $M_2 > 0$ large enough such that

$$2K(t,x) - (\nabla K(t,x).x) \geq -M_2, \ \forall (t,x) \in [0,T] \times \mathbb{R}^n. \tag{3.5}$$

By (H_3) and (3.3), one has

$$\int_{\Omega_{\delta_0}} [(\nabla W(t,u_m(t)).u_m(t)) - 2W(t,u_m(t))]\,dt \to +\infty, m \to +\infty. \tag{3.6}$$

According to (2.3), (3.2), (3.4), (3.5) and (3.6), there holds

$$
\begin{aligned}
3L &\geq 2\varphi(u_m) - (\varphi'(u_m).u_m) \\
&= \int_0^T [(\nabla W(t, u_m(t)).u_m(t)) - 2W(t, u_m(t))]\,dt \\
&\quad + \int_0^T [2K(t, u_m(t)) - (\nabla K(t, u_m(t)).u_m(t))]\,dt \\
&= \int_{\Omega_{\delta_0}} [(\nabla W(t, u_m(t)).u_m(t)) - 2W(t, u_m(t))]\,dt \\
&\quad + \int_{[0,T]\setminus\Omega_{\delta_0}} [(\nabla W(t, u_m(t)).u_m(t)) - 2W(t, u_m(t))]\,dt \\
&\quad + \int_0^T [2K(t, u_m(t)) - (\nabla K(t, u_m(t)).u_m(t))]\,dt \\
&\geq \int_{\Omega_{\delta_0}} [(\nabla W(t, u_m(t)).u_m(t)) - 2W(t, u_m(t))]\,dt - M_1T - M_2T \\
&\to +\infty, m \to +\infty,
\end{aligned}
$$

which yields a contradiction.

Case 2: $z(t) \equiv 0$. From (H_6), (2.1) and (2.2), one has

$$
\begin{aligned}
&\int_0^T W(t, u_m(t))dt - \int_0^T K(t, u_m(t))dt \\
&\quad = \frac{1}{2}\|u_m\|^2 + \frac{1}{2}\int_0^T (Au_m(t).\dot{u}_m(t))dt - \varphi(u_m).
\end{aligned}
$$

Divided by $\|u_m\|^2$ on both sides, together with (3.2), one has

$$\frac{1}{2}(1 - \|A\|) \leq \int_0^T \frac{W(t, u_m(t)) - K(t, u_m(t))dt}{\|u_m\|^2}dt \leq \frac{1}{2}(1 + \|A\|). \tag{3.7}$$

in the meantime, condition (H_1) implies that

$$K(t, x) \geq -d|x|^2 - \widetilde{M}, \forall(t, x) \in [0, T] \times \mathbb{R}^n. \tag{3.8}$$

where $\widetilde{M} = \max_{t\in[0,T]}\{\max_{|x|\leq L_1} |K(t, x)|\} > 0$. By (H_4), one has

$$W(t, x) \leq b(t)|x|^2 + \overline{M}, \forall(t, x) \in [0, T] \times \mathbb{R}^n, \tag{3.9}$$

where $\overline{M} = \max_{t\in[0,T]}\{\max_{|x|\leq L_3} |W(t, x)|\} > 0$.

According to (3.8), (3.9) and $z(t) \equiv 0$, there holds

$$\begin{aligned}\int_0^T \frac{W(t,u_m(t)) - K(t,u_m(t))}{\|u_m\|^2}dt &= \int_0^T \frac{b(t)|u_m(t)|^2 + d|u_m(t)|^2 + \overline{M} + \widetilde{M}}{\|u_m\|^2}dt \\ &\le \|z_m\|_\infty^2 . \int_0^T (b(t)+d)dt + \frac{(\overline{M}+\widetilde{M})T}{\|u_m\|^2} \\ &\to 0, m \to +\infty.\end{aligned}$$

which contradicts (3.7). Hence $(u_m)_{m\in\mathbb{N}}$ is bounded in E. By Proposition 4.3 in [12] we can assume that $\{u_m\}_{m\in\mathbb{N}}$ has a convergent subsequence in E. Hence φ satisfies the C condition. The proof is complete. □

Lemma 3.3 *If $W(t,x)$ satisfies (H_3) and (H_4), $K(t,x)$ satisfies (H_1) and (H_2) and (H_6) satisfied, then the functional φ satisfies (A_1) in Lemma 2.3.*

Proof Take $r_k = \beta_k^{-1}$, then Lemma 2.4 implies that $r_k \to +\infty$ as $k \to +\infty$. By Lemma 3.1, one has

$$K(t,x) \le M_3|x|^2 + M_4, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n, \tag{3.10}$$

where constants $M_3 = \frac{\max_{t\in[0,T]}\left\{\max_{|x|=L_2}|K(t,x)|\right\}}{L_2^2} > 0$, $M_4 = \max_{t\in[0,T]}$ $\left\{\max_{|x|\le L_2}|K(t,x)|\right\} > 0$.

By Lemma 3.1, for a certain constant $\sigma > \max\{L_2, L_3\}$ large enough, one has

$$W(t,x) \ge \frac{|x|^2}{\sigma^2}. \min_{|x|=\sigma} W(t,x), \quad |x| \ge \sigma \quad \forall t \in [0,T]. \tag{3.11}$$

Hence, by (H_4) and (3.11), one has

$$W(t,x) \ge B|x|^2 - C, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n, \tag{3.12}$$

where constants $B = \frac{\min_{t\in[0,T]}\left\{\min_{|x|=\sigma} W(t,x)\right\}}{\sigma^2} > 0$, $C = B\sigma^2 + \max_{t\in[0,T]}$ $\left\{\max_{|x|\le\sigma}|W(t,x)|\right\} > 0$.

From (3.8), (3.9), (3.10) and (3.12), there exists a function $\widetilde{b} \in L^1([0,T],\mathbb{R}_+)$ such that

$$|K(t,x)| + |W(t,x)| \le \widetilde{b}(t)|x|^2 + M_0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n, \tag{3.13}$$

where $\widetilde{b}(t) = b(t) + B + \max\{M_3, d\} > 0$, $M_0 = \max\{M_4, \widetilde{M}\} + \max\{C, \overline{M}\} > 0$. For $u_k \in Z_k \subseteq E$ with $\|u_k\| = r_k$, set $z_k(t) = \frac{u_k(t)}{\|u_k\|}$, then $\|z_k\| = 1$. By the definition of β_k, one has $\|z_k\|_\infty \le \beta_k$, which implies that $\|u_k\|_\infty \le \beta_k\|u_k\| = \beta_k.r_k = 1$. It follows from (H_6), (2.1) and (3.13) that

$$
\begin{aligned}
\varphi(u_k) &= \frac{1}{2}\int_0^T |\dot{u}_k(t)|^2 dt + \frac{1}{2}\int_0^T Au_k(t).\dot{u}_k(t)dt + \int_0^T K(t,u_k(t))dt \\
&\quad - \int_0^T W(t,u_k(t))dt \\
&\geq \frac{1}{2}\|u_k\|^2 - \frac{1}{2}\int_0^T A\dot{u}_k(t).u_k(t)dt - \int_0^T [|K(t,u_k(t))| + |W(t,u_k(t))|]dt \\
&\geq \frac{1}{2}\|u_k\|^2(1-\|A\|) - \|u_k\|_\infty^2 . \int_0^T \widetilde{b}(t)dt - M_0 T \\
&\geq \frac{1}{2}(1-\|A\|)r_k^2 - \int_0^T \widetilde{b}(t)dt - M_0 T,
\end{aligned}
$$

which implies that $\inf_{u\in Z_k,\|u\|=r_k} \varphi(u) \to +\infty$ as $k \to +\infty$. □

Lemma 3.4 *If $W(t,x)$ satisfies (H_3) and (H_4), $K(t,x)$ satisfies (H_2) and (H_6) satisfied, then the functional φ satisfies (A_2) in Lemma 2.3.*

Proof For every $k \in \mathbb{N}^*$, Y_k is a finite dimensional space, so there exists a constant $d_k > 0$ such that

$$
\|u_k\|_{L^2} \geq d_k\|u_k\|, \quad \forall u_k \in Y_k. \tag{3.14}
$$

By (3.12), $\forall \alpha \in (0,2)$, one has

$$
\frac{W(t,x)}{|x|^\alpha} \geq B|x|^{2-\alpha} - C|x|^{-\alpha} \to +\infty \text{ uniformly } \forall t \in [0,T] \text{ as } |x| \to +\infty. \tag{3.15}
$$

Then (3.15) implies that there exists a certain constant $L_4 \geq \sigma^{2-\alpha}(M_3 + \frac{2}{d_k^2}\}$ large enough such that

$$
\min_{|x|=\sigma} W(t,x) > L_4\sigma^\alpha, \quad \forall t \in [0,T]. \tag{3.16}
$$

By (3.11) and (3.16), there exists a constant $M_5 > 0$ such that

$$
W(t,x) > L_4\sigma^{\alpha-2}|x|^2 - M_5, \quad \forall (t,x) \in [0,T]\times\mathbb{R}^n. \tag{3.17}
$$

where $M_5 = L_4\sigma^\alpha + \max_{t\in[0,T]}\{\max_{|x|\leq\sigma}|W(t,x)|\} > 0$. For every $u_k \in Y_k$ with $\|u_k\| = \rho_k(\rho_k > r_k$ is determined later), by (H_6), (3.10), (3.14) and (3.17,) there holds

$$\varphi(u_k) = \frac{1}{2}\int_0^T |\dot{u}_k(t)|^2 dt + \frac{1}{2}\int_0^T Au_k(t).\dot{u}_k(t)dt + \int_0^T K(t, u_k(t))dt \tag{3.18}$$
$$- \int_0^T W(t, u_k(t))dt$$
$$\leq \frac{1}{2}\|u_k\|^2 + \frac{1}{2}\|A\|\|u_k\|^2 + M_3\|u_k\|_{L^2}^2 - L_4\sigma^{\alpha-2}\|u_k\|_{L^2}^2 + M_6$$
$$\leq \left[\frac{1}{2}(1 + \|A\|) - (L_4\sigma^{\alpha-2} - M_3)d_k^2\right]\|u_k\|^2 + M_6$$
$$\leq -\rho_k^2 + M_6, \tag{3.19}$$

where $M_6 = (M_4 + M_5)T > 0$. Therefore, if $\rho_k > \max\{r_k, \sqrt{2M_6}\}$ large enough, then (3.18) implies that $\max_{u\in Y_k, \|u\|=\rho_k} \varphi(u) < 0$. □

Proof of Theorem 1.1 In view of Lemma 2.2, $\varphi \in C^1(E, \mathbb{R})$ holds. Condition (H_5) shows that $\varphi(-u) = \varphi(u)$. Lemma 2.3 and Lemmas 3.2–3.4 imply that φ possesses a sequence of critical points $\{u_k\}$ such that

$$\varphi'(u_k) = 0 \quad \text{and} \quad c_k = \varphi(u_k) \to +\infty \quad \text{as} \quad k \to +\infty. \tag{3.20}$$

As is well known, $u \in E$ is a weak solution of problem (1.1) which corresponds to the critical points of the functional φ. Hence by Lemma 2.2, u is an odd classical solution of problem (1.1). Next, we claim that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$. If not, then there exists a constant $M_7 > 0$ such that

$$\varphi'(u_k) = 0 \quad \text{and} \quad \|u_k\|_\infty \leq M_7, \quad \forall k \in \mathbb{N}^*. \tag{3.21}$$

By a simple calculation, $K, W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and (3.21), there exists a constant $M_8 > 0$ independent of k such that

$$\varphi(u_k) - \frac{1}{2}(\varphi'(u_k), u_k) = \int_0^T \left[K(t, u_k(t)) - \frac{1}{2}(\nabla K(t, u_k(t)), u_k(t))\right] dt$$
$$- \int_0^T \left[W(t, u_k(t)) - \frac{1}{2}(\nabla W(t, u_k(t)), u_k(t))\right] dt$$
$$\leq M_8, \quad \forall k \in \mathbb{N}^*,$$

which contradicts $\varphi(u_k) - \frac{1}{2}(\varphi'(u_k), u_k) = c_k \to +\infty$ via (3.20). The proof is complete. □

Acknowledgements The author would like to express their sincere thanks to the referees for their valuable comments and suggestions.

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