

Set-valued Brownian motion

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Abstract Brownian motions, martingales, and Wiener processes are introduced and studied for set valued functions taking values in the subfamily of compact convex subsets of arbitrary Banach spaces X . The present paper is an application of the paper (Labuschagne et al. in *Quaest Math* 30(3):285–308, 2007) in which an embedding result is obtained which considers also the ordered structure of the family of compact

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convex subsets of a Banach space X and of Grobler and Labuschagne (J Math Anal Appl 423(1):797–819, 2015; J Math Anal Appl 423(1):820–833, 2015) in which these processes are considered in f -algebras. Moreover, in the space of continuous functions defined on a Stonian space, a direct Levy's result follows.

Keywords Brownian motion · Rådström embedding theorem · Vector lattices · Marginal distributions · Generalized Hukuhara difference

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1 Introduction

It is well known that the concept of Brownian motion is one of the most important in probability theory and its applications as Stock Market, Financial Mathematics, Mathematical Statistics, Uncertainty, Decision Making, Fractal Analysis in Medical Imaging.

The starting point of the present research are the papers [2, 13, 14, 16, 17, 22, 24, 27, 30] in which stochastic integration is studied in partially ordered spaces or in the fuzzy set valued case. The literature in this field is rich, we can cite for example [4–9, 15, 18–20, 23, 26, 28, 29, 33].

Here the notion of set valued Brownian motion is introduced and studied for the case of compact convex subsets of a Banach space X . By considering the scalar valued case, multiplication and subtraction have a prominent role in the definition of Brownian motion. In the set valued case, multiplication provides a problem as we are not aware of a canonical multiplication of convex compact sets that yields a set with the same properties. There are, however, known canonical subtraction operators for such sets ([25] and the references therein). Subtraction thus seems easier to deal with. Our study is motivated by the facts that very little is known about Brownian motion in the set valued case, and the known examples of set valued stochastic integration do not really deal with set valued Brownian motion, as would be a canonical analogue of the scalar valued case.

The paper is organized as follows: in Sect. 2 the basic properties of the hyperspace $ck(X)$ and its embedding in $C(K)$ are introduced. Since, in order to properly define a set valued Brownian motion, a difference and a multiplicative structure are needed, the embedding and the Riesz structure of $C(K)$ are used. For this reason the theory of integration in vector lattice is very important and useful, see for example [3, 11, 30–32].

In Sect. 3 examples of $ck(X)$ -valued Brownian motion are given together with some properties and with some characterizations involving pointwise martingales and Gaussian processes. Moreover the quadratic variation is introduced and a Levy's reverse result is obtained. In Sect. 4 a possible extension to arbitrary Banach lattices is given: this is done in the more abstract framework of [16, 17], with the purpose to compare the two types of construction in the particular case here discussed, where the Banach lattice is $C(K)$. In this last case, when K is Stonian, moreover a direct Levy's result follows. In the appendix a characterization of the generalized Hukuhara difference which extends [25] is introduced.

2 Preliminaries

We recall from [10, Chapter II] the following notations that will be used in the present paper. Let X be a Banach space with its dual X^* and let $ck(X)$ be the subfamily of $2^X \setminus \emptyset$ of all compact, convex subsets of X .

As in [10] for all $A, B \in ck(X)$ and $\lambda \in \mathbb{R}$ the Minkowski addition and scalar multiplication are defined as

$$A + B = \{a + b : a \in A, b \in B\}, \text{ and } \lambda A = \{\lambda a : a \in A\} \tag{1}$$

Let H be the corresponding Hausdorff metric on $ck(X)$, i.e.

$$H(A, B) = \max(e_d(A, B), e_d(B, A))$$

where the excess $e_d(A, B)$ of the set A over the set B is defined as

$$e_d(A, B) = \sup\{d(a, B) : a \in A\} = \sup\{\inf_{b \in B} d(a, b) : a \in A\}.$$

It is known that the family $ck(X)$ endowed with the Hausdorff metric is a complete metric space. For every $C \in ck(X)$, the *support function* of C is denoted by $s(\cdot, C)$ and is defined by $s(x^*, C) = \sup\{\langle x^*, c \rangle : c \in C\}$ for each $x^* \in X^*$.

Clearly, the map $x^* \mapsto s(x^*, C)$ is sublinear on X^* and

$$-s(-x^*, C) = \inf\{\langle x^*, c \rangle : c \in C\}, \text{ for each } x^* \in X^*.$$

The following theorem holds:

Theorem 2.1 ([21, Theorem 5.7]) *Let X be a Banach space; then there exist a compact (Stonian) Hausdorff space K and a map $j : ck(X) \rightarrow C(K)$ such that*

- (2.1.a) $j(\alpha A + \beta C) = \alpha j(A) + \beta j(C)$ for all $A, C \in ck(X)$ and $\alpha, \beta \in \mathbb{R}^+$,
- (2.1.b) $d_H(A, C) = \|j(A) - j(C)\|_\infty$ for every $A, C \in ck(X)$,
- (2.1.c) $j(ck(X))$ is norm closed in $C(K)$,
- (2.1.d) $j(\text{co}(A \cup B)) = \max\{j(A), j(C)\}$, for all $A, C \in ck(X)$.

The Rådström embedding $j(\widetilde{ck(X)})$ of $ck(X)$ is given by $j : ck(X) \rightarrow j(\widetilde{ck(X)})$, where $j(C) = s(\cdot, C)$ for all $C \in ck(X)$ and $j(\widetilde{ck(X)})$ is the closure of the span of $\{s(\cdot, C) : C \in ck(X)\}$ in $(C(B_{X^*}), \sigma(X^*, X))$. Here $C(B_{X^*}) = \{f : B_{X^*} \rightarrow \mathbb{R} : f \text{ is continuous}\}$, B_{X^*} denotes the unit ball of X^* and $\sigma(X^*, X)$ denotes the weak* topology on X^* .

The bounded-weak-star (bw*) topology is the strongest topology of B_{X^*} with coincides with the weak* topology of B_{X^*} on every ball $B_{X^*}^r := \{f \in B_{X^*} : \|f\| \leq r\}$. Let $\mathfrak{B}(ck(X))$ be the Borel σ -algebra on $(ck(X), d_H)$.

In order to define Brownian multivalued motion a multiplication and a difference in $ck(X)$ are needed. For what concerns the difference see the Appendix (however we shall always consider the difference $B_1 - B_2$ of two convex and compact sets as the element $j(B_1) - j(B_2)$ in $C(K)$), while to access the averaging properties of

conditional expectation operators a multiplicative structure is needed. In the Riesz space setting the most natural multiplicative structure is that of an f -algebra (see for example [1]). This gives a multiplicative structure that is compatible with the order and additive structures on the space.

The ideal, E^e , of E generated by e , where e is a weak order unit of E and E is Dedekind complete, has a natural f -algebra structure. This is constructed by setting $(Pe) \cdot (Qe) = PQe = (Qe) \cdot (Pe)$ for band projections P and Q , and extending to E^e by use of Freudenthal’s Theorem. In fact this process extends the multiplicative structure to the universal completion E^u , of E . This multiplication is associative, distributive and is positive in the sense that if $x, y \in E^+$ then $xy > 0$. Here e is the multiplicative unit.

Thus the multiplication operation $\cdot : ck(X) \times ck(X) \rightarrow C(K)$ can be defined by:

$$A \cdot B = j(A) \cdot j(B).$$

If X is finite dimensional then $j(B_X) \cdot j(B) = j(B)$, and $B_X \cdot B$ exists not only in $C(K)$ but also in $ck(X)$ (and of course coincides with B).

3 $ck(X)$ -valued Brownian motion

Now we shall introduce a Brownian motion taking values in the space $ck_r(X)$, where X is any general Banach space. (Here the notation $ck_r(X)$ means all the indicator functions of the type $r1_B$, as r varies in \mathbb{R} and B in $ck(X)$).

In order to do this, let us denote by e the unit function in $C(K)$. In case X is finite-dimensional, $e = j(B_X)$, the corresponding element of the unit ball of X .

Definition 3.1 Let S denote the hyperspace we are interested in, i.e. $ck_r(X)$, and let $(B_t)_t$ be a process taking values in S , namely for every $t \geq 0$ $B_t : \Omega \rightarrow S \subset C(K)$. This process will be called *set-valued Brownian motion* if the following conditions are satisfied:

- (3.1.1) There exists an f -algebra L such that $B_t(\omega) \in L$ for each $\omega \in \Omega$ and each $t > 0$;
- (3.1.2) B_t, B_t^2 are $C(K)$ -valued Bochner integrable functions for each $t > 0$;
- (3.1.3) For every evaluation functional $f \in C(K)^*$, the process $f(B_t)_t$ is a standard real Brownian motion.

We recall that an *evaluation functional* f associates to every $x \in C(K)$ the value $x(k)$ for some fixed $k \in K$.

Example 3.2 The following is an example of a set-valued Brownian motion, when X is finite-dimensional: $(B_t)_t = (W_t e)_t$ where $(W_t)_t$ is the standard scalar Brownian motion, and e is the unit ball in X . Then for every $f \in C(K)^*$ such that $f(e) = 1$ it is

$$f(W_t e)_t = (W_t f(e))_t = (W_t)_t.$$

So for every elementary event ω

$$B_t(\omega) = \begin{cases} W_t(\omega)j(B_X)j(S) & W_t(\omega) > 0 \\ -|W_t(\omega)|j(B_X) & \text{otherwise} \end{cases}$$

So $B_t(\omega) \in j(S)$ if $W_t(\omega) > 0$, while $B_t(\omega) \in -j(S)$ otherwise.

Next, for every real number t and every element $B \in ck(X)$, the notation tB represents the indicator function $t1_B$.

Finally, if $(W_t)_{t>0}$ denotes the standard Brownian motion, and if we set $V_t := W_t e$ for each positive t , then we have shown that $(V_t)_{t>0}$ is a Brownian motion taking values in $S := ck_r(X)$ (or in $j(S)$ after embedding).

From now on, let (Ω, \mathcal{A}, P) denote any fixed probability space, with a σ -algebra \mathcal{A} and a countably additive probability measure P .

Definition 3.3 Let $\Gamma : \Omega \rightarrow ck(X)$ be a measurable function. Define

$$P_\Gamma(B) = P(\Gamma(\omega) \subset B) \text{ for all } B \in \mathfrak{B}(ck(X))$$

and

$$F_\Gamma(Y) = P(\Gamma(\omega) \subset Y) \text{ for all } Y \in ck(X).$$

Then $P_\Gamma : \mathfrak{B}(ck(X)) \rightarrow [0, 1]$ is a probability measure (the *probability distribution* of Γ), and $F_\Gamma : ck(X) \rightarrow [0, 1]$ is its *distribution function*.

Proposition 3.4 Let $\Gamma : \Omega \rightarrow ck(X)$ be a measurable set-function. Then

$$F_\Gamma = F_{j \circ \Gamma}(j(\cdot)).$$

Proof It is

$$\begin{aligned} F_\Gamma(Y) &= P(\Gamma \subset Y) \text{ where } \Gamma \subset Y \iff j(\Gamma) \leq j(Y) \\ F_\Gamma(Y) &= P(\Gamma \subset Y) = P(j(\Gamma) \leq j(Y)) = F_{j(\Gamma)}(j(Y)). \end{aligned}$$

□

Example 3.5 Let us assume that X_1 and X_2 are two real-valued random variables, $X_1 \leq X_2$, and consider the variable $\Gamma := [X_1, X_2]$ taking values in the hyperspace $ck(\mathbb{R})$. Now, when Y is an element of $ck(\mathbb{R})$, i.e. $Y = [y_1, y_2]$, the condition $\Gamma \subset Y$ means $[X_1 \geq y_1, X_2 \leq y_2]$, and so

$$F_\Gamma(Y) = P([y_1 \leq X_1 \leq X_2 \leq y_2]).$$

On the other hand, in this situation, the *unit sphere* of the dual space of \mathbb{R} is simply the set $\{-1, 1\}$, and, for every set $[a, b] \in ck(\mathbb{R})$, one has

$$s(x^*, [a, b]) = \begin{cases} -a, & x^* = -1 \\ b, & x^* = 1 \end{cases}$$

Hence, one can write $j([a, b]) = (-a, b)$ as soon as $a, b \in \mathbb{R}$, $a \leq b$. Then $j(\Gamma) = (-X_1, X_2)$ and $j(Y) = (-y_1, y_2)$: the condition $j(\Gamma) \leq j(Y)$ now means

$$-X_1 \leq -y_1, \quad X_2 \leq y_2$$

and again one has

$$F_{j(\Gamma)}(j(Y)) = P([y_1 \leq X_1 \leq X_2 \leq y_2]) = F_{\Gamma}(Y).$$

Let X_1, Z be two independent random variables with distribution $\Gamma(1, \lambda)$, and denote $X_2 := X + Z$.

Then clearly $0 \leq X_1 \leq X_2$, and $\underline{X} = [X_1, X_2]$ defines a $ck(\mathbb{R})$ -valued variable. In order to compute its distribution function, fix arbitrarily y_1 and y_2 in \mathbb{R} , with $0 \leq y_1 \leq y_2$. Then

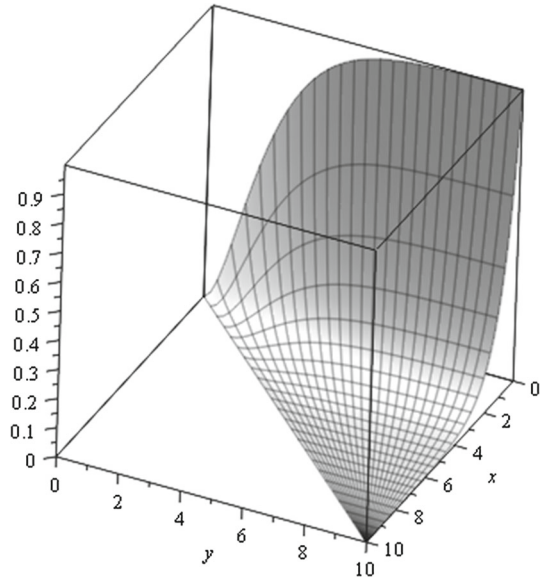
$$\begin{aligned} F_{\underline{X}}([y_1, y_2]) &= P([y_1 \leq X_1 \leq X_2 \leq y_2]) = \int_{y_1}^{y_2} \left(\int_0^{y_2-x} f_Z(z) dz \right) f_{X_1}(x) dx \\ &= \lambda^2 \int_{y_1}^{y_2} \int_0^{y_2-x} e^{-\lambda x} e^{-\lambda z} dz dx. \end{aligned}$$

Simple computations give finally

$$\begin{aligned} F_{\underline{X}}([y_1, y_2]) &= e^{-\lambda y_1} - e^{-\lambda y_2} + \lambda(y_1 - y_2)e^{-\lambda y_2} \\ &= F_{(-X_1, X_2)}(-y_1, y_2). \end{aligned}$$

In Figure Fig. 1 the plot of the the distribution function of $F_{\underline{X}}$ for $\lambda = 1$ is given.

Fig. 1 Plot of the distribution function of F_X



In [27] the set valued Gaussian distribution is defined to satisfy the condition $F_\Gamma = F_{j \circ \Gamma}(j(\cdot))$.

Theorem 3.6 Assume that $W_t : \Omega \rightarrow ck(X)$ is a weakly continuous L -valued function of $t \geq 0$ that satisfies $W_0 = 0$. Moreover suppose that W_t and W_t^2 are Bochner integrable for each t . Let $\{t_0, \dots, t_m\}$ be such that $0 = t_0 < t_1 < \dots < t_m$.

Then $(W_t)_i$ is a Brownian motion if and only if one of the following statements holds for any evaluation function $f \in C(K)^*$:

(3.6.i) The increments $f(W_{t_1} - W_{t_0}), f(W_{t_2} - W_{t_1}), \dots, f(W_{t_m} - W_{t_{m-1}})$ are independent and each of these increments is normally distributed with null mean and variance equal to $t_{i+1} - t_i$

(3.6.ii) The random variables $f(W_{t_1}), f(W_{t_2}), \dots, f(W_{t_m})$ are jointly normally distributed with means equal to zero and co-variance matrix V given by

$$V = \begin{pmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & \vdots & & \vdots \\ t_1 & t_m & \cdots & t_m \end{pmatrix}$$

(3.6.iii) The random variables $f(W_{t_1}), f(W_{t_2}), \dots, f(W_{t_m})$ have the joint moment-generating function given by

$$\varphi(u_1, \dots, u_m) = \exp \left\{ \frac{1}{2} u_m^2 (t_m - t_{m-1}) \right\} \cdots \exp \left\{ (u_1 + u_2 + \dots + u_m)^2 t_1 \right\},$$

for every $u_1, \dots, u_m \in \mathbb{R}$.

Proof Let $\{t_0, \dots, t_m\}$ be fixed with $0 = t_0 < t_1 < \dots < t_m$ and consider an evaluation function f . By 3.6.i) it is

$$\begin{aligned} t_{i+1} - t_i &= \text{Var}(f(W_{t_{i+1}}) - f(W_{t_i})) = \text{Var}(f(W_{t_{i+1}} - W_{t_i})) \\ &= \mathbb{E}[(f(W_{t_{i+1}} - W_{t_i}))^2] = \mathbb{E}[f^2(W_{t_{i+1}} - W_{t_i})] \\ &= \mathbb{E}[f^2(W_{t_{i+1}}) + f^2(W_{t_i}) - 2f(W_{t_{i+1}}) \cdot f(W_{t_i})] \\ &= \mathbb{E}[f^2(W_{t_{i+1}}) - f^2(W_{t_i})] \end{aligned}$$

Now, for every evaluation function f the process $f(W_t)_t$ is a scalar Brownian motion and so all the three conditions are equivalent thanks to [24, Theorem 3.3.2] since

$$V = \begin{pmatrix} \mathbb{E}[f(W(t_1))^2] & \mathbb{E}[f(W(t_1))f(W(t_2))] & \dots & \mathbb{E}[f(W(t_1))f(W(t_m))] \\ \mathbb{E}[f(W(t_2))f(W(t_1))] & \mathbb{E}[f(W(t_2))^2] & \dots & \mathbb{E}[f(W(t_2))f(W(t_m))] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[f(W(t_m))f(W(t_1))] & \mathbb{E}[f(W(t_m))f(W(t_2))] & \dots & \mathbb{E}[f(W(t_m))^2] \end{pmatrix}$$

and

$$\begin{aligned} \varphi(u_1, \dots, u_m) &= \mathbb{E}\left[\exp\left\{u_m f(W(t_m)) + u_{m-1} f(W(t_{m-1})) + \dots + u_1 f(W(t_1))\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{u_m f(W(t_m) - W(t_{m-1})) + (u_{m-1} + u_m) f(W(t_{m-1}) - W(t_{m-2}))\right.\right. \\ &\quad \left.\left.+ \dots + (u_1 + u_2 + \dots + u_m) f(W(t_1))\right\}\right] \\ &= \mathbb{E}\left[\exp\left\{u_m f(W(t_m) - W(t_{m-1}))\right\}\right] \dots \mathbb{E}\left[\exp\left\{(u_1 + u_2 + \dots + u_m) f(W(t_1))\right\}\right] \\ &= \exp\left\{\frac{1}{2}u_m^2(t_m - t_{m-1})\right\} \dots \exp\left\{(u_1 + u_2 + \dots + u_m)^2 t_1\right\}. \end{aligned}$$

So, by Definition 3.1, $(W_t)_t$ is a Brownian motion. □

Definition 3.7 For every Bochner integrable set-valued function W , the conditional expectation $E(W|\mathcal{F})$ of W with respect to a sub σ -algebra $\mathcal{F} \subset \mathcal{A}$ is a Bochner integrable function with respect to $(\Omega, \mathcal{F}, \lambda)$ such that for every evaluation function $f \in C(K)^*$ it is

$$\mathbb{E}(f(W)|\mathcal{F}) = f(\mathbb{E}(W|\mathcal{F})).$$

Definition 3.8 A set valued process $(M_t)_t$ is a pointwise martingale if

- (3.8.1) M_t is Bochner integrable for every t ;
- (3.8.2) $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$, for every $s < t$ where $(\mathcal{F}_s)_s$ is the natural filtration of $(M_t)_t$.

Then every Brownian motion $(M_t)_t$ is a pointwise martingale.

Theorem 3.9 Assume that $(B_t)_t$ is a set-valued Brownian motion, taking values in L . Then whenever $0 < s < t$ are fixed in \mathbb{R} , one has

$$\mathbb{E}(B_t^2|\mathcal{F}_s) = B_s^2 + (t - s)e.$$

Proof Let f be any evaluation functional. Then we have

$$\mathbb{E}(f(B_t^2)|\mathcal{F}_s) = \mathbb{E}((f(B_t))^2|\mathcal{F}_s) = f(B_s^2) + t - s = f(B_s^2 + (t - s)e)$$

by the usual properties of scalar Brownian motion and multiplicativity property of f . So, by arbitrariness of f , this leads to the assertion. \square

Clearly, this result means that, under the stated hypotheses, the sequence $(B_t^2 - te)_t$ is a pointwise martingale.

The last theorem can be reversed, in some sense: more precisely,

Theorem 3.10 *Let $(B_t)_t$ be a weak set-valued Gaussian process with homogeneous increments, such that $B_0 = 0$. If $(B_t^2 - te)_t$ is a pointwise martingale, then $(B_t)_t$ is a Wiener process (therefore, assuming also that the trajectories of $(B_t)_t$ are weakly continuous, one can conclude that $(B_t)_t$ is a set-valued Brownian motion).*

Proof Indeed, from the martingale condition, one can deduce that $\mathbb{E}(B_t^2 - te)$ is constant with respect to t , and therefore null, since $B_0 = 0$. So, $\mathbb{E}(B_t^2) = te$ for all t . Now, if $0 < s < t$, thanks to the homogeneity property:

$$\begin{aligned} \mathbb{E}(f(2B_t B_s)) &= \mathbb{E}\left(f\left[B_t^2 + B_s^2 - (B_t - B_s)^2\right]\right) \\ &= \mathbb{E}\left(\left[f(B_t^2) + f(B_s^2) - f(B_{t-s}^2)\right]\right) = t + s - t + s = 2s \end{aligned}$$

and this is precisely the defining property for a (weak) Wiener process. \square

Let $(M_t)_{t \geq 0}$ be an L -valued adapted process, then $\sum_{i=0}^{n-1} [M_{t_{j+1}} - M_{t_j}]^2 \in C(K)$ for every partition $\pi = \{0 = t_0 < t_1 < \dots < t_n = T\}$ of $[0, T]$, $T > 0$.

Definition 3.11 The quadratic variation $[M_t, M_t]$ of an L -valued adapted process $(M_t)_t$, when it exists, is given by the following limit

$$\lim_{\|\pi\| \rightarrow 0} \|f([M_t, M_t](T)) - f\left(\sum_{i=0}^{n-1} [M_{t_{j+1}} - M_{t_j}]^2\right)\|_2 = 0$$

for every evaluation function $f \in C(K)^*$ and every $T > 0$.

Theorem 3.12 (Theorem of Levy) *Let M_t be a martingale relative to a filtration \mathcal{F}_t with $M_0 = 0$. Assume that M_t has weakly continuous paths and $[M_t, M_t](T) = T$ for all $T \geq 0$. Then $(M_t)_t$ is a set-valued Brownian motion.*

Proof From the assumptions on M_t , we get that, for each evaluation function $f \in C(K)^*$, $f(M_t)$ is a martingale with $f(M_0) = 0$, $f(M_t)$ has continuous paths and $[f(M_t), f(M_t)](t) = t$ for all $t \geq 0$. Thus, $f(M_t)$ is a Brownian motion and so M_t is a set valued Brownian motion. \square

Definition 3.13 A set valued process $(W_t)_t$ is integrable with respect to a Brownian motion $(B_t)_t$ if for every $T > 0$ there exists an element $I_T \in C(K)$ such that:

- (3.13.1) $(I_T)_T$ is a martingale with respect to $(B_t)_t$;
- (3.13.2) for every evaluation function $f \in C(K)^*$ it is

$$f(I_T) = (I) \int_0^T f(W_t)d(f B_t)$$

where the last integral is in the Ito sense.

For instance, the process $(B_t)_t$ is integrable, with $I_T = \frac{B_T^2 - T}{2}$; more generally, if $(B_t)_t$ takes values in an f -algebra L , then the process $(B_t^k)_t$ is integrable for every positive integer k , and the usual Ito formula holds.

4 Brownian motion in vector lattices

In this section we generalize the notions of Brownian Motion introduced before, replacing the space $C(K)$ with a particular Riesz space E having an order unit e .

Definition 4.1 ([17, Definition 3.6]) Let (B_t, \mathfrak{F}_t) be an adapted stochastic process in the Dedekind complete Riesz space E with conditional expectation \mathbb{F} and unit element e . The process is called an \mathbb{F} -conditional Brownian motion in E if for all $0 \leq s < t$ we have

- (4.1.1) $B_0 = 0$;
- (4.1.2) the increment $B_t - B_s$ is \mathbb{F} -conditionally independent of \mathfrak{F}_s ;
- (4.1.3) $\mathbb{F}(B_t - B_s) = 0$;
- (4.1.4) $\mathbb{F}[(B_t - B_s)^2] = (t - s)e$;
- (4.1.5) $\mathbb{F}((B_t - B_s)^4) = 3(t - s)^2e$.

Remark 4.2 It was noted in [13, page 901] that the definition of a Brownian motion in the Riesz space setting yields a Brownian motion in the classical case of real valued Brownian motion; i.e., a real valued stochastic process satisfies conditions (4.1.1)–(4.1.5) if and only if it is a Brownian motion.

Theorem 4.3 Let B_t be a set valued stochastic process. Then B_t is a Brownian motion if and only if for any evaluation function $f \in C(K)^*$ and every pair (s, t) of positive real numbers with $s < t$:

- (4.3.1) $f(B_0) = 0$;
- (4.3.2) the increment $f(B_t) - f(B_s)$ is $(\mathbb{F}f)$ -conditionally independent of $f(\mathfrak{F}_s)$;
- (4.3.3) $\mathbb{F}(f(B_t) - f(B_s)) = 0$;
- (4.3.4) $\mathbb{F}[(f(B_t) - f(B_s))^2] = (t - s)f(e)$;
- (4.3.5) $\mathbb{F}((f(B_t) - f(B_s))^4) = 3(t - s)^2f(e)$.

Proof We note that B_t is a Brownian motion if and only if $f(B_t)$ is a Brownian motion for each evaluation function $f \in C(K)^*$ which is equivalent to the conditions (4.3.1)–(4.3.5) by the Remark 4.2. □

As a consequence of Theorem 4.3, when $E = C(K)$, with K Stonian and e is the unit function in $C(K)$, we have:

Corollary 1 *Let $(B_t)_t$ be a \mathbb{F} -conditional Brownian motion then, in the L^2 -norm,*

$$[B_t, B_t](T) = Te.$$

Proof Let $\Pi := \{t_0, \dots, t_m\}$ be fixed with $0 = t_0 < t_1 < \dots < t_m$ and let $\delta(\Pi) = \sup_{j < m} \{t_{j+1} - t_j\}$. Since

$$\mathbb{E} \left[(B_{t_{j+1}} - B_{t_j})^2 \right] = \text{Var} \left[(B_{t_{j+1}} - B_{t_j}) \right] = (t_{j+1} - t_j)e,$$

then

$$\mathbb{E} \left[\sum_{j=0}^{m-1} (B_{t_{j+1}} - B_{t_j})^2 \right] = \sum_{j=0}^{m-1} \mathbb{E} \left[(B_{t_{j+1}} - B_{t_j})^2 \right] = Te.$$

Moreover

$$\text{Var} \left[(B_{t_{j+1}} - B_{t_j})^2 \right] = \mathbb{E} \left[(B_{t_{j+1}} - B_{t_j})^4 \right] - (t_{j+1} - t_j)^2 e = 2(t_{j+1} - t_j)^2 e$$

and

$$\begin{aligned} \text{Var} \left[\sum_{j=0}^{m-1} (B_{t_{j+1}} - B_{t_j})^2 \right] &= \sum_{j=0}^{m-1} \text{Var} \left[(B_{t_{j+1}} - B_{t_j})^2 \right] = 2 \sum_{j=0}^{m-1} (t_{j+1} - t_j)^2 e \\ &\leq 2Te\delta(\Pi) \end{aligned}$$

and this implies that

$$\lim_{\delta(\Pi) \rightarrow 0} \text{Var} \left[\sum_{j=0}^{m-1} (B_{t_{j+1}} - B_{t_j})^2 \right] = 0$$

and so

$$\lim_{\delta(\Pi) \rightarrow 0} \left\| \sum_{j=0}^{m-1} (B_{t_{j+1}} - B_{t_j})^2 - Te \right\|_2 = 0.$$

□

5 Appendix

At the beginning of the paper we have claimed that, in order to introduce a notion of Brownian motion in this context, a kind of difference between sets is necessary. Here, following [25], for every $A \in ck(X)$ let $-A$ be the opposite of the set A , namely $-A = \{-a : a \in A\}$ and consider the following difference between sets:

Definition 5.1 ([25, Definition 1]) For every $A, B \in ck(X)$ the generalized Hukuhara difference of A and B (gH-difference for short), when exists, is the set $C \in ck(X)$ such that

$$A \ominus_g B := C \iff \begin{cases} (i) & A = B + C, \text{ or} \\ (ii) & B = A + (-C). \end{cases} \tag{2}$$

Remark 5.2 By [25, Propositions 1,6 and Remarks 2-5] if the set C exists it is unique and coincides with the Hukuhara difference between A and B . Moreover a necessary condition for the existence is that either A contains a translate of B or B contains a translate of A . If equations (2.i) and (2.ii) hold simultaneously then C is a singleton. Finally

(5.2.1) $A \ominus_g B \in ck(X)$ then $B \ominus_g A = -(A \ominus_g B)$;

(5.2.2) $A \ominus_g A = \{0\}$,

(5.2.3) $(A + B) \ominus_g B = A, A \ominus_g (A - B) = B, A \ominus_g (A + B) = -B$.

If A is compact and convex subset of X then it is characterized by its support function s_A by Hahn-Banach theorem (see for example [10, Proposition II.16]). It is possible to express the gH-difference of convex compact sets using support functions.

Given $A, B, C \in ck(X)$ let $s(\cdot, A), s(\cdot, B), s(\cdot, C), s(\cdot, -C)$ be the support functions of $A, B, C, -C$ respectively.

Again by [10, Propositions II-19] the map $A \mapsto s(\cdot, A)$ is injective, $s(x^*, A + B) = s(x^*, A) + s(x^*, B), s(x^*, \lambda A) = \lambda s(x^*, A)$ for every non negative λ , while

$$\begin{aligned} s(x^*, -A) &= \sup\{ \langle x^*, -x \rangle, x \in A \} = \sup\{ \langle -x^*, x \rangle, x \in A \} \\ &= s(-x^*, A) \geq -s(x^*, A) \end{aligned}$$

And the equality in the last line holds when the opposite of A is a set $C \in ck(X)$ such that $A + C = \{0\}$, namely $s(x^*, \ominus_g A) = -s(x^*, A)$. So in general $s(-x^*, A) \geq -s(x^*, A)$ and the equality holds when equation (2.i) holds.

We recall some well-known facts concerning Banach spaces.

Theorem 5.3 [12, Theorem 2.3] *Let X be any Banach space and $H : B_{X^*} \rightarrow \mathbb{R}$ be any mapping. Then H is the support function of a convex compact subset of X if and only if H is bw^* -continuous, subadditive and positively homogeneous.*

A consequence of this result can be stated in the following way.

Proposition 5.4 *Let $(B_t)_t := (W_t e)_t$ be the example of Brownian motion in a finite-dimensional space X , given above in the Example 3.2. Then, for each positive t , the function $j(B_t^2 - \int_0^t 2B_\tau dB_\tau)$ is (bw^*) -continuous, subadditive and positively homogeneous.*

Proof Clearly, since $\int_0^t 2B_\tau dB_\tau = B_t^2 - te$, it follows that the difference $B_t^2 - \int_0^t 2B_\tau dB_\tau$ is a positive multiple of e , i.e. a convex compact set. The conclusion then follows from Theorem 5.3. □

The generalized Hukuhara difference can be expressed by means of the support functions as in [25, Proposition 8] in the following way:

Proposition 5.5 *Let $s(\cdot, A), s(\cdot, B)$ be the support functions of $A, B \in ck(X)$ and denote by $s_1 := s(\cdot, A) - s(\cdot, B), s_2 = s(\cdot, B) - s(\cdot, A)$. Then only four cases may occur:*

- (5.5.a) *if s_1, s_2 are bw^* -continuous and subadditive then $A \ominus_g B \in ck(X)$ and $A \ominus_g B$ is a singleton;*
- (5.5.b) *if only s_1 is bw^* -continuous and subadditive then equation (2i) holds and $s(\cdot, C) := s_1$;*
- (5.5.c) *if only s_2 is bw^* -continuous and subadditive then equation (2ii) holds and $s(\cdot, C) := s(\cdot, B) - s(\cdot, -A)$;*
- (5.5.d) *if none of them is bw^* -continuous or subadditive then $A \ominus_g B$ does not exist.*

Proof The proof of the first statement is the same as in [25, Proposition 8] since it depends only on the subadditivity of $s_i, i = 1, 2$ and the fact that in this case A, B are each one a translate of the other, the bw^* -continuity of s_1, s_2 implies again by Theorem 5.3, that $A \ominus_g B \in ck(X)$.

As to the second statement observe that by [12, Theorem 2.3] there exists $C \in ck(X)$ such that $s_1 = s(\cdot, C)$, so (2.i) is valid.

In the third case the same can be done for s_2 , so there exists $D \in ck(X)$ such that $s_2 = s(\cdot, D)$. Set now $C = -D$, the rest of the proof follows as in the quoted [25, Proposition 8].

Finally since [12, Theorem 2.3] is a necessary and sufficient condition that there exist no $C \in ck(X)$ such that $A = B + C$ or $D \in ck(X)$ such that $B = A + D$, so $A \ominus_g B$ does not exist. \square

References

- Boulabiar, K., Buskes, G., Triki, A.: Results in f-algebras. In: Boulabiar, K., Buskes, G., Triki, A. (eds.) Positivity. Trends in Mathematics, pp. 73–96. Birkhäuser, Basel (2007)
- Boccuto, A., Candeloro, D., Kubinska, E.: Kondurar theorem and Ito formula in Riesz spaces. J. Concr. Appl. Math. **4**, 67–90 (2006)
- Boccuto, A., Candeloro, D., Sambucini, A.R.: A Fubini theorem in Riesz spaces for the Kurzweil-Henstock integral. J. Funct. Spaces Appl. **9**(3), 283–304 (2011). <https://doi.org/10.1155/2011/158412>
- Boccuto, A., Candeloro, D., Sambucini, A.R.: Vitali-type theorems for filter convergence related to vector lattice-valued modulars and applications to stochastic processes. J. Math. Anal. Appl. **419**(2), 818–838 (2014). <https://doi.org/10.1016/j.jmaa.2014.05.014>
- Boccuto, A., Candeloro, D., Sambucini, A.R.: Henstock multivalued integrability in Banach lattices with respect to pointwise non atomic measures, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **26**(4), 363–383 (2015). <https://doi.org/10.4171/RLM/710>. [arXiv:1503.08285v1](https://arxiv.org/abs/1503.08285v1)
- Boccuto, A., Candeloro, D., Sambucini, A.R.: L^p spaces in vector lattices and applications. Math. Slov. **67**(6), 1409–1426 (2017). <https://doi.org/10.1515/ms-2017-0060>
- Boccuto, A., Minotti, A.M., Sambucini, A.R.: Set-valued Kurzweil-Henstock integral in Riesz space setting. PanAm. Math. J. **23**(1), 57–74 (2013)
- Candeloro, D., Sambucini, A.R.: Order-type Henstock and Mc Shane integrals in Banach lattice setting. In: Proceedings of the SISY 2014—IEEE 12th International Symposium on Intelligent Systems and Informatics, pp. 55–59 (2014). ISBN 978-1-4799-5995-2. <https://doi.org/10.1109/SISY.2014.6923557>
- Candeloro, D., Sambucini, A.R.: Comparison between some norm and order gauge integrals in Banach lattices. PanAmerican Math. J. **25**(3), 1–16 (2015). [arXiv:1503.04968](https://arxiv.org/abs/1503.04968) [math.FA]

10. Castaing, C., Valadier, M.: *Convex Analysis and Measurable Multifunctions*. Lecture Notes Math, vol. 580. Springer, Berlin (1977)
11. Costarelli, D., Vinti, G.: Order of approximation for sampling Kantorovich operators. *J. Int. Equ. Appl.* **26**(3), 345–368 (2014)
12. Giné, E., Hahn, M.G.: Characterization and domains of attraction of p -stable random compact sets. *Ann. Probab.* **13**(2), 447–468 (1985)
13. Grobler, J.J.: The Kolmogorov-Čentsov theorem and Brownian motion in vector lattices. *J. Math. Anal. Appl.* **410**(2), 891–901 (2014)
14. Grobler, J.J.: Corrigendum to “The Kolmogorov-Čentsov theorem and Brownian motion in vector lattices” (*J. Math. Anal. Appl.* 410 (2014) 891–901). *J. Math. Anal. Appl.* **420**(1), 878 (2014)
15. Grobler, J.J.: Jensen’s and martingale inequalities in Riesz spaces. *Indag. Math.* **25**(2), 275–295 (2014)
16. Grobler, J.J., Labuschagne, C.C.A.: The Ito integral for Brownian motion in vector lattices: part 1. *J. Math. Anal. Appl.* **423**(1), 797–819 (2015)
17. Grobler, J.J., Labuschagne, C.C.A.: The Ito integral for Brownian motion in vector lattices: part 2. *J. Math. Anal. Appl.* **423**(1), 820–833 (2015)
18. Grobler, J.J., Labuschagne, C.C.A., Marraffa, V.: Quadratic variation of martingales in Riesz spaces. *J. Math. Anal. Appl.* **410**(1), 418–426 (2014)
19. Labuschagne, C.C.A., Marraffa, V.: Operator martingale decomposition and the Radon–Nikodym property in Banach spaces. *J. Math. Anal. Appl.* **363**(2), 357–365 (2010)
20. Labuschagne, C.C.A., Marraffa, V.: On set-valued cone absolutely summing map. *Cent. Eur. J. Math.* **8**(1), 148–157 (2010)
21. Labuschagne, C.C.A., Pinchuck, A.L., van Alten, C.J.: A vector lattice version of Rådström’s embedding theorem. *Quaest. Math.* **30**(3), 285–308 (2007)
22. Labuschagne, C.C.A., Watson, B.A.: Discrete stochastic integration in Riesz spaces. *Positivity* **14**(4), 859–875 (2010)
23. Sambucini, A.R.: The Choquet integral with respect to fuzzy measures and applications. *Math. Slov.* **67**(6), 1427–1450 (2017). <https://doi.org/10.1515/ms-2017-0049>
24. Shreve, S.: *Stochastic Calculus for Finance II: Continuous Time Models*. Springer, Berlin (2004)
25. Stefanini, L.: A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. *Fuzzy Set Syst.* **161**, 1564–1584 (2010)
26. Li, Shoumei, Ogura, Yukio: A convergence theorem of fuzzy-valued martingales in the extended Hausdorff metric H_∞ . *Fuzzy Sets Syst.* **135**, 391–399 (2003)
27. Li, Shoumei, Guan, Li: Fuzzy set-valued Gaussian processes and Brownian motions. *Inf. Sci.* **177**, 3251–3259 (2007)
28. Stoica, G.: On some stochastic-type operators. *Analele Universității București, Matematică* **39**, 58–62 (1990)
29. Stoica, G.: Vector valued quasi-martingales. *Stud. Cerc. Mat.* **42**, 73–79 (1990)
30. Stoica, G.: The structure of stochastic processes in normed vector lattices. *Stud. Cerc. Mat.* **46**, 477–486 (1994)
31. Vardy, J., Watson, B.A.: Markov processes on Riesz spaces. *Positivity* **16**(2), 373–391 (2012)
32. Vardy, J., Watson, B.A.: Erratum to: Markov processes on Riesz spaces. *Positivity* **16**(2), 393 (2012)
33. Vardy, J., Watson, B.A.: On the decompositions of T-quasi-martingales on Riesz spaces. *Positivity* **18**(3), 425–437 (2014)