



On solutions to a FitzHugh–Rinzel type model

Fabio De Angelis¹ · Monica De Angelis²

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Abstract

A ternary autonomous dynamical system of FitzHugh–Rinzel type is analyzed. The system, at start, is reduced to a nonlinear integro differential equation. The fundamental solution $H(x, t)$ is explicitly determined and the initial value problem is analyzed in the whole space. The solution is expressed by means of an integral equation involving $H(x, t)$. Moreover, adding an extra control term, explicit solutions are achieved.

Keywords FitzHugh–Rinzel model · Fundamental solution · Laplace transform · Exact solutions

Mathematics Subject Classification 35K47 · 35K25 · 78A70 · 35E05 · 44A10

1 Introduction

The FitzHugh–Rinzel (FHR) system [1–4] is a three-dimensional model deriving from the FitzHugh–Nagumo (FHN) model [5–12] developed to incorporate bursting phenomena of nerve cells. Indeed, a number of different cell types exhibit a behaviour characterized by brief bursts of oscillatory activity alternated by quiescent periods during which the membrane potential only changes slowly, and this behaviour is called bursting, see e.g. [13]. Accordingly, bursting oscillations are characterized by a variable of the system that changes periodically from an active phase of rapid spike oscillations to a silent phase. These phenomena are becoming increasingly important as they are being investigated in many scientific fields. Indeed, phenomena of bursting

✉ Monica De Angelis
modeange@unina.it

Fabio De Angelis
fabio.deangelis@unina.it

¹ Department of Structures for Engineering and Architecture, University of Naples Federico II, Via Claudio 21, 80125 Naples, Italy

² Department of Mathematics and Applications “R. Caccioppoli”, University of Naples Federico II, Via Cinthia 26, 80126 Naples, Italy

have been observed as electrical behaviours in many nerve and endocrine cells such as hippocampal and thalamic neurons, mammalian midbrain and pancreatic in β -cells, see e.g. [1] and references therein. Also, in the cardiovascular system, bursting oscillations are generated by the electrical activity of cardiac cells that excite the heart membrane to produce the contraction of ventricles and auricles [14]. Furthermore, bursting oscillations represent a topic of potential interest in dynamics and bifurcation mechanisms of devices and structures and in the analysis of nonlinear problems in mechanics [15–19]. Recent studies proved that the development of this field helps in the studying of the restoration of synaptic connections. Indeed, it seems that nanoscale memristor devices have potential to reproduce the behaviour of a biological synapse [20,21]. This would lead in the future, also in case of traumatic lesions, to the introduction of electronic synapses to connect neurons directly.

The paper is organized as follows. In Sect. 1.1 the mathematical problem is defined and the state of the art with the aim of the paper are discussed. In Sect. 2, the explicit expression of the fundamental solution $H(x, t)$ is achieved and some properties are proved. In Sect. 3 the integral solution for the initial value problem is given. In Sect. 4 the insertion of an extra term allows us to obtain explicit solutions for the model.

1.1 Mathematical considerations, state of the art and aim of the paper

Generally, denoting by D, ε, β, c constant parameters, the (FHN) model is a p.d.e. system such that

$$\begin{cases} \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - w + f(u) \\ \frac{\partial w}{\partial t} = \varepsilon(-\beta w + c + u), \end{cases} \quad (1)$$

where function $f(u)$ depends on the reaction kinetics of the model. In the literature $f(u)$ can assume a piecewise linear form, see, e.g. [22] and reference therein, or $f(u) = u - u^3/3$ [12]. However, in general, one has [5,13]:

$$f(u) = u(a - u)(u - 1) \quad (0 < a < 1). \quad (2)$$

As for the FitzHugh–Rinzel model, most of the articles consider the following system characterized by three o.d.e.:

$$\begin{cases} \frac{\partial u}{\partial t} = u - u^3/3 + I_{ext} - w + y \\ \frac{\partial w}{\partial t} = \varepsilon(-\beta w + c + u) \\ \frac{\partial y}{\partial t} = \delta(-u + h - dy) \end{cases} \quad (3)$$

where $I_{ext}, \varepsilon, \beta, c, d, h, \delta$ indicate arbitrary constants.

In this paper, in order to evaluate the contribute of a diffusion term, the following FitzHugh–Rinzel type system is considered:

$$\begin{cases} \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - w + y + f(u) \\ \frac{\partial w}{\partial t} = \varepsilon(-\beta w + c + u) \\ \frac{\partial y}{\partial t} = \delta(-u + h - dy). \end{cases} \tag{4}$$

Indeed, the second order term with $D > 0$ represents just the diffusion contribution and it can be associated to the axial current in the axon. It derives from the Hodgkin-Huxley (HH) theory for nerve membranes where, if b represents the axon diameter and r_i is the resistivity, the spatial variation in the potential V gives the term $(b/4r_i)V_{xx}$ from which term $D u_{xx}$ derives [23].

Moreover it is also assumed $\beta, d, \varepsilon, \delta$ as positive constants that together with c, h , characterize the model’s kinetic.

Model (4) can be considered as a two time-scale slow-fast system with two fast variables (u, w) and one slow variable (y) . However, if for instance, $\varepsilon = \delta$ the system can be considered as a two time-scale with one fast variable u and two slow variables (w, y) . Otherwise, if δ and ε have significant difference, it can also be considered as a three-time-scale system with the fast variable u , the intermediate variable and the slow variable [24].

As for function $f(u)$ one considers the non-linear form expressed in formula (2). As a consequence, it results

$$f(u) = -a u + \varphi(u) \quad \text{with} \quad \varphi(u) = u^2 (a + 1 - u) \quad 0 < a < 1 \tag{5}$$

Then, the system (4) becomes

$$\begin{cases} \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - au - w + y + \varphi(u) \\ \frac{\partial w}{\partial t} = \varepsilon(-\beta w + c + u) \\ \frac{\partial y}{\partial t} = \delta(-u + h - dy). \end{cases} \tag{6}$$

Indicating by means of

$$u(x, 0) = u_0, \quad w(x, 0) = w_0 \quad y(x, 0) = y_0, \quad (x \in \mathfrak{R}) \tag{7}$$

the initial values, from (6)_{2,3} it follows:

$$\begin{cases} w = w_0 e^{-\varepsilon\beta t} + \frac{c}{\beta} (1 - e^{-\varepsilon\beta t}) + \varepsilon \int_0^t e^{-\varepsilon\beta(t-\tau)} u(x, \tau) d\tau \\ y = y_0 e^{-\delta d t} + \frac{h}{d} (1 - e^{-\delta d t}) - \delta \int_0^t e^{-\delta d(t-\tau)} u(x, \tau) d\tau. \end{cases} \quad (8)$$

Consequently, denoting the source term by

$$F(x, t, u) = \varphi(u) - w_0(x)e^{-\varepsilon\beta t} + y_0(x)e^{-\delta d t} - \frac{c}{\beta}(1 - e^{-\varepsilon\beta t}) + \frac{h}{d}(1 - e^{-\delta d t}), \quad (9)$$

problem (6)–(7) can be modified into the following initial value problem \mathcal{P} :

$$\begin{cases} u_t - Du_{xx} + au + \int_0^t [\varepsilon e^{-\varepsilon\beta(t-\tau)} + \delta e^{-\delta d(t-\tau)}] u(x, \tau) d\tau = F(x, t, u) \\ u(x, 0) = u_0(x) \quad x \in \mathfrak{R}. \end{cases} \quad (10)$$

As for the state of art, mathematical considerations allow to assert that the knowledge of the fundamental solution $H(x, t)$ related to the linear parabolic operator L :

$$Lu \equiv u_t - Du_{xx} + au + \int_0^t [\varepsilon e^{-\varepsilon\beta(t-\tau)} + \delta e^{-\delta d(t-\tau)}] u(x, \tau) d\tau, \quad (11)$$

leads to determine the solution of \mathcal{P} . Indeed, if $F(x, t, u)$ verifies appropriate assumptions, through the fixed point theorem, solution can be expressed by means of an integral equation, see f.i. [25,26].

Moreover, according to [26], when operator L assumes a similar but simpler form, many properties and inequalities are achieved.

The aim of the paper is to explicitly determine the fundamental solution $H(x, t)$ which involves naturally the diffusion constant D . Then, the initial value problem in all the space is analyzed and the solution is deduced by means of an integral equation. Moreover, using a method of travelling wave, solutions of a modified FitzHugh–Rinzler type system have been explicitly determined pointing out the influence of the diffusion parameter D .

2 Fundamental solution and its properties

Indicating by T an arbitrary positive constant, let us consider the initial-value problem (10) defined in the whole space Ω_T :

$$\Omega_T = \{(x, t) : x \in \mathfrak{R}, \quad 0 < t \leq T\},$$

and let us denote by

$$\hat{u}(x, s) = \int_0^\infty e^{-st} u(x, t) dt, \quad \hat{F}(x, s) = \int_0^\infty e^{-st} F[x, t, u(x, t)] dt,$$

the Laplace transform with respect to t . If $\hat{H}(x, s)$ expresses the \mathcal{L}_t transform of the fundamental solution $H(x, t)$, from (10) it follows:

$$\hat{u}(x, s) = \int_{\Re} \hat{H}(x - \xi, s) [u_0(\xi) + \hat{F}(\xi, s)] d\xi, \tag{12}$$

and *formally* it follows that

$$u(x, t) = \int_{\Re} H(x - \xi, t) u_0(\xi) d\xi + \int_0^t d\tau \int_{\Re} H(x - \xi, t - \tau) F[\xi, \tau, u(\xi, \tau)] d\xi. \tag{13}$$

So that, denoting by $J_1(z)$ the Bessel function of first kind and order 1, let us consider the following functions:

$$H_1(x, t) = \frac{e^{-\frac{x^2}{4Dt}}}{2\sqrt{\pi Dt}} e^{-at} - \frac{1}{2} \int_0^t \frac{e^{-\frac{x^2}{4Dy} - ay}}{\sqrt{t - y}} \frac{\sqrt{\varepsilon} e^{-\beta\varepsilon(t-y)}}{\sqrt{\pi D}} J_1(2\sqrt{\varepsilon y(t - y)}) dy, \tag{14}$$

$$H_2 = \int_0^t H_1(x, y) e^{-\delta d(t-y)} \sqrt{\frac{\delta y}{t - y}} J_1(2\sqrt{\delta y(t - y)}) dy. \tag{15}$$

Besides, by setting

$$\sigma^2 = s + a + \frac{\delta}{s + \delta d} + \frac{\varepsilon}{s + \beta\varepsilon}, \tag{16}$$

and by denoting

$$H = H_1 - H_2, \tag{17}$$

the following theorem holds:

Theorem 1 *In the half-plane $\Re s > \max(-a, -\beta\varepsilon, -\delta d)$ the Laplace integral $\mathcal{L}_t H$ converges absolutely for all $x > 0$, and it results:*

$$\mathcal{L}_t H \equiv \hat{H} = \int_0^\infty e^{-st} H(x, t) dt = \frac{1}{\sqrt{D}} \frac{e^{-\frac{|x|}{\sqrt{D}}\sigma}}{2\sigma}. \tag{18}$$

Moreover, function $H(x,t)$ satisfies some properties typical of the fundamental solution of heat equation, such as:

- (a) $H(x, t) \in C^\infty, \quad t > 0, \quad x \in \mathfrak{R},$
- (b) for fixed $t > 0, H$ and its derivatives are vanishing esponentially fast as $|x|$ tends to infinity.
- (c) In addition, it results $\lim_{t \rightarrow 0} H(x, t) = 0,$ for any fixed $\eta > 0,$ uniformly for all $|x| \geq \eta.$

Proof Since for all real z one has $|J_1(z)| \leq 1,$ the Fubini–Tonelli theorem assures that

$$\begin{aligned} &L_t \left(\frac{1}{2} \int_0^t \frac{e^{-\frac{x^2}{4Dy} - ay}}{\sqrt{t-y}} \frac{\sqrt{\varepsilon} e^{-\beta\varepsilon(t-y)}}{\sqrt{\pi D}} J_1(2\sqrt{\varepsilon y(t-y)}) dy \right) \\ &= \frac{\sqrt{\varepsilon}}{2\sqrt{\pi D}} \int_0^\infty e^{-(s+a)y - \frac{x^2}{4Dy}} dy \int_0^\infty e^{-(s+\beta\varepsilon)t} J_1(2\sqrt{\varepsilon y t}) \frac{dt}{\sqrt{t}} \end{aligned}$$

and being

$$\int_0^\infty e^{-pt} \sqrt{\frac{c}{t}} J_1(2\sqrt{ct}) dt = 1 - e^{-c/p} \quad (\Re p > 0), \tag{19}$$

$$\int_0^\infty \frac{e^{-x^2/4t}}{\sqrt{\pi t}} e^{-(s+\alpha)t} dt = \frac{e^{-x\sqrt{s+\alpha}}}{\sqrt{s+\alpha}} \tag{20}$$

it results:

$$\hat{H}_1(x, s) = \frac{1}{2\sqrt{\pi D}} \int_0^\infty e^{-\frac{x^2}{4Dy} - (s+a+\frac{\varepsilon}{s+\beta\varepsilon})y} \frac{dy}{\sqrt{y}} = \frac{1}{2\sqrt{D}} \frac{e^{-\frac{|x|}{\sqrt{D}}r}}{r} \tag{21}$$

where

$$r^2 = s + a + \frac{\varepsilon}{s + \beta\varepsilon}. \tag{22}$$

Besides, since Fubini–Tonelli theorem and (19) one has:

$$\hat{H}_2 = \hat{H}_1 - \frac{1}{2\sqrt{\pi D}} \int_0^\infty e^{-\frac{x^2}{4Dy} - (s+a+\frac{\varepsilon}{s+\beta\varepsilon} + \frac{\delta}{s+d\delta})y} \frac{dy}{\sqrt{y}} \tag{23}$$

from which, since (20),

$$\hat{H}(x, s) = \frac{1}{2\sqrt{D}} \frac{e^{-\frac{|x|}{\sqrt{D}}\sigma}}{\sigma} \tag{24}$$

is deduced.

Besides, by means of property of convolution for which $f * g = g * f$, since (14) and (15), property a) is evident. Moreover, properties b) and c) are proved following Theorem 3.2.1 of [25]. In particular, as for property c), for $|x| \geq \eta$ and since $|J_1(z)| \leq 1$, it results:

$$|H_1| = \frac{e^{-\frac{\eta^2}{4Dt}}}{2\sqrt{\pi Dt}} e^{-at} + \sqrt{\frac{\varepsilon t}{\pi D}}; \tag{25}$$

$$|H_2| \leq \sqrt{\frac{\delta t}{4\pi D}} + 2\sqrt{\frac{\varepsilon\delta}{\pi D}} t \tag{26}$$

from which property follows. □

Now, introducing the following functions:

$$\begin{cases} \varphi(x, t) = \frac{e^{-\frac{x^2}{4Dt}}}{2\sqrt{\pi Dt}} e^{-at}; \\ \psi_\varepsilon(y, t) = \frac{\sqrt{\varepsilon y} e^{-\beta\varepsilon(t-y)}}{\sqrt{t-y}} J_1(2\sqrt{\varepsilon y(t-y)}); \\ \psi_\delta(y, t) = \frac{\sqrt{\delta y} e^{-\delta d(t-y)}}{\sqrt{t-y}} J_1(2\sqrt{\delta y(t-y)}); \end{cases} \tag{27}$$

it results:

$$H_1(x, t) = \varphi(x, t) - \int_0^t \varphi(x, y) \psi_\varepsilon(y, t) dy \tag{28}$$

$$H_2(x, t) = \int_0^t H_1(x, y) \psi_\delta(y, t) dy. \tag{29}$$

Moreover, by denoting

$$g_1(x, t) * g_2(x, t) = \int_0^t g_1(x, t - \tau)g_2(x, \tau) d\tau \tag{30}$$

the convolution with respect to t , for $t > 0$, as proved in [26] by means of formula (20),(21) and (24), it results:

$$(\partial_t + a - D\partial_{xx})H_1 = -\varepsilon e^{-\varepsilon\beta t} * H_1(x, t) = -\varepsilon K_\varepsilon(x, t) \tag{31}$$

where K_ε is given by

$$K_\varepsilon(x, t) = \frac{1}{2\sqrt{\pi D}} \int_0^t e^{-\frac{x^2}{4Dy} - ay - \beta\varepsilon(t-y)} J_0(2\sqrt{\varepsilon y(t-y)}) \frac{dy}{\sqrt{y}}. \quad (32)$$

Hence, the following theorem can be proved:

Theorem 2 For $t > 0$, it results $LH = 0$, i.e.

$$H_t - DH_{xx} + aH + \int_0^t [\varepsilon e^{-\varepsilon\beta(t-\tau)} + \delta e^{-\delta d(t-\tau)}] H(x, \tau) d\tau = 0. \quad (33)$$

Proof Let us consider that:

$$\begin{aligned} (\partial_t + a - D\partial_{xx})H_2 &= H_1(x, t)\psi_\delta(t, t) + \int_0^t H_1(x, y) [\partial_t\psi_\delta(y, t) + a\psi_\delta(y, t)] \\ &\quad - D \int_0^t \psi_\delta(y, t) \partial_{xx}H_1(x, y) dy. \end{aligned} \quad (34)$$

Accordingly, given relation (31), one has

$$\begin{aligned} (\partial_t + a - D\partial_{xx})H_2 &= \int_0^t [H_1(x, y) \partial_t\psi_\delta(y, t) - \psi_\delta(y, t) \partial_y H_1(x, y)] dy + \\ &\quad + H_1(x, t)\psi_\delta(t, t) - \varepsilon \int_0^t K_\varepsilon(x, y)\psi_\delta(y, t) dy. \end{aligned} \quad (35)$$

Besides, considering that:

$$\int_0^t \psi_\delta(y, t) \partial_y H_1(x, y) dy = H_1(x, t)\psi_\delta(t, t) - \int_0^t H_1(x, y) \partial_y \psi_\delta(y, t) dy, \quad (36)$$

one has:

$$\begin{aligned} (\partial_t + a - D\partial_{xx})H_2 &= \int_0^t H_1(x, y) [\partial_t\psi_\delta(y, t) + \partial_y\psi_\delta(y, t)] dy \\ &\quad - \varepsilon \int_0^t K_\varepsilon(x, y)\psi_\delta(y, t) dy \end{aligned} \quad (37)$$

where it results:

$$\partial_t\psi_\delta(y, t) + \partial_y\psi_\delta(y, t) = \delta e^{-\delta d(t-y)} J_0(2\sqrt{\delta y(t-y)}). \quad (38)$$

So that, denoting by

$$K_\delta(x, t) \equiv \int_0^t e^{-\delta d(t-y)} H_1(x, y) J_0(2\sqrt{\delta y(t-y)}) dy, \quad (39)$$

one has:

$$(\partial_t + a - D\partial_{xx})H_2 = \delta K_\delta - \varepsilon \int_0^t K_\varepsilon(x, y)\psi_\delta(y, t)dy.$$

Consequently, since for equation (31) one has $K_\varepsilon(x, t) = H_1(x, t) * e^{-\varepsilon\beta t}$, by means of Fubini–Tonelli theorem and (29), it is proved that:

$$(\partial_t + a - D\partial_{xx})H_2 = \delta K_\delta - \varepsilon e^{-\varepsilon\beta t} * H_2(x, t). \tag{40}$$

On the other hand, the convolution $e^{-\delta dt} * H(x, t)$ is given by

$$e^{-\delta dt} * H(x, t) = e^{-\delta dt} * H_1(x, t) - \int_0^t H_1(r, y)dy \int_y^t e^{-\delta d(t-\tau)}\psi_\delta(y, \tau) d\tau$$

with

$$\begin{aligned} \int_y^t e^{-\delta d(t-\tau)}\psi_\delta(y, \tau)d\tau &= e^{-\delta d(t-y)} \int_y^t \sqrt{\frac{\delta y}{\tau-y}} J_1(2\sqrt{\delta y(\tau-y)})d\tau \\ &= e^{-\delta d(t-y)} \left[1 - J_0(2\sqrt{\delta y(t-y)}) \right]. \end{aligned} \tag{41}$$

As a consequence, it results:

$$e^{-\delta dt} * H = K_\delta. \tag{42}$$

Therefore, given relations (31), (40), (42), theorem holds. □

3 Solution related to the FitzHugh–Rinzel problem

To provide the solution by means of the integral expression (13), some convolutions need to be determined.

In order to evaluate $\int_0^t d\tau \int_{\mathbb{R}} H(\xi, \tau) d\xi$, let us start to observe that

$$\int_{\mathbb{R}} d\xi \int_0^t H_2(\xi, \tau)d\tau = \int_{\mathbb{R}} d\xi \int_0^t H_1(\xi, y)dy \int_y^t \psi_\delta(y, \tau)d\tau$$

with

$$\begin{aligned} \int_y^t \psi_\delta(y, \tau)d\tau &= 1 - e^{-\delta d(t-y)} J_0(2\sqrt{\delta y(t-y)}) + \\ &\quad - \delta d \int_y^t e^{-\delta d(t-y)} J_0(2\sqrt{\delta y(t-y)})d\tau. \end{aligned} \tag{43}$$

Consequently, for (39), one has:

$$\int_{\mathfrak{R}} d\xi \int_0^t H_2(\xi, \tau) d\tau = \int_{\mathfrak{R}} d\xi \int_0^t H_1(\xi, \tau) d\tau - \int_{\mathfrak{R}} K_\delta(\xi, t) d\xi + \delta d \int_{\mathfrak{R}} d\xi \int_0^t K_\delta(\xi, \tau) d\tau \quad (44)$$

So that, according to (17), it results:

$$\int_0^t d\tau \int_{\mathfrak{R}} H(\xi, \tau) d\xi = \int_{\mathfrak{R}} K_\delta(\xi, t) d\xi + \delta d \int_{\mathfrak{R}} d\xi \int_0^t K_\delta(\xi, \tau) d\tau. \quad (45)$$

Now, let us evaluate

$$e^{-\varepsilon\beta t} * H_2 = \int_0^t H_1(x, y) dy \int_y^t e^{-\beta\varepsilon(t-\tau)} \psi_\delta(y, \tau) d\tau.$$

Considering (27)₃, after an integration by parts, one obtains:

$$e^{-\varepsilon\beta t} * H_2 = e^{-\beta\varepsilon t} * H_1(x, t) - \int_0^t e^{-\delta d(t-y)} H_1(x, y) J_0(2\sqrt{\delta y(t-y)}) dy + (\varepsilon\beta - \delta d) \int_0^t d\tau \int_0^\tau H_1(x, y) e^{-\delta d(\tau-y)} e^{-\varepsilon\beta(t-\tau)} J_0(2\sqrt{\delta y(\tau-y)}) dy, \quad (46)$$

and, for (39), it results:

$$e^{-\varepsilon\beta t} * H_2 = e^{-\varepsilon\beta t} * H_1 - K_\delta + (\varepsilon\beta - \delta d) e^{-\beta\varepsilon t} * K_\delta. \quad (47)$$

Moreover, since (17) and (47), one deduces:

$$e^{-\varepsilon\beta t} * H = K_\delta + (\delta d - \varepsilon\beta) e^{-\beta\varepsilon t} * K_\delta. \quad (48)$$

Now, let us denote by

$$f_1(x, t) \star f_2(x, t) = \int_{\mathfrak{R}} f_1(\xi, t) f_2(x - \xi, t) d\xi \quad (49)$$

the convolution with respect to the space x , and let

$$H \otimes F = \int_0^t d\tau \int_{\mathfrak{R}} H(x - \xi, t - \tau) F[\xi, \tau, u(\xi, \tau)] d\xi. \quad (50)$$

Considering (42) and (48) one has:

$$\begin{cases} H \otimes e^{-\delta t} = \int_{\mathfrak{R}} K_{\delta}(\xi, t) d\xi, \\ H \otimes e^{-\beta \varepsilon t} = \int_{\mathfrak{R}} [K_{\delta} + (\delta d - \varepsilon \beta) e^{-\beta \varepsilon t} * K_{\delta}] d\xi. \end{cases} \tag{51}$$

Moreover, it results:

$$\begin{cases} H \otimes (y_0(x) e^{-\delta t}) = y_0 * K_{\delta} \\ H \otimes (w_0(x) e^{-\beta \varepsilon t}) = w_0 * [K_{\delta} + (\delta d - \varepsilon \beta) e^{-\beta \varepsilon t} * K_{\delta}] \end{cases} \tag{52}$$

where

$$w_0 * (\delta d - \varepsilon \beta) e^{-\beta \varepsilon t} * K_{\delta} = (\delta d - \varepsilon \beta) w_0 \otimes (e^{-\beta \varepsilon t} K_{\delta}). \tag{53}$$

Consequently, given (9) and (13), for (45), (51), (52) and (53), it results:

$$\begin{aligned} u(x, t) = & u_0(x) * H + (y_0(x) - w_0(x)) * K_{\delta} + \varphi(u) \otimes H \\ & + (\varepsilon \beta - \delta d) w_0(x) \otimes e^{-\beta \varepsilon t} K_{\delta} + \frac{c}{\beta} (\delta d - \varepsilon \beta) e^{-\beta \varepsilon t} \otimes K_{\delta} \\ & + \left(\frac{h}{d} - \frac{c}{\beta} \right) \delta d \otimes K_{\delta} \end{aligned} \tag{54}$$

and this formula, together with relations (8), allow us to determine also $v(x, t)$ and $y(x, t)$ in terms of the data.

4 Explicit solutions

Several methods have been developed to find exact solutions related to partial differential equations [27–31]. In this case, by referring to [7], an extra term is added in order to achieve some solutions. Accordingly, let us consider

$$\begin{cases} \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - w + y + f(u) \\ \frac{\partial w}{\partial t} = \varepsilon(-\beta w + c + u) + k u^2 \\ \frac{\partial y}{\partial t} = \delta(-u + h - dy) \end{cases} \tag{55}$$

where $k \neq 0$ and let us assume $\varepsilon \beta = \delta d$ and $f(u) = 2u(a - u)(u - 1)$.

Under these conditions, problem (10) turns into:

$$\begin{cases} u_t - Du_{xx} + 2au + (\varepsilon + \delta) \int_0^t e^{-\varepsilon\beta(t-\tau)} u(x, \tau) d\tau = F(x, t, u) \\ u(x, 0) = u_0(x) \quad x \in \mathfrak{R}, \end{cases} \quad (56)$$

where, by denoting $\varphi_1 = 2u^2(a+1-u) + k \int_0^t e^{-\varepsilon\beta(t-\tau)} u^2(x, \tau) d\tau$, it results:

$$F = \varphi_1(u) - w_0 e^{-\varepsilon\beta t} + y_0 e^{-\delta dt} - \frac{c}{\beta}(1 - e^{-\varepsilon\beta t}) + \frac{h}{d}(1 - e^{-\delta dt}) \quad (57)$$

and

$$\begin{cases} w = w_0 e^{-\varepsilon\beta t} + \frac{c}{\beta}(1 - e^{-\varepsilon\beta t}) + \int_0^t e^{-\varepsilon\beta(t-\tau)} [\varepsilon u(x, \tau) + k u^2(x, \tau)] d\tau \\ y = y_0 e^{-\delta dt} + \frac{h}{d}(1 - e^{-\delta dt}) - \delta \int_0^t e^{-\delta d(t-\tau)} u(x, \tau) d\tau. \end{cases} \quad (58)$$

In order to find explicit solutions, let us introduce

$$z = x - Ct,$$

obtaining, from system (55), the following equation:

$$DC u_{zzz} + (C^2 - \varepsilon\beta D)u_{zz} - 6Cu^2 u_z + 4C(a+1)u u_z + 2\varepsilon\beta u^3 + k u^2 - C(2a + \varepsilon\beta)u_z - 2\varepsilon\beta(a+1)u^2 + 2\varepsilon\beta a u + (\varepsilon + \delta)u + \varepsilon c - \delta h = 0. \quad (59)$$

Now, let us consider

$$f(z) = \sqrt{y} \tanh(\sqrt{y}(z - z_0)), \quad (60)$$

that is a solution of Riccati type equation:

$$f_z + f^2 - y = 0,$$

and let us assume

$$u(z) = A f(z) + b. \quad (61)$$

Since

$$\begin{aligned}
 u_z &= -A f^2(z) + A y \\
 u_{zz} &= 2 A f^3 - 2 A f y \\
 u_{zzz} &= -6 A f^4(z) + 8 A y f^2(z) - 2 A y^2 \\
 u u_z &= -A^2 f^3(z) - A b f^2 + A^2 y f + A b y \\
 u^2 u_z &= -A^3 f^4 - 2 A^2 b f^3 + (A^3 y - A b^2) f^2 + 2 A^2 b y f + A b^2 y, \quad (62)
 \end{aligned}$$

in order to satisfy Eq. (59), constant b needs to assume the following expression:

$$b = \frac{\varepsilon \beta A}{2C} - \frac{\varepsilon + \delta}{2k} \quad (63)$$

and, moreover, it has to be

$$\begin{aligned}
 A^2 &= D \\
 a + 1 &= 3b + \frac{CA}{2D} \\
 D y &= 3b^2 + \left(\frac{C}{A} - 3 - \frac{k}{\varepsilon \beta} \right) b - \frac{C}{2A} - \frac{\varepsilon + \delta}{2\varepsilon \beta} + 1
 \end{aligned}$$

and

$$\begin{aligned}
 \varepsilon c - \delta h &= 2 C D A y^2 + \varepsilon \beta C A y - 2 \varepsilon \beta b^3 + 2 \varepsilon \beta (a + 1) b^2 + \\
 &- 2 \varepsilon \beta a b + 6 C A y b^2 - 4 C y (a + 1) A b + 2 a C A y - (\varepsilon + \delta) b - k b^2.
 \end{aligned}$$

So, if for instance, it is assumed $\varepsilon \beta = 1$, $z_0 = 0$, $\varepsilon + \delta = 0.04$, $k = 0.01$ and $C = 1$ with $D > 0.019$, for $A = \sqrt{D}$, by introducing

$$g(D) = \sqrt{3900\sqrt{D} - 1501 D + 150 D \sqrt{D} - 500}, \quad (64)$$

equation (61) gives

$$u(z) = \frac{\sqrt{2}}{20 D^{1/4}} g(D) \tanh \left(\frac{\sqrt{2} z g(D)}{20 D^{3/4}} \right) + \frac{\sqrt{D}}{2} - 2. \quad (65)$$

In Fig. 1, solutions $u(z)$ expressed by means of formula (65) are illustrated for different values of D , by showing that the amplitude increases as $0 < D < 1$ increases.

Remark When fast variable u simulates the membrane potential of a nerve cell, while slow variable w and superslow variable y determine the corresponding ion number densities, model (55) with its solutions can be of interest in applications to understand

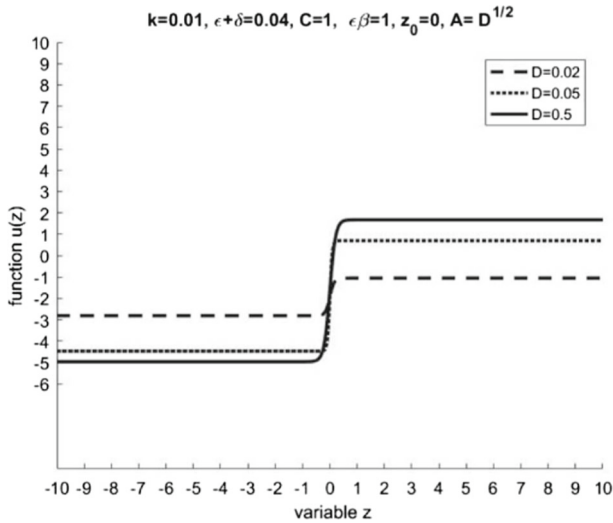


Fig. 1 Solution $u(z)$ when $\epsilon\beta = 1$, $z_0 = 0$, $\epsilon + \delta = 0.04$, $k = 0.01$, $C = 1$, $A = \sqrt{D}$ and for $D = 0.02$, $D = 0.05$, $D = 0.5$.

how impulses are propagated from one neuron to another. Moreover, as similarly underlined in [4], the knowledge of exact solutions can help in testing different applications of models in neuroscience.

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