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# Nonlinear models and bifurcation trees in quantum mechanics: a review of recent results

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**Abstract**
In this talk we discuss some recent results I obtained for a class of nonlinear models in quantum mechanics. In particular we focus our attention to the nonlinear one-dimensional Schrodinger equation with a periodic potential and a Stark-type perturbation. In the limit of large periodic potential the Stark–Wannier ladders of the linear equation become a dense energy spectrum because a cascade of bifurcations of stationary solutions occurs; for a detailed treatment we refer to Sacchetti (Phys Rev E 95:062212, 2017, SIAM J Math Anal 50(6):5783–5810, 2018) where this model has been studied.

**Keywords** Gross–Pitaevskii equation · Bose–Einstein condensates · Bifurcation tree

**Mathematics Subject Classification** 35Q55 · 81Qxx · 81T25

## 1 Introduction

The nonlinear Schrodinger equation (NLSE) received an increasing attention from mathematicians since the seminal papers by Ginibre and Velo [26], we refer to the research monographs [14] and [16]. Such an attention was initially motivated because of applications of NLSE to nonlinear optics; indeed, Maxwell equations in a Kerr medium may lead to certain NLSEs [60]. More recently, a large interest for NLSE

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also arose in quantum mechanics; in fact, the dynamics of a Bose–Einstein condensate (BEC), that is a state of matter of a dilute gas of bosons cooled to temperatures very close to absolute zero (that is, very near to 0 K), may be described by means of a NLSE. We should mention that BECs were predicted by Satyendra Nath Bose and Albert Einstein and only recently, in 1995, E. Cornell and C. Wieman at the University of Colorado gave the first experimental evidence of BECs, they received with Ketterle the 2001 Nobel price in Physics for such a result. The Bose gas is governed by Bose–Einstein statistics which describes the statistical distribution of bosons allowed to share the same quantum state. Under such conditions macroscopic quantum phenomena become apparent. In other words, a BEC is a *Schrodinger's cat*, that is a *macroscopic object* which obeys to the laws of quantum mechanics. In 1961 Eugene P. Gross [28] and Lev Petrovich Pitaevskii [35] independently obtained a NLSE, called now Gross–Pitaevkii equation (GPE), which describes the dynamics of a BEC under some circumstances, e.g. when only binary collisions are taken into account. The rigorous derivation of the GPE from a N-body problem in a suitable scale limit has been the object of a large interest in the last few years (see [9] and the reference therein). We should also mention that NLSE represents a versatile and interesting model for understanding molecular physics too (see [37] and the references therein).

In the present paper I briefly review some results I have obtained in this last decade and finally I describe with more details the cascade of bifurcations that occur in the model of accelerated BECs in a periodic lattice. The NLSE I consider has the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi + \alpha|\psi|^{2\sigma}\psi \, , \ \sigma > 0 \, , \tag{1}$$

where $m$ is the mass of the quantum particle, $\hbar$ is the Planck constant, $V$ is the external potential and $\sigma$ is the power nonlinearity; in fact, $\sigma = 1$ in the GPE. One can study such an equation by reducing it to a discrete NLSE by making use of semiclassical techniques (the basic ideas are developed in the papers [43,44]) or canonical perturbation theory (the basic ideas are developed by [6,7]).

Concerning the potential $V(x)$ I have considered different situations.

### 1.1 Singular potential

In this case the external potential $V$ is represented by means of a singular "function", typically it is a Dirac's $\delta$ or a derivative of the Dirac's $\delta$ [1,2,8,18,19,30,45,62]. In fact, Dirac's delta potentials provide a general and idealized model for short-range interactions. They have been introduced by Enrico Fermi in 1936 in three dimensions in order to investigate the scattering of slow neutrons by atoms; such potentials were later recognized to provide the simplest example of exactly solvable quantum models and have been widely employed in toy models.

### 1.2 Double-well potential

The spontaneous symmetry breaking (SSB) phenomenon is a very important effect that arises in a wide range of physical systems modeled by nonlinear equations. For instance, in optics SSB has been experimentally observed for laser beams in Kerr media and focusing nonlinearity [13,29]. Another natural setting in which SSB phenomenon may arise is for Bose–Einstein condensates with a double well potential [4,17,38]. Also, the study of gases of pyramidal molecules such as the ammonia $NH_3$ is a topic where SSB phenomenon plays a crucial role. A nonlinear mean field model of a gas of pyramidal molecules has been introduced [37,61]; in this model spontaneous symmetry breaking explaining the presence of two asymmetrical degenerate ground states, corresponding to the different localization of the molecules, has been predicted with the full agreement with experimental data. The bifurcation picture is quite complicated, and it depends on the spatial dimension as well as on the nonlinearity power; see, e.g. [3,5,15,22,27,31,34,46–49].

### 1.3 Periodic potential

Trapped atoms in optical lattices is an emerging field for applications in quantum optics and quantum information processing; furthermore, it is a model system for solid state physics, too [10,11]. A lattice potential can be formed by overlapping counter-propagating laser beams, where the atoms are trapped by the optical standing wave produced by the interference between the laser beams. By interfering more laser beams, one can obtain one-, two- and three-dimensional lattice potentials. In a periodic lattice BECs exhibit a quantum phase transition from a superfluid phase to a Mott-insulator phase when the depth of the lattice potential, which can be tuned with great accuracy, changes from shallow to deep [12]. This fact has been predicted to occur in the framework of the Bose–Hubbard model [21], and recently observed experimentally (see,e.g., [24,25,58,59]). The basic argument which leads to phase transition for nonlinear models is the SSB, this effect has been deeply discussed by [23,32,50], see also [33].

### 1.4 External symmetric potential plus a Stark-type field

Here, we consider the case where the trapping external potential has a symmetric double well or a periodic shape and where an external Stark-type perturbation breaks the symmetry. The external Stark-type perturbation term may represent the effect of gravity on a BEC. In this case one speaks of accelerated BECs in an external potential. It is well known that when the Stark-type perturbation is absent then a periodic behavior of the wave-function is expected; in fact, in the case of double-well potential a beating motion is observed, while in the case of a periodic potential Bloch oscillations are expected. The point is to understand what happen to these periodic behaviors when a Stark-type potential is switched on. The problem has some interesting aspects connected with applications. In fact, accelerated ultracold atoms moving in an optical lattice have opened the field to multiple applications [10,11,39,42,57], as well as the measurements of the value of the gravity acceleration $g$ using ultracold strontium atoms confined in a vertical optical lattice [20,36], and direct measurement of the universal Newton gravitation constant $G$ [40] and of the gravity-field curvature [41]. This problem has been the main object of my research activity in these last few years [51–56].

## 2 Accelerated BECs in a periodic lattice

The dynamics of quantum particles in a one-dimensional periodic potential under an homogeneous external field is one of the most important problems in solid-state physics. Because of the periodicity of the potential, it is expected the existence of families of quantum resonances with associated energies displaced on regular ladders, the so-called Stark–Wannier ladders, and the wavefunction would perform Bloch oscillations with period

$$T = \frac{2\pi \hbar}{mga}, \tag{2}$$

where $g$ is the gravity acceleration and $a$ is the lattice period. In fact, if one look for the spectrum of the one-dimensional operator $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V + mgx$, where $V$ is a smooth periodic potential with period $a$, it turns out that it covers the whole real axis (and it is absolutely continuous). Thus stationary states are no admissible. However, the existence of quantum resonances are proved; where a quantum resonances is, roughly speaking, a complex pole of the kernel of the resolvent operator and it is associated to (meta)stable stationary states. In such a model these complex poles $\lambda_{\ell,n}$ are labeled by the indexes $\ell \in \mathbb{N}$ and $n \in \mathbb{Z}$ and they are displaced on a regular ladders:

$$\lambda_{\ell,n} = \lambda_{\ell,0} + nmga, \quad \text{where} \quad \Im \lambda_{n,0} < 0.$$

The study of accelerated ultracold atoms moving in an optical lattice has opened the field to multiple applications, as well as the measurement of the value of the gravity acceleration $g$ using ultracold Strontium atoms confined in a vertical optical lattice. Determination of $g$ has been obtained by measuring the period $T$ of the Bloch oscillations of the atoms in the vertical optical lattice; indeed, recalling that the period $T$ is connected to the gravity acceleration $g$ by (2), then a precise value of the constant $g$ has been obtained by means of the experimental measurements of the oscillating period. Since Bloch oscillations with period $T$ have been predicted by the Bloch Theorem only for a one-body particle in a periodic field and under the effect of a Stark potential then it has been chosen, in the experiments above, a particular Strontium's isotope $^{88}Sr$; in fact, the scattering length $a_s$ of atoms $^{88}Sr$ is very small and thus it has been assumed that the effects of the atomic binary interactions are negligible [20,36].

Motivated by such physical applications we study, as a model for a confined accelerated BECs in a periodic optical lattice under the effect of the gravitational force, the nonlinear one-dimensional time-dependent Schrodinger equation with a periodic potential $V$ and an accelerating Stark-type potential $W$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{\epsilon}V\psi + \alpha W\psi + \gamma|\psi|^{2\sigma}\psi,\ \sigma > 0, \tag{3}$$

in the limit of large periodic potential, i.e. $0 < \epsilon \ll 1$. Here, $\gamma$ is the strength of the nonlinearity term; the real valued parameters $m$, $\hbar$, $\alpha$ and $\gamma$ are assumed to be fixed. In particular $W(x)$ is a Stark-type potential with strength $\alpha$, that is it is locally a linear function: $W(x) = x$ for any $x$ belonging to a fixed interval large enough.

By means of a simple recasting we swap the limit of large potential $\epsilon \ll 1$ to a semiclassical equation. If we set

$$F = \epsilon\alpha,\ h = \hbar\sqrt{\epsilon/2m},\ \tau = t/\sqrt{\epsilon/2m}\ \text{ and }\ \eta = \epsilon\gamma$$

then the above equation takes the form

$$ih\frac{\partial\psi}{\partial\tau} = -h^2\frac{\partial^2\psi}{\partial x^2} + V\psi + FW\psi + \eta|\psi|^{2\sigma}\psi \tag{4}$$

and the limit of large periodic potential $\epsilon \to 0^+$ is equivalent to the semiclassical limit $h \to 0^+$ where

$$\eta \sim F \sim h^2\ \text{ as }\ h \text{ goes to zero.} \tag{5}$$

In the semiclassical limit we will show that the time-independent NLSE may be approximated by means of a system of discrete time-independent NLSEs which stationary solutions may be explicitly calculated. In particular, a cascade of bifurcations occurs when the ratio between the nonlinearity strength and the strength of the Stark-type potential increases; in the opposite situation, that is when this ratio goes to zero, we recover a local Wannier–Stark ladders picture.

### 2.1 Notation

By $\ell^p_{\mathbb{R}}$ we denote the space of vectors $\mathbf{c} = \{c_n\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ such that $c_n \in \mathbb{R}$ are real valued. Similarly

$$L^p_{\mathbb{R}} = \{\psi \in L^p\ :\ \psi\ \text{ is a real-valued function}\}.$$

Let $f$ and $g$ two vectors belonging to a normed space with norm $\|\cdot\|$, and depending on the semiclassical parameter $h$. By the notation $f = g + \tilde{\mathcal{O}}\left(e^{-S_0/h}\right)$, as $h \to 0$, we mean that for any $\rho \in (0, S_0)$ there exist a positive constant $C := C_\rho > 0$ (independent of $h$) such that

$$\|f - g\| \leq Ce^{-(S_0-\rho)/h},\ \forall h \in (0, h^\star),$$

for some $h^\star > 0$. By the notation $f \sim g$, as $h \to 0$, we mean that $\lim_{h\to 0^+} \frac{f}{g} = C$ for some $C \in (0, +\infty)$. By $C$ we denote a generic positive constant independent of $h$ whose value may change from line to line.

## 3 Description of the model and assumptions

Here we consider the nonlinear Schrodinger equation (4) where $V(x)$ is a smooth, real-valued, periodic and non negative function with period $a$, i.e.

$$V(x) = V(x+a),\ \forall x \in \mathbb{R},$$

and with minimum point $x_0 \in \left[-\frac{1}{2}a, +\frac{1}{2}a\right)$ such that

$$V(x) > V(x_0),\ \forall x \in \left[-\frac{1}{2}a, +\frac{1}{2}a\right) \setminus \{x_0\}.$$

For argument's sake we assume that $V(x_0) = 0$ and $x_0 = 0$.

In the following let us denote by $x_n = x_0 + na$.

Concerning the perturbation $W(x)$ we assume that it is a smooth, real-valued and bounded function such that

$$W(x) = x \ \text{ if } \ |x| \leq Na,$$

for some $N \in \mathbb{N}$. Furthermore $W$ has compact support $\Omega \supset [-Na, Na]$. In fact, a Stark potential is such that $W(x) \equiv x$ for any $x$; in such a case $W(x)$ is not a bounded operator and this fact is a source of serious technical problems. For this kind of reasons we have restricted ourselves to a bounded and compactly supported Stark-type potential.

Let $H_B$ be the Bloch operator formally defined on $L^2(\mathbb{R}, dx)$ as

$$H_B := -h^2 \frac{d^2}{dx^2} + V. \tag{6}$$

The linear operator $H$, formally defined as

$$H = H_B + FW$$

on the Hilbert space $L^2(\mathbb{R}, dx)$, admits a self-adjoint extension, still denoted by $H$. Here, we look for stationary solutions to Eq. (4) of the form

$$\psi(x,\tau) = e^{-i\lambda\tau/h}\psi(x)$$

for some *energy* $\lambda \in \mathbb{R}$ and wave function $\psi(x)$. Hence, Eq. (4) takes the form

$$H\psi + \eta|\psi|^{2\sigma}\psi = \lambda\psi. \tag{7}$$

We must underline that when a stationary solution $\psi$ to Eq. (7) is regular enough then $\psi$ is, up to a phase factor, a real-valued function. Hence, Eq. (7) can be replaced by the following equation

$$H\psi + \eta\psi^{2\sigma+1} = \lambda\psi,\ \ \psi \in L^2_{\mathbb{R}}. \tag{8}$$

where $\psi$ is real-valued.

Finally, we state our last assumption; it concerns the nonlinearity power: we assume that

$$\sigma = 1.$$

That is we restrict ourselves to the case of a cubic nonlinear equation where (4) becomes the so called Gross–Pitaevskii equation.

Our aim is to look for real-valued stationary solutions $\psi \in H^1$ to (8) with associated energy $\lambda \in \mathbb{R}$.

The following results have been proved in forthcoming paper [56] and already discussed from a physical point of view by [55].

## 4 Construction of the discrete nonlinear Stark–Wannier equation

We consider low energies: let $\Pi$ the projection operator associated to the first band $[E_1^b, E_1^t]$ of $H_B$ and let $\Pi_\perp = \mathbf{1} - \Pi$. Let

$$\psi = \psi_1 + \psi_\perp \ \text{ where } \psi_1 = \Pi\psi \ \text{ and } \ \psi_\perp = \Pi_\perp\psi. \tag{9}$$

We may write $\psi_1$ by means of a linear combination of a suitable orthonormal base $\{u_n\}_{n\in\mathbb{Z}}$ of the space $\Pi\left[L^2(\mathbb{R})\right]$, that is

$$\psi_1(x) = \sum_{n\in\mathbb{Z}} c_n u_n(x), \tag{10}$$

where $u_n \in H^1(\mathbb{R})$ and

$$\mathbf{c} = \{c_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}(\mathbb{Z})$$

since $\psi$, and then $\psi_1$, is a real-valued function by (8) and $u_n$ are real valued too. Roughly speaking each vector $u_n$ is approximated by the ground state of the operator $-h^2\frac{\partial^2}{\partial x^2} + V_n$, where $V_n$ is obtained by filling all the wells of the periodic potential $V$, but the one with index $n$ and center at $x_n$.

By inserting (9) and (10) in Eq. (8) then it takes the form

$$\begin{cases} \lambda c_n &= \langle u_n, H_B\psi\rangle + F\langle u_n, W\psi\rangle + \eta\langle u_n, \psi^{2\sigma+1}\rangle,\ \ n \in \mathbb{Z} \\ \lambda\psi_\perp &= \Pi_\perp H_B\psi + F\Pi_\perp W\psi + \eta\Pi_\perp\psi^{2\sigma+1} \end{cases}, \tag{11}$$

where $\mathbf{c} \in \ell^2_{\mathbb{R}}$ and $\psi_\perp$ are such that

$$\|\psi\|^2_{L^2} = \|\mathbf{c}\|^2_{\ell^2} + \|\psi_\perp\|^2_{L^2}.$$

If one expands the r.h.s of the first equation of (11) then it finally takes the form

$$\begin{cases} \lambda c_n &= (\Lambda_1 + FC_0)c_n - \beta(c_{n+1} + c_{n-1}) + F\xi(n)ac_n + \eta C_1 c_n^{2\sigma+1} + r_n, \\ \lambda\psi_\perp &= H_B\psi_\perp + F\Pi_\perp W\psi + \eta\Pi_\perp\psi^{2\sigma+1} \end{cases} \tag{12}$$

where $C_1 = \|u_0\|^{2\sigma+2}_{2\sigma+2} \sim h^{-\frac{\sigma}{2}}$ as $h$ goes to zero, $\xi(n) = n$, $|n| \leq N$, is a bounded function with compact support; $\mathbf{r} = \{r_n\}_{n\in\mathbb{Z}}$ is a remainder term such that for any $\delta_0 > 0$ fixed

$$\sup_{\|\mathbf{c}\|_{\ell^1} \leq \delta_0} \|\mathbf{r}\|_{\ell^1} \leq C\hbar^{\frac{5}{2}}$$

and

$$\beta = \tilde{O}(e^{-S_0/h}) \ \text{ where } \ S_0 = d_A(x_n, x_{n+1}) = \int_{x_n}^{x_{n+1}} \sqrt{V(x)}dx$$

is the Agmon distance between two adjacent wells.

## 5 Anticontinuous limit of the DNLSWE

Let us set

$$\mu^S := \lambda - (\Lambda_1 + FC_0), \ \nu := \eta C_1, \ f := Fa, \tag{13}$$

hence

$$f \sim h^2 \ \text{ and } \ \nu \sim h^{2-\frac{1}{2}\sigma}. \tag{14}$$

For argument's sake, we assume that $f, \ \nu \geq 0$. If we neglect the remainder term $\mathbf{r}$ then (12) takes the form

$$\mu^S g_n = -\beta\left(g_{n+1} + g_{n-1}\right) + f\xi(n)g_n + \nu g_n^{2\sigma+1}. \tag{15}$$

and in the *anticontinuous limit* $\beta \to 0$ then (15) becomes

$$\left(\mu^S - \nu d_n^{2\sigma}\right) d_n = f\xi(n)d_n, \mathbf{d} = \{d_n\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}(\mathbb{Z}). \tag{16}$$

### 5.1 Finite-mode solutions to the anticontinuous limit equation (16)

Here, we look for stationary solutions $\mathbf{d} \in \ell^2_{\mathbb{R}}$ to (16) under the normalization condition

$$\|\mathbf{d}\|^2_{\ell^2} = \sum_{n\in\mathbb{Z}} d_n^2 = 1.$$

We say that the anticontinuous limit equation (16) has a finite-mode solution if there exists a solution-set $S \subset \mathbb{Z}$ with finite cardinality, a real value $\mu^S$ and a normalized vector $\mathbf{d}^S = \{d_n^S\}_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{R}}(\mathbb{Z})$ where $\mu^S$ and $\mathbf{d}^S$ solve

$$\left(\mu^S - \nu d_n^{2\sigma}\right) d_n = f\xi(n) d_n, \quad \text{with } d_n \neq 0 \text{ if } n \in S, \tag{17}$$

and where $d_n^S = 0$ if $n \notin S$. The real value $\mu^S$ is hereafter called the *energy* associated to the stationary solution $\mathbf{d}^S$.

In Fig. 1 an example of finite-mode solutions is given; both solution sets are associated to the same value of the energy $\mu = \frac{1}{3}\nu + \frac{7}{3}f$.

We consider solution-sets $S$ associated to a given *rung* of the (kind of) Stark–Wannier ladder satisfying the condition $S \subseteq [-N, +N]$ where $\xi(n)$ is a linear function. That is we consider energies $\mu^S$ in the interval $[\nu - fN, \nu + fN]$. We can see that stationary solutions to Eq. (17) associated to such solution-sets $S$ may bifurcate when the ratio $\nu/f$ is a positive integer number.

In order to count how many stationary solutions we have let us introduce the following function denoted by $Q(n)$, where $n \in \mathbb{N}$, and defined by the number of ways of writing the integer number $n$ as a sum of positive integers without regard to order, with the constraint that all integers in a given partition are distinct. E.g.: $Q(1) = 1$, $Q(2) = 1$, $Q(3) = 2$ and $Q(4) = 2$.

**Theorem 1** *When $\nu/f$ takes the value of a positive integer number then stationary solutions to* (17)*, associated to solution-sets $S \subset [-N, N]$, bifurcate. Furthermore, the total number of solutions-sets S associated to a given* rung *of the (kind of) Wannier–Stark ladder, assuming that all these sets S are contained in the interval* $[-N, +N]$*, is given by*

$$M(\nu/f) = \sum_{0<n<\nu/f} Q(n). \tag{18}$$

A cascade of bifurcation points, when $\nu/f$ takes the value of any positive integer, occurs; indeed, when the ratio $\nu/f$ becomes larger than a positive integer $n$ then $Q(n)$ new stationary solutions appear. This fact can be seen in Fig. 2, where we plot the values of $\mu/f$, when $\nu/f$ belongs to the interval [0, 10], associated to solution-sets $S$ such that $\min S = 0$, that is we plot the value of energies associated to the 0-th *rung* of the (kind of) Wannier–Stark ladder. By translation this picture occurs for each *rung* of the ladder and then the collection of values of $\mu$ associated to stationary solutions is going to densely cover intervals of the real axis.

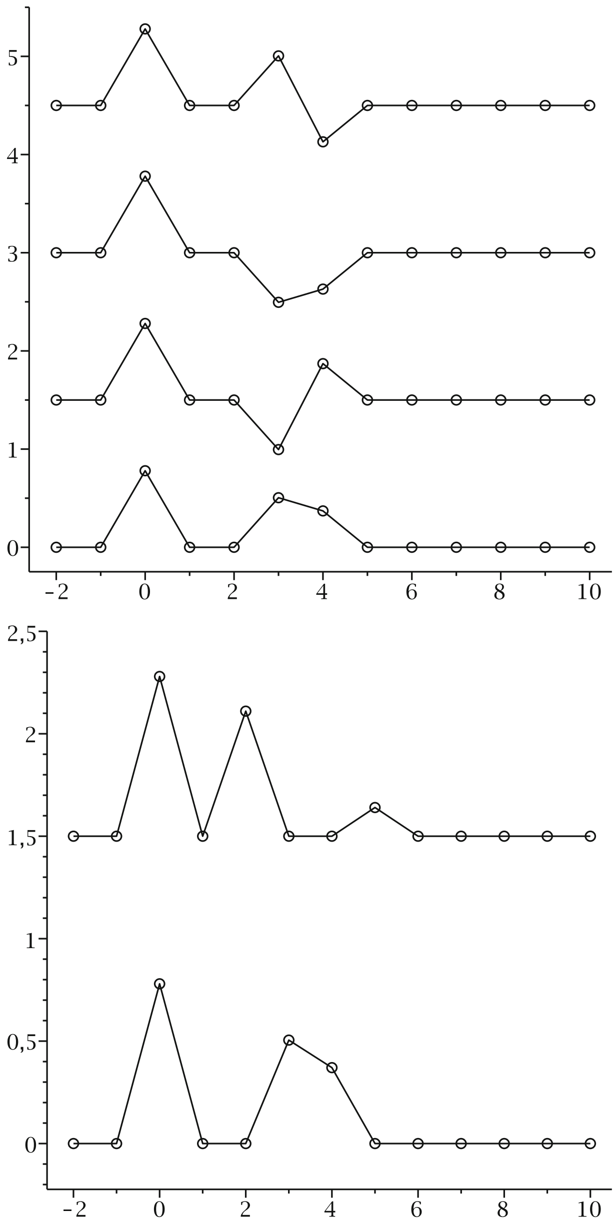


**Fig. 1** In the left panel we plot the 4 solutions $\mathbf{d}^{S_1}$ corresponding to the solution-set $S_1 = \{0, 3, 4\}$. In the right panel we plot the solutions $\mathbf{d}^{S_1}$ and $\mathbf{d}^{S_2}$ with sign $+$, corresponding to the solution-sets $S_1$ and $S_2 = \{0, 2, 5\}$; both solutions are associated to the same value of the energy $\mu$. Reprinted from [56]

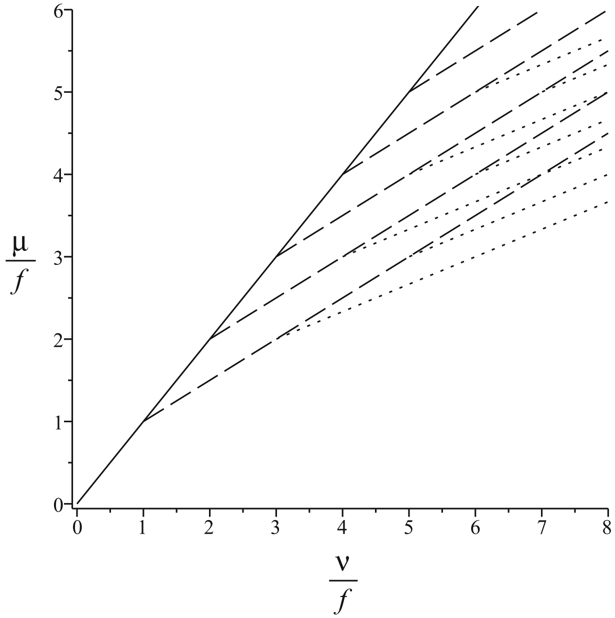


**Fig. 2** Here we plot the values of $\mu/f$ associated to stationary solution-sets $S$ such that $\min S = 0$ and where $\mathcal{N} = 1, 2, 3$; we can see a cascade of bifurcations when $\nu/f$ increases. Full line represents the solution corresponding to the 0-th *rung* of the Stark–Wannier ladder localized on the 0-th cell ($\mathcal{N} = 1$), broken lines represent the solutions of the same *rung* of the Stark–Wannier ladder localized on two cells ($\mathcal{N} = 2$), and finally point lines represent the solutions of the same *rung* of the Stark–Wannier ladder localized on three cells ($\mathcal{N} = 3$). Reprinted from [56]

In order to understand why Theorem 1 holds true we assume now that the effective nonlinearity strength is not zero, that is $\nu > 0$ for argument's sake. In such a case, Eq. (17) has finite mode solutions $\mathbf{d}^S = \{d_n^S\}_{n\in\mathbb{Z}}$, associated to sets $S \subset \mathbb{Z}$ with finite cardinality $\mathcal{N} = \sharp S < \infty$, given by

$$d_n^S = \begin{cases} 0 & \text{if } n \notin S \\ \pm\left[\frac{\mu^S - f\xi(n)}{\nu}\right]^{1/2\sigma} & \text{if } n \in S \end{cases}, \tag{19}$$

with the condition

$$\mu^S - f\xi(n) > 0, \; n \in S, \tag{20}$$

since we have assumed that $d_n^S \in \mathbb{R}$ and $\nu > 0$. The normalization condition reads

$$1 = \|\mathbf{d^S}\|_{\ell^2} = \sum_{n\in S} (d_n^S)^2 = \sum_{n\in S} \left[\frac{\mu^S - f\xi(n)}{\nu}\right]^{1/\sigma}. \tag{21}$$

In the case $\mathcal{N} = 1$ then $S = \{j\}$ again for any $j \in \mathbb{Z}$ and (21) reduces to

$$\mu^S = \nu + f\xi(j)$$

where condition (20) holds true because we have assumed that $\nu > 0$; the associated stationary solution $\mathbf{d}^S$ takes the form:

$$d_n^S = \begin{cases} 0 & \text{if } n \neq j \\ \pm 1 & \text{if } n = j \end{cases}.$$

That is we recover a kind of (perturbed) Stark–Wannier ladder.

From this fact we can conclude that the anticontinuous limit (16) always admits a ladder-type family of normalized one-mode solutions.

### 5.2 Finite-mode solutions to equation (17) associated to solution-sets *S* with finite cardinality bigger that 1

In order to look for finite-mode solutions with $\mathcal{N} > 1$ we restrict ourselves to the case of cubic nonlinearity (i.e. $\sigma = 1$) as we have previously assumed; in such a case it follows that the normalization condition (21) implies that

$$\mu^S = \frac{\nu}{\mathcal{N}} + \frac{f}{\mathcal{N}} \sum_{n \in S} \xi(n) \text{ with } \max_{n \in S} \xi(n) < \frac{\mu^S}{f}. \tag{22}$$

**Theorem 2** *Let* $S = \{j + \ell_0, j + \ell_1, \ldots, j + \ell_{\mathcal{N}-1}\}$, *with* $j \in \mathbb{Z}$ *and* $0 = \ell_0 < \ell_1 < \ell_2 < \ldots < \ell_{\mathcal{N}-1}$ *positive integer numbers such that*

$$\xi\left(j + \ell_{\mathcal{N}-1}\right) < \frac{\nu}{f\mathcal{N}} + \frac{1}{\mathcal{N}} \sum_{k=0}^{\mathcal{N}-1} \xi\left(j + \ell_k\right) \tag{23}$$

*Then S is a solution-set connected to the j-th* rung *of a (kind of) Stark–Wannier ladder and Eq.* (17) *has a* $\mathcal{N}$*-mode solution*

$$\mu^S = \frac{\nu}{\mathcal{N}} + \frac{f}{\mathcal{N}} \sum_{k=0}^{\mathcal{N}-1} \xi\left(j + \ell_k\right). \tag{24}$$

First of all, since the stationary problem (17) is translation invariant $n \to n + \ell$ and $\mu^S \to \mu^S - f\ell$, provided that the solution-sets are contained in the interval $[-N, N]$, then we can always restrict ourselves to the 0-th *rung* of the ladder such that $\min S = 0$, that is the solution-set has the form $S = \{0, \ell_1, \ldots, \ell_{\mathcal{N}-1}\}$ with $0 < \ell_1 < \ell_2 < \cdots < \ell_{\mathcal{N}-1} < N$ positive and integer numbers. Hence, (22) becomes

$$\mu^S = \frac{\nu}{\mathcal{N}} + \frac{f}{\mathcal{N}} \sum_{\ell \in S} \ell$$

and condition (20) implies the following condition on the solution-set $S$

$$\frac{\nu}{f} > \mathcal{N} \max S - \sum_{\ell \in S} \ell = \sum_{\ell \in S} [\max S - \ell] > \sum_{\ell^\star \in S^\star} \ell^\star \tag{25}$$

where

$$S^\star = \{\ell^\star := \max S - \ell \ : \ \ell \in S\}.$$

Let $\mathcal{S}^\star(\nu/f)$ be the collection of sets $S^\star$ satisfying (25), and let $\mathcal{Q}^\star(n)$ be the collection of sets of all non negative integer numbers, including the number 0, which sum is equal to $n$, without regard to order with the constraint that all integers in a given partition are distinct; e.g. $\mathcal{Q}^\star(1) = \{\{0, 1\}\}$, $\mathcal{Q}^\star(2) = \{\{0, 2\}\}$ and $\mathcal{Q}^\star(3) = \{\{0, 3\}, \{0, 1, 2\}\}$. Hence, by construction

$$\mathcal{S}^\star(n+1) = \mathcal{S}^\star(n) \cup \mathcal{Q}^\star(n).$$

In conclusion, we have shown that the counting function $M(\nu/f)$ defined as the number of solution-sets $S$ of integer numbers satisfying the conditions (25) and such that $\min S = 0$, is given by

$$M(\nu/f) = \sum_{0<n<\nu/f} Q(n).$$

Theorem 1 then follows.

One can see that $M(\nu/f)$ grows quite fast, indeed the following asymptotic behavior holds true:

$$Q(n) \sim \frac{e^{\pi\sqrt{n/3}}}{4 \cdot 3^{1/4} n^{3/4}} \ \text{ as } \ n \to \infty.$$

Hence

$$M(n) \sim \frac{1}{2} \text{erfi}\left[\sqrt{\pi}(n/3)^{1/4}\right] \sim \frac{\exp\left[\pi(n/3)^{1/2}\right]}{2\pi(n/3)^{1/4}}$$

as $n$ goes to infinity, where $\text{erfi}(x) = -i\,\text{erf}(ix)$ is the imaginary error function. In particular, since $\frac{\nu}{f} \sim C_1 \sim h^{-\sigma/2}$ with $\sigma = 1$, then we have that the energy $\mu$ lies in an interval $[\nu - fN, \nu + fN]$ with center at $\nu \sim h^{3/2}$ and amplitude of order $h^2$, and the number of stationary solutions is of order

$$M\left(\frac{\nu}{f}\right) \sim h^{1/8} e^{Ch^{-1/4}} \ \text{ as } \ h \ \text{ goes to zero,}$$

for some positive constant $C$. That is the energy spectrum densely fill the interval $[\nu - fN, \nu + fN]$ when $h$ goes to zero.

### 5.3 When do $\mathcal{N}$-mode stationary solutions arise from $(\mathcal{N} - 1)$-mode stationary solutions?

If one looks with more detail the bifurcation cascade one can see that we have $\mathcal{N}$-mode solutions for any value of $\mathcal{N}$, provided that $S \subset [-N, N]$ for some $N$ large enough.

**Theorem 3** *If $\nu/f < \mathcal{N}(\mathcal{N} - 1)/2$ then stationary solutions to* (16)*, associated to solution-sets $S \subset [-N, +N]$, are localized on a number of sites less than $\mathcal{N}$, at $\nu/f = \mathcal{N}(\mathcal{N} - 1)/2$ a stationary solution localized on $\mathcal{N} - 1$ sites bifurcates and a new stationary solution localized on $\mathcal{N}$ sites arises.*

## 6 Conclusion

Now, we are ready to close our analysis by connecting the stationary solutions to the discrete NLSE in the anticontinuous limit (16) with the stationary solutions to the GP equation (4). In particular, by making use of some fixed point arguments developed with Reika Fukuizumi [23] we are able to prove that:

**Theorem 4** *Let $\frac{\nu}{f} \notin \mathbb{N}$ and let $h > 0$ small enough. Let $\sigma = 1$. Let $\mathbf{d}^S$ be a finite-mode normalized solution to the discrete nonlinear Schrodinger equation in the anticontinuous limit associated to a solution-set $S$ satisfying the assumption of Theorem 1. Then there exists a stationary solution $\psi^S$ to the nonlinear Schrodinger equation* (8) *such that*

$$\left\| \psi^S - \sum_{n\in S} d_n^S u_n \right\|_{H^1} \le C h^{1/4}.$$

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