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Stability and existence results for a class of nonlinear parabolic equations with three lower order terms and measure data using Lorentz spaces

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Abstract

We study both existence and stability of *renormalized* solutions for nonlinear parabolic problems with three lower order terms that have, respectively, growth with respect to u and to the gradient, whose model

$$(\mathcal{P})\quad \begin{cases} u_t - \Delta_p u - \operatorname{div}[c(t,x)|u|^{\gamma-1}u] + b(t,x)|\nabla u|^{\lambda} + d(t,x)|u|^{\iota} = \mu - \operatorname{div}(E) & \text{in } Q,\\ u(0,x) = u_0(x) \quad \text{in } \Omega,\ u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases}$$

where $Q := (0,T)\times\Omega$ (with Ω is an open bounded subset of $\mathbb{R}^N$ ($N \geq 2$) and $T > 0$), $1 < p < N$, Δ_p is the usual p-Laplace operator, and $\mu \in \mathbf{M}(Q)$ is a (general) measure with bounded total variation on Q. As a consequence of our main results, we prove that the conditions $\gamma = \frac{(N+2)(p-1)}{N+p}$, $\lambda = \frac{N(p-1)+p}{N+2}$, $0 \leq \iota \leq p - \frac{N-p}{N}$, $c \in L^{\tau=\frac{N+p}{p-1}}(Q)^N$, $b \in L^{N+2,1}(Q)$ and $d \in L^{z',1}(Q)$ (with $z = \frac{pN-N-p}{\iota N}$) are necessary and sufficient for the existence and the stability of solutions for every sufficiently regular $u_0 \in L^2(\Omega)$, $E \in L^{p'}(Q)^N$ and irregular $\mu \in \mathbf{M}(Q)$.

Keywords Capacity · Noncoercive Cauchy problems · Parabolic PDEs · Regularity · Existence · Singular measure

Mathematics Subject Classification 35B45 · 35B65 · 35B35 · 28A12 · 35R06 · 46E30

Résumé

Résultats de stabilité et d'existence pour une classe d'équations non-linéaires paraboliques avec trois termes d'ordre inférieur et une donnée mesure utilisant des espaces de Lorentz. Nous étudions l'existence et la stabilité des solutions *renor-*

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malisées pour des problèmes paraboliques non-linéaires avec trois termes d'ordre inférieur qui ont, respectivement, une croissance par rapport à u et au gradient, de modèle ($\mathcal{P}$), où Ω est un sous-ensemble ouvert borné de $\mathbb{R}^N$ ($N \geq 2$), $T > 0$, $1 < p < N$, Δ_p est l'opérateur usuel p-Laplacien, et $\mu \in \mathbf{M}(Q)$ est une mesure (générale) avec une variation totale dans Q. Comme conséquence de nos résultas, nous montrons que les conditions $\gamma = \frac{(N+2)(p-1)}{N+p}$, $\lambda = \frac{N(p-1)+p}{N+2}$, $0 \leq \iota \leq p - \frac{N-p}{N}$, $c \in L^{\tau=\frac{N+p}{p-1}}(Q)^N$, $b \in L^{N+2,1}(Q)$ et $d \in L^{z',1}(Q)$ (avec $z = \frac{pN-N-p}{\iota N}$) sont nécessaires et suffisantes pour l'existence et la stabbilité des solutions pour $u_0 \in L^2(\Omega)$, $E \in L^{p'}(Q)^N$ suffisament réguliers et $\mu \in \mathbf{M}(Q)$ irrégulière.

Contents

Version française abrégée

Dans le domaine des EDP's beaucoup de travaux sont focalisés sur le cas des problèmes elliptiques et paraboliques à données mesures. Les modèles des EDP's "classiques" définissent l'importance de la notion de capacité par rapport à la décomposition de la donnée en utilisant des mesures telles que les mesures "diffuses" ou "singulières". Celles-ci déterminent l'importance de la décomposition en fonction de l'apparition des termes définis dans le problème approché. Cependant, ces méthodes ne permettent pas de vérifier si les solutions délivrées par le problème sont uniques, ou, si les nouveaux termes tels que les "termes d'ordre inférieur" sont bien définis. Nous traitons dans cet article un de ces problèmes à donnée mesure avec trois terms d'ordre inférieur. L'idée c'est de mettre des approximations adaptées lorsque ces nouveaux termes apparaissent. Nous devons pour cela déterminer: Quelles écritures de solutions permettant d'obtenir des meilleures approximations du problème en question pour retrouver la solution du problème initial, et définir la méthode permettant d'établir, et d'estimer, le lien entre les solutions contenues dans le problème approché et les solutions du problème de base. Cette problématique mathématique s'inscrit donc parfaitement dans la problématique posée dans ce travail avec différentes phases que l'on peut retrouver comme la recherche des *estimations a priori*, l'extraction des *convergences* et le *passage à la limite*. Nous citons dans la suite les contributions scientifiques qui ont été apportées jusqu'à aujourd'hui par différents auteurs ainsi que les points que nous avons traité. Notons qu'un grand nombre d'articles a déjà été consacré à l'étude d'existence et de stabilité des solutions des problèmes paraboliques sous des multiples hypothèses et des différents contextes: pour plus de détails sur des résultats classiques, voir [1,83,86,94] et les références incluses. Plus récemment, dans [2–4], nous avons considéré le cas des opérateurs monotones non-linéaires et des termes d'ordre inférieur (avec une croissance par rapport au gradient); en particulier, dans [2], nous avons étudié le comportement asymptotique des solutions pour des opérateurs paraboliques non-linéaires, un terme de croissance naturelle et une mesure positive μ absolument continue par rapport à la p-capacité parabolique. Ici, nous analysons le cas d'un problème à trois termes d'ordre inférieur avec des mesures générales, éventuellement singulières, et sans condition de signe sur les données. Nous nous intéressons à l'étude d'existence et de stabilité, dans un sens convenable, des solutions généralisées (*renormalisées*) pour une classe d'équations non linéaires et non coercives de modèle $(\mathcal{P})$. Notre résultat principal sera:

Theorem 0.1 *Supposant que $a(t,x,s,\zeta)$, $K(t,x,s)$, $H(t,x,s,\zeta)$, $G(t,x,s)$, u_0, E et μ satisfont les hypothèses* (1.4)–(1.11). *Donc, pour tout $p > 1$, il existe une solution renormalisée du problème $(\mathcal{P})$.*

1 Introduction

A large number of papers has been devoted to the study of *existence/stability* of solutions for parabolic problems under different assumptions and various contexts: for a review on classical results see [1,83,86,94], and references therein. More recently, in [2–4], the case of nonlinear monotone operators with nonlinear lower order terms (having growth with respect to the gradient) have been considered; in particular, in [2], we deal with nonnegative measures μ *absolutely continuous* with respect to the parabolic *p-capacity* (the so called "*diffuse*" measures). Here we analyze the case of three nonlinear lower order terms with general, possibly *singular*, measure data with no sign assumption. We are interested in the study of existence and stability, in a suitable sense, of "generalized" (*renormalized*) solutions of a class of nonlinear and "*noncoercive*" parabolic equations whose model is

$$\begin{cases} u_t - \Delta_p u - \operatorname{div}[c(t,x)|u|^{\gamma-1}u] + b(t,x)|\nabla u|^{\lambda} + d(t,x)|u|^{\iota} = \mu - \operatorname{div}(E) \quad \text{in } Q, \\ u(0,x) = u_0(x) \quad \text{in } \Omega, \quad u(t,x) = 0 \text{ on } (0,T) \times \partial\Omega, \end{cases} \tag{1.1}$$

where $Q := (0,T) \times \Omega$ (with $\Omega \subset \mathbb{R}^N$ is a bounded regular domain and $T > 0$), $c(t,x) \in L^{\tau}(Q)^N$ with $\tau = \frac{N+p}{p-1}$, $\gamma = \frac{(N+2)(p-1)}{N+p}$, $b(t,x)$ belongs to the *Lorentz* space $L^{N+2,1}(Q)$ with $\lambda = \frac{N(p-1)+p}{N+2}$, $d(t,x)$ belongs to the *Lorentz* space $L^{z',1}(Q)$ with $z = \frac{pN-N-p}{\iota N}$ $(0 \le \iota \le p - \frac{N-p}{N})$, the *initial* datum $u_0 \in L^2(\Omega)$, $E \in L^{p'}(Q)^N$ and μ is a general, possibly *singular*, data satisfying some hypothesis that we will specify later. In the case where $b \equiv c \equiv d \equiv 0$, this parabolic equation appears in the weak theory where it is known as the *Boccardo–Gallouët* equation [21] for $p > 2-\frac{1}{N+1}$, see also [18]. A modification of the problem above is studied by *Porzio* as a model for some *noncoercive* equations, see [88] for nonconstant b (with $c \equiv d \equiv 0$) and L^1-data. Existence results for problem (1.1) in the whole $Q = (0, T) \times \Omega$, where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$, $T > 0$, with a regular data u_0 and μ is a general *Radon* measure is well-known, we refer to [96], where equations of the form

$$u_t - \Delta_p u = \mu \quad \text{in } Q, \tag{1.2}$$

are studied; we refer also to the paper [98], where problem (1.2) is studied in the framework of "*duality*" solutions and where some regularity properties of the solutions are obtained in that case, i.e., $u \in L^q(0, T; W_0^{1,q}(\Omega))$ for every $q < \frac{N-2}{N+1}$. It is not difficult to obtain an existence result for problem (1.1) in the case where the data are bounded: it suffices to use an "*approximate*" technique with problems having regular data, and using "*compactness*" arguments, also known, as "*distributional*" approach, to transform the equation into a regularized problem (a weak one if the data $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$) which can then be solved by *Leray–Lions*'s methods. In the case where the data is "*more general*", or in the case where the data is singular, this weak/distributional approach can not be done due to the lack of regularity of the solutions (the weak/distributional formulation is not strong enough to provide uniqueness as it can be proved by adapting the counterexample of *J. Serrin* [99], for stationary problem, to the parabolic case [94]). But, this approach can be replaced with the use of dependent-solution test functions whose role is again to overcome the lack of regularity and to deal with the unbounded terms appearing in distributional formulation, see [85,98] for the linear case. The case where the "*Laplace*" operator is replaced by a nonlinear operator like the "*p-Laplace*" operator has been studied in [5,34,84], and some references therein, where test functions of the form $S(u)\varphi$, where $S \in W^{2,\infty}(\mathbb{R})$ is such that S' has compact support on $\mathbb{R}$, $S(0) = 0$, and $\varphi \in C_c^\infty(Q)$ are considered. We first show that there is a natural notion of *generalized* solution, of problem (1.1) above, in spite of its nonlinear characters. A major consequence of this notion is that, if the range of p is greater than $\frac{2N+1}{N+1}$, then the solution is such that $|\nabla u|^{p-1} \in L^q(Q)$ for all $q < 1 + \frac{1}{(N+1)(p-1)}$ (even if its gradient may not belong to any Lebesgue space); we prove that there exists exactly a *generalized* (very weak) solution u of the problem such that the *truncation*[1] function $T_k(u) \in L^\infty(0, T; L^1(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ for every $k > 0$, and the energy of the solution u, where it is "*large*", goes to construct the *singular* (concentrated) part of the measure μ (with respect to the (b, p)-*capacity*). Moreover, using a family of real bounded Lipschitz continuous functions on $\mathbb{R}$, one can justifies, in some sense, that the absolutely continuous (*diffuse*) part of μ in the distributional formulation is well-defined. Finally, a first main result is proved, as we give compactness results (depending on the data of the problem) to ensure that the solution of problems (1.1) are, in fact, stable when passing to the limit. We do not consider only the model problem (1.1),

[1] One of the principal tools which will be used to define solutions: let $G_k(v) = (|v| - k)_+ \text{sign}(v)$ be the level-set function, then for any $k > 0$, the truncation function T_k is defined by $T_k(v) = v - G_k(v) = \max\{-k, \min\{k, v\}\}$ (see if necessary, the Sect. 2.1 for more details).

but we prove the *existence/stability* results for the general form

$$\begin{cases} u_t - \mathrm{div}[a(t,x,u,\nabla u) - K(t,x,u)] + H(t,x,u,\nabla u) + G(t,x,u) \\ \quad = \mu - \mathrm{div}(E) \text{ in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega, \quad u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{1.3}$$

where the vector field $a\colon (0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}^N$ is a *Carathéodory*[2] function such that for every $s\in\mathbb{R}$, $\zeta,\zeta'\in\mathbb{R}^N$ with $\zeta\neq\zeta'$, the following properties

$$\begin{cases} a(t,x,s,\zeta)\zeta \geq \alpha|\zeta|^p, & (1.4)\\ |a(t,x,s,\zeta)| \leq c[a_0(t,x) + |s|^{p-1} + |\zeta|^{p-1}], & (1.5)\\ \big[a(t,x,s,\zeta) - a(t,x,s,\zeta')\big]\cdot(\zeta-\zeta') > 0, & (1.6) \end{cases}$$

holds for two fixed positive constants $c,\alpha>0$, and a nonnegative function $a_0(t,x)\in L^{p'}(Q)$. Hence the nonlinear parabolic operator is defined on $L^p(0,T;W_0^{1,p}(\Omega))$, and $\mathcal{L}u := -\mathrm{div}[a(t,x,u,\nabla u)]$ maps $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual space $L^{p'}(0,T;W^{-1,p'}(\Omega))$ surjectively, see [67,69]. As the lower order terms are concerned, $K(t,x,s)\colon (0,T)\times\Omega\times\mathbb{R}\mapsto\mathbb{R}$ is a *Carathéodory* function that satisfy assumption

$$\begin{cases} |K(t,x,s)| \leq c_0(t,x)|s|^\gamma + c_1(t,x), \\ c_0(t,x)\in L^\tau(Q)^N, \quad c_1\in L^{p'}(Q), \\ \text{with } \tau = \dfrac{N+p}{p-1} \text{ and } \gamma = \dfrac{N+2}{N+p}(p-1), \end{cases} \tag{1.7}$$

for almost every $(t,x)\in Q$, and for every $s\in\mathbb{R}$. Moreover, $H\colon (0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^N\mapsto\mathbb{R}$ and $G\colon(0,T)\times\Omega\times\mathbb{R}\mapsto\mathbb{R}$ are *Carathéodory* functions satisfying

$$\begin{cases} |H(t,x,s,\zeta)| \leq b_0(t,x)|\zeta|^\lambda + b_1(t,x), \\ \lambda = \dfrac{N(p-1)+p}{N+2}, \; b_0\in L^{N+2,1}(Q), \; b_1\in L^1(Q), \end{cases} \tag{1.8}$$

and

$$\begin{cases} G(t,x,s)s \geq 0, \\ G(t,x,s) \leq d_1(t,x)|s|^\iota + d_2(t,x), \\ 0\leq\iota\leq p-\dfrac{N-p}{N}, \; d_i\in L^{z',1}(Q), \; d_2(t,x)\in L^1(Q), \\ \text{with } z = \dfrac{pN-N-p}{N}\dfrac{1}{\iota} \text{ and } \dfrac{1}{z}+\dfrac{1}{z'} = 1, \end{cases} \tag{1.9}$$

[2] $a(\cdot,\cdot,s,\zeta)$ is measurable on Q for every (s,ζ) in $\mathbb{R}\times\mathbb{R}^N$, and $a(t,x,\cdot,\cdot)$ is continuous on $\mathbb{R}\times\mathbb{R}^N$ for a.e. (t,x) in Q.

for almost every $(t, x) \in Q$ and for every $\zeta \in \mathbb{R}^N$. Finally, μ is a (general) measure in $\mathbf{M}_b(Q)$ decomposed as

$$\mu = \mu_{d,p} + \mu^+_{c,p} - \mu^-_{c,p}, \tag{1.10}$$

according to the *Radon–Nikodym* and to the *Banach* decompositions, and

$$E \in L^{p'}(Q)^N, \quad u_0 \in L^1(\Omega). \tag{1.11}$$

It is worth pointing that problem (1.3) has two main features: firstly; since the standard subjectivity theorem for *Leray–Lions* operators can not be applied, we should reason by means of the approximate theory introduced in [51,72,84,95], by using truncations of solutions in order to get a *pseudo-monotone* and *coercive* differential operator in $L^p(0, T; W_0^{1,p}(\Omega))$, then establish some a priori estimates on u, $T_k(u)$ and ∇u. Thus, a technical result on the a.e. convergence of gradients leads to pass to the limit. Secondly, the right-hand side μ of problem (1.3), which contains a measure term, is not an element of the dual space $L^{p'}(0, T; W^{-1,p'}(\Omega))$, therefore the solution can not be expected to belong to the energy space $L^p(0, T; W_0^{1,p}(\Omega))$, so it is necessary to change the functional setting in order to prove existence/stability results; to overcome this problem, a concept of "*generalized*" solution should be considered in this specific class. Now, we have to specify what we mean by "*generalized solution*"; let us recall that for equations with singular datum (say $L^1(Q)$, or more in general, measures), several notions of solutions have been introduced. A notion of "*renormalized*" solution when μ is a diffuse measure was introduced in [54], and in the same paper the existence and uniqueness of such a solution are proved (when $b \equiv c \equiv 0$). In [53], a similar notion of "*entropy*" solution is also defined and proved to be equivalent to that of renormalized solution. A new definition of "*renormalized-entropy*" solution which, in contrast with the previous ones, is closer to the one used for conservation laws in [17] and to the one existing in the elliptic case in [50], is established in [90,91]. The case of general measure in established in a similar way in [84,89]. The main idea in such formulations is to move the attention from the solution u to its truncations $T_k(u)$ and to use a "truncated" version of the equation. This is advantageous both in order to obtain a priori estimates and because requiring $T_k(u)$ to belong to the energy space allows to get informations about the solution. Observe that the above problem does not admit, in general, a solution in the sense of distribution since we can not expect to have the fields $a(t, x, u, \nabla u)$, $K(t, x, u)$ in $L^1_{\mathrm{loc}}(Q)^N$ and $H(t, x, u, \nabla u)$ in $L^1_{\mathrm{loc}}(Q)$. Indeed the last assumptions are in fact crucial in order to obtain renormalized solution that allow a priori estimates to hold true. In the present paper, we shall consider the same test functions to deal with questions of regularity, existence and stability of "*generalized*" solutions of problems of the form (1.3). The first point of the paper is devoted to the case $b \equiv d \equiv 0$ by developing technical a priori estimates for u and its gradients in appropriate "*Lorentz*"[3] spaces, which appear in the generalized formulation, and which simply the proof for the general case ($b \not\equiv 0$). In this first case, we will state all necessary assumptions which seemed to be important to obtain existence of a solution.

[3] We refer to Sect. 2.4 for more details on *Lorentz* spaces and their properties.

More precisely, we will show that the solution u and its gradients ∇u satisfy

$$\begin{cases} u^{\frac{N(p-1)+p}{N+p}} \in L^{\frac{N+p}{N},\infty}(Q), \quad |\nabla u|^{\frac{N(p-1)+p}{N+2}} \in L^{\frac{N+2}{N+1},\infty}(Q), \quad \forall p > 1, \\ \left\| |u|^{\frac{N(p-1)+p}{N+p}} \right\|_{L^{\frac{N+p}{N},\infty}(Q)} \leq C(N,p)\left[M + |Q|^{\frac{Np}{N+2}} L^{\frac{N(p-1)+p}{(N+2)p}} \right], \\ \left\| |\nabla u|^{\frac{N(p-1)+p}{N+2}} \right\|_{L^{\frac{N+2}{N+1},\infty}(Q)} \leq C(N,p)\left[M + |Q|^{\frac{N}{(N+2)p}} \right] L^{\frac{N(p-1)+p}{(N+2)p}}, \end{cases} \tag{1.12}$$

where M, L are constants to be defined, see [44,46]. The result (1.12) resembles the corresponding one for elliptic equations with Radon measure term, proved by authors in [28], and has in difference with it the presence of the time derivative of u in the parabolic equation and which applies a modification in the "*control*" of u with respect to p. More precisely, these main estimates coming from using "*Gagliardo–Nirenberg's*" inequality, lead to a control of $|u|^{\frac{N(p-1)+p}{N+p}}$ with respect to the lower order terms and to the boundedness of the right-hand side. Moreover, as in the elliptic case, no regularity/sign-condition on the datum μ are assumed, only $\mu \in \mathbf{M}(Q)$, space of Radon measures, is required. However, the proof of the parabolic result is more complicated since one has to estimate the term with time derivative of u. Then, we proceed in performing a precise analysis in what happens in the pioneering works [5,84] (see also [89] for a different approach); particularly, we do not assume that the right-hand side μ admits a derivative part (with respect to the time variable). The first difficulty is to obtain some a priori estimates for $|\nabla u|^{\lambda}$ (where u is the solution of problem (1.3), this is done by proving uniform estimates of $|u|^{\frac{N(p-1)+p}{N+p}}$, see [88], which allow to obtain an estimate of u in $L^{\infty}(0,T;L^1(\Omega))$, and an estimate of the *truncate* functions $T_k(u)$ in $L^{\infty}(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$, of the type

$$\sup_{t\in[0,T]} \int_{\Omega} |T_k(u)(t)|^2 dx + \int_Q |\nabla T_k(u)|^p dxdt \leq Mk + L, \quad \forall k > 0, \tag{1.13}$$

for some positive constants M and L. We will show using the strict monotone character of the vector filed "a", a "*generalized*" stability result (this main feature of this "generalized" result is due to the term $-\mathrm{div}[c(t,x)|u|^{\gamma-1}u]$, which, in general, does not belong to the dual space $L^{p'}(0,T;W^{-1,p'}(\Omega)))$. Since $p \leq N$, we have $L^{p'}(Q)^N \subset L^{\frac{N+2}{N+1},\infty}(Q)^N$, which implies that $-\mathrm{div}[c(t,x)|u|^{\gamma-1}u]$ does not in general belong to $L^{p'}(Q)^N$, which can be proved by using the same arguments of [84], but actually the given proof is slightly, and is inspired by the works [61,72,73], by using a direct correspondence between solutions of problem (1.1), and solutions of "*truncation*" problems with measure data, that is, by considering the following truncation nonlinear problems

$$\begin{cases} T_k(u)_t - \mathrm{div}[a(t,x,T_k(u),\nabla T_k(u)) - K(t,x,T_k(u))] + H(t,x,T_k(u),\nabla T_k(u)) \\ \quad + G(t,x,T_k(u)) = \mu - \mathrm{div}(E) \text{ in } Q := (0,T)\times\Omega, \\ T_k(u(0,x)) = T_k(u_0)(x) \text{ in } \Omega, \; T_k(u(t,x)) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{1.14}$$

where $a\colon (0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}^N$, $K\colon (0,T)\times\Omega\times\mathbb{R}\mapsto\mathbb{R}^N$, $H\colon (0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^N\mapsto\mathbb{R}$ and $G\colon (0,T)\times\Omega\times\mathbb{R}\mapsto\mathbb{R}$ are *Carathéodory*[4] functions, satisfying, respectively, the assumptions (1.4)–(1.9). Here $T_k(u_0)\in L^1(\Omega)$, $E\in L^{p'}(Q)^N$ and μ is a, possibly singular, measure data with respect to the parabolic capacity (this coincides exactly with the stability result of [5] when the terms $-\text{div}[K(t,x,u)]$ and $H(t,x,u,\nabla u)$ and $G(t,x,u)$ does not appear). Recall that the stability result is sometimes rather called "*weak-stability*" in the theory of renormalized solutions introduced by *Diperna and Lions* [48,49], this terminology is sometimes used to prove global existence and weak stability for some large-data Cauchy problems, it consists in proving that the sequence of solutions which satisfy only the physically natural a priori bounds converge weakly in L^1 to a solution where we are able to deduce a global existence result of the solution, and which allows to overcome the lack of strong a priori estimates (in the context of renormalized solutions: we shall prove that sequences of classical solutions of the Cauchy problem with uniform a priori bounds obtained from the standard physical identities associated with the equation converge weakly in L^1 to a renormalized solution of the problem in order to deduce an existence of a global renormalized solution), this new approach is different and is based on general techniques giving the possibility of writing the equation in a form provided certain bounds are satisfied. Roughly speaking, the *weak-stability* results in L^1-weak reveal several unexpected forms of weak L^1-continuity for approximate sequences of solutions in normalized sense. These facts are reminiscent of various results in the weak topology arising in the basic work of *Ball* in elasticity, see [14,15], *Murat and Tartar* in the theory of compensated compactness [77,78,101,102] (a general question in this area: Is the weak limit of a sequence of solutions again a solution? In general, nonlinear maps are not continuous in the weak topology and the answer is negative, but in the case of some problems, like Boltzmann equations, the special structure of the operators leads to a positive result in the form of the *weak-stability* theorem). In particular, the weak limit of a sequence of classical solutions is a *renormalized* (or, equivalently, *mild*) solution and the set of renormalized solutions is closed in the weak topology (the weak L^1 limit of approximate solutions generated by mass normalization is a renormalized solution); finally, we mention that the constructed global solutions satisfy the entropy inequality (this fact has implication for various asymptotic problems such as the hydrodynamic limit and the large-time behavior), we refer the interested reader to [48] for a complete account on *weak-stability/global-existence* results for Boltzmann equations. Nevertheless, we use a method which is slightly different, and more easy, than the one used in [61]. Observe that, the term "*singular*" means that it is, possibly, concentrated on a set with zero capacity (where by "*capacity*" we mean the parabolic capacity introduced by *Pierre* [82], and developed in [54]). More precisely, under appropriate assumptions on the lower order term K, on the data u_0, F and μ, and applying a "*specific*" approximation on the decomposition of μ, we prove that u satisfies a stability result (Theorem 3.1) with $H\equiv G\equiv E\equiv 0$. We also show, (Theorem 3.2), that problem (1.3) admits exactly a "*generalized*" solution under appropriate assumptions on $H\not\equiv 0$, $G\not\equiv 0$ and $E\not\equiv 0$. We could summarize

[4] I.e., it is continuous with respect to s and ζ for almost every $(t,x)\in Q$, and measurable with respect to (t,x) for every $s\in\mathbb{R}$ and $\zeta\in\mathbb{R}^N$.

these stability/existence results by saying that there exists a correspondence between solutions of problem (1.3) and sets of the cylinder $Q = (0, T) \times \Omega$ where the measure is concentrated with respect to the parabolic capacity. Therefore, problem (1.3) admits a solution without any smallness on the coefficients b and/or c (which is different from analogous assumptions for elliptic equation, see [28,61]). The idea behind the result is very simple: if one makes formally an approximate problem, then one can derive a priori estimates on the solutions and their gradients following the ideas contained in [88] (see also [26,28]). Of course, these estimates are formally calculated in each cylinder $Q_{t_i} = [t_{i-1}, t_i] \times \Omega$, where $([t_{i-1}, t_i])_{i\in\mathbb{N}}$ is a partition of the entire interval $[0, T]$, but they will be justified rigorously, using some technical lemmas and a passage to the limit, in order to get estimates on the entire cylinder Q. The stability result (Theorem 3.1) can also be reads as: every "*generalized*" solution of approximate problems, with $H \equiv G \equiv F \equiv 0$, corresponds, via a "*Kernel*" regularization, to μ_n (a regular measure in $\mathbf{M}(Q)$ which converges to μ in the *narrow*[5] topology of measures) converges almost everywhere (a.e.) to the solution u of the corresponding problem with measure μ, this function u is such that: $u \in L^\infty(0, T; L^1(\Omega))$ with $T_k(u) \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$, and where all the gradients satisfy ∇u_n converges to ∇u a.e. in Q, and $T_k(u_n)$ converges strongly to $T_k(u)$ in $L^p(0, T; W_0^{1,p}(\Omega))$, for all $k > 0$ and for every $n \in \mathbb{N}$. It is interesting to point out that, we also get many "*asymptotic behavior*" results satisfied by the solution with respect to the nonnegative parts of the singular term of the measure data and with respect to lower order terms. More precisely, if u is a "*generalized*" solution, we get

$$\begin{cases} \lim\limits_{n\to+\infty} \dfrac{1}{n} \displaystyle\int_{\{n<u<2n\}} a(t, x, u, \nabla u) \cdot \nabla u \varphi dxdt = \int_Q \varphi d\mu_c^+, \\ \lim\limits_{n\to+\infty} \dfrac{1}{n} \displaystyle\int_{\{-2n<u<-n\}} a(t, x, u, \nabla u) \cdot \nabla u \varphi dxdt = \int_Q \varphi d\mu_c^-, \\ \limsup\limits_{n\to+\infty} \dfrac{1}{n} \displaystyle\int_{\{n<|u|<2n\}} |K(t, x, u)|\, |\nabla u|\, dxdt = 0, \\ \limsup\limits_{n\to+\infty} \dfrac{1}{n} \displaystyle\int_{\{n<|u|<2n\}} |H(t, x, u, \nabla u)|\, |u|\, dxdt = 0, \end{cases} \tag{1.15}$$

where φ is a positive function in $C^1(\overline{Q})$ with $\varphi \geq 0$ and $\mu_c^\pm$ are the two (nonnegative) singular parts of μ, we refer to Sect. 4 for more details. Recall that the stationary (elliptic) case was studied by authors in [55,98] when $p = 2, \gamma = \lambda = 1$, and in [28,29,41] with only the term $b(x)|\nabla u|^\lambda$, and [19,20,30] with only the term $-\mathrm{div}[c(t, x)|u|^\gamma]$. A similar connection between the stationary solution and solutions of linear/nonlinear problems with two lower order terms and measure data is proved in [40,42,43,59,61,62]. Therefore, similar problems are expected to occur in the evolution case, see for example [25] where a lower order term of the type $\mathrm{div}(\phi(u))$ appears, with ϕ is continuous in $\mathbb{R}^N$, in [31] when $p = 2$, $b = 0$ and $c(t, x) \in L^2(Q)^N$ using the framework of "*entropy*" solutions, in [44] when $b = 0$, $\mu \in L^1(Q)$ and $u_0 \in$

[5] Also called "*weak*" convergence where test functions are taken in the set $C_b^0(Q)$ of all bounded and continuous functions in Q and where the total mass is conserved (see Definition 2.1 for more details).

$L^1(\mathbb{R})$, and in [46] when μ is a *diffuse* measure ($\mu \in \mathbf{M}_{d,p}(Q)$) in the framework of "*renormalized*" solutions. However, these parabolic works aren't proved when one deals with general, possibly singular, measure, as stated in [28], which is the main result of the present paper. Another interesting result is contained in (Theorem 3.2) where one can apply the same stability method to ensure the existence of a generalized solution of (1.1) for more general lower order and forcing terms: an explicit example is given when considering the datum as $\mu + \operatorname{div}(E)$, with $E \in L^{p'}(Q)^N$ and $\mu \in \mathbf{M}(Q)$. Our main result reads as follows:

Theorem 1.1 *Assume that $a(t,x,s,\zeta)$, $K(t,x,s)$, $H(t,x,s,\zeta)$, $G(t,x,s)$, u_0, E and μ satisfy assumptions* (1.4)–(1.11)*. Then, for every $p > 1$, there exists a renormalized solution u of problem* $(\mathcal{P})$.

The paper is organized as follows. In Sect. 2.1, we give some notations and we recall some well-known results as they are used to define our main results. The definition and some properties of the functional Sobolev spaces are given in Sect. 2.2. In Sect. 2.3, we define the parabolic (b,p)-*capacity*, we give its properties and its relation to measure spaces, compared with the well-known *Bessel* (b,p)-capacity defined on $\mathbb{R}^N$. In Sect. 2.4, we use Lebesgue spaces to characterize *Lorentz* spaces, and we give necessary and sufficient conditions for embedding theorems to hold. In Sect. 3.1, we introduce the main assumptions, we specify what we wean by "*generalized*" solutions, and we characterize the statements of the main results. In Sect. 3.2, using the above mentioned measure spaces, we define an approximation of data with "*Kernel type mollifers*", we also show that each of these approximations generate a priori estimates (several other properties on the solution are also given). In Sect. 4, we investigate and prove existence/stability and regularity results of generalized solutions to the parabolic boundary value problems (1.1). The case $b \equiv d \equiv 0$ is considered in Sect. 4.1 where we assume, for simplicity, that μ is general and $E \equiv 0$. Under these hypotheses, we prove the "*stability*" of a distributional solution [*Step 1*], the asymptotic behaviour results are obtained in [*Steps 2–3*], with a slightly different version of [84, Theorem 5], which lead to a limit-solution using a "*near/far from*" approach [*Step 4–5*]. Existence of a solution in connection with three lower order terms is proved in Sect. 4.2 where a more general datum ($E \not\equiv 0$) is considered. It is worth to point out that the uniqueness view-point opens a "*large*" quantity of questions when a general measure datum is considered.

2 Preliminaries

Throughout this paper, Ω will be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, with smooth boundary, and p and p' will be two real numbers with $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. In what follows, $|\xi|$ and $\xi \cdot \xi'$, will denote respectively, the Euclidean norm of a vector $\xi \in \mathbb{R}^N$ and the scalar product between ξ and $\xi' \in \mathbb{R}^N$. In the first section, we give some notations/preliminary tools and we introduce some functional spaces. In particular, we recall several useful properties of parabolic capacities in connection with measure spaces, parabolic "*Lorentz*" spaces and "*Kernel*" regularization.

2.1 Notations and tools

We set by $\mathbb{R}^N$ the N-euclidean (simply $\mathbb{R}$ if $N = 1$, while $\mathbb{R}^+ = (0, +\infty)$) on which the standard Lebesgue measure is concentrated, as defined on the σ-algebra of Lebesgue measurable sets. Given $\Omega \subseteq \mathbb{R}^N$ be an open set, with boundary $\partial\Omega$, and $Q = (0, T) \times \Omega$ whose boundary $(0, T) \times \partial\Omega$. We denote by $C_c(\overline{Q})$ the space of continuous functions on $\overline{Q}$ with compact support in $\overline{Q}$, and $C_c^\infty(Q)$ (also denoted by $\mathcal{D}(Q)$) will designate the space of test functions on Q, that is, the space of infinitely continuously differentiable functions in Q with compact support in Q, and $\mathcal{D}'(Q)$ the space of continuous linear functionals from $C_c^\infty(Q)$ into $\mathbb{R}$. Considering $C_c(Q)$ with the topology of locally uniform convergence, we denote by $\mathbf{M}(Q)$ the space of Radon measures whose elements μ are identified with the associated real valued additive set functions, defined on the σ-algebra of Borelian subsets of Q, and which are finite on compact subsets ($\mu^\pm$ denotes the positive measures, mutually orthogonal, of the Hahn decomposition of μ, that is, $\mu = \mu^+ - \mu^-$). By $\mathbf{M}_b(Q)$, we mean the subspace of measures in $\mathbf{M}(Q)$ whose total variation $|\mu| = \mu^+ + \mu^-$ is finite on Q, that is, $|\mu|(Q) < +\infty$ with respect to the measure $|\cdot|$. For $p \in [1, \infty)$ and for a function $u \in L^p(Q)$, we denote $\|u\|^p_{L^p(Q)} = \int_Q |u|^p dxdt$, and for measurable functions u, v: $Q \mapsto \mathbb{R}$, we set

$$\begin{cases} u^+ := \max\{u, 0\}, \;\; u^- := \max\{-u, 0\}, \\ u \vee v = \max\{u, v\}, \;\; u \wedge v = \min\{u, v\}. \end{cases} \tag{2.1}$$

We define, for $k > 0$, the truncation function $T_k(s) = (-k) \vee [k \wedge s]$; we also consider its auxiliary function $G_k(s) = s - T_k(s) = (|s| - k)^+\text{sign}(s)$. Following [10], we introduce $\mathcal{T}_0^{1,p}(Q)$ as the set of all measurable functions u: $Q \mapsto \mathbb{R}$ such that $T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega))$ for all $k > 0$, we point out that $\mathcal{T}_0^{1,p}(Q) \cap L^\infty(Q) = L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$. The following lemma, see [10, Lemma 2.1], which is of analytic nature will be useful in defining the "*gradient*" for functions, that may not belong to Sobolev spaces, enjoying some properties, and is important in deriving a priori estimates of weak solutions.

Lemma 2.1 *Let $u \in \mathcal{T}_0^{1,p}(Q)$. Then, there exists a unique measurable function v: $Q \mapsto \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \text{ almost everywhere (abbreviated a.e.,) in } Q, \text{ for every } k > 0. \tag{2.2}$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$. If u belongs to $L^\infty(0, T; W_0^{1,1}(\Omega))$, this gradient coincides with the usual gradient in distributional sense.

In what follows we will indicate by $\omega(\cdot)$ a generalized sequence that converges to zero as $(\cdot)$ goes to its limit. Moreover, we will denote $C(\cdot)$ several (possibly different) constants which depend on the parameter $(\cdot)$ but not on the sequence indices. We will set

$$\Theta_k(s) = T_1(s - T_k(s)), \quad h_n(s) = 1 - |\Theta_n(s)|, \quad S_n(s) = \int_0^s h_n(\tau)d\tau, \quad \forall s \in \mathbb{R}, \tag{2.3}$$

and in particular, we will exploit their useful properties: Note that, $h_n(s)$ converges to 1 as n tends to infinity and has compact support; so that, $S_n(s)$ is a sequence of $W^{2,\infty}(\mathbb{R})$ having a derivative with compact support, and converging, as n tends to infinity, to the identity function $I(s) = s$. Let us recall a useful lemma we will apply during the proof of the main results (it is a well-known tool about the strong convergence for monotone operators).

Lemma 2.2 *Let $a(t, x, s, \zeta)$ satisfy Leray–Lions assumptions* [*i.e.,* (1.4)–(1.6)], *and suppose that w_n converges weakly to w in $L^p(0, T; W_0^{1,p}(\Omega))$. Moreover, if*

$$\lim_{n\to+\infty} \int_Q [a(t, x, w_n, \nabla w_n) - a(t, x, w_n, \nabla w)] \cdot \nabla(w_n - w) = 0, \tag{2.4}$$

then

$$w_n \to w \text{ strongly in } L^p(0, T; W_0^{1,p}(\Omega)) \text{ and a.e. in } Q. \tag{2.5}$$

Proof See [27, Lemma 5] (see also [86, Lemma 2.4]). □

2.2 Some properties of functional parabolic spaces

Given a real Banach space V_0, and two numbers a, b in $\mathbb{R}$; the space $C_c^\infty([a, b]; V_0)$ will be the restrictions to $[a, b]$ of functions in $C_c^\infty(\mathbb{R}; V_0)$ (the space of functions in $C^\infty(\mathbb{R}; V_0)$ having compact support), and $C([a, b]; V_0)$ the space of continuous functions from $[a, b]$ into V_0. Then, for $1 \leq p < +\infty$, $L^p(a, b; V_0)$ is defined as the space of measurable functions $u: [a, b] \mapsto V_0$ such that

$$\|u\|_{L^p(a,b;V_0)} = \left(\int_a^b \|u\|_{V_0}^p dt\right)^{\frac{1}{p}} < +\infty, \tag{2.6}$$

while $L^\infty(a, b; V_0)$ is the space of measurable functions such that

$$\|u\|_{L^\infty(a,b;V_0)} := \operatorname*{sup-ess}_{[a,b]} \|u\|_{V_0} < +\infty. \tag{2.7}$$

Of course both spaces are meant to be quotiented, as usual, with respect to the a.e. equivalence, we denote by $\mathbb{W}$ the functional space

$$\mathbb{W} = \left\{u \in L^p(0, T; V);\ u_t \in L^p(0, T; V')\right\}, \tag{2.8}$$

being $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ endowed with its natural norm $\|\cdot\|_{W_0^{1,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$ and V' its dual space. As usual, this functional space $\mathbb{W}$ is endowed with the norm

$$\|u\|_{\mathbb{W}} = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}. \tag{2.9}$$

Let us consider

$$\widehat{\mathbb{W}} = \left\{ u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(0,T; L^2(\Omega));\ u_t \in L^{p'}(0,T; W^{-1,p'}(\Omega)) \right\}. \tag{2.10}$$

It is well-known, since $W^{-1,p'}(\Omega) \hookrightarrow V'$, we have $\widehat{\mathbb{W}}$ is continuously embedded in $\mathbb{W}$, and it is the natural space that appears in the study of the parabolic problem (1.1) with time-dependent measure data, but it is not lost in in working with $\mathbb{W}$ instead of $\widehat{\mathbb{W}}$ (observe that $\widehat{\mathbb{W}} \subset \mathbb{W}$) since the sets of null-capacity with regards to $\mathbb{W}$ coincide with the sets of null capacity coming from $\widehat{\mathbb{W}}$, see [54, Remark 2.18] (see also [56] fore more details). Moreover, since $V \hookrightarrow L^2(\Omega) \hookrightarrow V'$, we notice that $\mathbb{W}$ is continuously embedded in $C([0,T]; L^2(\Omega))$, see [47], which means that there exists $C > 0$ such that $\|u\|_{L^\infty(0,T;L^2(\Omega))} \leq C\|u\|_{\mathbb{W}}$, for all $u \in \mathbb{W}$. For further properties of these spaces, we refer to the monographs [6,7] and to the papers [54,82,83] (see also references therein). Given two measures μ and ν, the measure μ is said to be "*singular with respect to* ν" if there exists a Borel set E such that $\mu = \mu_E$ and $\nu(E) = 0$, where the "*restriction*" μ_E of μ to E is defined by

$$(\mu_E)(A) := \mu(E \cap A) \text{ for every Borel set } A \subseteq Q \text{ (abbreviated, } \mu \llcorner E), \tag{2.11}$$

with $\mu_\emptyset = 0$. For any $\mu \in \mathbf{M}_b(Q)$ and any Borel set $E \subseteq Q$, we denote by $\mathbf{M}_s(Q)$ the set of measures which are "*singular with respect to the Lebesgue measure*", namely

$$\mathbf{M}_s(Q) := \left\{ \mu \in \mathbf{M}(Q) |\ \exists \text{ a Borel set } E \subseteq Q \text{ such that (s.t) } |E| = 0 \text{ and } \mu = \mu \llcorner E \right\}. \tag{2.12}$$

Similarly, we denote by $\mathbf{M}_{ac}(Q)$ the set of measures "*absolutely continuous with respect to the Lebesgue measure*", namely

$$\mathbf{M}_{ac}(Q) := \{\mu \in \mathbf{M}(Q) |\ \mu(E) = 0 \text{ for any Borel set } E \subseteq Q \text{ s.t } |E| = 0\}. \tag{2.13}$$

Recall that $\mathbf{M}_s(Q) \cap \mathbf{M}_{ac}(Q) = \{0\}$. Moreover, by the Lebesgue decomposition and the "*Radon–Nikodym*" theorem, see [63], for any $\mu \in \mathbf{M}_b(Q)$:

($\mathcal{I}_1$) there exists a unique couple $(\mu_{ac}, \mu_s) \in \mathbf{M}_{ac}(Q) \times \mathbf{M}_s(Q)$ such that

$$\mu := \mu_{ac} + \mu_s; \tag{2.14}$$

($\mathcal{I}_2$) there exists a unique $\mu_r \in L^1(Q)$ (called the "*density*" of the measure μ_{ac}) such that

$$\mu_{ac}(E) = \int_E \mu_r(t,x)dxdt, \text{ for any Borel set } E \subseteq Q. \tag{2.15}$$

It is worth observing that the following relations hold

$$\mu_s = \mu_s^+ - \mu_s^-, \quad [\mu_s]^\pm = [\mu^\pm]_s; \tag{2.16}$$

thus, we will use the notation

$$[\mu]_s^\pm := [\mu_s]^\pm = [\mu^\pm]_s. \tag{2.17}$$

Further relevant subsets of $\mathbf{M}_b(Q)$ (beside $\mathbf{M}_{ac}(Q)$ and $\mathbf{M}_s(Q)$) arise, if we replace the Lebesgue measure of Borel sets with their parabolic "*capacity*". This is the content of the next part where we characterize "*good*" measures, in the sense that, a "*not good*" measure is *singular* (or more exactly, it is "*concentrated with respect to the Choquet-capacity*").

2.3 Parabolic capacity and related measure spaces

Throughout this part, we introduce the parabolic capacity with respect to Q, and we state some of its properties. First, we give the definition of the so-called "*Choquet capacity*": Let $\mathcal{T}$ be a topological space, and let $\mathcal{P}(\mathcal{T})$ be the power set of $\mathcal{T}$. A mapping $C: \mathcal{P}(\mathcal{T}) \mapsto [0, +\infty)$ is called a *Choquet capacity* on $\mathcal{T}$ if the following properties are satisfied:

($\mathcal{C}_0$) $\mathcal{C}(\emptyset) = 0$,
($\mathcal{C}_1$) $A \subset B \subset \mathcal{T}$ implies $\mathcal{C}(A) \leq \mathcal{C}(B)$,
($\mathcal{C}_2$) $(A_n)_n \subset \mathcal{T}$ an increasing sequence implies $\lim_{n\to\infty} C(A_n) = C(\cup_{n=1}^\infty A_n)$,
($\mathcal{C}_3$) $(K_n)_n \subset \mathcal{T}$ a decreasing sequence, K_n compact, implies $\lim_{n\to\infty} C(K_n) = C(\cap_{n=1}^\infty K_n)$.

Fore more details on the *Choquet* capacity, we refer the reader to [35] (see also [52, A.II.1]). Let $\text{cap}_{b,p}$ denotes the classical "*Bessel*" capacity, it is defined for open sets $U \subseteq \mathbb{R}^{N+1}$ by

$$\text{cap}_{b,p}(U) = \inf\{\|u\|_{\mathbb{W}}: u \in \mathbb{W} \text{ s.t. } u \geq 1 \text{ a.e. on } U\}. \tag{2.18}$$

For an arbitrary set $A \subset \mathbb{R}^{N+1}$,

$$\text{cap}_{b,p}(A) = \inf\left\{\text{cap}_{b,p}(U): U \text{ is an open set in } \mathbb{R}^{N+1} \text{ containing } A\right\}. \tag{2.19}$$

A set $P \subset \mathbb{R}^{N+1}$ is called "*polar*" if $\text{cap}_{b,p}(P) = 0$, and a function $u \in \mathbb{W}$ is said to be "*quasi-continued*" (abbreviated, q.c.) if for every $\epsilon > 0$, there exists an open set $U \subseteq \mathbb{R}^{N+1}$ such that $\text{cap}_{b,p}(U) < \epsilon$ and $\restriction u\mathbb{R}^{N+1}\backslash U$ is continuous. It is well-known, see [7, Section 2.2] and [54, Section 2], that $\text{cap}_{b,p}$ is a *Choquet* capacity on $\mathbb{R}^{N+1}$, and for every $u \in \mathbb{W}$ there exists a unique (up to a polar set) q.c. function $\widetilde{u}: \mathbb{R}^{N+1} \mapsto \mathbb{R}$ such that $\widetilde{u} = u$ a.e. on $\mathbb{R}^{N+1}$. Moreover, if $K \subseteq \mathbb{R}^{N+1}$ is a compact set, then $\text{cap}_{b,p}(K)$ can also be defined by

$$\text{cap}_{b,p}(K) = \inf\left\{\|u\|_{\mathbb{W}}: u \in \mathbb{W} \cap C_c(\mathbb{R}^{N+1}) \text{ s.t. } u \geq 1 \text{ on } K\right\}. \tag{2.20}$$

Since $\text{cap}_{b,p}$ is a *Choquet* capacity on $\mathbb{R}^{N+1}$, we have for every Borel set $B \subset \mathbb{R}^{N+1}$

$$\text{cap}_{b,p}(B) = \sup\left\{\text{cap}_{b,p}(K): K \subseteq B \subseteq \mathbb{R}^{N+1} \text{ compact}\right\}. \tag{2.21}$$

For more details on the classical Bessel capacity and its connection to measures, we refer to the monographs [7,75] and references therein. Note that, one can also define the "*relative*" capacity for relatively open sets $U \subset \overline{Q}$ with respect to the relative topology of $\overline{Q}$ (for further applications of this type of capacity we refer the reader to [23] and references therein). Next, we give several useful properties of parabolic capacity.

Theorem 2.1 *Let B be a Borel subset of Q. Then, one has $cap_{b,p}(B) = 0$ if and only if $cap_p(E) = 0$, where*

$$cap_p(B) = inf\left\{\|u\|_{\mathbb{W}}: u \in C_c^{\infty}([0,T] \times \Omega) \text{ s.t. } u \geq \chi_U\right\}. \tag{2.22}$$

The alternative definition given in property (2.22) follows directly from the definition of $C_c^{\infty}([0,T] \times \Omega)$ (space of restrictions to Q of such functions in $\mathbb{R} \times \mathbb{R}^N$ with compact support in $\mathbb{R} \times \Omega$) and the fact that $C_c^{\infty}([0,T] \times \Omega)$ is dense in $\mathbb{W}$ (the inverse implication holds since cap_p satisfies the sub-additivity property, see [54, Proposition 2.14]). The following lemmas are useful in order to estimate the capacity on the level sets of the solution u.

Lemma 2.3 *Let $u \in \mathbb{W}$ be a $cap_{b,p}$-q.c. function, then for every $k > 0$*

$$cap_{b,p}(\{|u| > k\}) \leq \frac{C}{k} max\left\{\|u\|_{\mathbb{W}}^{\frac{p}{p'}}, \|u\|_{\mathbb{W}}^{\frac{p'}{p}}\right\}. \tag{2.23}$$

Proof See [54, Proposition 2.19]. □

The following result shows that functions in $\mathbb{W}$ has a quasi-continuous representative (q.c.r).

Lemma 2.4 *For every $u \in \mathbb{W}$, there exists a unique (up to a polar set) q.c. function $\widetilde{u}: \overline{Q} \mapsto \mathbb{R}$ such that $\widetilde{u} = u$ a.e. in Q.*

Proof See [54, Lemma 2.20]. □

Lemma 2.5 *Let (u_n) be a sequence of q.c. functions in $\mathbb{W}$ which converges to a q.c. function $u \in \mathbb{W}$. Then, there exists a subsequence which converges q.e. to u on $\overline{Q}$.*

Proof See [54, Lemma 2.1]. □

As mentioned above, we would like to characterize measures in $\mathbf{M}_b(Q)$ in terms of capacity (in this case, we will denote $\mathbf{M}_{b,p}(Q)$ (instead of $\mathbf{M}_b(Q)$) to refer that measures are defined with respect to (b, p)-*capacities* (instead of Lebesgue measures). For this reason, we denote by $\mathbf{M}_{d,p}(Q)$, for $p \in [1, \infty)$, the set of measures on Q which are "*diffuse with respect to the* (b, p)-*capacity*", namely

$$\mathbf{M}_{d,p}(Q) := \left\{\mu \in \mathbf{M}(Q) | \; \mu(E) = 0 \text{ for every Borel set } E \subseteq Q \text{ s.t. } \operatorname{cap}_{b,p}(E) = 0\right\}. \tag{2.24}$$

Similarly, we denote by $\mathbf{M}_{c,p}(Q)$ the set of measures on Q which are "*concentrated with respect to the* (b, p)-*capacity*", namely

$$\mathbf{M}_{c,p}(Q) := \left\{\mu \in \mathbf{M}(Q) | \; \exists \text{ a Borel set } E \subseteq Q \text{ s.t. } \operatorname{cap}_{b,p}(E) = 0 \text{ and } \mu = \mu_E\right\}. \tag{2.25}$$

Clearly, $\mathbf{M}_{d,p}(Q) \cap \mathbf{M}_{c,p}(Q) = \{0\}$ and $\mathbf{M}_{d,p_1}(Q) \subseteq \mathbf{M}_{d,p_2}(Q)$ if $p_1 < p_2$. Recall that every subset $E \subseteq Q$ such that $\mathbf{M}_{c,p}(E) = 0$, $p \in [1, \infty)$, is Lebesgue measurable and there holds $|E| = 0$, see [39, Proposition 7.3]. This plainly implies

$$\mathbf{M}_{ac}(Q) \subseteq \mathbf{M}_{d,p}(Q) \text{ and } \mathbf{M}_{c,p}(Q) \subseteq \mathbf{M}_s(Q), \quad \forall p \in [1, \infty). \tag{2.26}$$

It is known that a measure $\mu \in \mathbf{M}_{b,p}(Q)$ belongs to $\mathbf{M}_{d,p}(Q)$ if and only if $\mu \in L^1(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega))$, where $L^{p'}(0, T; W^{-1,p'}(\Omega))$ denotes the dual space of $L^p(0, T; W_0^{1,p}(\Omega))$, see [22], and we say that a Radon measure μ belongs to $L^{p'}(0, T; W^{-1,p'}(\Omega))$ if there exists $F \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ such that

$$\langle F, \varphi\rangle_{L^{p'}(0,T;W^{-1,p'}(\Omega)),L^p(0,T;W_0^{1,p}(\Omega))} = \int_Q \varphi d\mu, \quad \forall \varphi \in C_c^\infty(Q). \tag{2.27}$$

In this case, we also say that F is a Radon measure by identifying F with μ, see [39, Subsection 1.13] for further details. Thus, one can extend the duality symbol $\langle \cdot, \cdot \rangle$ to any $\mu \in \mathbf{M}_{d,p}(Q)$ and $\varphi \in \mathbb{W} \cap L^\infty(Q)$ (recall that, if $\mu \in \mathbf{M}_{d,p}(Q)$, every function $v \in \mathbb{W} \cap L^\infty(Q)$ also belongs to $L^\infty(Q, \mu)$) and it results

$$\langle \mu, v\rangle \leq \left| \int_Q v d\mu \right| \leq \|v\|_{L^\infty(Q,\mu)} |\mu(Q)|. \tag{2.28}$$

As in (2.14), there exists a unique couple $(\mu_{d,p}, \mu_{c,p})$ of (mutually singular) measures such that $\mu_{d,p} \in \mathbf{M}_{d,p}(Q)$ and $\mu_{c,p} \in \mathbf{M}_{c,p}(Q)$, and there holds

$$\mu := \mu_{d,p} + \mu_{c,p} \tag{2.29}$$

(the measures $\mu_{d,p}$ and $\mu_{c,p}$ will be called, respectively, "*the diffuse*" and "*the concentrated*" parts of μ "*with respect to the* (b, p)*-capacity*"). Combining the decompositions (2.14) and (2.29), it can be seen that for every $\mu \in \mathbf{M}_b(Q)$,

$$\mu_{c,p} = [\mu_s]_{c,p}, \quad \mu_{d,p} = \mu_{ac} + [\mu_s]_{d,p}. \tag{2.30}$$

Moreover, there also holds

$$[\mu^{\pm}]_{d,p} = [\mu_{d,p}]^{\pm} := [\mu]^{+}_{d,p}; \quad [\mu^{\pm}_s]_{d,p} = \left[[\mu_s]_{d,p}\right]^{\pm} := [\mu_s]^{\pm}_{d,p}. \tag{2.31}$$

Finally, from (2.29)–(2.31), we obtain the decomposition

$$\mu = \mu_{ac} + [\mu_s]_{d,p} + \mu_{c,p}. \tag{2.32}$$

Remark 2.1 In connection with the first inclusion in (2.26), observe that if $N = 1$, then $\mathbf{M}^{+}_{c,p}(Q) = \emptyset$ for every $p \in [1, \infty)$. In fact, for *singletons* $E = \{(t, x)\}$, $(t, x) \in Q$, there holds

$$\text{cap}_{b,p}\left(\{(t, x))\}\right) \geq 0 \quad \text{if either } p > N \text{ or } p = N = 1, \tag{2.33}$$

see [57]. Therefore, by monotonicity there holds $\text{cap}_{b,p}(E) > 0$ for every nonempty Borel set $E \subseteq Q$; hence the claim follows, and (2.32) implies, in the case $N = 1$, that

$$\mu := \mu_{ac} + [\mu_{d,p}]. \tag{2.34}$$

Proposition 2.1 *Let Ω be a bounded open subset of $\mathbb{R}^N$. Assume that ρ_ϵ is a sequence of $L^1(Q)$ functions converging to ρ weakly in $L^1(Q)$ and assume that σ_ϵ is a sequence of $L^\infty(Q)$ functions which is bounded in $L^\infty(Q)$ and converges to σ a.e. in Q. Then*

$$\lim_{\epsilon \to 0} \int_Q \rho_\epsilon \sigma_\epsilon dx dt = \int_Q \rho \sigma dx dt. \tag{2.35}$$

The natural convergence in $\mathbf{M}_b(Q)$ is defined by the rule that μ_n converges to μ if $\lim_{n\to\infty} \int_Q \varphi d\mu_n = \int_Q \varphi d\mu$, for all $\varphi \in C_c(Q)$. Technically, this is the *weak-** convergence (it is also refereed to as "*vague*" convergence). The problem is that the limit measure can be "*defective*", i.e., may have less masses than the limit of the masses in the convergent family (some masses can go to infinity or to the boundary). This is avoided by "*weak convergence*" (also called "*narrow convergence*") where test functions are taken in $C^0_b(Q)$, the set of all bounded and continuous functions on Q, and which is stricter than vague convergence and the total mass is conserved in the limit.

Definition 2.1 We say that a sequence $\{\mu_n\}$ converges "*in the narrow topology*" to a measure μ in $\mathcal{M}_{b,p}(Q)$ if

$$\lim_{n\to\infty}\int_Q \varphi d\mu_n = \int_Q \varphi d\mu, \quad \forall \varphi \in C_b^0(Q). \tag{2.36}$$

The following result holds if one can consider nonnegative Radon measures with finite and fixed total mass.

Proposition 2.2 *We recall that, if μ_n is a nonnegative measure in $\mathcal{M}_{b,p}(Q)$, then $\{\mu_n\}$ converges in the narrow topology to a measure μ if and only if $\mu_n(Q)$ converges to $\mu(Q)$ and (2.36) holds for every $\varphi \in C_0^\infty(Q)$. It follows that if μ_n is a nonnegative measure, μ_n converges in the narrow topology to μ if and only if (2.36) holds for any $\varphi \in C^\infty(\overline{Q})$.*

To conclude, the next part contains some well-known results on the characterizations of *Lorentz* spaces, we refer the reader to [64,65,70,80] and references therein for more details.

2.4 Lorentz spaces and embedding theorems

Recently, there is a great deal with the topic of *Lorentz* spaces and their regularities to solve various PDEs, see [9,11–13,76,104], by using decreasing rearrangements, see [8,37,66]. The *Lorentz* spaces can be considered as two-parameter scale of spaces, and which refined, in some sense, Lebesgue spaces to "*more*" general spaces. They are extensions of Lebesgue spaces where the classical theory still valid. This "*specific*" kind of spaces is introduced when dealing with the interpolation theory: recall that the averaging operator $(\mathcal{T} f)(s) = \frac{1}{s}\int_0^s f(\tau)d\tau$ (defined in $L^1([0,1])$ for $0 < s < 1$) is proved as a bounded linear functional on L^p into itself (for $1 < p \leq \infty$) by using the *Riesz–Thorin interpolation* theorem, see [33, Corollary 2.3], and Hardy-inequality. But, this interpolation result can't be applied in order to prove the boundedeness in L^1 (it suffices to consider the decreasing function $f(s) = s^{-1}[\log(s)]^{-2}$ near the origin as a counter-example). A quite different technique, called "*Marcinkiewicz interpolation Theorem*", see [74], formulated in a larger class of two parameter family of spaces $L^{p,q}$ is the desired interpolation to accomplish the L^1-boundedness. To be more precise, let $\mathcal{M}$ be the cone of μ-measurable functions on $\mathbb{R}$ whose values lie in $[0,\infty]$, and $\mathcal{M}_0$ be the class of functions in $\mathcal{M}$ that are finite μ-a.e., this class of two parameter family of spaces (called *Lorentz* spaces) can be derived by using decreasing rearrangements as follows:

Definition 2.2 Let $\mathcal{M}_0(Q,\mu)$ be the totally σ-finite measure space, and suppose $0 < p, q \leq \infty$. The *Lorentz* space $L^{p,q} = L^{p,q}(Q,\mu)$ consists of all $u \in \mathcal{M}_0(Q,\mu)$ for which the quantity

$$\|u\|_{p,q} = \begin{cases} \left\{\int_0^\infty [\tau^{\frac{1}{p}} u^\star(\tau)]^q \frac{d\tau}{\tau}\right\}^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \sup\limits_{0<\tau<\infty} \left\{\tau^{\frac{1}{p}} u^\star(\tau)\right\} & \text{if } q = \infty, \end{cases} \tag{2.37}$$

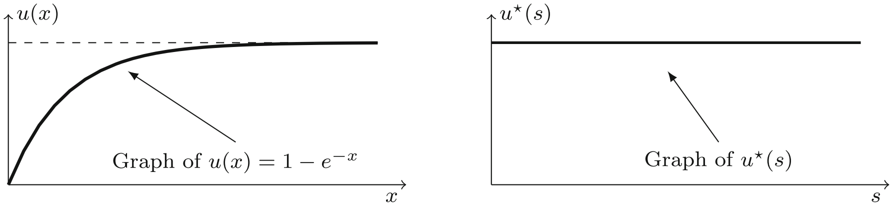

Fig. 1 Example of a decreasing rearrangement $u^\star(s)$

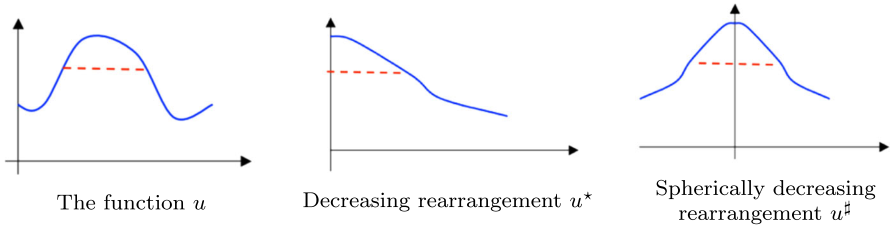

Fig. 2 Decreasing rearrangements

is finite, where the function $u^\star$ is the decreasing rearrangement of u.

Some comments about this definition are in order to be given: note that the construction of a decreasing right-continuous function $u^\star$ on the interval $(0, \infty)$, i.e., "*equi-measurable*", is analogous to rearranging the terms of a finite sequence in decreasing order [in the sense that: two nonnegative functions f and g will be "*rearrangements*" of each other (or, in a more precise terminology, will be asked to be "*equi-measurable*") if their distribution functions coincide]; this notion, which is clearly symmetric, also allows for equi-measurability of functions defined in different measure spaces. Moreover, the concept of measure-preserving transformation for nonnegative measurable functions u, v (that is to say, v is a "*rearrangement*" of u if $v = u \circ \sigma$ for some measure-preserving transformation σ, which coincides with the notion of rearrangement for finite sequences of nonnegative numbers $[(b_i)_{i=1}^n$ is a rearrangement of (a_i) if $b_i = a_{\sigma(i)}$, for some permutation σ of the numbers $1, 2, \ldots, n]$ but in more general measure spaces. This concept, while valid, is not broad enough for our purpose since the symmetry fails (v may be rearrangement of u in this sense without u being a rearrangement of v), then we shall adopt here the first broader definition. Recall that for $u \in \mathcal{M}_0(Q, \mu)$, the decreasing rearrangement of u is the function $u^\star$ defined on $[0, \infty)$ by

$$u^\star(s) = \inf\{\lambda\colon \mu_u(\lambda) \leq s\}, \quad \forall s \geq 0, \tag{2.38}$$

with the convention that $\inf(\emptyset) = \infty$. Thus, if $\mu_u(\lambda) > s$ for all $\lambda \geq 0$, then $u^\star(s) = \infty$, see Figs. 1, 2 and 3.

Notice also that if μ_u is continuous and strictly deceasing, then $u^\star$ is simply the inverse of μ_u on the appropriate interval. In fact, for general u, if we first form the distribution function μ_u and then form the distribution function m_{μ_u} of μ_f (with respect to Lebesgue measure m on $[0, \infty)$), we obtain precisely the decreasing rearrangement $u^\star$. This is an immediate consequence of the identities

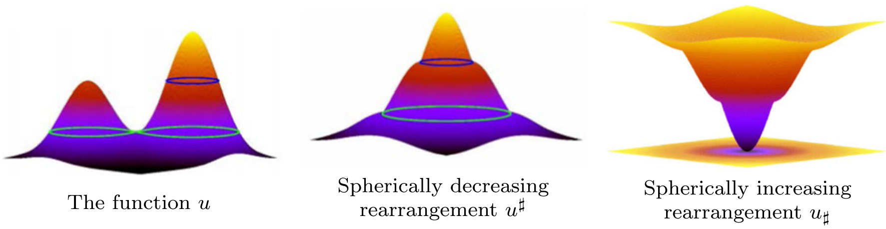

Fig. 3 Decreasing/increasing rearrangements

$$u^\star(s) = \sup\{\lambda\colon \mu_u(\lambda) > s\} = m_{\mu_u}(s) \quad \forall s \geq 0, \tag{2.39}$$

which follow from (2.38), the fact that μ_u is decreasing and the definition of the distribution function. In addition, for every u, v in $\mathcal{M}_0(Q, \mu)$, we have the following "*Hardy–Littlewood inequality*":

$$\int_Q |uv| d\mu \leq \int_0^\infty u^\star(\tau) v^\star(\tau) d\tau, \tag{2.40}$$

which reduces, when v is the characteristic function, to

$$\frac{1}{\mu(E)} \int_E |u| d\mu \leq \frac{1}{s} \int_0^s u^\star(\tau) d\tau, \quad \forall u \in \mathcal{M}_0(Q, \mu). \tag{2.41}$$

Notice that, the average of $|u|$ (over any set of measure s) is dominated by the corresponding average of $u^\star$ (over the interval $(0, s)$), which is also maximal among all averages of $u^\star$ (over sets of measure s). For this reason, the function $u^{\star\star}$ defined by

$$u^{\star\star}(s) = \frac{1}{s} \int_0^s u^\star(\tau) d\tau, \quad \forall s > 0. \tag{2.42}$$

is called "*the maximal function*" of $u^\star$. It is clear that the *Lorentz* space $L^{p,p}(Q)$, $0 < p \leq \infty$, coincides with the *Lebesgue* space $L^p(Q)$, and

$$\|u\|_{p,p} = \|u\|_p \quad \forall u \in L^p(Q). \tag{2.43}$$

Note that also the space $L^{\infty,q}$, for finite q, is trivial in the sense that it contains only the zero-function, and for any fixed p, the *Lorentz* space $L^{p,q}$ increases as the secondary exponent q increases.

Proposition 2.3 *Suppose that $1 \leq p \leq \infty$, and $1 \leq q \leq r \leq \infty$. Then*

$$\|u\|_{p,r} \leq C \|u\|_{p,q} \quad \forall u \in \mathcal{M}_0(Q, \mu), \tag{2.44}$$

where C is a constant depending only on p, q, s, and r. In particular, $L^{p,q} \hookrightarrow L^{p,r}$, $L^{r,s} \hookrightarrow L^{p,q}$ on finite measure spaces, and $l^{p,q} \hookrightarrow l^{r,s}$ on discrete measure spaces.

Proof See [33, Proposition 4.2]. □

The *Lorentz* space $L^{p,q}$ is reduced to the *Lebesgue* spaces L^1 or L^∞, respectively, when $p = q = 1$ or $p = q = \infty$. If $1 \le q \le p < \infty$ or $p = q = \infty$, the $(L^{p,q}, \|\cdot\|_{p,q})$ is a *rearrangement-invariant* Banach space, but the functional $u \mapsto \|u\|_{p,q}$ is not always a norm even when $p, q \ge 1$. Although the restriction $q \le p$ is necessary, $\|\cdot\|_{p,q}$ can be replaced, in the case $p > 1$, with an equivalent functional which is a norm for all $q \ge 1$. The trick is simple, it suffices to replace $u^\star$ with $u^{\star\star}$ in the definition (2.37) of $\|u\|_{p,q}$.

Definition 2.3 Suppose that $1 < p \le \infty$ and $0 < q \le \infty$. If $u \in \mathcal{M}_0(Q, \mu)$, let

$$\|u\|_{(p,q)} = \begin{cases} \left\{\int_0^\infty [\tau^{\frac{1}{p}} u^{**}(\tau)]^q \frac{d\tau}{\tau}\right\}^{1/q} & \text{if } 0 < q < \infty, \\ \sup\limits_{0<\tau<\infty} \left\{\tau^{1/p} f^{**}(\tau)\right\} & \text{if } q = \infty, \end{cases} \tag{2.45}$$

where $u^{\star\star}$ is the maximal function of $u^\star$ defined in (2.42).

It is worth noting that if $1 \le p < \infty$, the space $L^{p,1}$ is a *Lorentz* space, equipped with the norm $\|\cdot\|_{p,1}$ defined by

$$\|u\|_{p,1} = \int_0^\infty \tau^{\frac{1}{p}} u^\star(\tau) \frac{d\tau}{\tau}. \tag{2.46}$$

On the other hand, if $1 < p \le \infty$, the space $L^{p,\infty}$ is also a *Lorentz* space, equipped with the "*modified*" norm $\|\cdot\|_{(p,\infty)}$ defined by

$$\|f\|_{(p,\infty)} = \sup_{\tau>0} \tau^{1/p} f^{\star\star}(\tau). \tag{2.47}$$

In particular, if $1 < p < \infty$, $L^{p,1}$ and $L^{p,\infty}$, when suitably normed, are respectively the smallest and the largest of all rearrangement-invariant spaces having the same fundamental function as L^p, and the associate space of $L^{p,q}(Q, \mu)$, up to equivalence of norms, is the Lorentz space $L^{p',q'}(Q, \mu)$ where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. To conclude, these spaces play a particularly important role in the weak type interpolation theory ("*Marcinkiewcz interpolation*") and the details of these facts require various and hard analytic tools like theory of *Fourier* multipliers, Sobolev embedding and Marcinkiewicz interpolation theorems, etc (this is why we restrict ourselves to some classical properties). Let us now turn to settle the tools we need, for $1 < p < \infty$, $1 < q < \infty$ and for $1 < r < \infty$, the parabolic *Lorentz* spaces $L^{p,q}(Q)$ and $L^{r,\infty}(Q)$ are the spaces of Lebesgue measurable functions such that

$$\begin{cases} \|f\|_{L^{p,q}(Q)} = \left(\int_0^{|Q|} [u^\star(\tau) r^{\frac{1}{p}}]^q \frac{d\tau}{\tau})^{\frac{1}{q}}\right) < +\infty, & (2.48) \\ \|f\|_{L^{r,\infty}(Q)} = r[\underset{r>0}{\text{meas}} \{(t, x) \in Q \colon |f(t, x)| > r\}]^{\frac{1}{r}} < +\infty. & (2.49) \end{cases}$$

In general (2.48)–(2.49) does not define a norm in Lorentz spaces, but one can define a "modified" norm [see (2.46)–(2.47)]. Parabolic *Lorentz* spaces can be considered as "*intermediate spaces*" between the parabolic *Lebesgue* spaces, in the sense that, for every $1 < s < r < \infty$ we have

$$L^{r,1}(Q) \subset L^{r,r}(Q) = L^r(Q) \subset L^{r,\infty}(Q) \subset L^{s,1}(Q). \tag{2.50}$$

The space $L^{r,\infty}(Q)$ is the *dual* space of $L^{r',1}(Q)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, and we have the generalized *Hölder*'s inequality

$$\begin{cases} \forall f \in L^{r,\infty}(Q),\ \forall g \in L^{r',1}(Q), \\ \displaystyle\int_Q |fg| dxdt \leq \|f\|_{L^{r,\infty}(Q)} \|g\|_{L^{r',1}(Q)}. \end{cases} \tag{2.51}$$

More generally, if $1 < p < \infty$ and $1 \leq q < \infty$, we get

$$\begin{cases} \forall f \in L^{p_1,q_1}(Q),\ \forall g \in L^{p_2,q_2}(Q), \\ \|fg\|_{L^{p,q}(Q)} \leq \|f\|_{L^{p_1,q_1}(Q)} \|g\|_{L^{p_2,q_2}}(Q) \\ \dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{p_2}, \quad \dfrac{1}{q} = \dfrac{1}{q_1} + \dfrac{1}{q_2}. \end{cases} \tag{2.52}$$

Recall that different classes of functional spaces are natural in the study of symmetrization, for instance the "*Marcinkiewicz* spaces". The *Marcinkiewicz* space $\mathtt{M}^p(\mathbb{R}^{N+1})$, $1 < p < \infty$, is defined as the set of functions $u \in L^1_{loc}(\mathbb{R}^{N+1})$ such that

$$\int_K |u(t,x)| dxdt \leq C|K|^{(p-1)/p}, \tag{2.53}$$

for all subsets K of finite measure, see [16]. The minimal C is (2.53) gives a norm in this space, i.e.,

$$\|u\|_{\mathtt{M}^p(\mathbb{R}^{N+1})} = \sup\left\{ \text{meas}(K)^{-(p-1)/p} \int_K |u| dxdt \colon K \subset \mathbb{R}^{N+1},\ \text{meas}(K) > 0 \right\}. \tag{2.54}$$

Since functions in $L^p(\mathbb{R}^{N+1})$ satisfy inequality (2.53) with $C = \|f\|_{L^p}$ (by Hölder's inequality), we conclude that $L^p(\mathbb{R}^{N+1}) \subset \mathtt{M}^p(\mathbb{R}^{N+1})$ and $\|f\|_{\mathtt{M}^p} \leq \|f\|_{L^p}$. The *Marcinkiewicz* space $\mathtt{M}^p(\mathbb{R}^{N+1})$ (also called *weak* L^p*-space*) is a particular case of *Lorentz* space (more precisely, it is the space $L^{p,\infty}(\mathbb{R}^{N+1})$). We recall that for every $0 < s < \infty$, a *Marcinkiewicz* space $\mathtt{M}^s(Q)$ (or a *weak Lebesgue* space) is the space of measurable functions $v \colon Q \mapsto \mathbb{R}$ such that there exists $C > 0$, with

$$\text{meas}\,\{(t,x) \in Q \colon |v(t,x)| \geq k\} \leq \frac{C}{k^s}, \quad \forall k > 0. \tag{2.55}$$

The space $\mathtt{M}^s(Q)$ turns out to be a *Banach* space with respect to the norm

$$\|v\|_{\mathtt{M}^s(Q)} = \inf\left\{C > 0\colon \text{meas}\{(t,x) \in Q\colon |v(t,x)| \geq k\} \leq \frac{C}{k^s} \text{ holds}, \forall k > 0\right\}. \tag{2.56}$$

and, for $s > 1$, we have the following *continuous* embedding (since Ω is bounded)

$$\mathcal{L}^s(Q) \hookrightarrow \mathtt{M}^s(Q) \hookrightarrow \mathcal{L}^{s-\epsilon}(Q), \quad \forall \epsilon \in (0, s-1]. \tag{2.57}$$

Finally, let us define, for every $p > 1$, the space S^p (needed to construct the convergence of "*cut-off*" functions)

$$S^p = \left\{u \in L^p(0,T; W_0^{1,p}(\Omega));\ u_t \in L^1(Q) + L^{p'}(0,T; W^{-1,p'}(\Omega))\right\}, \tag{2.58}$$

endowed with its natural norm $\|u\|_{S^p} = \|u\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$, and satisfying the following trace result.

Theorem 2.2 *Let $p > 1$, we have the following continuous injection*

$$S^p \hookrightarrow C(0,T; L^1(\Omega)). \tag{2.59}$$

Proof [87, Theorem 1.1]. □

The two following embedding theorems will play a central role in our work: the first one is an "*Aubin–Simon*" type result that we state in a form general enough to our purpose, while the second one is a well-known "*Gagliardo–Nirenberg embedding*" theorem.

Theorem 2.3 *Let v_n be a bounded sequence in $L^q(0,T; W_0^{1,q}(\Omega))$ such that $(v_n)_t$ is bounded in $L^1(Q)+L^{s'}(0,T; W^{-1,s'}(\Omega))$ with $q, s > 1$, then v_n is relatively strongly compact in $L^1(Q)$, that is, up to subsequences, v_n strongly converges in $L^1(Q)$ to some function $v \in L^1(Q)$.*

Proof [100, Corollary 4]. □

Theorem 2.4 (Gagliardo–Nirenberg inequality) *Let v be a function in $W_0^{1,q}(\Omega) \cap L^\rho(\Omega)$ with $q \geq 1$ and $\rho \geq 1$. Then, there exists a positive constant C, depending on N, q and ρ, such that*

$$\|v\|_{L^\gamma(Q)} \leq C\|\nabla v\|^\theta_{L^q(Q)^N}\|v\|^{1-\theta}_{L^\rho(Q)}, \tag{2.60}$$

for every θ and γ satisfying

$$0 \leq \theta \leq 1, \quad 1 \leq \gamma \leq +\infty, \quad \frac{1}{\gamma} = \theta\left(\frac{1}{q} - \frac{1}{N}\right) + \frac{1-\theta}{\rho}. \tag{2.61}$$

Proof See [79, Lecture II]. □

The following embedding results are consequences of the previous theorem. We will use these results in the next sections but we give here the statements for completeness.

Corollary 2.1 (i) *Let* $v \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$, *with* $q \geq 1$, $\rho \geq 1$. *Then* $v \in L^\sigma(Q)$ *with* $\sigma = q\frac{N+\rho}{N}$ *and*

$$\int_Q |v|^\sigma dxdt \leq C\|v\|_{L^\infty(0,T;L^\rho(\Omega))}^{\frac{\rho q}{N}} \int_Q |\nabla v|^q dxdt. \tag{2.62}$$

(ii) *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open subset of* $\mathbb{R}^N$ *(*$N \geq 2$*),* $T > 0$, $1 < p < N$, *and let* $w \in L^\infty(0,T;L^p(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$. *Then, there exists a positive constant* C *depending only on* N *and* p *such that*

$$\left[\left(\int_0^T \int_\Omega |w|^\sigma dxdt\right)^{\frac{\mu}{\sigma}}\right]^{\frac{p}{\mu}} \leq C\left(\sup_{t\in[0,T]} \int_\Omega |w|^p dx + \int_Q |\nabla w|^p dxdt\right), \tag{2.63}$$

for all μ *and* σ *satisfying*

$$p \leq \sigma \leq p^\star, \quad p \leq \mu \leq \infty, \quad \frac{N}{p\sigma} + \frac{1}{\mu} = \frac{N}{p^2}. \tag{2.64}$$

Proof See [45, Proposition 3.1]. □

The next result is a useful result in the sense that it allows to handle functions which do not have time derivatives belonging to the energy space $L^p(0,T;W_0^{1,p}(\Omega))$; in fact, it consists in a generalized "*integration by parts*" formula where its proof can also be found in [38,53].

Lemma 2.6 *Let* Ω *be a bounded open subset of* $\mathbb{R}^N$, $N \geq 2$, *and let* $\phi : \mathbb{R} \mapsto \mathbb{R}$ *be a continuous piecewise* C^1*-function such that* $\phi(0) = 0$ *and* ϕ' *has compact support; let us define* $\Phi(s) = \int_0^s \phi(\tau)d\tau$. *If* $v \in L^p(0,T;W_0^{1,p}(\Omega))$ *is such that* $v_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ *and if* $\psi \in C^\infty(\overline{Q})$, *then we have*

$$\int_0^T \langle v_t, \phi(v)\psi\rangle dt = \int_\Omega \Phi(v(T))\psi(T)dx - \int_\Omega \Phi(v(0))\psi(0)dx - \int_Q \psi_t \Phi(v)dxdt. \tag{2.65}$$

Proof See [86, Lemma 6.10]. □

We observe that $v_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$, which implies that there exist $\eta_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and $\eta_2 \in L^1(Q)$ such that $u_t = \eta_1 + \eta_2$. Even if η_1

and η_2 are not uniquely determined, the *integration by parts* formula turns out to be independent of the representation of v_t; moreover, according with the notation introduced before, $\langle \cdot, \cdot \rangle$ will also indicate the duality between $L^{p'}(0, T; W^{-1,p'}(\Omega))+L^1(Q)$ and $L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.

3 Definitions of solutions, intermediary lemmas and mains results

3.1 Definitions of generalized solutions

Initial value problems for quasilinear/nonlinear parabolic equations having Radon measures as right-hand side has been widely investigated looking for solutions which for positive times take values in some functional spaces. Their studies are motivated by some engineering problems, see [36,58,60,68] for applications in electromagnetic induction heating, modeling of wells in porous media flow, and the $k-\epsilon$ model of turbulence. In contrast, it is the purpose of this section to define and investigate solutions that for positive times take values in more general spaces when the data is considered in the space of Radon measures. We call such solutions "*generalized weak solutions*", in contrast to weak/distributional solutions previously considered in the literature. Following [28,61], we state the definition of renormalized solution for problem (1.3) were we give in the general case.

Definition 3.1 Assume (1.4)–(1.11), let $\mu \in \mathbf{M}(Q)$, and $u_0 \in L^1(\Omega)$. A measurable function u is a renormalied solution of problem (1.3) if, there exists a decomposition $(\mu_{d,p}, \mu_{c,p})$ of μ such that $u\colon Q \mapsto \overline{\mathbb{R}}$ is measurable on Q and $T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ for every $k > 0$, $|u|^{p-1} \in L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q)$, $|\nabla u|^{p-1} \in L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q)$, and for every $S \in W^{2,\infty}(\mathbb{R})$ $(S(0) = 0)$ such that S' has compact support on $\mathbb{R}$, we have

$$
\begin{aligned}
&-\int_\Omega S(u_0)\varphi(0)dx - \int_0^T \langle \varphi_t, S(u)\rangle dt + \int_Q S'(u)a(t,x,u,\nabla u)\cdot\nabla\varphi dxdt \\
&+ \int_Q S''(u)a(t,x,u,\nabla u)\cdot\nabla u\varphi dxdt + \int_Q K(t,x,u)\cdot\nabla\varphi S'(u)dxdt \\
&+ \int_Q K(t,x,u)\cdot\nabla u S''(u)\varphi dxdt + \int_Q H(t,x,u,\nabla u)S'(u)\varphi dxdt \\
&+ \int_Q G(t,x,u)S'(u)\varphi dxdt = \int_Q S'(u)\varphi d\mu_d \\
&+ \int_Q E\cdot\nabla\varphi S'(u)dxdt + \int_Q E\cdot\nabla u S''(u)\varphi dxdt, \qquad (3.1)
\end{aligned}
$$

for every $\varphi \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, $\varphi_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))$, with $\varphi(T, x) = 0$, such that $S'(u)\varphi \in L^p(0, T; W_0^{1,p}(\Omega))$. Moreover,

$$\begin{cases} \lim_{n\to+\infty} \frac{1}{n} \int_{\{n<|u|<2n\}} |K(t,x,u)|\,|\nabla u|\,dxdt = 0, \\ \lim_{n\to+\infty} \frac{1}{n} \int_{\{n<|u|<2n\}} |H(t,x,u,\nabla u)|\,|\nabla u|\,dxdt = 0, \end{cases} \tag{3.2}$$

and, for every $\psi \in C(\overline{Q})$, we have

$$\begin{cases} \lim_{n\to+\infty} \frac{1}{n} \int_{\{n\leq u<2n\}} a(t,x,u,\nabla u)\cdot\nabla u\psi\,dxdt = \int_Q \psi\,d\mu_{c,p}^{+}, \\ \lim_{n\to+\infty} \frac{1}{n} \int_{\{-2n<u\leq n\}} a(t,x,u,\nabla u)\cdot\nabla u\psi\,dxdt = \int_Q \psi\,d\mu_{c,p}^{-}, \end{cases} \tag{3.3}$$

where $\mu_{c,p}^{+}$ and $\mu_{c,p}^{-}$ are, respectively, the positive and the negative forms of the singular part $\mu_{c,p}$ of μ.

Remark 3.1 Notice that the distributional meaning of each term in (3.1) is well defined thanks to the fact that $T_k(u)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$ and since S' has compact support. Indeed, by taking M such that $\mathrm{Supp}(S')\subset]-M,M[$, since $S'(u)=S''(u)=0$ as soon as $|u|\geq M$, we can replace, in (3.1), ∇u by $\nabla T_M(u)\in L^p(Q)^N$ (recall that $a(t,x,0,0)=0$). Moreover, according to Lemma 2.1, ∇u is well defined. We also have, for all S as above $S(u)=S(T_M(u))\in L^p(0,T;W_0^{1,p}(\Omega))$. Furthermore, since $S(u)_t\in L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$, we can use as test function in (3.1) not only functions in $C_c^\infty(Q)$ but also in $L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$. By our regularity assumptions, $S(u)_t$ belongs to the space $L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$ which implies that $S(u)$ belongs to $C(0,T;L^1(\Omega))$, see [87, Theorem 1.1], thus, initial condition is achieved in a weak sense, that is, $S(u)(0)=S(u_0)$ in $L^1(\Omega)$ for every S. Finally, observe also that assumptions on S and u_0 imply that

$$S(u)(0)=S(u_0)\text{ in }L^1(\Omega). \tag{3.4}$$

Remark 3.2 We want to stress that, thanks to our definition and the choice of S, we have $S(u)\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$ and $S(u)_t\in L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)$; this would prove that $S(u)$ has a C_p-q.c.r. Observe also that (3.1) implies that equation

$$\begin{aligned} S(u)_t &- \mathrm{div}[a(t,x,u,\nabla u)S'(u)] + S''(u)a(t,x,u,\nabla u)\cdot\nabla u \\ &+ \mathrm{div}[K(t,x,u)S'(u)] - S''(u)K(t,x,u)\cdot\nabla u + H(t,x,u,\nabla u)S'(u) \\ &+ G(t,x,u)S'(u) \\ &= S'(u)f + G\cdot S''(u)\nabla u - \mathrm{div}(GS'(u)) - \mathrm{div}[ES'(u)] + S''(u)E\cdot\nabla u, \end{aligned} \tag{3.5}$$

holds. Observe that by Definition 3.1, and since $T_k(u)$ is a renormalized truncated solution for problem (1.3), there exists a sequence (ν_n) in $\mathbf{M}(Q)$ such that $T_k(u)_t - \mathrm{div}[a(t,x,T_k(u),\nabla T_k(u))+K(t,x,T_k(u)]+H(t,x,T_k(u),\nabla T_k(u))+G(t,x,T_k(u))$ is a finite measure, and moreover

$$T_k(u)_t - \operatorname{div}[a(t,x,T_k(u),\nabla T_k(u)) + K(t,x,T_k(u)]$$
$$+ H(t,x,T_k(u),\nabla T_k(u)) + G(t,x,T_k(u)) = \mu + \nu_k \text{ in } \mathbf{M}(Q). \quad (3.6)$$

It is important to note that if we consider the case that $\mu_{c,p} \equiv 0$, which recovers the problem (1.3) to a classical one, one can define the following notion of "*entropy*" solution which is equivalent, in this case, to the Definition 3.1, see [53] for more details. To this end, we define

$$\mathbb{E} = \left\{ \varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q) \text{ s.t. } \varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q) \right\}. \quad (3.7)$$

According to [87], one has $\mathbb{E} \subset C([0,T];L^1(\Omega))$.

Definition 3.2 Under hypothesis (1.4)–(1.11), and for $\mu \in \mathbf{M}_{d,p}(Q)$ and $u_0 \in L^2(\Omega)$. A function u is an "*entropy*" solution of problem (1.3) if the following conditions hold:

(i) u is a.e. finite such that $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$ (abbreviated, $u \in \mathcal{T}_0^{1,p}(Q)$).
(ii) $H(t,x,u,\nabla u) \in L^1(Q)$, $G(t,x,u) \in L^1(Q)$.
(iii) For all $\varphi \in \mathbb{E}$, and for all $k > 0$, we have

$$t \in [0,T] \mapsto \int_\Omega \Theta_k(u-\varphi)(t,x)dx \text{ is (a.e. equal to) a continuous function,} \quad (3.8)$$

(iv) For every $\varphi \in \mathbb{E}$, it holds

$$\int_0^T \langle u_t, T_k(u-\varphi)\rangle dt + \int_Q [a(t,x,u,\nabla u) + K(t,x,u)] \cdot \nabla T_k(u-\varphi)dxdt$$
$$+ \int_Q H(t,x,u,\nabla u)T_k(u-\varphi)dxdt + \int_Q G(t,x,u)T_k(u-\varphi)dxdt$$
$$\leq \langle \mu - \operatorname{div}(E), T_k(u-\varphi)\rangle, \quad \forall k > 0. \quad (3.9)$$

Since $\mu = f - \operatorname{div}(F)$ is diffuse: every terms in (3.9) is well defined (remark that the set $\mathcal{T}_0^{1,p}(Q)$ is the minimal requirement to give a meaning to the entropy or renormalized formulations). In fact, the right-hand side is well-defined since f belongs to $L^1(Q)$ and $T_k(u-\varphi)$ is in $L^\infty(Q)$, and G belongs to $L^{p'}(Q)^N$ while $T_k(u-\varphi)$ is in $\mathbb{W}$. The left-hand side is also well-defined since the integral is only on the set $\{|u-\varphi| \leq k\}$, and in this set $|u| \leq k + \|\varphi\| := M$, it is equal to write

$$\int_Q [a(t,x,u,\nabla u) + K(t,x,u)] \cdot \nabla T_k(u-\varphi)dxdt$$
$$= \int_{\{|u-\varphi|\leq k\}} [a(t,x,u,\nabla u) + K(t,x,u)] \cdot \nabla(u-\varphi)dxdt$$

$$= \int_{\{|u-\varphi|\le k\}} [a(t,x,T_M(u),\nabla T_M(u)) + K(t,x,T_M(u))]\cdot \nabla(T_M(u)-\varphi)dxdt, \tag{3.10}$$

which is finite by the growth assumption (1.5) on "a". Now, if μ is general, this definition of entropy solution is not suitable to tackle singular terms, since $\int_Q T_k(u-\varphi)d\mu$ may not be well defined when μ is a Radon measure. However, this notion of entropy solution can be extended for the new concept of "*measure-valued*" solution for $\mu \in \mathbf{M}(Q)$, see [81,92,93,103] (and also [97]), but this notion of solution is still in progress and need further works to apply it. Actually, the renormalized solution have some what more regularity, this is the content of the following proposition where the proof is a particular consequence of a more general result stated in [46, Lemma 1.1], see also [44, Lemma 2.2]; thus it is omitted, where we will assume that $T_k(u) \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$, for all $k>0$, and where a crucial role is played by Lorentz spaces.

Proposition 3.1 *Let u be any measurable solution of (1.3) such that $T_k(u) \in L^\infty(0,T;L^2(\Omega) \cap L^p(0,T;W_0^{1,p}(\Omega))$ for every $k>0$, and suppose that*

$$\sup_{t\in[0,T]} \int_\Omega |T_k(u)(t)|^2 dx + \int_Q |\nabla T_k(u)|^p dxdt \le Mk + L, \tag{3.11}$$

where M and L are two positive constants. Then $u^{\frac{N(p-1)+p}{N+p}} \in L^{\frac{N+p}{N},\infty}(Q)$ and $|\nabla u|^{\frac{N(p-1)+p}{N+2}} \in L^{\frac{N+2}{N+1},\infty}(Q)$ such that

$$\begin{cases} \left\| |u|^{\frac{N(p-1)+p}{N+p}} \right\|_{L^{\frac{N+p}{N},\infty}(Q)} \le C(N,p)\left[M + |Q|^{\frac{Np}{N+2}} L^{\frac{N(p-1)+p}{(N+2)p}}\right], & (3.12) \\ \left\| |\nabla u|^{\frac{N(p-1)+p}{N+2}} \right\|_{L^{\frac{N+2}{N+1},\infty}(Q)} \le C(N,p)\left[M + |Q|^{\frac{N}{(N+2)p}} L^{\frac{N(p-1)+p}{(N+2)p}}\right], & (3.13) \end{cases}$$

where $C(N,p)$ is a constant which depends on p and N.

Proof See [46, Lemma A.1 (Appendix)]. □

We are now in position to show that u satisfies some other useful estimates.

Lemma 3.1 *Let $u \in L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q)$, $p>1$, and $|\nabla u| \in L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q)$. Then u belongs to the Lebesgue space $L^m(Q)$ with $m < \frac{p(N+1)-N}{N(p-1)}$ and ∇u belongs to the Lebesgue space $L^s(Q)$ with $s < \frac{p(N+1)-N}{(N+1)(p-1)}$.*

Proof The proofs are similar to those of [10,84] (see also [32]). □

3.2 Approximate problems and a priori estimates

As indicated before, the main tool, in order to prove the stability/existence of renormalized solution (Theorems 3.1, 3.2) relies on approximating our problems with more regular ones in bounded domains and in proving the existence/stability of a solution via a priori estimates and strong convergence of truncatures in $L^p(0, T; W_0^{1,p}(\Omega))$. In this part we consider a family of approximating problems to be used in the prof of the main result. Let us first define a technical notion of parabolic "*mollifiers*".

Definition 3.3 A "*mollifier*" is a function $\rho_\epsilon \in C_c^\infty(\mathbb{R}^{N+1})$ such that

$$\begin{aligned} &\text{supp } \rho_\epsilon = \mathcal{B}(0, \epsilon) = \left[\left\{(t, x) \in \mathbb{R}^{N+1}: \|(t, x)\| < \epsilon\right\}\right], \\ &\rho_\epsilon \geq 0 \text{ and } \int_{\mathbb{R}^{N+1}} \rho_\epsilon(t, x) dx dt = 1. \end{aligned} \tag{3.14}$$

So the mollifier ρ_ϵ is a positive test function, with support that decreases as $\epsilon \downarrow 0$, but the volume under the graph is preserved. As $\epsilon \downarrow 0$, these functions are concentrated at the origin (i.e., approximate the Dirac functional). The mollifiers and the operation of convolution $\star$ provide the best tools to approximate initial/source data by smooth C_c^∞-functions.

Proposition 3.2 *Let $\mu \in \mathbf{M}(Q)$ and ρ_n is a mollifier. Let us fix $\theta \in C_c^\infty(Q)$ and set $\widetilde{\mu} = \theta\mu$. We extend $\widetilde{\mu}$ to all $\mathbb{R}^{N+1}$ by setting it equal to zero outside Q, and then define $\widetilde{\mu}_n = \rho_n \star \widetilde{\mu}$, i.e.,*

$$\widetilde{\mu}_n(t, x) := \int_{\mathbb{R}^{N+1}} \rho_n(t - s, x - z) d\widetilde{\mu}(s, z) = \int_Q \rho_n(t - s, x - z) d\widetilde{\mu}(s, z). \tag{3.15}$$

Then, $\widetilde{\mu}_n = \rho_n \star \widetilde{\mu} \in C_c^\infty(Q)$, and

$$\begin{cases} \widetilde{\mu}_n = \rho_n \star \widetilde{\mu} \to \widetilde{\mu} \text{ strongly in } L^{p'}(0, T; W^{-1,p'}(\Omega)), \\ \|\widetilde{\mu}_n\|_{L^1(Q)} = \|\rho_n \star \widetilde{\mu}\|_{L^1(Q)} \leq \|\widetilde{\mu}_n\|_{\mathcal{M}(Q)} = \|\rho_n \star \widetilde{\mu}\|_{\mathcal{M}(Q)} \leq \|\mu\|_{\mathcal{M}(Q)}. \end{cases} \tag{3.16}$$

Proof See [54, Lemma 2.25]. □

It is worth observing that the class in which we study problem (1.3) is "natural", since the solution in the main result is obtained as a limit of a family (u_ϵ) of solutions of the regularized problems

$$\begin{cases} (u_\epsilon)_t - \operatorname{div}\left[a(t, x, u_\epsilon, \nabla u_\epsilon) + K(t, x, u_\epsilon)\right] + H(t, x, u_\epsilon, \nabla u_\epsilon) + G(t, x, u_\epsilon) \\ \quad = \mu_\epsilon - \operatorname{div}(E) \text{ in } (0, T) \times \Omega, \\ u_\epsilon(t, x) = 0 \text{ on }]0, T[\times \partial\Omega, \; u_\epsilon(0, x) = u_0^\epsilon(x) \text{ in } \Omega. \end{cases} \tag{3.17}$$

As far as the lower order terms are concentrated, we assume that $K_\epsilon(t,x,u,s)$, $H_\epsilon(t,x,s,\zeta)$ and $G_\epsilon(t,x,s)$ are *Carathéodory*[6] functions such that

$$\begin{aligned} &K_\epsilon(t,x,s) = K(t,x,T_{\frac{1}{\epsilon}}(s));\ \ H_\epsilon(t,x,s,\zeta) = T_{\frac{1}{\epsilon}}(H(t,x,s,\zeta)) \\ &\text{and } G_\epsilon(t,x,s) = T_{\frac{1}{\epsilon}}(G(t,x,s)), \end{aligned} \tag{3.18}$$

satisfying the standard growth conditions:

$$\begin{cases} |K_\epsilon(t,x,s)| \le |K(t,x,s)| \le c_0(t,x)|s|^\gamma + c_1(t,x), & (3.19)\\ |H_\epsilon(t,x,s,\zeta)| \le |H(t,x,s,\zeta)| \le b_0(t,x)|\zeta|^\lambda + b_1(t,x), & (3.20)\\ |G_\epsilon(t,x,s)| \le |G(t,x,s)| \le d_1(t,x)|s|^\iota + d_2(t,x), & (3.21) \end{cases}$$

the boundedness assumptions with respect to ϵ:

$$\begin{cases} |K_\epsilon(t,x,s)| \le c_0(t,x)\dfrac{1}{\epsilon^\gamma} + c_1(t,x), & (3.22)\\ |H_\epsilon(t,x,s,\zeta)| \le \dfrac{1}{\epsilon}, & (3.23)\\ |G_\epsilon(t,x,s)| \le \dfrac{1}{\epsilon}, & (3.24) \end{cases}$$

and the sign-condition:

$$G_\epsilon(t,x,s)s \ge 0. \tag{3.25}$$

Recall that the first three assumptions are in fact crucial in order to obtain a priori estimates; indeed the boundedness-conditions on the lower order terms are in fact a consequence of their definitions. Finally, we suppose that the sequences $\{u_0^\epsilon\} \subseteq C^\infty(\overline{\Omega})$ and $\{\mu_\epsilon\} \subseteq C^\infty(\overline{Q})$ are sequences satisfying

$$\begin{cases} \mu_\epsilon \rightharpoonup^* \mu \text{ in } \mathbf{M}(Q)\colon \|\mu_\epsilon\|_{L^1(Q)} \le \|\mu\|_{\mathbf{M}(Q)},\\ u_0^\epsilon \to u_0 \text{ in } L^1(\Omega)\colon \|u_0^\epsilon\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}, \end{cases} \tag{3.26}$$

for any $\epsilon > 0$. For instance, the sequence $\{\mu_\epsilon\}$ can be defined by "*convolution*", i.e.,

$$\mu_\epsilon(t,x) := \widetilde{\mu}\star\rho_\epsilon(t,x), \quad (t,x) \in \mathbb{R}^{N+1}, \tag{3.27}$$

where $\widetilde{\mu} \in \mathbf{M}(\mathbb{R}^{N+1})$ denotes the trivial extension of μ to $\mathbb{R}^{N+1}$ and $\{\rho_\epsilon\}$ is a sequence of parabolic mollifiers. We notice that it is easy to exhibits a "trivial" measure approximation proceeding as follows: since $\mu := \mu_{d,p} + \mu_{c,p}$, we can construct

[6] On the regularized functions K_ϵ, H_ϵ and G_ϵ we assume, besides continuity with respect to $s \in \mathbb{R}$ and $\zeta \in \mathbb{R}^N$ for a.e. $(t,x) \in (0,T) \times \Omega$ and measurability with respect to $(t,x) \in (0,T) \times \Omega$ for every fixed $s \in \mathbb{R}$ and $\zeta \in \mathbb{R}^N$, the same assumptions (1.7)–(1.9).

$\widetilde{\mu} \in \mathbf{M}(\mathbb{R}^{N+1})$ by setting $\widetilde{\mu} := \widetilde{\mu}_{d,p} + \widetilde{\mu}_{c,p}$, where

$$\widetilde{\mu}_{d,p}(t,x) = \begin{cases} \mu_{d,p} & \text{if } (t,x) \in Q, \\ 0 & \text{otherwise}, \end{cases} \tag{3.28}$$

and

$$\widetilde{\mu}_{c,p}(B) := \mu_{c,p}(B \cap Q) \text{ for any Borel set } B \subseteq \mathbb{R}^{N+1}. \tag{3.29}$$

Observe that by definition

$$\widetilde{\mu} = \widetilde{\mu} \llcorner Q, \quad \widetilde{\mu}(B) = \mu(B) \text{ for every Borel set } B \subseteq Q. \tag{3.30}$$

Hence, if $\varphi \in C_c(Q)$ and $\widetilde{\varphi} \in C_c(\mathbb{R}^{N+1})$, which denotes its trivial extension to $\mathbb{R}^{N+1}$, there holds $\langle \widetilde{\mu}, \widetilde{\varphi} \rangle_{\mathbb{R}^{N+1}} = \langle \mu, \varphi \rangle_Q$. Now, consider the sequence $\{\widetilde{\mu}_\epsilon\} \subset C_c^\infty(\mathbb{R}^{N+1})$ where $\widetilde{\mu}_\epsilon := \widetilde{\mu} \star \rho_\epsilon$ with $(\rho_\epsilon) \subset C_c^\infty(\mathbb{R}^{N+1})$ being a regularizing sequence, one can also define

$$\widetilde{\mu}^\epsilon_{d,p} := \widetilde{\mu}_{d,p} \star \rho_\epsilon, \quad \widetilde{\mu}^\epsilon_{c,p} := \widetilde{\mu}_{c,p} \star \rho_\epsilon, \tag{3.31}$$

with ρ_ϵ is defined as above. To be more specific, one can choose

$$\rho_\epsilon(t,x) = \frac{1}{\epsilon^N \int_{\mathbb{R}^{N+1}} \rho(t,x)\,dx\,dt} \rho\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right), \quad \forall (t,x) \in \mathbb{R}^{N+1}, \tag{3.32}$$

where $\rho \in C_c^\infty(\mathbb{R}^{N+1})$, $\rho(t,x) = \rho(t,|x|)$ is a standard mollifier. Next, since $\mu_{d,p} = f - \operatorname{div}(F)$, one can choose any sequence of functions $\{f_\epsilon\} \subseteq C_c^\infty(Q)$ and $\{F_\epsilon\} \subseteq C_c^\infty(Q)$ such that f_ϵ strongly converges to f in $L^1(Q)$ and F_ϵ strongly converges to F in $L^{p'}(Q)^N$. Finally, one can set $u_0^\epsilon := \widetilde{u}_0^\epsilon \eta_\epsilon$ in $\mathbb{R}^N$ where $\{\eta_\epsilon\} \subseteq C_c^\infty(\mathbb{R}^N)$ such that $\eta_\epsilon \in C_c^\infty(\Omega_{n+1})$, $0 \le \eta_\epsilon \le 1$, $\eta_\epsilon = 1$ in $\overline{\Omega_n}$; here Ω_n is open, $\overline{\Omega_n} \subset \Omega_{n+1} \subset \Omega$ for every $n \in \mathbb{N}$, and $\cup_{n=1}^\infty \Omega_n = \Omega$, observe that $\{u_0^\epsilon\} \subset C_c^\infty(\Omega)$ and $0 \le u_0^\epsilon(x) \le \widetilde{u}_0^\epsilon(x)$ in $\mathbb{R}^N$. By standard convolution arguments, it is easily seen that

$$\begin{cases} \|\mu_\epsilon\|_{L^1(Q)} \leqq \|\widetilde{\mu}_\epsilon\|_{L^1(\mathbb{R}^{N+1})} \leqq \|\widetilde{\mu}\|_{\mathbf{M}(\mathbb{R}^{N+1})} = \|\mu\|_{\mathbf{M}(Q)}, \\ \|u_0^\epsilon\|_{L^2(\Omega)} \leqq \|\widetilde{u}_0^\epsilon\|_{L^2(\mathbb{R}^N)} \leqq \|\widetilde{u}_0\|_{L^2(\mathbb{R}^N)} = \|u_0\|_{L^2(\Omega)}. \end{cases} \tag{3.33}$$

Moreover, for any $\varphi \in C_b(Q)$, with extension $\widetilde{\varphi} \in C_b(\mathbb{R}^{N+1})$, there holds

$$\lim_{\epsilon \to 0} \int_Q \mu_\epsilon \varphi\,dx\,dt = \lim_{\epsilon \to 0} \int_{\mathbb{R}^{N+1}} \widetilde{\mu}_\epsilon \widetilde{\varphi}\,dx\,dt = \langle \widetilde{\mu}_\epsilon, \widetilde{\varphi} \rangle_{\mathbb{R}^{N+1}} = \langle \mu, \varphi \rangle_Q. \tag{3.34}$$

We notice explicitly that, if $\mu_{c,p} \neq 0$, the solution constructed in the main result, is different from the trivial solution constructed here and hence, in general, the solution is not unique, this is not surprising being such a notion of solution is weaker than the notion of distributional solution for which, see [95,99], the results do not hold; one can solve the problem of uniqueness, when $\mu \in \mathbf{M}_{d,p}$, by introducing the notion of "*weak renormalized-entropy*" solution, see [84,89], which, by definition, coincides with the "trivial" solution defined here, and coincides, for $\mu_{c,p} \equiv 0$, with the classical "entropy" solution introduced in [96], see also [53]. By classical results, see for instance [69,71], there exists a (weak) solution $u_\epsilon \in C^\infty(\overline{Q})$, $\epsilon > 0$, of problem (3.17). Now, to let $\epsilon \to 0$ in (3.17) we need a priori estimates for approximate sequences (u_ϵ) and (∇u_ϵ). The next estimates, which are well-known in the literature, immediately follows, from Proposition 3.1, by taking test functions depending on T_k and by using assumptions (3.19)–(3.25) and (3.26), it is the main tool in order to establish fundamental a priori estimates for the solutions and their gradients.

Lemma 3.2 *Let u_ϵ be defined as before, and assume that there exists $M, L > 0$ such that*

$$\sup_{t\in[0,T]} \int_\Omega |T_k(u_\epsilon)|^2 dx + \int_Q |\nabla T_k(u_\epsilon)|^p dxdt \leq Mk + L, \tag{3.35}$$

for every $k > 0$. Then, there exists $C(N, M, p) > 0$ (the constants M and L to be defined) such that

$$\begin{cases} \left[k_1 \operatorname{meas}\left\{|u_\epsilon|^{\frac{N(p-1)+p}{N+p}} > k_1\right\}\right]^{\frac{p+N}{N}} \leq C\left[M + Lk_1^{-\frac{N+p}{N(p-1)+p}}\right], & \forall k_1 > 0, \\ k_2\left[\operatorname{meas}\left\{|\nabla u_\epsilon|^{\frac{N(p-1)+p}{N+2}} > k_2\right\}\right]^{\frac{N+1}{N+2}} \leq C\left[M + L^{\frac{N+1}{N+2}} k_2^{-\frac{N}{N(p-1)+p}}\right], & \forall k_2 > 0. \end{cases} \tag{3.36}$$

Proof $\boxed{(i)}$ We can improve this kind of estimate by using a suitable Gagliardo–Nirenberg type inequality (Theorem 2.4) which asserts that, if $w \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, with $p > 1$, then $w \in L^\sigma(Q)$ with $\sigma = p\frac{N+2}{N}$ and

$$\int_Q |w|^\sigma dxdt \leq C\|w\|^{\frac{2p}{N}}_{L^\infty(0,T;L^2(\Omega))} \int_Q |\nabla w|^p dxdt. \tag{3.37}$$

Indeed, in this way we obtain

$$\begin{aligned} \int_Q |T_k(u_\epsilon)|^{\frac{p(N+2)}{N}} dxdt &\leq C\|T_k(u_\epsilon)\|^{\frac{2p}{N}}_{L^\infty(0,T;L^2(\Omega))} \int_Q |\nabla T_k(u_\epsilon)|^p dxdt \\ &\leq C\left[\operatorname{Sup}_{t\in[0,T]} \int_\Omega |T_k(u_\epsilon)|^2 dx\right]^{\frac{p}{N}} \int_\Omega |\nabla T_k(u_\epsilon)|^p dxdt \\ &\leq C[Mk + L]^{\frac{p}{N}+1}, \end{aligned} \tag{3.38}$$

and so, we can write

$$k^{\frac{p(N+2)}{N}} \operatorname{meas}\{|u_\epsilon| > k\} \leq \int_{\{|u|\geq k\}} |T_k(u_\epsilon)|^{\frac{p(N+2)}{N}} dxdt \leq \int_Q |T_k(u_\epsilon)|^{\frac{p(N+2)}{N}} dxdt$$
$$\leq C[Mk+L]^{\frac{p}{N}+1} \tag{3.39}$$

Then, taking, for every $k > 0$, $k = k_1^{\frac{N+p}{N(p-1)+p}}$, we get

$$\begin{aligned}
\operatorname{meas}\left\{|u_\epsilon|^{\frac{N(p-1)+p}{N+p}} > k_1\right\} &\leq C\left[Mk_1^{\frac{N+p}{N(p-1)+p}} + L\right]^{\frac{p}{N}+1} k_1^{-\frac{p(N+2)}{N}\times\frac{N+p}{N(p-1)+p}} \\
&\leq C\left[Mk_1^{\frac{N+p}{N(p-1)+p}} + L\right]^{\frac{p}{N}+1} \times [k_1^{-\frac{p(N+2)}{N(p-1)+p}}]^{\frac{p}{N}+1} \\
&\leq C\left[Mk_1^{-1} + Lk_1^{-\frac{p(N+2)}{N(p-1)+p}}\right]^{\frac{p}{N}+1} \\
&\leq C\left[k_1^{-1}(M + Lk_1^{-\frac{N+p}{N(p-1)+p}})\right]^{\frac{p+N}{N}},
\end{aligned} \tag{3.40}$$

then, we deduce that

$$\left[k_1 \operatorname{meas}\left\{|u_\epsilon|^{\frac{N(p-1)+p}{N+p}} > k_1\right\}\right]^{\frac{p+N}{N}} \leq C\left[M + Lk_1^{-\frac{N+p}{N(p-1)+p}}\right], \quad \forall k_1 > 0. \tag{3.41}$$

(ii) We are interested about a similar estimate on the gradients of functions u_ϵ. First of all, observe that

$$\operatorname{meas}\{|\nabla u_\epsilon| \neq \lambda\} \leq \operatorname{meas}\{|\nabla u_\epsilon| \neq \lambda;\ |u_\epsilon| \leq k\} + \operatorname{meas}\{|\nabla u_\epsilon| \neq \lambda;\ |u_\epsilon| > k\} \tag{3.42}$$

with regard to the first term in the right-hand side, we have

$$\begin{aligned}
\operatorname{meas}\{|\nabla u_\epsilon| \neq \lambda;\ |u_\epsilon| \leq k\} &\leq \frac{1}{\lambda^p}\int_{\{|\nabla u_\epsilon|\geq\lambda;\ |u_\epsilon|\leq k\}} |\nabla u|^p dxdt \\
&\leq \frac{1}{\lambda^p}\int_{\{|u_\epsilon|\leq k\}} |\nabla u_\epsilon|^p dxdt = \frac{1}{\lambda^p}\int_Q |\nabla T_k(u_\epsilon)|^p dxdt \\
&\leq \frac{Mk+L}{\lambda^p}.
\end{aligned} \tag{3.43}$$

Then, taking $\lambda = k_2^{\frac{N+2}{N(p-1)+p}}$, we get

$$k_2^{\frac{p(N+2)}{N(p-1)+p}} \operatorname{meas}\left\{|\nabla u_\epsilon|^{\frac{N(p-1)+p}{N+2}} > k_2;\ |u_\epsilon| < k\right\} \le Mk + L, \tag{3.44}$$

while for the last term, thanks to (i), one can write

$$\operatorname{meas}\{|\nabla u_\epsilon| \ge \lambda;\ |u_\epsilon| > k\} \le \operatorname{meas}\{|u_\epsilon| \ge k\} \le \frac{\overline{C}[kM + L]^{\frac{p}{N}+1}}{k^\sigma}, \tag{3.45}$$

with $\sigma = \frac{p(N+2)}{N}$. So finally, we obtain

$$\operatorname{meas}\left\{|\nabla u_\epsilon|^{\frac{N(p-1)+p}{N+2}} \ge k_2\right\} \le \frac{\overline{C}[kM + L]^{\frac{p}{N}+1}}{k^\sigma} + \frac{Mk + L}{k_2^{\frac{p(N+2)}{N(p-1)+p}}}, \tag{3.46}$$

and we obtain a better estimate taking the minimum over k_2 of the right-hand side, the minimum is achieved for the value

$$k_2^* = \left[\frac{L}{|Q|}\right]^{\frac{N(p-1)+p}{N+2}} \tag{3.47}$$

that is,

$$\operatorname{meas}\left\{|\nabla u_\epsilon|^{\frac{N(p-1)+p}{N+2}} > k_2\right\} \le C\left[M + \frac{L^{\frac{N+1}{N+2}}}{k_2^{\frac{N}{N(p-1)+p}}}\right]^{\frac{N+2}{N+1}} k_2^{\frac{N+2}{N+1}}, \tag{3.48}$$

we also have the estimate (see [46, Lemma A.1, *Step 4*])

$$k_2\left[\operatorname{meas}\{|\nabla u_\epsilon|^{\frac{N(p-1)+p}{N+2}} > k_2\}\right]^{\frac{N+1}{N+2}} \le C\left[M + |Q|^{\frac{N}{(N+2)p}} L^\gamma\right] \tag{3.49}$$

with $\gamma = \frac{N(p-1)+p}{(N+2)p}$. Then, we obtain that u_ϵ (resp. $|u_\epsilon|^{p-1}$) is uniformly bounded in the Marcinkiewicz space $\mathtt{M}^{\frac{p(N+1)-N}{N}}(Q)$ (resp. $\mathtt{M}^{\frac{p(N+1)-N}{N(p-1)}}$), and $|\nabla u_\epsilon|$ (resp. $|\nabla u_\epsilon|^{p-1}$) is equibounded in $\mathtt{M}^{p-\frac{N}{N+1}}(Q)$ (resp. $\mathtt{M}^{\frac{p-\frac{N}{N+1}}{p-1}} = \mathtt{M}^{\frac{p(N+1)-N}{(N+1)(p-1)}}$). □

As a first step, we get a function u such that $T_k(u) \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ which is the limit, up to subsequences, of (u_ϵ) in suitable topology.

Proposition 3.3 *Let $\mu_\epsilon \in \mathbf{M}(Q)$, $(u_0^\epsilon) \in L^2(\Omega)$ with sup $|\mu_\epsilon(Q)| < \infty$ and $\|u_0^\epsilon\|_{L^2(\Omega)} < \infty$. Let (u_ϵ) be a sequence of renormalized solutions of* (3.17). *Then,* *there exists* $M, L > 0$ *such that*

$$\sup_{t\in[0,T]} \int_\Omega |T_k(u_\epsilon)|^2 dx + \int_Q |\nabla T_k(u_\epsilon)|^p dxdt \le Mk + L, \tag{3.50}$$

for every ϵ *and for every* $k > 0$. *Moreover, there exists a subsequence still denoted by* u_n *and a measurable function* u *such that the following convergence results hold:*

(i) u_ϵ *converges to* u *a.e. in* Q;
(ii) u *belongs to* $L^\infty(0, T; L^2(\Omega))$ *and for every* $k > 0$, *the sequence* $T_k(u_\epsilon)$ *converges to* $T_k(u) \in L^p(0, T; W_0^{1,p}(\Omega))$ *in the weak topology of* $L^p(0, T; W_0^{1,p}(\Omega))$;
(iii) ∇u_ϵ *converges to* ∇u *a.e. in* Q;
(iv) $a(t, x, u_\epsilon, \nabla u_\epsilon)$ *converges to* $a(t, x, u, \nabla u)$ *in the weak topology of* $L^{p'}(Q)^N$ *for every* $k > 0$.

Proof Let us begin by proving Proposition 3.3 under assumptions (1.4)–(1.6), (3.19)–(3.24) and condition (3.26). Observe that from now on, such a condition (3.50) will be used only to obtain a priori estimates for $|u_\epsilon|^{\frac{N(p-1)+p}{N+p}}$ and $|\nabla u_\epsilon|^{\frac{N(p-1)+p}{N+2}}$ in the "correct" Lorentz spaces. In the first step below we prove a priori estimates on u_ϵ and $T_k(u_\epsilon)$, while the second step for the corresponding convergence results.

Step 1. A priori estimates In this step we prove the estimate (3.50) on the truncation functions $T_k(u_\epsilon)$ given in Sect. 2.1. It is performed throughout a multiplication by admissible test function. Define the function $\Psi: \mathbb{R} \mapsto \mathbb{R}$ by $\Psi(s) = \int_0^s T_k(\tau)d\tau$, for all $s \in \mathbb{R}$. Observe that Ψ satisfies the following property

$$\frac{1}{2}|T_k(s)|^2 \le \frac{1}{2}T_k(s)s \le \Psi_k(s) \le k|s|, \quad \forall s \in \mathbb{R}. \tag{3.51}$$

Observe also that $T_k(s)$ is a Lipschitz function such that $T_k(0) = 0$. Therefore, since $u_\epsilon \in \mathbb{W}$, the function $T_k(u_\epsilon)$ belongs to $\mathbb{W} \cap L^\infty(Q)$. This allows us to use $T_k(u_\epsilon)$ as test function in (3.17). Then, we get

$$\begin{aligned} &\int_0^T \langle (u_\epsilon)_t, T_k(u_\epsilon)\rangle dt + \int_Q a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla T_k(u_\epsilon) dxdt \\ &+ \int_Q K_\epsilon(t, x, u_\epsilon) \cdot \nabla T_k(u_\epsilon) dxdt + \int_Q H_\epsilon(t, x, u_\epsilon, \nabla u_\epsilon) T_k(u_\epsilon) dxdt \\ &+ \int_Q G_\epsilon(t, x, u_\epsilon) T_k(u_\epsilon) dxdt = \int_Q f_\epsilon T_k(u_\epsilon) dxdt \\ &+ \int_Q (F_\epsilon + E) \cdot \nabla T_k(u_\epsilon) dxdt + \int_Q T_k(u_\epsilon) d\mu^\oplus_{c,\epsilon} - \int_Q T_k(u_\epsilon) d\mu^\ominus_{c,\epsilon}, \end{aligned} \tag{3.52}$$

where $\mu^\oplus_{c,\epsilon}$ and $\mu^\ominus_{c,\epsilon}$ approximate $\mu^+_{c,p}$ and $\mu^-_{c,p}$ in the sense of (3.27). Now, we evaluate the various integrals in (3.52): by the definition of $\Psi(s)$, property (3.51) and the integration by parts method, we have

$$\int_0^T \langle (u_\epsilon)_t, T_k(u_\epsilon)\rangle dt = \int_\Omega \Psi_k(u_\epsilon(t))dx - \int_\Omega \Psi_k(u_0^\epsilon)dx$$
$$\geq \frac{1}{2}\int_\Omega u_\epsilon(t)T_k(u_\epsilon(t))dx - k\int_\Omega |u_0^\epsilon(x)|dx, \quad (3.53)$$

for almost every $t \in [0, T]$. Now, by the ellipticity condition (1.4), we obtain

$$\int_Q a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla T_k(u_\epsilon)dxdt \geq \alpha \int_Q |\nabla T_k(u_\epsilon)|^p dxdt. \quad (3.54)$$

Let us estimate $|\int_Q K_\epsilon(t, x, u_\epsilon) \cdot \nabla T_k(u_\epsilon)|$: by the growth condition (3.20) on K_ϵ, Hölder and Gagliardo–Nirenberg inequalities together with Young's inequality, we get

$$\left|\int_Q K_\epsilon(t, x, u_\epsilon) \cdot \nabla T_k(u_\epsilon)dxdt\right|$$
$$\leq \int_Q c_0(t, x)|u_\epsilon|^\gamma |\nabla T_k(u_\epsilon)|dxdt + \int_Q c_1(t, x)|\nabla T_k(u_\epsilon)|dxdt$$
$$\leq \left[\int_Q c_0^{p'}(t, x)|T_k(u_\epsilon)|^{\gamma p'}\right]^{\frac{1}{p'}} \left[\int_Q |\nabla T_k(u_\epsilon)|^p dxdt\right]^{\frac{1}{p}}$$
$$+ \frac{3^{\frac{p'}{p}}}{p'\alpha^{\frac{p'}{p}}}\|c_1\|_{L^{p'}(Q)}^{p'} + \frac{\alpha}{3p}\int_Q |\nabla T_k(u_\epsilon)|^p dxdt$$
$$\leq \left[\int_Q c_0^\tau(t, x)dxdt\right]^{\frac{1}{\tau}} \left[\int_0^T |T_k(u_\epsilon)|^{\frac{(N+2)p}{N}}\right]^{\frac{N(p-1)}{p(N+1)}} \left[\int_Q |\nabla T_k(u_\epsilon)|^p dxdt\right]^{\frac{1}{p}}$$
$$+ \frac{\alpha}{3p}\int_Q |\nabla T_k(u_\epsilon)|^p dxdt + \frac{3^{\frac{p'}{p}}}{p'\alpha^{\frac{p'}{p}}}\|c_1\|_{L^{p'}(Q)}^{p'}$$
$$\leq C\|c_0\|_{L^\tau(Q)}\left[\frac{1}{\tau}\sup_{\tau\in[0,T]}\int_\Omega |T_k(u_\epsilon(t))|^2 dx + \frac{N+1}{N+p}\int_Q |\nabla T_k(u_\epsilon)|^p dxdt\right]$$
$$+ \frac{\alpha}{3p}\int_Q |\nabla T_k(u_\epsilon)|^p dxdt + \frac{3^{\frac{p'}{p}}}{p'\alpha^{\frac{p'}{p}}}\|c_1\|_{L^{p'}(Q)}^{p'}. \quad (3.55)$$

Let us estimate $|\int_Q H_\epsilon(t, x, u_\epsilon, \nabla u_\epsilon)T_k(u_\epsilon)dxdt|$: by definition of $T_k(s)$, the growth assumption (3.21) on H_ϵ and the generalized Hölder's inequality (2.51), we have

$$\begin{aligned}
&\left|\int_Q H_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)T_k(u_\epsilon)dxdt\right| \\
&\quad \le \int_Q b_0|\nabla u_\epsilon|^\lambda T_k(u_\epsilon)dxdt + \int_Q b_1 T_k(u_\epsilon)dxdt \\
&\quad \le k\left[\int_Q b_0|\nabla u_\epsilon|^\lambda dxdt + \int_Q b_1 dxdt\right] \\
&\quad \le k\left[\|b_0\|_{L^{N+2,1}(Q)}\, \||\nabla u_\epsilon|\|_{L^{\frac{N+2}{N+1},\infty}(Q)} + \|b_1\|_{L^1(Q)}\right].
\end{aligned} \tag{3.56}$$

Moreover, by the sign condition (3.25) on G_ϵ, we get

$$\int_Q G_\epsilon(t,x,u_\epsilon)T_k(u_\epsilon) \ge 0. \tag{3.57}$$

Finally, we have

$$\begin{aligned}
\int_Q [F_\epsilon + E]\cdot\nabla T_k(u_\epsilon)dxdt \le \frac{2\alpha}{3p}\int_Q |\nabla T_k(u_\epsilon)|^p dxdt \\
+ \frac{3^{\frac{p'}{p}}}{p'\alpha^{\frac{p'}{p}}}\left[\|F\|^{p'}_{L^{p'}(Q)} + \|E\|^{p'}_{L^{p'}(Q)}\right],
\end{aligned} \tag{3.58}$$

and, by the boundedness of $T_k(s)$, we also get

$$\begin{cases}
\displaystyle\int_Q f_\epsilon T_k(u_\epsilon)dxdt \le k\|f_\epsilon\|_{L^1(Q)}, & (3.59)\\
\displaystyle\left|\int_Q T_k(u_\epsilon)d\mu^+_{c,n}\right| \le k\mu^+_{c,n}(Q), & (3.60)\\
\displaystyle\left|\int_Q T_k(u_\epsilon)d\mu^-_{c,n}\right| \le k\mu^-_{c,n}(Q). & (3.61)
\end{cases}$$

Combining (3.52)–(3.61), we get [by observing that $|T_k(u_\epsilon)|^2 \le u_\epsilon(t)T_k(u_\epsilon(t))$]

$$\begin{aligned}
&\left[\tfrac{1}{1} - \tfrac{1}{\tau}C\|c_0\|_{L^\tau(Q)}\right]\sup_{t\in[0,T]}\int_\Omega |T_k(u_\epsilon)(t)|^2dx \\
&\quad + \left[\tfrac{\alpha}{p'} - C\tfrac{N+1}{N+p}\|c_0\|_{L^\tau(Q)}\right]\int_Q |\nabla T_k(u_\epsilon)|^p dxdt \\
&\le k\left[\|b_0\|_{L^{N+2,1}(Q)}\,\left\||\nabla u_\epsilon|^\lambda\right\|_{L^{\frac{N+2}{N+1}}(Q)} + M_0\right] + L,
\end{aligned} \tag{3.62}$$

where

$$\begin{cases} M_0 = \|b_1\|_{L^1(Q)} + \sup_{n\in\mathbb{N}} \|f_\epsilon\|_{L^1(Q)} + \sup_{\epsilon>0} \|u_0^\epsilon\|_{L^1(\Omega)} + \sup_{n\in\mathbb{N}} [\mu^{\oplus}_{c,\epsilon}(Q) + \mu^{\ominus}_{c,\epsilon}(Q)], \\ L = \dfrac{3^{\frac{p'}{p}}}{p'\alpha^{\frac{p'}{p}}} \left[\|c_1\|^{p'}_{L^{p'}(Q)} + \|F\|^{p'}_{L^{p'}(Q)} + \|E\|^{p'}_{L^{p'}(Q)}\right]. \end{cases} \tag{3.63}$$

Define

$$M = \|b_0\|_{L^{N+2,1}(Q)} \left\| |\nabla u_\epsilon|^\lambda \right\|_{L^{\frac{N+2}{N+1}}(Q)} + M_0. \tag{3.64}$$

Observe that if $T = T_1$ be such that

$$\begin{cases} C_1 = \dfrac{1}{2} - \dfrac{1}{\tau} C \|c_0\|_{L^\tau([0,T_1]\times\Omega)} > 0, \\ C_2 = \dfrac{\alpha}{p'} - C \dfrac{N+1}{N+2} \|c_0\|_{L^\tau([0,T_1]\times\Omega)} > 0, \end{cases} \tag{3.65}$$

it implies that

$$\sup_{t\in[0,T]} \int_\Omega |T_k(u_\epsilon)(t)|^2 dx + \int_Q |\nabla T_k(u_\epsilon)|^p dxdt \le [\min(C_1, C_2)]^{-1} [Mk + L]. \tag{3.66}$$

On the other hand, by Proposition 3.1, we get

$$\begin{aligned} &\left\| |\nabla u_\epsilon|^\lambda \right\|_{L^{\frac{N+2}{N+1},\infty}(Q)} \\ &= \left\| |\nabla u_\epsilon|^{p-1} \right\|^{\frac{\lambda}{p-1}}_{L^{\frac{(N+1)p-N}{N+1)(p-1)},\infty}(Q)} \\ &\le C \left[M_0 + \left\| |\nabla u_\epsilon|^\lambda \right\|_{L^{\frac{N+2}{N+1},\infty}(Q)} \right] \|b_0\|_{L^{N+2,1}(Q)} + |Q|^{\frac{N}{(N+2)p}} L^{\frac{N(p-1)+p}{(N+2)p}} \end{aligned} \tag{3.67}$$

which implies, by defining $C_3 = 1 - \frac{\|b_0\|_{L^{N+2,1}(Q)}}{\|b_0\|_{L^{N+2,1}(Q)}}$, the estimate of $|\nabla u_\epsilon|^{p-1}$ in $L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q)$, or more precisely

$$\left\| |\nabla u_\epsilon|^\lambda \right\|_{L^{\frac{N+2}{N+1},\infty}(Q)} \le C, \tag{3.68}$$

we repeat the same argument to get the estimate of $|u_\epsilon|^{p-1}$ in $L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q)$: we have

$$\left\| |u_\epsilon|^{\frac{N(p-1)+p}{N+p}} \right\|_{L^{\frac{N+p}{N},\infty}(Q)}$$
$$\leq C\left[M_0 + \left\| |\nabla u_\epsilon|^\lambda \right\|_{L^{\frac{N+2}{N+1},\infty}(Q)} \|b\|_{L^{N+2,1}(Q)} + |Q|^{\frac{Np}{N+2}} L^{\frac{N(p-1)+p}{(N+2)p}} \right], \quad (3.69)$$

and using (3.68), we obtain (this estimate is useful to prove that M is bounded)

$$\left\| |u_\epsilon|^{\frac{N(p-1)+p}{N+p}} \right\|_{L^{\frac{N+p}{N},\infty}(Q)} \leq C. \quad (3.70)$$

As a consequence of (3.66) and (3.70), we obtain the estimates of u_ϵ in $L^\infty(0, T; L^1(\Omega))$ and $T_k(u_\epsilon)$ in $L^p(0, T; W_0^{1,p}(\Omega))$, that is

$$\begin{cases} \|u_\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \\ \int_Q |\nabla T_k(u_\epsilon)|^p dxdt \leq Ck. \end{cases} \quad (3.71)$$

Up to a subsequence, we are going to prove that u_ϵ converges[7] a.e. in Q towards a measurable function u. Lemma 3.2 gives the usual estimates for parabolic problem (3.17) with general measure data, that is to say, u_ϵ is bounded in $L^q(0, T; W_0^{1,q}(\Omega))$ for every $q < p - \frac{N}{N+1}$, and in $L^\infty(0, T; L^1(\Omega))$, then we can deduce that

$$\lim_{k\to+\infty} \text{meas}\,\{(t, x) \in Q: |u_\epsilon| > k\} = 0 \text{ uniformly with respect to } u. \quad (3.72)$$

From (3.71), we have $T_k(u_\epsilon)$ is bounded in $L^p(0, T; W_0^{1,p}(\Omega))$ for every $k > 0$. Now, if we multiply[8] the approximating equation by $S'(u_\epsilon)$, where S is a non-deceasing $W^{2,\infty}(\mathbb{R})$-function, we obtain

$$\begin{aligned} &S(u_\epsilon)_t - \text{div}(a(t, x, u_\epsilon, \nabla u_\epsilon))S'(u_\epsilon) + a(t, x, u_\epsilon, \nabla u_\epsilon) \cdot \nabla u_\epsilon S''(u_\epsilon) \\ &\quad - \text{div}(K_\epsilon(t, x, u_\epsilon)S'(u_\epsilon)) + K_\epsilon(t, x, u_\epsilon) \cdot \nabla u_\epsilon S''(u_\epsilon) + H_\epsilon(t, x, u_\epsilon, \nabla u_\epsilon)S'(u_\epsilon) \\ &= S'(u_\epsilon)f_\epsilon + S''(u_\epsilon)[F_\epsilon + E] \cdot \nabla u_\epsilon - \text{div}(F_\epsilon + E)S'(u_\epsilon)) + (\mu^\oplus_{c,s} - \mu^\ominus_{c,s})S'(u_\epsilon), \end{aligned} \quad (3.73)$$

in the sense of distributions. This implies, thanks to the last equality and the fact that S' has compact support, that $T_k(u_\epsilon)$ is bounded in $L^p(0, T; W_0^{1,p}(\Omega))$ while its time derivative $S(u_\epsilon)_t$ is bounded in $L^{p'}(0, T; W^{-1,p'}(\Omega)) + L^1(Q)$.

Step 2. Convergence results In particular, we have found out that there exists a measurable function u in $L^\infty(0, T; L^1(\Omega)) \cap L^q(0, T; W_0^{1,q}(\Omega))$ for every $q < p - \frac{N}{N+1}$ such that $T_k(u)$ belongs to $L^p(0, T; W_0^{1,p}(\Omega))$ for every $k > 0$, and for a subsequence,

7 Arguing as in [84].

8 We borrow the argument from [5].

not relabeled, see [5, Proposition 5.2] for more details,

$$T_k(u_\epsilon) \rightharpoonup T_k(u) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \text{ strongly in } L^p(Q) \text{ and a.e. in } Q, \tag{3.74}$$

we deduce that

$$u_\epsilon \to u \text{ a.e. in } Q, \tag{3.75}$$

where the estimate (3.71) also imply that $u \in L^\infty(0,T;L^1(\Omega))$, and in addition

$$\int_Q \nabla u_\epsilon \chi_{\{|u_\epsilon|\leq k\}} \leq Ck, \quad \forall k > 0, \tag{3.76}$$

that is (*ii*) holds. One can prove using the ideas of [5, Proposition 5.2 (*Step 3*)] that ∇u_ϵ is a Cauchy sequence in measure, which yields that

$$\nabla u_\epsilon \to \nabla u \quad \text{a.e. in Q}, \tag{3.77}$$

and then, by (1.5) and Lemma 3.2, $a(t,x,u_\epsilon,\nabla u_\epsilon)$ is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ for every $q < p - \frac{N}{N+1}$. Moreover, by (*i*) and (*iii*), $a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)$ converges to $a(t,x,u,\nabla u)$ in the strong topology of $L^q(0,T;W_0^{1,q}(\Omega))$, $1 \leq q < p - \frac{N}{N+1}$. Finally, by (*ii*) and (2.2), the sequence $a(t,x,u_\epsilon,\nabla T_k(u_\epsilon))$ is bounded in $L^{p'}(Q)$, which easily implies that it converges to $a(t,x,u,\nabla T_k(u))$ in the weak topology of $L^{p'}(Q)$. Let us observe that, thanks to the assumption (1.4) on "a" and Vitali's theorem, we easily deduce that $a(t,x,u_\epsilon,\nabla u_\epsilon)$ is strongly compact in $L^1(Q)$. □

Actually, in the sequel we will prove that the renormalized solutions and their gradients satisfy somewhat more regularity and energy estimates. Let us first show the following interesting properties.

Proposition 3.4 *Let u be a renormalized solution of problem* (1.3). *Then, for every $k > 0$, we have*

$$\begin{cases} \lim\limits_{n\to\infty} \dfrac{1}{n}\displaystyle\int_Q |K(t,x,u)||\nabla T_n(u)|dxdt = 0, & (3.78)\\ \lim\limits_{n\to\infty} \dfrac{1}{n}\displaystyle\int_Q |H(t,x,u,\nabla u)||T_n(u)|dxdt = 0, & (3.79)\end{cases}$$

Proof Obviously, we can prove it without loss of generality for $n \in \mathbb{N}$. First of all, observe that thanks to (1.4), (3.3) and Proposition 3.3 (*ii*), using coercivity condition one can easily show that there exists a positive constant M such that

$$\sup_{t\in[0,T]}\int_{\Omega}|u_n(t)|dx+\int_0^t\int_{\Omega}|\nabla T_n(u)|^p dxdt\leq Mn+L,\quad \forall k>0,\quad \forall t\in[0,T]. \tag{3.80}$$

Therefore, arguing as in [46, Lemma A.1], this leads to a control of $|u_n|^{\frac{N(p-1)+p}{N+p}}$ with respect to M and L, which implies that $|\nabla u_n|^{\frac{N(p-1)+p}{N+2}}\in L^{\frac{N+2}{N+1},\infty}(Q)$, while from Hölder and Gagliardo inequalities we have $H(t,x,u,\nabla u)\in L^1(Q)$, we can improve this result by using the Lebesgue dominated convergence theorem and the fact that u is a.e. finite. Indeed, in this way we get (3.79). We are interested about a similar asymptotic behavior result on K; let us emphasize that assumption (1.7) leads to

$$\int_Q |K(t,x,u)||\nabla T_n(u)|dxdt\leq\int_Q c_0(t,x)|u|^{\gamma}|\nabla T_n(u)|dxdt$$
$$+\int_Q c_1(t,x)|\nabla T_{\epsilon}(u)|dxdt, \tag{3.81}$$

we can improve this estimate by using the Gagliardo–Nirenberg result, and so we can write

$$\int_Q c_0(t,x)|T_{\epsilon}(u)|^{\gamma}|\nabla T_{\epsilon}(u)|dxdt$$
$$\leq\left(\int_Q c_0^r(t,x)dxdt\right)^{\frac{1}{r}}\left(\int_Q |T_n(u)|^{\frac{(N+2)\rho}{N}}dxdt\right)^{\frac{N(p-1)}{p(N+p)}}\left(\int_Q|\nabla T_n(u)|^p dxdt\right)^{\frac{1}{p}}$$
$$\leq n^{\frac{1}{r}}C\|c_0\|_{L^r(Q)}\|u\|^{\frac{1}{r}}_{L^{\infty}(0,T;L^r(Q))}\left(\int_Q|\nabla u|^p dxdt\right)^{\frac{N+1}{N+p}}, \tag{3.82}$$

while $1-\frac{1}{r}=\frac{N+1}{N-p}$. So, finally, the energy condition (3.3), with assumption (1.4), imply that

$$\begin{cases}\lim\limits_{n\to+\infty}\dfrac{1}{n}\displaystyle\int_Q c_0(t,x)|T_n(u)|^{\gamma}|\nabla T_n(u)|dxdt=0,\\ \lim\limits_{n\to+\infty}\dfrac{1}{n}\displaystyle\int_Q c_1(t,x)|\nabla T_n(u)|dxdt=0,\end{cases} \tag{3.83}$$

and so we get the desired asymptotic behaviour result for the function K. □

3.3 Main results and comments

We explicitly note that the stability result in [5, Theorem 3.8] can be adapted to our case, but the focus in this study is on a new method.

Stability result Int this part, we consider a nonlinear parabolic problem which can be formally written as

$$\begin{cases} u_t - \operatorname{div}[a_\epsilon(t,x,u,\nabla u) + K_\epsilon(t,x,u)] = \mu_\epsilon \text{ in } Q := (0,T)\times\Omega, \\ u_0(t,x) = u_0(x) \text{ in } \Omega, \quad u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{3.84}$$

where ϵ belongs to a sequence of positive numbers that converges to zero and the function $a_\epsilon\colon (0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}^N$ is a *Carathéodory*[9] function which satisfies assumptions (1.4)–(1.6). Assume that there exists a function $a_0\colon (0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}^N$ satisfying hypotheses (1.4)–(1.6), and such that

$$\lim_{\epsilon\to 0} a_\epsilon(t,x,s_\epsilon,\zeta_\epsilon) = a_0(t,x,s,\zeta), \tag{3.85}$$

for every $(s_\epsilon,\zeta_\epsilon)\in\mathbb{R}\times\mathbb{R}^N$ converging to (s,ζ) and for a.e. $(t,x)\in Q$. Moreover, $K_\epsilon\colon (0,T)\times\Omega\times\mathbb{R}\mapsto\mathbb{R}^N$ is a *Carathéodory* function which satisfies the growth condition (3.20), i.e.,

$$K_\epsilon(t,x,s) \le c_0(t,x)|s|^\gamma + c_1(t,x), \tag{3.86}$$

for almost every $(t,x)\in Q$, and for every $s\in\mathbb{R}$, where c_0 and c_1 satisfy conditions of (3.20). Denote by $K\colon (0,T)\times\Omega\times\mathbb{R}\mapsto\mathbb{R}^N$ a *Carathéodory* function such that

$$\lim_{\epsilon\to 0} K_\epsilon(t,x,s_\epsilon) = K(t,x,s), \tag{3.87}$$

for every sequence $s_\epsilon\in\mathbb{R}$ such that s_ϵ tends to s a.e. Finally, we assume that μ_ϵ has a splitting $(f_\epsilon, F_\epsilon, \lambda_\epsilon^\oplus, \lambda_\epsilon^\ominus)$ converging to μ in the sense that, for every $\epsilon > 0$, the measure μ_ϵ can be decomposed as

$$\mu_\epsilon = f_\epsilon - \operatorname{div}(F_\epsilon) + \lambda_\epsilon^\oplus - \lambda_\epsilon^\ominus, \tag{3.88}$$

where the following convergences hold true:

(i) (f_ϵ) is a sequence of $C_c^\infty(Q)$-functions converging to f weakly in $L^1(Q)$;
(ii) (F_ϵ) is a is a sequence of $C_c^\infty(Q)^N$-functions converging to F strongly in $L^{p'}(Q)^N$;
(iii) $(\lambda_\epsilon^\oplus)$ is a sequence of nonnegative measures in $\mathbf{M}_b(Q)$ such that

[9] I.e., it is continuous with respect to s and ζ for a.e. $(t,x)\in Q$, and measurable with respect to (t,x) for every $s\in\mathbb{R}$ and $\zeta\in\mathbb{R}^N$.

$$\lambda_\epsilon^{\oplus} = \lambda_{\epsilon,d}^{1,\oplus} - \operatorname{div}(\lambda_{\epsilon,d}^{2,\oplus}) + \lambda_{\epsilon,c}^{\oplus}$$
$$\text{with } \lambda_{\epsilon,d}^{1,\oplus} \in L^1(Q),\ \lambda_{\epsilon,d}^{2,\oplus} \in L^{p'}(Q)^N \text{ and } \lambda_{\epsilon,c}^{\oplus} \in \mathbf{M}_c^+(Q), \tag{3.89}$$

that converges to μ_c^+ in the narrow topology of measures;

(iv) $(\lambda_\epsilon^{\ominus})$ is a sequence of nonnegative measures in $\mathbf{M}_b(Q)$ such that

$$\lambda_\epsilon^{\ominus} = \lambda_{\epsilon,d}^{1,\ominus} - \operatorname{div}(\lambda_{\epsilon,d}^{2,\ominus}) + \lambda_{\epsilon,c}^{\ominus}$$
$$\text{with } \lambda_{\epsilon,d}^{1,\ominus} \in L^1(Q),\ \lambda_{\epsilon,d}^{2,\ominus} \in L^{p'}(Q)^N \text{ and } \lambda_{\epsilon,c}^{\ominus} \in \mathbf{M}_c^-(Q), \tag{3.90}$$

that converges to μ_c^+ in the narrow topology of measures.

Moreover, let $u_0^\epsilon \in C_0^\infty(\Omega)$ that approaches u_0 in the sense of (3.26). Recall that these approximations can be easily obtained via the standard convolution arguments stated in Sect. 3.2.

Remark 3.3 If we decompose the measures μ_ϵ, $\lambda_\epsilon^{\oplus}$ and $\lambda_\epsilon^{\ominus}$ respectively as $\mu_\epsilon = \mu_{\epsilon,d} + \mu_{\epsilon,c}$, $\lambda_\epsilon^{\oplus} = \mu_{\epsilon,d}^{\oplus} + \mu_{\epsilon,c}^{\oplus}$ $(\lambda_{\epsilon,d}^{\oplus} = \lambda_{\epsilon,d}^{1,\oplus} - \operatorname{div}(\lambda_{\epsilon,d}^{2,\oplus}))$, $\lambda_\epsilon^{\ominus} = \lambda_{\epsilon,d}^{\ominus} + \lambda_{\epsilon,c}^{\ominus}$ $(\lambda_{\epsilon,d}^{\ominus} = \lambda_{\epsilon,d}^{1,\ominus} - \operatorname{div}(\lambda_{\epsilon,d}^{2,\ominus}))$, with $\mu_{\epsilon,d}, \lambda_{\epsilon,d}^{\oplus}, \lambda_{\epsilon,d}^{\ominus}$ in $\mathbf{M}_d(Q)$, and $\mu_{\epsilon,c}, \lambda_{\epsilon,c}^{\oplus}, \lambda_{\epsilon,c}^{\ominus}$ in $\mathbf{M}_c(Q)$, then clearly $\lambda_{\epsilon,d}^{\oplus}, \lambda_{\epsilon,d}^{\ominus}, \lambda_{\epsilon,c}^{\oplus}, \lambda_{\epsilon,c}^{\ominus}$ are nonnegative, $\mu_{\epsilon,d} = f_\epsilon - \operatorname{div}(F_\epsilon) + \lambda_{\epsilon,d}^{\oplus} - \lambda_{\epsilon,d}^{\ominus}$ and $\mu_{\epsilon,c} = \lambda_{\epsilon,c}^{\oplus} - \lambda_{\epsilon,c}^{\ominus}$. In particular, we have

$$0 \le \mu_{\epsilon,c}^+ \le \lambda_{\epsilon,c}^{\oplus}, \quad 0 \le \mu_{\epsilon,c}^- \le \lambda_{\epsilon,c}^{\ominus}. \tag{3.91}$$

Our first main result reads as follows.

Theorem 3.1 *Let (a_ϵ), a be functions satisfying (1.4)–(1.6) and (3.85), and (μ_ϵ) be a sequence of measures in $\mathbf{M}_b(Q)$ having a splitting $(f_\epsilon, F_\epsilon, \lambda_\epsilon^{\oplus}, \lambda_\epsilon^{\ominus})$ converging to μ. Assume that u_ϵ is a renormalized solution of*

$$\begin{cases} (u_\epsilon)_t - \operatorname{div}[a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon) + K_\epsilon(t,x,u_\epsilon)] = \mu_\epsilon \text{ in } Q := (0,T)\times\Omega, \\ u_\epsilon(0,x) = u_0^\epsilon(x) \text{ in } \Omega, \quad u_\epsilon(t,x) = 0 \text{ on } (0,T)\times\partial\Omega. \end{cases} \tag{3.92}$$

Then, up to a subsequence still denoted by ϵ, u_ϵ converges a.e. to u renormlized solution of problem

$$\begin{cases} u_t - \operatorname{div}[a_0(t,x,u,\nabla u) + K(t,x,u)] = \mu \text{ in } Q := (0,T)\times\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega, \quad u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega. \end{cases} \tag{3.93}$$

Moreover

$$T_k(u_\epsilon) \to T_k(u) \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)),\ \forall k > 0. \tag{3.94}$$

Remark 3.4 Note that:

(i) The stability result given by Theorem 3.1 is an extension of the stability result proved in [5, Theorem 3.8] (see also [89, Theorem 2] for a different proof). Indeed our result coincides exactly with the stability result of [84] where the term $-\text{div}(K(t, x, u))$ does not appear. Nevertheless the method we use to prove Theorem 3.1 is slightly different and more simplest.

(ii) By the growth assumption (3.86) and the convergence assumption (3.87) on K_ϵ, we deduce

$$|K(t, x, s)| \leq c_0(t, x)|s|^\gamma + c_1(t, x), \tag{3.95}$$

for a.e. $(t, x) \in Q$ and for every $s \in \mathbb{R}$.

(iii) If we replace the right-hand side by a more general datum $\mu - \text{div}(E)$, with $E \in L^{p'}(Q)^N$, Theorem 3.1 holds true under the same assumptions. Indeed $K_\epsilon(t, x, s)$ (resp. $K(t, x, s)$) can be replaced by $K_\epsilon(t, x, s) - E(t, x)$ (resp. $K(t, x, s) - E$) which satisfy conditions (3.86)–(3.87) (with c_1 replaced by $c_1 + |E|$).

(iv) The proof of Theorem 3.1 heavily needs conditions (3.85)–(3.88) [for example, (3.86)–(3.87) are crucial to obtain (4.24)–(4.25)]. Remark that the same assumptions are needed if one follows the proof of [84].

(v) We could prove Theorem 3.1 under the assumptions $u_0 \in L^1(\Omega)$ and $\mu \in \mathbf{M}_d(Q)$. Therefore, we have $\mu_c^+ = \mu_c^- = 0$, which imply

$$\lim_{n\to\infty}\lim_{\epsilon\to 0}\frac{1}{n}\int_{\{n<|u_\epsilon|<2n\}} a_\epsilon(t, x, u_\epsilon, \nabla u_\epsilon)\cdot\nabla u_\epsilon dxdt = 0. \tag{3.96}$$

Furthermore, we can state a result which concerns right-hand sides $\mu + \text{div}(E)$, which belong to $\mathbf{M}_b(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega))$ ($E \in L^{p'}(Q)^N$), by using similar arguments to those used in [50,72]. Our second main result of the present paper is the following existence result which is a generalization of the existence result of [46].

Theorem 3.2 *Under assumptions* (1.4)–(1.11), *there exists a renormalized solution u of problem* (1.3).

Remark 3.5 The stationary version of such existence result is studied under three conditions:

1. $\gamma = \lambda = p - 1$ with $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$, $r < +\infty$ and $\|b_0\|_{L^{N,1}(\Omega)}$ is small enough.
2. $\gamma = p - 1$, $\lambda < p - 1$ with $c_0 \in L^{\frac{N}{p-1},r}(\Omega)$ and $r < +\infty$.
3. $\gamma < p - 1$, $\lambda < p - 1$ with $c_0 \in L^{\frac{N}{p-1},\infty}(\Omega)$.

It is worth observing that the class in which the stationary problem is studied is "natural", since

(i) If $1 < p \leq 2$, we have

$$\inf\left\{\frac{(N+2)(p-1)}{N+p}, \frac{N(p-1)+p}{N+2}\right\} \geq p - 1. \tag{3.97}$$

(ii) If $p \geq 2$, we have

$$\sup \left\{ \frac{(N+2(p-1))}{N+p}, \frac{N(p-1)+p}{N+2} \right\} \leq p-1, \tag{3.98}$$

then $|u|^{p-1}$ belongs to $L^{\frac{N}{N-p},\infty}(\Omega)$ and $|\nabla u|^{p-1}$ belongs to $L^{\frac{N}{N-p},\infty}(\Omega)$, and therefore the a priori estimate is depending on M, i.e.,

$$\left\| |u|^{p-1} \right\|_{L^q(Q)} \leq CM, \quad \forall q < \frac{N}{N-p}, \tag{3.99}$$

which in our parabolic case is equivalent to the control of $|u|^{\frac{N(p-1)+p}{N+p}}$ with respect to M and L, which means that in the evolution case smallness conditions on b and c when $\gamma = \delta = p-1$ seems to be unnecessary to obtain the existence of a solution.

(iii) Actually, in the last case, we can improve a little bit the complexity of the right-hand side; indeed we are able to take a derivative part $g \in L^p(0,T; W_0^{1,p}(\Omega) \cap L^2(\Omega))$ in the decomposition of μ where the proof relies on a change of unknown $w = u - g$, see [53,54].

4 Proofs of stability/existence results (Theorems 3.1, 3.2)

4.1 Proof of stability result (Theorem 3.1)

As before, the main tool, in order to prove the existence of a renormalized solution relies on approximating our problem with a more regular one [i.e., (3.84)] in bounded domains and in proving the existence of a solution via a priori estimates and strong convergence of truncations in $L^p(0,T; W_0^{1,p}(\Omega))$. In what follows we will indicate by ϵ_n a generic sequence that converges to zero as n goes to infinity. We need to define, for any $\delta > 0$, the two "*cut-off*" functions ψ_δ^+ and ψ_δ^- belonging to $C_0^\infty(Q)$ in order to localize some integrals near the support of $\mu_c \in \mathbf{M}_c(Q)$. This is possible by virtue of the following lemmas provided in [84, Lemma 5], and introduced in [51].

Lemma 4.1 *Let μ_c be a measure in $\mathbf{M}_c(Q)$, and let μ_c^+, μ_c^- be respectively the positive and the negative parts of μ_c. Then for every $\delta > 0$, there exist two functions ψ_δ^+, ψ_δ^- in $C_0^1(Q)$, such that the following assertions hold true:*

(i) $0 \leq \psi_\delta^+ \leq 1$ *and* $0 \leq \psi_\delta^- \leq 1$ *on* Q*;*
(ii) $\lim_{\delta \to 0} \psi_\delta^+ = \lim_{\delta \to 0} \psi_\delta^- = 0$ *strongly in* $L^p(0,T; W_0^{1,p}(\Omega))$ *and weakly* in* $L^\infty(Q)$*;*
(iii) $\lim_{\delta \to 0} (\psi_\delta^+)_t = \lim_{\delta \to 0} (\psi_\delta^-)_t = 0$ *strongly in* $L^{p'}(0,T; W^{-1,p'}(\Omega)) + L^1(Q)$*;*
(iv) $\int_Q \psi_\delta^- d\mu_s^+ \leq \delta$ *and* $\int_Q \psi_\delta^+ d\mu_s^- \leq \delta$*;*
(v) $\int_Q (1 - \psi_\delta^+ \psi_\eta^+) d\mu_s^+ \leq \delta + \eta$ *and* $\int_Q (1 - \psi_\delta^- \psi_\eta^-) d\mu_s^- \leq \delta + \eta$ *for all* $\eta > 0$.

Lemma 4.2 *Let μ_c be a measure in $\mathbf{M}_c(\Omega)$, decomposed as $\mu_c = \mu_c^+ - \mu_c^-$, with μ_s^+ and μ_c^- are concentrated on two disjoint subsets E^+ and E^- of zero (b, p)-capacity. Then, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_c^+(E^+ \backslash K_\delta^+) \leq \delta, \quad \mu_c^-(E^- \backslash K_\delta^-) \leq \delta, \tag{4.1}$$

and there exist ψ_δ^+, $\psi_\delta^- \in C_0^1(Q)$, such that

$$\psi_\delta^+, \psi_\delta^- \equiv 1 \text{ respectively on } K_\delta^+, K_\delta^-, \tag{4.2}$$

$$0 \leq \psi_\delta^+, \psi_\delta^- \leq 1, \tag{4.3}$$

$$\operatorname{supp}(\psi_\delta^+) \cap \operatorname{supp}(\psi_\delta^-) \equiv \emptyset. \tag{4.4}$$

Moreover

$$\|\psi_\delta^+\|_S \leq \delta, \quad \|\psi_\delta^-\|_S \leq \delta, \tag{4.5}$$

and, in particular, there exists a decomposition of $(\psi_\delta^+)_t$ and a decomposition of $(\psi_\delta^-)_t$ such that

$$\|(\psi_\delta^+)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^+)_t^2\|_{L^1(Q)} \leq \frac{\delta}{3}, \tag{4.6}$$

$$\|(\psi_\delta^-)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^2\|_{L^1(Q)} \leq \frac{\delta}{3}, \tag{4.7}$$

*and both ψ_δ^+ and ψ_δ^- converge to zero *weakly in $L^\infty(Q)$, in $L^1(Q)$, and up to subsequences, a.e. as δ vanishes. Moreover, if $\lambda_\epsilon^\oplus$ and $\lambda_\epsilon^\ominus$ are as in (3.89)–(3.90) we have*

$$\int_Q \psi_\delta^- d\lambda_\epsilon^\oplus = \omega(\epsilon, \delta), \quad \int_Q \psi_\delta^- d\mu_c^+ \leq \delta, \tag{4.8}$$

$$\int_Q \psi_\delta^+ d\lambda_\epsilon^\ominus = \omega(\epsilon, \delta), \quad \int_Q \psi_\delta^+ d\mu_c^- \leq \delta, \tag{4.9}$$

$$\int_Q (1 - \psi_\delta^+ \psi_\eta^+) d\lambda_\epsilon^\oplus = \omega(\epsilon, \delta, \eta), \quad \int_Q (1 - \psi_\delta^+ \psi_\eta^+) d\mu_c^+ \leq \delta + \eta, \tag{4.10}$$

$$\int_Q (1 - \psi_\delta^- \psi_\eta^-) d\lambda_\epsilon^\ominus = \omega(\epsilon, \delta, \eta), \quad \int_Q (1 - \psi_\delta^- \psi_\eta^-) d\mu_c^- \leq \delta + \eta. \tag{4.11}$$

Proof See [84, Lemma 5]. □

Remark 4.1 If $\lambda_\epsilon^\oplus$ and $\lambda_\epsilon^\ominus$ satisfy (3.89)–(3.90), and ψ_δ^- and ψ_δ^+ are the functions defined in Lemma 4.1, as an easy consequence of the narrow convergence we obtain

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q \psi_\delta^- d\lambda_\epsilon^\oplus = 0, \quad \lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q \psi_\delta^+ d\lambda_\epsilon^\ominus = 0, \tag{4.12}$$

$$\lim_{\eta\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q (1-\psi_\delta^+\psi_\eta^+) d\lambda_\epsilon^\oplus = 0, \quad \lim_{\eta\to 0}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_Q (1-\psi_\delta^-\psi_\eta^-) d\lambda_\epsilon^\ominus = 0. \tag{4.13}$$

Proof of Theorem 3.1 At this point, u_ϵ is a renormalized solution of (3.92) and u is a measurable function u such that $T_k(u) \in L^2(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$ where the convergences of Proposition 3.3 hold. We have to prove that u is a renormalized solution to (3.93). By proposition 3.3 (ii), the first condition of Definition 3.1 is satisfied while by (3.71) and Lemma 3.2, we obtain that u satisfies the second condition of Definition 3.1. Hence, it is enough to prove (3.1)-(3.3). Let $S \in W^{2,\infty}(\mathbb{R})$ and $\varphi \in C_0^1([0,T]\times\Omega)$, we choose $S'(u_\epsilon)\varphi$ as test function in the equation solved by u_ϵ, obtaining

$$\begin{aligned}
&-\int_\Omega S(u_0^\epsilon)\varphi(0)dx - \int_0^T \langle u_\epsilon^t, S(u_\epsilon)\rangle dt + \int_Q S'(u_\epsilon)a_\epsilon(t,x,u_\epsilon,\nabla_\epsilon)\cdot\nabla\varphi dxdt \\
&+\int_Q S''(u_\epsilon)a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon\varphi dxdt + \int_Q S''(u_\epsilon)K_\epsilon(t,x,u_\epsilon)\cdot\nabla u_\epsilon\varphi dxdt \\
&+\int_Q S'(u_\epsilon)K_\epsilon(t,x,u_\epsilon)\cdot\nabla\varphi dxdt = \int_Q f_\epsilon S'(u_\epsilon)\varphi dxdt \\
&+\int_Q F_\epsilon\cdot\nabla u_\epsilon S''(u_\epsilon)\varphi dxdt \\
&+\int_Q S'(u_\epsilon)F_\epsilon\cdot\nabla\varphi dxdt + \int_Q S'(u_\epsilon)\varphi d\lambda_{\epsilon,d}^\oplus - \int_Q S'(u_\epsilon)\varphi d\lambda_{\epsilon,d}^\ominus,
\end{aligned} \tag{4.14}$$

for every $\varphi \in \mathbb{W}\cap L^\infty(Q)$, for all $S \in W^{2,\infty}(\mathbb{R})$ with compact support in $\mathbb{R}$, which are such that $S'(u_\epsilon)\varphi \in \mathbb{W}$. It suffices to follow the lines of the long and not easy proof [51, Section 5-8] for the elliptic case, [83, Section 7] and [5, Section 6] for the parabolic case. The assumptions on a_ϵ and the choice of $B_n(u_\epsilon)T_k(u_\epsilon)$, for every $k>0$, as test function in (3.92) where B_n is defined by (see Fig. 4)

$$B_n(s) = \begin{cases} 0 & \text{if } |s| > 2n, \\ \frac{2n-|s|}{n} & \text{if } n < |s| \le 2n, \\ 1 & \text{if } |s| \le n. \end{cases} \tag{4.15}$$

Therefore, using similar calculations to those of Proposition 3.3, we get

$$\|T_k(u_\epsilon)\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla T_k(u_\epsilon)\|^p_{L^p(Q)^N} \le \widetilde{M}k + \widetilde{L}, \quad \forall k>0, \tag{4.16}$$

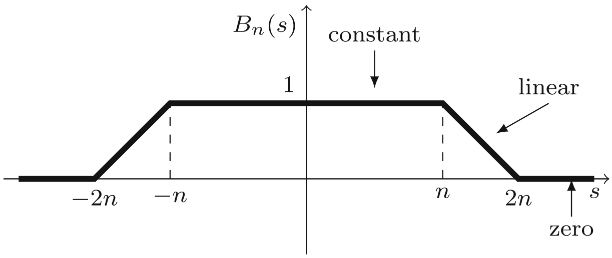

Fig. 4 The function $B_n(s)$

for some $\widetilde{M} > 0$ and $\widetilde{L} > 0$. This implies by, Proposition 3.1, the following a priori estimates for the renormalized solutions: u_ϵ

$$\begin{cases} \left\| |u_\epsilon|^{p-1} \right\|_{L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q)} \leq C, \\ \left\| |\nabla u_\epsilon|^{p-1} \right\|_{L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q)} \leq C, \end{cases} \tag{4.17}$$

some constant C independent of ϵ but depending on the data of the problem. Estimate (4.16) and the growth assumption on K_ϵ, since the operator is strictly monotone, allow us to use standard techniques of [5,24,84] which imply that there exists a measurable function $u\colon Q \mapsto \overline{\mathbb{R}}$, finite a.e. in Q and such that, up to a subsequence still denoted by ϵ,

$$\begin{cases} T_k(u_\epsilon) \rightharpoonup T_k(u) \text{ weakly in } L^p(0,T; W_0^{1,p}(\Omega)), \quad \forall k > 0, , & (4.18) \\ u_\epsilon \to u \text{ a.e. in } Q, & (4.19) \\ \nabla u_\epsilon \to \nabla u \text{ a.e. in } Q, & (4.20) \end{cases}$$

as ϵ tends to zero.

Step 1. The function u is a solution of (3.93) *in the sense of distributions* By assumption (3.87) and (4.19), it follows that $K_\epsilon(t,x,u_\epsilon)$ converges to $K(t,x,u)$ a.e. in Q. Moreover, the growth assumption (3.95) on K_ϵ and the estimate (4.17) on $|u_\epsilon|^{p-1}$ imply that $|K_\epsilon(t,x,u_\epsilon)|$ is bounded in $L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q)$. Thanks to the a.e. convergence of u_ϵ in Q and to the Lebesgue theorem, we get

$$K_\epsilon(t,x,u_\epsilon) \to K(t,x,u) \text{ strongly in } L^{p'}(Q), \quad \forall p > 1. \tag{4.21}$$

Proceeding in a similar way, by using (1.5), (4.17) and (4.19)–(4.20), we get

$$a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon) \to a_0(t,x,u,\nabla u) \text{ strongly in } L^{p'}(Q)^N, \quad \forall p > 1. \tag{4.22}$$

Moreover, since u_ϵ is also a solution of (3.6) in distributional sense, this gives

$$\int_0^T \langle (u_\epsilon)_t, \varphi\rangle dt + \int_Q a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla\varphi dxdt \\ + \int_Q K_\epsilon(t,x,u_\epsilon)\cdot\nabla\varphi dxdt = \int_Q \varphi d\mu_\epsilon, \tag{4.23}$$

for all $\varphi \in C_0^\infty(Q)$. So, using the convergence results (4.21)–(4.22), we can pass to the limit in (4.23) obtaining that u is a solution of (3.93) in the distributional sense (for the convergence of the first and last terms, we refer to [5]).

Step 2. Asymptotic behaviour results onK_ϵ Our next purpose is to prove that

$$\begin{cases} \lim\limits_{n\to+\infty} \limsup\limits_{\epsilon\to 0} \dfrac{1}{n}\displaystyle\int_{\{n<|u_\epsilon|<2n\}} |K_\epsilon(t,x,u_\epsilon)|\,|\nabla u_\epsilon|\,dxdt = 0, & (4.24)\\ \lim\limits_{n\to+\infty} \dfrac{1}{n}\displaystyle\int_{\{n<|u|<2n\}} |K(t,x,u)|\,|\nabla u|\,dxdt = 0. & (4.25)\end{cases}$$

Using the Gagliardo–Nirenberg inequality and the growth condition (3.85) on K_ϵ, we get

$$\begin{aligned} &\frac{1}{n}\int_{\{n<|u_\epsilon|<2n\}} |K_\epsilon(t,x,u_\epsilon)||\nabla u_\epsilon|dxdt \\ &\quad\le \frac{1}{n}\int_{\{n<|u_\epsilon|<2n\}} c_0|u_\epsilon|^\gamma|\nabla u_\epsilon|dxdt + \int_{\{n<|u_\epsilon|<2n\}} c_1|\nabla u_\epsilon|dxdt \\ &\quad\le \frac{1}{n}\int_{\{n<|u_\epsilon|<2n\}} c_0|T_{2n}(u_\epsilon)|^\gamma|\nabla T_{2n}(u_\epsilon)|dxdt + \int_{\{n<|u_\epsilon|<2n\}} c_1|\nabla T_{2n}(u_\epsilon)|dxdt \\ &\qquad + \frac{1}{n^{\frac{1}{p'}}}\|c_1\|_{L^{p'}(Q)}\frac{1}{n^{\frac{1}{p}}}\|\nabla T_{2n}(u_\epsilon)\|_{L^p(Q)^N} \\ &\quad\le C\|c_0\|_{L^\tau(\{n<|u_\epsilon|<2n\})} + C\frac{1}{n^{\frac{1}{p'}}}\|c_1\|_{L^{p'}(Q)}. \end{aligned} \tag{4.26}$$

On the other hand, since $\|c_0\|_{L^\tau(\{n<|u_\epsilon|<2n)}$ tends to zero when firstly ϵ goes to zero and then n goes to infinity, we use Fatou's lemma to obtain (4.24). While, since u_ϵ converges to u a.e. and u is finite a.e. in Q, we get (4.25).

Step 3. Asymptotic behaviour results onμ_c Now, we are able to prove that

$$\limsup_{n\to\infty}\limsup_{\epsilon\to 0}\frac{1}{n}\int_{\{n<u_\epsilon<2n\}} a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon\varphi dxdt \le \int_Q \varphi d\mu_c^+, \tag{4.27}$$

$$\limsup_{n\to\infty}\limsup_{\epsilon\to 0}\frac{1}{n}\int_{\{-2n<u_\epsilon<-n\}} a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon\varphi dxdt \le \int_Q \varphi d\mu_c^-, \tag{4.28}$$

for every nonnegative $\varphi \in C^1(\overline{Q})$ by using a technical argument slightly different from [84, Lemma 6] and [5, Lemma 6.6 (ii)]. To this aim, we define, for $n \geq 1$, the functions $s_n : \mathbb{R} \mapsto \mathbb{R}$ and $h_\eta : \mathbb{R} \mapsto \mathbb{R}$, by

$$s_n(s) = \frac{T_{2n}(s) - T_n(s)}{n}, \quad h_\eta(s) = 1 - |s_\eta(s)|. \tag{4.29}$$

Choosing $h_\eta(u_\epsilon)s_n(u_\epsilon^+)\varphi$ ($\varphi \in C^1(Q)$ nonnegative) as test function in (4.14), by using the notations of (2.1), and letting η tends to infinity, we get

$$\begin{aligned}
&\int_0^T \langle (u_\epsilon)_t, s_n(u^+)\varphi \rangle dt && (A)\\
&\quad + \int_Q a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot \nabla\varphi s_n(u_\epsilon^+)dxdt && (B)\\
&\quad + \frac{1}{n}\int_{\{n<u_\epsilon<2n\}} a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot \nabla u_\epsilon \varphi dxdt && (C)\\
&\quad + \int_Q K_\epsilon(t,x,u_\epsilon)\cdot \nabla\varphi s_n(u_\epsilon^+)dxdt && (D)\\
&\quad + \frac{1}{n}\int_{\{n<u_\epsilon<2n\}} K_\epsilon(t,x,u_\epsilon)\cdot \nabla u_\epsilon \varphi dxdt && (E)\\
&= \int_Q f_\epsilon s_n(u_\epsilon^+)\varphi dxdt && (F)\\
&\quad + \int_Q F_\epsilon \cdot \nabla\varphi s_n(u_\epsilon^+)dxdt && (G)\\
&\quad + \frac{1}{n}\int_{\{n<u_\epsilon<2n\}} F_\epsilon \cdot \nabla u_\epsilon \varphi dxdt && (H)\\
&\quad + \int_Q s_n(u_\epsilon^+)\varphi d\lambda_{\epsilon,d}^{\oplus} && (I)\\
&\quad + \int_Q \varphi d\mu_{\epsilon,c}^{+} && (J)\\
&\quad - \int_Q s_n(u_\epsilon^+)\varphi d\lambda_{\epsilon,d}^{\ominus} && (K)
\end{aligned} \tag{4.30}$$

for every nonnegative $\varphi \in C^1(\overline{Q})$. It remains to pass to the limit in (4.30), first as ϵ tends to zero and then as n goes to infinity, since $s_n(u_\epsilon^+)$ is bounded by 1, we get

$$s_n(u_\epsilon^+) \to s_n(u^+) \text{ a.e. and weakly* in } L^\infty(Q). \tag{4.31}$$

Hence, by (4.21)–(4.22) and Lebesgue convergence theorem, we can pass to the limit in (B) and (D) using the fact that $a(t,x,u,\nabla u)$ and $K(t,x,u)$ belong to $L^{p'}(Q)^N$ for $p > 1$, $s_n(u_\epsilon^+)$ is bounded by 1 and tends to zero a.e. in Q, and the fact that

$\varphi \in C^1(\overline{Q})$, we get

$$\begin{aligned}\lim_{n\to\infty}\lim_{\epsilon\to 0}(B) &= \lim_{n\to\infty}\lim_{\epsilon\to 0}\int_Q a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla\varphi s_n(u_\epsilon^+)dxdt\\ &= \lim_{n\to\infty}\int_Q a_0(t,x,u,\nabla u)\cdot\nabla\varphi s_n(u^+)dxdt\\ &= 0,\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}\lim_{n\to\infty}\lim_{\epsilon\to 0}(D) &= \lim_{n\to\infty}\lim_{\epsilon\to 0}\int_Q K_\epsilon(t,x,u_\epsilon)\cdot\nabla\varphi s_n(u_\epsilon^+)dxdt\\ &= \lim_{n\to\infty}\int_Q s_n(u^+)K(t,x,u)\cdot\nabla\varphi dxdt\\ &= 0.\end{aligned} \tag{4.33}$$

By (4.24), as n, ϵ tend to their limits, we obtain

$$\lim_{n\to\infty}\lim_{\epsilon\to 0}(E) = \lim_{n\to\infty}\lim_{\epsilon\to 0}\frac{1}{n}\int_{\{n<|u_\epsilon|<2n\}}|K_\epsilon(t,x,u_\epsilon)||\nabla u_\epsilon|\varphi dxdt = 0. \tag{4.34}$$

On the other hand, by properties (3.88) (i)–(ii) and in virtue of the a.e. convergence of $s_n(u_\epsilon^+)$ to $s_n(u^+)$, $|s_n(u_\epsilon^+)| \leq 1$ a.e. in Q and Proposition 2.1, we obtain

$$\begin{aligned}\lim_{n\to\infty}\lim_{\epsilon\to 0}(F) &= \lim_{n\to\infty}\lim_{\epsilon\to 0}\int_Q f_\epsilon s_n(u_\epsilon^+)\varphi dxdt\\ &= \lim_{n\to\infty}\int_Q f_\epsilon s_n(u^+)\varphi dxdt = 0,\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}\lim_{n\to\infty}\lim_{\epsilon\to 0}(G) &= \lim_{n\to\infty}\lim_{\epsilon\to 0}\int_Q F_\epsilon\cdot\nabla\varphi s_n(u_\epsilon^+)dxdt\\ &= \lim_{n\to\infty}\int_Q F\cdot\nabla\varphi s_n(u^+)dxdt = 0.\end{aligned} \tag{4.36}$$

Indeed, by Hölder's inequality, we write

$$\begin{aligned}&\frac{1}{n}\left|\int_{\{n<u_\epsilon<2n\}}F_\epsilon\cdot\nabla u_\epsilon dxdt\right|\\ &\leq \|\varphi\|_{L^\infty(Q)}\frac{1}{n^{\frac{1}{p'}}}\|F_\epsilon\|_{L^{p'}(Q)^N}\left(\frac{1}{n}\int_{\{n<u_\epsilon<2n\}}|\nabla u_\epsilon|^p\right)^{\frac{1}{p}},\end{aligned} \tag{4.37}$$

so, using (4.16), we get

$$\lim_{n\to\infty}\lim_{\epsilon\to 0}(H) = \lim_{n\to\infty}\lim_{\epsilon\to 0}\frac{1}{n}\int_{\{n<u_\epsilon<2n\}} F_\epsilon \cdot \nabla u_\epsilon \varphi dxdt = 0. \tag{4.38}$$

Recalling that $\mu^{\oplus}_{\epsilon,d}$, $\lambda^{\oplus}_{\epsilon,c}$, $\lambda^{\ominus}_{\epsilon,d}$ and φ are nonnegative and using (3.91) (observe that $0 \leq s_n(u^+_\epsilon) \leq 1$), we get

$$\begin{cases} (I) + (J) = \int_Q s_n(u^+_\epsilon)\varphi d\lambda^{\oplus}_{\epsilon,d} + \int_Q \varphi d\mu^+_{c,\epsilon} \leq \int_Q \varphi d\lambda^{\oplus}_{\epsilon,d} + \int_Q \varphi d\lambda^{\oplus}_{\epsilon,c}, \\ (K) = -\int_Q s_n(u^+_\epsilon)\varphi d\lambda^{\ominus}_{\epsilon,d} \leq \int_Q \varphi d\lambda^{\oplus}_\epsilon. \end{cases} \tag{4.39}$$

Collecting together (4.30)–(4.39), we conclude for every nonnegative $\varphi \in C^1(\overline{Q})$, that

$$\frac{1}{n}\int_{\{n<u_\epsilon<2n\}} a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\varphi dxdt \leq \int_Q \varphi d\lambda^{\oplus}_\epsilon + \omega(\epsilon,n). \tag{4.40}$$

Since $\lambda^{\oplus}_\epsilon$ converges to μ^+_c in the narrow topology of measures, we obtain (4.27). The second asymptotic estimate (4.28) is obtained by using a similar argument with test functions $h_\eta(u_\epsilon)s_n(u^-_\epsilon)\varphi$ for any $\varphi \in C^1(Q)$ nonnegative.

Step 4. The function u is a renormalized solution of (3.93) Since $\mu_c = \mu^+_c - \mu^-_c$ is a concentrated measure, for every $\delta > 0$, there exist two compacts sets $K^\pm_\delta \subseteq E^\pm$ and two sequences of $C^\infty_c(Q)$-functions $\{\psi^\pm_\delta\}$ with properties of Lemmas 4.1 and 4.2. Now, consider a function $S \in W^{2,\infty}$ such that S' has compact support on $\mathbb{R}$ ($S(0) = 0$), and choose $B_n(u_\epsilon)S'(u)\varphi(1-\psi^+_\delta-\psi^-_\delta)$ as test function in (4.14) where $\varphi \in \mathbb{W} \cap L^\infty(Q)$, $S'(u)\varphi \in \mathbb{W}$ and B_n is defined in (4.15), we can write

$$\begin{aligned}
&\int_0^T \langle (u_\epsilon)_t, B_n(u_\epsilon)S'(u)\varphi(1-\psi^+_\delta-\psi^-_\delta)\rangle dt && (A^\epsilon_\delta)\\
&\quad+\int_Q B'(u_\epsilon)S''(u)\varphi(1-\psi^+_\delta-\psi^-_\delta)[a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)+K_\epsilon(t,x,u_\epsilon)]\cdot\nabla u_\epsilon dxdt && (B^\epsilon_\delta)\\
&\quad+\int_Q [a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)+K_\epsilon(t,x,u_\epsilon)]\cdot\nabla\varphi S''(u)B_n(u_\epsilon)\varphi(1-\psi^+_\delta-\psi^-_\delta)dxdt && (C^\epsilon_\delta)\\
&\quad+\int_Q [a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)+K_\epsilon(t,x,u_\epsilon)]S'(u)B_n(u_\epsilon)(1-\psi^+_\delta-\psi^-_\delta)dxdt && (D^\epsilon_\delta)\\
&\quad+\int_Q [a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)+K_\epsilon(t,x,u_\epsilon)]\cdot\nabla(1-\psi^+_\delta-\psi^-_\delta)S'(u)B_n(u_\epsilon)\varphi dxdt && (E^\epsilon_\delta)\\
&=\int_Q f_\epsilon B_n(u_\epsilon)S'(u)\varphi(1-\psi^+_\delta-\psi^-_\delta)dxdt && (F^\epsilon_\delta)\\
&\quad+\int_Q F_\epsilon\cdot\nabla[B_n(u_\epsilon)S'(u)\varphi(1-\psi^+_\delta-\psi^-_\delta)]dxdt && (G^\epsilon_\delta)
\end{aligned}$$

$$+\int_Q B_n(u_\epsilon)S'(u)\varphi(1-\psi_\delta^+-\psi_\delta^-)d\lambda_{\epsilon,d}^\oplus \qquad (H_\delta^\epsilon)$$
$$+\int_Q B_n(u_\epsilon)S'(u)\varphi(1-\psi_\delta^+-\psi_\delta^-)d\lambda_{\epsilon,d}^\ominus. \tag{4.41}$$

Now, we want to pass to the limit (as ϵ tends to zero, n goes to infinity and δ tends to zero). It is easy to deal with the first integral (A_δ) using an integration by parts formula and the fact that $\varphi(T,x)=0$. Since (u_0^ϵ) converges to u_0 in $L^1(\Omega)$, $S(u_\epsilon)$ converges to $S(u)$ strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ and weakly* in $L^\infty(Q)$, it follows that

$$\int_\Omega \varphi(0)S(u_0)dx-\int_Q \varphi_t S(u)dt=(A_\delta^\epsilon)+\omega(n,\delta). \tag{4.42}$$

Moreover, from (4.24), we have

$$\left|\int_Q B_n'(u_\epsilon)S'(u)\varphi(1-\psi_\delta^+-\psi_\delta^-)K_\epsilon(t,x,u_\epsilon)\cdot\nabla u_\epsilon dxdt\right|=\omega(\epsilon,n). \tag{4.43}$$

Hence, by (4.27) and the fact that $a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon(1-\psi_\delta^+-\psi_\delta^-)$ is positive, we deduce

$$\begin{aligned}0&\le \limsup_{n\to+\infty}\limsup_{\epsilon\to 0}\frac{1}{n}\int_{\{n<|u_\epsilon|<2n\}} a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)|S'(u)\varphi|(1-\psi_\delta^+-\psi_\delta^-)dxdt\\&\le \|S'\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(Q)}\left[\int_Q(1-\psi_\delta^+-\psi_\delta^-)d\mu_c^+ +\int_Q(1-\psi_\delta^+-\psi_\delta^-)d\mu_c^-\right].\end{aligned} \tag{4.44}$$

we can use properties (4.8)–(4.11) to conclude that

$$\frac{1}{n}\int_{\{n<|u_\epsilon|<2n\}} a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\cdot\nabla u_\epsilon|S'(u)\varphi|(1-\psi_\delta^+-\psi_\delta^-)dxdt=\omega(\epsilon,n,\delta). \tag{4.45}$$

Since, as the sequence $T_{2n}(u_\epsilon)$ converges weakly to $T_{2n}(u)$ in $L^p(0,T;W_0^{1,p}(\Omega))$, we have $a_\epsilon(t,x,T_{2n}(u_\epsilon),\nabla T_{2n}(u_\epsilon))$ converges to $a_0(t,x,T_{2n}(u),\nabla T_{2n}(u))$ weakly in $L^{p'}(Q)^N$, and recall that $B_n(u_\epsilon)$ is bounded by 1 and converges a.e. to $B_n(u)$, we get

$$\int_Q[a_0(t,x,u,\nabla u)+K(t,x,u)]S''(u)B_n(u)\varphi(1-\psi_\delta^+-\psi_\delta^-)dxdt=(B_\delta^\epsilon)+\omega(\epsilon). \tag{4.46}$$

Now, we can use properties of the function S to replace the first integral in (4.46) by $a_0(t,x,T_M(u),\nabla T_M(u))\in L^{p'}(Q)^N$, and applying convergence properties of $\psi_\delta^\pm$ to obtain

$$\int_Q S'(u)[a_0(t,x,u,\nabla u)+K(t,x,u)]\cdot\nabla\varphi dxdt=(B_\delta^\epsilon)+\omega(\epsilon,n,\delta). \quad (4.47)$$

In a similar way

$$\int_Q [a_0(t,x,u,\nabla u)+K(t,x,u)]\cdot\nabla\varphi dxdt=(C_\delta^\epsilon)+\omega(\epsilon,n,\delta), \quad (4.48)$$

and

$$\int_Q [a_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)+K_\epsilon(t,x,u_\epsilon)]\cdot\nabla(1-\psi_\delta^+-\psi_\delta^-)S'(u)B_n(u_\epsilon)dxdt$$
$$=\omega(\epsilon,n,\delta). \quad (4.49)$$

Furthermore Proposition 2.1, the a.e. convergence of u_ϵ to u, property (4.5) on $\psi_\delta^\pm$ and the definition of $S'(u_\epsilon)$, imply

$$\int_Q S'(u)f\varphi dxdt=(F_\delta^\epsilon)+\omega(\epsilon,n,\delta). \quad (4.50)$$

In a similar way

$$\int_Q F\cdot\nabla[S'(u)\varphi]dxdt=(G_\delta^\epsilon)+\omega(\epsilon,n,\delta). \quad (4.51)$$

Using the fact that $\lambda_{\epsilon,d}^\oplus$ is nonnegative, we have

$$|(H_\delta^\epsilon)|\le\|S'\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(Q)}\int_Q(1-\psi_\delta^+-\psi_\delta^-)d\mu_{d,\epsilon}^\oplus. \quad (4.52)$$

Therefore, by (4.8)–(4.11) and the fact that $0\le\lambda_{\epsilon,d}^\oplus\le\lambda_\epsilon^\oplus$, we get

$$|(H_\delta^\epsilon)|=\omega(\epsilon,n,\delta). \quad (4.53)$$

Similarly

$$|(I_\delta^\epsilon)|=\omega(\epsilon,n,\delta). \quad (4.54)$$

So that, we can pass to the limit in each term of (4.41) to obtain that u satisfies (3.1) of Definition 3.1. To conclude the proof of our main result, it remains to check condition (3.3) of Definition 3.1. Since $a_0(t,x,u,\nabla u)\cdot\nabla u$ is positive and using Proposition 2.2, we get that condition (3.3) holds for $\varphi \in C^\infty(\overline{Q})$. On the other hand (4.27), the a.e. convergence of u_ϵ to u and Fatou's Lemma imply

$$\begin{cases} \limsup_{n\to+\infty} \frac{1}{n}\int_{\{n<u<2n\}} a_0(t,x,u,\nabla u)\cdot\nabla u\varphi dxdt \le \int_Q \varphi d\mu_c^+, \\ \limsup_{n\to+\infty} \frac{1}{n}\int_{\{-2n<u<-n\}} a_0(t,x,u,\nabla u)\cdot\nabla u\varphi dxdt \le \int_Q \varphi d\mu_c^-, \end{cases} \tag{4.55}$$

for every $\varphi \in C^1(\overline{Q})$ nonnegative. Moreover, since u satisfies (3.92) in the sense of distributions, one can use $B_n(u)\psi$, with $\psi \in C_0^\infty(Q)$ and B_n is defined in (4.15), as test function in (4.35) and let n tends yo infinity, to get

$$\lim_{n\to\infty}\int_Q B_n'(u)a_0(t,x,u,\nabla u)\cdot\nabla u\psi dxdt$$
$$= -\int_Q \psi d\mu_s^+ + \int_Q \psi d\mu_s^-, \quad \forall\psi \in C_0^\infty(Q). \tag{4.56}$$

Now, let $\varphi \in C^1(\overline{Q})$ be nonnegative, we have $0 \le \varphi(1-\psi_\delta^-)\psi_\delta^+ \le \varphi$ (since $0 \le \psi_\delta^\pm \le 1$) and $\varphi(1-\psi_\delta^-)\psi_\delta^+ \in C_0^\infty(Q)$ (since $\psi_\delta^\pm \in C_0^\infty(Q)$). Since B_n' can be splitted as $B_n'(s) = \frac{1}{n}(-\chi_{\{n<s<2n\}} + \chi_{\{-2n<s<-n\}})$ a.e. in $\mathbb{R}$, and using (4.56), we obtain

$$\liminf_{n\to\infty}\frac{1}{n}\int_{\{n<u<2n\}} a_0(t,x,u,\nabla u)\cdot\nabla u\varphi dxdt$$
$$\ge \liminf_{n\to\infty}\frac{1}{n}\int_{\{n<u<2n\}} a_0(t,x,u,\nabla u)\cdot\nabla u\varphi(1-\psi_\delta^-)\psi_\delta^+ dxdt$$
$$\ge \lim_{n\to\infty}\left[-\int_Q B_n'(u)a_0(t,x,u,\nabla u)\cdot\nabla u\varphi(1-\psi_\delta^-)\psi_\delta^+\right]dxdt$$
$$= \int_Q \varphi(1-\psi_\delta^-)\psi_\delta^+ d\mu_c^+ - \int_Q \varphi(1-\psi_\delta^-)\psi_\delta^+ d\mu_c^-. \tag{4.57}$$

It is easy to pass to the limit, as s tends to zero, using (4.8)–(4.11), to obtain

$$\liminf_{n\to\infty}\frac{1}{n}\int_{\{n<u<2n\}} a_0(t,x,u,\nabla u)\cdot\nabla u\varphi dxdt \ge \int_Q \varphi d\mu_s^+, \tag{4.58}$$

for every nonnegative $\varphi \in C^1(\overline{Q})$. Then (4.55) implies that condition (3.3) of Definition 3.1 holds for every nonnegative $\varphi \in C^1(\overline{Q})$ and, by a standard density argument, for every $\varphi \in C^\infty(\overline{Q})$. Similarly, one can use $\varphi(1-\psi_\delta^+)\psi_\delta^-$ to obtain the asymptotic behaviour result for μ_c^-. Finally, in order to prove the last asymptotic behaviour result on K in condition (3.2) it suffices to use a similar argument of (4.25).

Step 6. Strong convergence of truncations Now, we are able to prove that $T_k(u_\epsilon)$ converges to $T_k(u)$ in $L^p(0,T;W_0^{1,p}(\Omega))$, $k>0$, as ϵ goes to zero. By using a standard argument, see [5,24,84] for more details. We just need to remark that by the coercivity of the vector filed "a", the a.e. convergence of u_ϵ, ∇u_ϵ and Fatou's lemma, that

$$\begin{aligned}&\int_Q a_0(t,x,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u)dxdt\\&\quad\le \liminf_{\epsilon\to 0}\int_Q a_\epsilon(t,x,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)dxdt.\end{aligned}\tag{4.59}$$

Therefore, the choice of $B_n(u_\epsilon)T_k(u_\epsilon)$ as test function in (4.14) and letting n then ϵ tend to their limits, we get

$$\begin{aligned}&\limsup_{\epsilon\to 0}\int_Q a_\epsilon(t,x,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)dxdt\\&\quad\le \int_Q a_0(t,x,u,\nabla u)\cdot\nabla T_k(u)dxdt.\end{aligned}\tag{4.60}$$

Finally, assumptions (1.4)–(1.5) on "a_ϵ", the a.e. convergence of $\nabla T_k(u_\epsilon)$ and Vitali's theorem imply

$$\nabla T_k(u_\epsilon)\to\nabla T_k(u) \text{ strongly in } L^{p'}(Q)^N,\tag{4.61}$$

which completes the proof of Theorem 3.1. □

4.2 Proof of existence result (Theorem 3.2)

Until now, we have assumed that $H\equiv G\equiv E\equiv 0$ mainly to simplify our exposition. Using a change of the form of problem (3.17) one can prove that the terms $H_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)$ and $G_\epsilon(t,x,u_\epsilon)$ converges strongly in $L^1(Q)$ with similar arguments of Theorem 3.1.

Proof of Theorem 3.1 It suffices to check that the solution u_ϵ of (3.17) belongs to $\mathbb{W}$ and satisfies

$$(u_\epsilon)_t-\mathrm{div}[a(t,x,u_\epsilon,\nabla u_\epsilon)+K_\epsilon(t,x,u_\epsilon)]=\Phi_\epsilon-\mathrm{div}(F)+\mathrm{div}(E),\tag{4.62}$$

in the sense of distributions, where

$$\Phi_\epsilon=f_\epsilon-H_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)-G_\epsilon(t,x,u_\epsilon)+\lambda_\epsilon^\oplus-\lambda_\epsilon^\ominus \text{ is bounded in } L^1(Q).\tag{4.63}$$

Indeed, the growth assumption (3.20) on H_ϵ, Proposition 3.1 and the generalized Hölder inequality (2.51), imply that

$$\begin{aligned}\|H_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)\|_{L^1(Q)} &= \int_Q |H_\epsilon(t,x,u_\epsilon,\nabla u_\epsilon)|dxdt \\ &\le \int_Q b_0|\nabla u_\epsilon|^\lambda dxdt + \int_Q b_1(t,x)dxdt \\ &\le \|b_0\|_{L^{N+2,1}(Q)} \left\||\nabla u|^\lambda\right\|_{L^{\frac{N+2}{N+1},\infty}(Q)} + \|b_1\|_{L^1(Q)} \\ &\le C. \end{aligned} \tag{4.64}$$

Moreover, using the growth condition (3.21) on G_ϵ, the generalized Hölder inequality (2.51) and the fact that $0 \le r \le p - \frac{N}{N+1}$, we get

$$\begin{aligned}\|G_\epsilon(t,x,u_\epsilon)\|_{L^1(Q)} &= \int_Q |G_\epsilon(t,x,u_\epsilon)|dxdt \\ &\le \int_Q d_1(t,x)|u_\epsilon|^\iota dxt + \int_Q d_2(t,x)dxdt \\ &\le \|d_1\|_{L^{z',1}(Q)} \left\||u_\epsilon|^\iota\right\|_{L^{z,\infty}(Q)} + \|d_2\|_{L^1(Q)} \\ &\le C\|d_1\|_{L^{z',1}(Q)} \left\||u_\epsilon|^\iota\right\|_{L^{z,\infty}(Q)} + \|d_2\|_{L^1(Q)} \\ &\le C\|d_1\|_{L^{z',1}(Q)} \left\||u_\epsilon|^{p-1}\right\|_{L^{\frac{pN+p-N}{(N+1)(p-1)},\infty}(Q)} + \|d_2\|_{L^1(Q)}. \end{aligned} \tag{4.65}$$

The use of $T_k(u_\epsilon)$ as test function in (4.62) implies, in virtue of the argument of Proposition 3.3, that there exist two constants M and L such that

$$\sup_{t\in[0,T]} \int_\Omega |T_k(u_\epsilon)|^2 dxdt + \int_Q |\nabla T_k(u_\epsilon)|^p dxdt \le Mk + L, \tag{4.66}$$

for every $k > 0$ and every $\epsilon > 0$. Moreover, the a priori estimates and the growth assumption (3.19) on K_ϵ imply, by using the technical results of [5,84,96], that, up to a subsequence still denoted by ϵ, there exists a function u_ϵ and a measurable function u, a.e. finite in Q such that $T_k(u) \in L^\infty(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$, satisfying

$$\begin{cases} u_\epsilon \to u \text{ a.e. in } Q, \\ \nabla u_\epsilon \to \nabla u \text{ a.e. in } Q, \\ \nabla T_k(u_\epsilon) \rightharpoonup \nabla T_k(u) \text{ weakly in } L^{p'}(Q)^N, \end{cases} \tag{4.67}$$

for every fixed $k \in \mathbb{N}$. The estimate (4.66) imply, by Fatou's Lemma, that

$$\sup_{t\in[0,T]} \int_\Omega |T_k(u)|^2 dxdt + \int_Q |\nabla T_k(u)|^p dxdt \le Mk + L, \tag{4.68}$$

which gives, by using Proposition 3.1, that

$$|u|^{p-1} \in L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q) \text{ and } |\nabla u|^{p-1} \in L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q). \tag{4.69}$$

Then, we conclude, by (4.67), that

$$H_\epsilon(t, x, u_\epsilon, \nabla u_\epsilon) \to H(t, x, u, \nabla u) \text{ a.e. in } Q. \tag{4.70}$$

In particular, it is enough to remark by (3.20)–(3.21), (2.51), Proposition 3.1 and (4.64)–(4.65), that $H_\epsilon(t, x, u_\epsilon, \nabla u_\epsilon)$ and $G_\epsilon(t, x, u_\epsilon)$ are equi-integrable, which imply by using Vitali's theorem that

$$\begin{cases} H_\epsilon(t, x, u_\epsilon, \nabla u_\epsilon) \to H(t, x, u, \nabla u) \text{ strongly in } L^1(Q), \\ G_\epsilon(t, x, u_\epsilon) \to G(t, x, u) \text{ strongly in } L^1(Q), \end{cases} \tag{4.71}$$

that is to say, the function $u_\epsilon \in \mathbb{W}$, solution of problem (4.62), satisfies the "*modified*" problem

$$\begin{aligned} (u_\epsilon)_t &- \operatorname{div}[a(t, x, u_\epsilon, \nabla u_\epsilon) + K_\epsilon(t, x, u_\epsilon)] \\ &= f_\epsilon - \Psi_\epsilon - \operatorname{div}(F) + \operatorname{div}(E) + \lambda_\epsilon^\oplus - \lambda_\epsilon^\ominus, \end{aligned} \tag{4.72}$$

in the sense of distributions and the convergence results (4.67) hold, where

$$\begin{aligned} \Psi_\epsilon = H_\epsilon(t, x, u_\epsilon, \nabla u_\epsilon) + G_\epsilon(t, x, u_\epsilon) &\to H(t, x, u, \nabla u) \\ &+ G(t, x, u) \text{ strongly in } L^1(Q), \end{aligned} \tag{4.73}$$

and F, E, f_ϵ, $\lambda_\epsilon^\oplus$ and $\lambda_\epsilon^\ominus$ are defined as before. Since the weak solution u_ϵ of (4.73) is also a renormalized solution of the same problem, then by virtue of the stability result (Theorem 3.1), the function u is a renormalized solution of

$$\begin{cases} u_t - \operatorname{div}[a(t, x, u, \nabla u) + K(t, x, u)] + H(t, x, u, \nabla u) + G(t, x, u) \\ \quad = f - \operatorname{div}(F) + \mu_c^+ - \mu_c^- + \operatorname{div}(E) \text{ in } Q, \\ u(0, x) = u_0(x) \text{ in } \Omega, \quad u(t, x) = 0 \text{ on } (0, T) \times \partial\Omega, \end{cases} \tag{4.74}$$

which concludes the proof of Theorem 3.2. □

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Compliance with ethical standards

Conflict of interest The corresponding author states that there is no conflict of interest.

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