

# Some combinatorial properties of the Hurwitz series ring

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**Abstract** We study some properties and perspectives of the Hurwitz series ring  $H_R[[t]]$ , for an integral domain  $R$ , with multiplicative identity and zero characteristic. Specifically, we provide a closed form for the invertible elements by means of the complete ordinary Bell polynomials, we highlight some connections with well-known transforms of sequences, and we see that the Stirling transforms are automorphisms of  $H_R[[t]]$ . Moreover, we focus the attention on some special subgroups studying their properties. Finally, we introduce a new transform of sequences that allows to see one of this subgroup as an ultrametric dynamic space.

**Keywords** Hurwitz series ring · Binomial convolution · Bell polynomials · Transforms of sequences

**Mathematics Subject Classification** 11B75 · 11B73

## 1 The Hurwitz series ring, transformations of sequences and automorphisms

Given an integral domain  $R$  with multiplicative identity and zero characteristic, let  $H_R[[t]]$  denote the Hurwitz series ring whose elements are the formal series

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$$A(t) := \sum_{n=0}^{+\infty} \frac{a_n}{n!} t^n,$$

equipped with the standard sum and the binomial convolution product. Given two formal series  $A(t)$  and  $B(t)$ , the binomial convolution product is defined as follows:

$$A(t) \star B(t) := C(t),$$

where

$$c_n := \sum_{h=0}^n \binom{n}{h} a_h b_{n-h}.$$

The Hurwitz series ring has been organizationally studied by Keigher [9] and in the recent years it has been extensively studied, see, e.g., [3–5, 7, 10, 11, 13].

The Hurwitz series ring is trivially isomorphic to the ring  $H_R$  whose elements are infinite sequences of elements of  $R$ , with operations  $+$  and  $\star$ . In the following, when we consider an element  $a \in H_R$ , we refer to a sequence  $(a_n)_{n=0}^{+\infty} = (a_0, a_1, a_2, \dots)$ ,  $a_i \in R$  for all  $i \geq 0$ , having exponential generating function (e.g.f.)  $A(t)$ . Clearly, given two sequences  $a, b \in H_R$ , with exponential generating functions  $A(t)$  and  $B(t)$  respectively, the sequence  $c = a \star b$  has e.g.f.  $C(t) = A(t)B(t)$ . Moreover, fixed any positive integer  $n$ , we can also consider the rings  $H_R^{(n)}$  whose elements are sequences of elements of  $R$  with length  $n$ .

*Remark 1* The binomial convolution is a commutative product and the identity in  $H_R$  is the sequence

$$(1, 0, 0, \dots).$$

Moreover,  $H_R$  can be also considered as an  $R$ -algebra with respect to the map

$$\pi : R \rightarrow H_R, \quad \pi(r) := (r, 0, 0, \dots),$$

for any  $r \in R$ .

**Proposition 1** *An element  $a \in H_R$  is invertible if and only if  $a_0 \in R$  is invertible, i.e.,*

$$H_R^* = \{a \in H_R : a_0 \in R^*\}.$$

*Proof* The proof is straightforward. □

Given  $a \in H_R^*$ , we can recursively evaluate the terms of  $b = a^{-1}$ . Indeed,  $b_0 = a_0^{-1}$  and for all  $n \geq 1$  we have

$$b_n = -a_0^{-1} \sum_{h=1}^n \binom{n}{h} a_h b_{n-h},$$

since the equality  $a \star b = (1, 0, 0, \dots)$  implies

$$a_0 b_0 = 1, \quad \sum_{h=0}^n \binom{n}{h} a_h b_{n-h} = 0, \quad \forall n \geq 1.$$

On the other hand, we can find a closed form for the elements of  $b$  by means of the complete ordinary Bell polynomials [2]. First of all, we recall their definition as given in [14].

**Definition 1** Let us consider the sequence  $x = (x_1, x_2, \dots)$ , whose terms  $x_i$  are some indeterminates over the domain  $R$ . The *complete ordinary Bell polynomials* are defined by

$$B_0(x) = 1, \quad \forall n \geq 1 \quad B_n(x) = B_n(x_1, x_2, \dots, x_n) = \sum_{k=1}^n B_{n,k}(x),$$

where  $B_{n,k}(x)$  are the *partial ordinary Bell polynomials*, with

$$B_{0,0}(x) = 1, \quad \forall n \geq 1 \quad B_{n,0}(x) = 0, \quad \forall k \geq 1 \quad B_{0,k}(x) = 0,$$

and

$$\begin{aligned} \forall n \geq k \geq 1 \quad B_{n,k}(x) &= B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \\ &= \sum_{\substack{i_1+2i_2+\dots+(n-k+1)i_{n-k+1}=n \\ i_1+i_2+\dots+i_{n-k+1}=k}} \frac{k!}{i_1!i_2! \dots i_{n-k+1}!} x_1^{i_1} x_2^{i_2} \dots x_{n-k+1}^{i_{n-k+1}}, \end{aligned}$$

satisfying the equality

$$\left( \sum_{n \geq 1} x_n z^n \right)^k = \sum_{n \geq k} B_{n,k}(x) z^n.$$

Then, we introduce the Invert transform (see, e.g., [6] for a detailed survey).

*The Invert transform*

The Invert transform  $\mathcal{I}$  maps a sequence  $a = (a_n)_{n=0}^{+\infty}$  into a sequence  $\mathcal{I}(a) = b = (b_n)_{n=0}^{+\infty}$  whose ordinary generating function satisfies

$$\sum_{n=0}^{+\infty} b_n t^n = \frac{\sum_{n=0}^{+\infty} a_n t^n}{1 - t \sum_{n=0}^{+\infty} a_n t^n}.$$

Barbero et al. [1] highlighted the closed relation between the Invert transform and the complete ordinary Bell polynomials: given  $g \in H_R$  and  $h = \mathcal{I}(g)$ , we have, for all  $n \geq 0$ , that

$$h_n = B_{n+1}(g_0, g_1, g_2, \dots, g_n). \tag{1}$$

Now these tools allow us to explicitly find the terms of  $b = a^{-1}$  for every  $a \in H_R^*$ .

**Theorem 1** *Let  $a, b = a^{-1} \in H_R^*$  be sequences with e.g.f.  $A(t)$  and  $B(t)$ , respectively. Then we have  $b_0 = \frac{1}{a_0}$  and*

$$b_n = \frac{n! B_n(g_0, g_1, g_2, \dots, g_{n-1})}{a_0},$$

for all  $n \geq 1$ , where

$$g = (g_n)_{n=0}^{+\infty} = \left( -\frac{a_{n+1}}{a_0(n+1)!} \right)_{n=0}^{+\infty}.$$

*Proof* The ordinary generating function of the sequence  $g$  is

$$\bar{G}(t) = \frac{1}{t} \left( 1 - \frac{A(t)}{a_0} \right)$$

since

$$\bar{G}(t) = \frac{1}{t} \left( 1 - \frac{1}{a_0} \left( a_0 + \sum_{n=1}^{+\infty} \frac{a_n}{n!} t^n \right) \right) = -\frac{1}{t} \sum_{n=1}^{+\infty} \frac{a_n}{a_0 n!} t^n = \sum_{n=0}^{+\infty} \left( -\frac{a_{n+1}}{a_0(n+1)!} \right) t^n.$$

Moreover, considering  $b = a^{-1}$ , we have

$$B(t) = \frac{1}{A(t)} = \frac{1}{a_0(1 - t\bar{G}(t))} = \frac{1}{a_0} \left( 1 + \frac{t\bar{G}(t)}{1 - t\bar{G}(t)} \right) = \frac{1}{a_0} (1 + t\bar{H}(t)),$$

where  $\bar{H}(t)$  is the ordinary generating function of the sequence  $h = \mathcal{I}(g)$ . Thus relation (1) holds and, since  $B_0(g) = 1$ , we obtain

$$\begin{aligned} B(t) &= \frac{1}{a_0} (1 + t\bar{H}(t)) = \frac{1}{a_0} \left( 1 + \sum_{n=0}^{+\infty} B_{n+1}(g_0, g_1, g_2, \dots, g_n) t^{n+1} \right) \\ &= \frac{1}{a_0} + \sum_{n=1}^{+\infty} \frac{B_n(g_0, g_1, g_2, \dots, g_{n-1})}{a_0} t^n \end{aligned}$$

and the thesis easily follows. □

Let us note that the recursive formula

$$b_n = -a_0^{-1} \sum_{h=1}^n \binom{n}{h} a_h b_{n-h},$$

giving the inverse  $b = a^{-1}$  of  $a$ , is much easier than the one established in Theorem 1. But the second one has a theoretical importance. It uses the Bell polynomials and gives the  $b_n$  in terms of  $a_n$  only.

We point out that some well-studied transforms acting on sequences can be considered in  $H_R[[t]]$  as the product (i.e., the binomial convolution) between a suitable fixed sequence and any sequence belonging to  $H_R$ . We present two enlightening and interesting examples.

*The Binomial interpolated transform*

The *Binomial interpolated transform*  $\mathcal{L}^{(y)}$ , with parameter  $y \in R$ , maps any sequence  $a \in H_R$  into a sequence  $b = \mathcal{L}^{(y)}(a) \in H_R$ , whose terms are

$$b_n = \sum_{h=0}^n \binom{n}{h} y^{n-h} a_h.$$

For a survey and a detailed study of the action of  $\mathcal{L}^{(y)}$  on recurrence sequences we refer the reader to [1]. The definition of this transform by means of the binomial convolution is straightforward. Indeed, considering the sequence

$$\lambda = (y^n)_{n=0}^{+\infty},$$

we have for any  $a \in H_R$  with e.g.f.  $A(t)$

$$\mathcal{L}^{(y)}(a) = \lambda \star a.$$

with the corresponding e.g.f. given by the product  $e^{yt} A(t)$ .

*The Boustrophedon transform*

If we consider  $R = \mathbb{Z}$ , the *Boustrophedon transform*  $\mathcal{B}$ , introduced and studied in [12], maps any sequence  $a \in H_{\mathbb{Z}}$ , with e.g.f.  $A(t)$ , into a sequence  $b = \mathcal{B}(a) \in H_{\mathbb{Z}}$  with e.g.f.

$$B(t) = (\sec t + \tan t)A(t).$$

This transform is closely related to the sequence  $\beta = (\beta_n)_{n=0}^{+\infty}$  of the Euler zigzag numbers (see [12]), with e.g.f.

$$\sum_{n=0}^{+\infty} \frac{\beta_n}{n!} t^n := \sec(t) + \tan(t),$$

since for any  $a \in H_{\mathbb{Z}}$  clearly

$$\mathcal{B}(a) = \beta \star a.$$

The Hurwitz series ring is strictly connected to other well-known transforms. We consider further two examples: the alternating sign transform, which is a little bit trivial, and the Stirling transform. We also show that are examples of  $H_R$ -automorphisms.

*The alternating sign transform*

The alternating sign transform  $\mathcal{E}$  maps any sequence  $a \in H_R$  into a sequence  $b = \mathcal{E}(a) \in H_R$ , whose terms are

$$b_n = (-1)^n a_n.$$

The transform  $\mathcal{E}$  often appears in studying properties of integer sequences combined with other transforms. Clearly, we have  $\mathcal{E} = \mathcal{E}^{-1}$  and it is straightforward to see that, given any  $a \in H_R$  with e.g.f.  $A(t)$ , then  $\mathcal{E}(a)$  has e.g.f.  $A(-t)$ . Moreover it is easy to verify that for all sequences  $a, b \in H_R$

$$\mathcal{E}(a + b) = \mathcal{E}(a) + \mathcal{E}(b),$$

and

$$\mathcal{E}(a \star b) = \mathcal{E}(a) \star \mathcal{E}(b),$$

showing that  $\mathcal{E}$  is an automorphism of  $H_R$ .

*The Stirling transform*

The Stirling transform  $\mathcal{S}$  maps any sequence  $a \in H_R$  into a sequence  $b = \mathcal{S}(a) \in H_R$ , whose terms are

$$b_n = \sum_{h=0}^n \left\{ \begin{matrix} n \\ h \end{matrix} \right\} a_h,$$

where  $\left\{ \begin{matrix} n \\ h \end{matrix} \right\}$  are the Stirling numbers of the second kind (see e.g. [8], chapter 6, for definition and properties of Stirling numbers of first and second kinds). Some properties of this transform are exposed in [6], here we observe that  $\mathcal{S}$  is a bijection from  $H_R$  to itself. The inverse  $\mathcal{S}^{-1}$  maps any sequence  $a \in H_R$  into a sequence  $b = \mathcal{S}^{-1}(a) \in H_R$ , whose terms are

$$b_n = \sum_{h=0}^n (-1)^{n-h} \left[ \begin{matrix} n \\ h \end{matrix} \right] a_h,$$

where  $\left[ \begin{matrix} n \\ h \end{matrix} \right]$  are the (unsigned) Stirling numbers of the first kind. Moreover, we recall that for all  $a \in H_R$  with e.g.f.  $A(t)$ , then  $b = \mathcal{S}(a)$  has e.g.f.  $B(t) = A(e^t - 1)$ . It is very interesting to observe that, for all  $a, b \in H_R$ ,  $\mathcal{S}$  obviously satisfies

$$\mathcal{S}(a + b) = \mathcal{S}(a) + \mathcal{S}(b),$$

but also

$$\mathcal{S}(a \star b) = \mathcal{S}(a) \star \mathcal{S}(b).$$

Indeed, remembering that  $\left\{ \begin{smallmatrix} n \\ h \end{smallmatrix} \right\} = 0$  when  $n < h$ , and that the e.g.f. of the Stirling numbers of the second kind is  $\frac{(e^t - 1)^n}{n!}$  (see [8]), if we consider the e.g.f.  $\mathcal{S}(t)$  of  $\mathcal{S}(a \star b)$ , we have

$$\begin{aligned} \mathcal{S}(t) &= \sum_{n=0}^{+\infty} \left( \sum_{h=0}^n \left\{ \begin{smallmatrix} n \\ h \end{smallmatrix} \right\} \left( \sum_{j=0}^h \binom{h}{j} a_j b_{h-j} \right) \right) \frac{t^n}{n!} \\ &= \sum_{h=0}^{+\infty} \left( \sum_{j=0}^h \binom{h}{j} a_j b_{h-j} \right) \sum_{n=0}^{+\infty} \left\{ \begin{smallmatrix} n \\ h \end{smallmatrix} \right\} \frac{t^n}{n!} \\ &= \sum_{h=0}^{+\infty} \left( \sum_{j=0}^h \binom{h}{j} a_j b_{h-j} \right) \frac{(e^t - 1)^h}{h!} = A(e^t - 1) B(e^t - 1) \end{aligned}$$

and  $A(e^t - 1) B(e^t - 1)$  is the e.g.f. of  $\mathcal{S}(a) \star \mathcal{S}(b)$ . Hence  $\mathcal{S}$  is an automorphism of  $H_R$ .

## 2 Special subgroups of $H_R^*$

The purpose of this section is to highlight some properties of two interesting subgroups of  $H_R^*$  with respect to the binomial convolution product  $\star$  operation. We also study their relationship with the transforms presented in the previous section and with other transforms which we will define in the next.

**Definition 2** Let us denote  $U_R$  and  $B_R$  the subgroups of  $H_R^*$  defined as

$$U_R = \{a \in H_R^* : a_0 = 1\}, \quad B_R = \{a \in U_R : \mathcal{E}(a) = a^{-1}\}.$$

We start considering the subgroup  $U_R$  and observing that, for all  $a \in H_R$ , we can find sequences in  $U_R$  closely related with  $a$ , obtained by prepending to  $a$  a finite sequence of 1. Hence, it is natural to consider these sequences as the images of  $a$  under the iteration of the following transform.

The *1-prepending* transform  $\mathcal{V}$  maps a sequence  $a = (a_0, a_1, a_2, \dots) \in H_R$  into the sequence  $b = \mathcal{V}(a) = (1, a_0, a_1, a_2, \dots) \in U_R$ . We denote by  $\mathcal{V}^k$  the  $k$ -times iteration of  $\mathcal{V}$  and obviously  $\mathcal{V}^k(a) \in U_R$ , for all  $k \geq 1$ . We observe that the action of  $\mathcal{V}^k$  on a sequence  $a \in H_R^*$  corresponds to the  $k$ -th iteration of the integral operator

$$\mathcal{J}(\cdot)(t) := 1 + \int_0^t (\cdot) du$$

on the e.g.f.  $A(t)$  of  $a$ . In particular,

$$\mathcal{J}^k(A)(t) = \underbrace{\mathcal{J} \circ \dots \circ \mathcal{J}}_{k\text{-times}}(A)(t) = \sum_{h=0}^{k-1} \frac{t^h}{h!} + \sum_{h=k}^{+\infty} a_{h-k} \frac{t^h}{h!} = V(t),$$

where  $V(t)$  is the e.g.f. of  $\mathcal{V}^k(a)$ , and clearly

$$V(0) = V'(0) = \dots = V^{(k-1)}(0) = 1, \quad V^{(k)}(t) = A(t),$$

being  $V^{(k)}(t)$  the  $k$ -th derivative of  $V(t)$ .

Now, we explore some interesting properties of the subgroup  $B_R$ . We recall that a formal power series  $f(t)$  is called *even* if  $f(-t) = f(t)$  and *odd* when  $f(-t) = -f(t)$ . First of all, we characterize all the elements in  $B_R$ .

**Theorem 2** *All the elements in  $B_R$  corresponds to the sequences of  $U_R$  whose e.g.f.  $A(t)$  is the solution of*

$$\begin{cases} A'(t) = g(t)A(t) \\ A(0) = 1 \end{cases} \tag{2}$$

where  $g(t)$  is any fixed even formal power series. Hence, if we consider the formal exponential operator  $\exp$  such that  $\exp(f(t)) = \sum_{n=0}^{+\infty} \frac{(f(t))^n}{n!}$ , the e.g.f. of  $a \in B_R$  is  $A(t) = \exp(h(t))$  where  $h(t)$  is an odd formal power series.

*Proof* It is immediate to see that  $a \in B_R$  if and only if  $A(t)A(-t) = 1$ . If we differentiate this relation with respect to  $t$  we obtain

$$A'(t)A(-t) - A(t)A'(-t) = 0$$

which is equivalent to

$$\frac{A'(t)}{A(t)} = \frac{A'(-t)}{A(-t)}.$$

Thus we have  $g(t) = \frac{A'(t)}{A(t)}$  where, from the previous relation,  $g(t) = g(-t)$ , i.e.  $g(t)$  is even, and we must have  $A(0) = 1$ , since, for all sequences  $a \in U_R$ ,  $A(0) = a_0 = 1$ . It is straightforward to verify that, given  $g(t)$ , a formal integration term by term of its power series corresponds to an odd power series  $h(t)$ , and consequently  $A(t) = \exp(h(t))$  satisfies (2). □

The transforms  $\mathcal{L}^y$ ,  $\mathcal{E}$ , and  $\mathcal{B}$  (when  $R = \mathbb{Z}$ ), act on  $B_R$  preserving the closure, as we point out in the following proposition.

**Proposition 2** *The group  $B_R$  is closed with respect to the transforms  $\mathcal{E}$  and  $\mathcal{L}^{(y)}$ , for any  $y \in R$ . Moreover, if  $R = \mathbb{Z}$ ,  $B_{\mathbb{Z}}$  is closed with respect to  $\mathcal{B}$ .*



*Proof* By definition of  $B_R$ , it is immediate to check that  $\mathcal{E}(B_R) = B_R$  (with this notation, we say that given any  $a \in B_R$ , then  $\mathcal{E}(a)$  is still in  $B_R$ ). Given any  $a \in B_R$ , with e.g.f.  $A(t)$ , we have that  $b = \mathcal{L}^{(y)}(a)$  has e.g.f.  $e^{yt}A(t)$  and  $b \in B_R$  since

$$(e^{yt}A(t))(e^{-yt}A(-t)) = A(t)A(-t) = 1.$$

Finally, when  $R = \mathbb{Z}$ , let us recall that the Euler zig-zag numbers  $\beta$  have e.g.f.  $B(t) = \sec(t) + \tan(t)$  which satisfies  $B(t)B(-t) = 1$  as a simple calculation shows

$$B(t) - \frac{1}{B(-t)} = \frac{1 + \sin(t)}{\cos(t)} - \frac{\cos(-t)}{1 + \sin(-t)} = \frac{1 - \sin^2(t) - \cos^2(t)}{\cos(t)(1 - \sin(t))} = 0,$$

i.e.,  $\beta \in B_{\mathbb{Z}}$ . Hence, given any  $a \in B_{\mathbb{Z}}$ ,  $\mathcal{B}(a) = \beta \star a \in B_{\mathbb{Z}}$ . □

*Remark 2* The group  $B_R$  is not closed with respect to the transform  $\mathcal{S}$ . Indeed, if  $A(t)$  is the e.g.f. of  $a \in B_R$ , the e.g.f. of  $S(a)$  is  $A(e^t - 1)$ , while the e.g.f. of  $\mathcal{E}(S(a))$  is  $A(e^{-t} - 1)$  and in general  $A(e^{-t} - 1)A(e^t - 1) \neq 1$ . It would be interesting to characterize the group  $\mathcal{S}(B_R)$ .

Let us examine the structure of a sequence  $a \in B_R$ . From the definition of  $B_R$  and from Theorem 2, we can observe that the elements of a sequence  $a \in B_R$  are constrained to severe restrictions, since the equality  $\mathcal{E}(a) = a^{-1}$  must hold. If we pose  $b = a^{-1}$  and  $c = \mathcal{E}(a)$ , we have, for instance,

$$b_0 = 1, \quad b_1 = -a_1, \quad b_2 = 2a_1^2 - a_2$$

and

$$c_0 = 1, \quad c_1 = -a_1, \quad c_2 = a_2,$$

i.e., the element  $a_1$  of the sequence  $a$  can be arbitrary, while  $a_2$  must satisfy

$$a_2 = 2a_1^2 - a_2,$$

i.e.,  $a_2 = a_1^2$ . By continuing in this way, we can also see, e.g., that  $a_3$  can be arbitrary, while  $a_4 = -3a_1^4 + 4a_1a_3$ . Thus, any sequence  $a \in B_R$  is completely determined when we fix the values of  $a_{2k-1}, k = 1, 2, \dots$ . Indeed, the following theorem shows how to evaluate the terms with even positive index as functions of the ones with odd index, by means of the partial ordinary Bell polynomials described in Definition 1.

**Theorem 3** *Given any  $a \in B_R$ , we have*

$$a_{2n} = (2n)! \sum_{k=0}^n \binom{\frac{1}{2}}{k} B_{n+k, 2k}(x_1, \dots, x_{n-k+1}), \quad \forall n \geq 1,$$

where  $x_i = \frac{a_{2i-1}}{(2i-1)!}, \binom{\frac{1}{2}}{k} = \frac{\prod_{j=0}^{k-1} (\frac{1}{2} - j)}{k!}$ .

*Proof* Let  $A(t)$  be the e.g.f. of  $a$ . Clearly,  $A(t) = P(t) + D(t)$ , where

$$P(t) = \sum_{n=0}^{+\infty} \frac{a_{2n}}{(2n)!} t^{2n} \tag{3}$$

and

$$D(t) = \sum_{n=1}^{+\infty} \frac{a_{2n-1}}{(2n-1)!} t^{2n-1} = \frac{1}{t} \sum_{n=1}^{+\infty} \frac{a_{2n-1}}{(2n-1)!} t^{2n}. \tag{4}$$

Moreover, we have

$$1 = A(t)A(-t) = (P(t) + D(t))(P(t) - D(t)) = (P(t))^2 - (D(t))^2,$$

since  $A(-t) = P(-t) + D(-t) = P(t) - D(t)$  and  $\mathcal{E}(a) = a^{-1}$ . Now, observing that  $P(0) = a_0 = 1$  and  $D(0) = 0$ , we obtain from the formal Maclaurin power series of  $(1 + X)^{\frac{1}{2}}$  that

$$P(t) = \left(1 + (D(t))^2\right)^{\frac{1}{2}} = \sum_{k=0}^{+\infty} \binom{\frac{1}{2}}{k} (D(t))^{2k}.$$

By definition of partial ordinary Bell polynomials we have

$$(D(t))^{2k} = \sum_{m=2k}^{+\infty} B_{m,2k}(x_1, \dots, x_{m-2k+1}) t^{2m-2k}.$$

If we set  $n = m - k$ , we get

$$\begin{aligned} P(t) &= \sum_{k=0}^{+\infty} \binom{\frac{1}{2}}{k} \sum_{n=k}^{+\infty} B_{n+k,2k}(x_1, \dots, x_{n-k+1}) t^{2n} \\ &= \sum_{n=0}^{+\infty} \left( \sum_{k=0}^n \binom{\frac{1}{2}}{k} B_{n+k,2k}(x_1, \dots, x_{n-k+1}) \right) t^{2n}. \end{aligned}$$

From this equality, comparing the coefficients of the respective even powers of  $t$  in (3) we finally obtain

$$a_{2n} = (2n)! \sum_{k=0}^n \binom{\frac{1}{2}}{k} B_{n+k,2k}(x_1, \dots, x_{n-k+1}), \quad \forall n \geq 1.$$

□

By Definition 1 and observing that  $B_{0,0} = 1$ ,  $B_{h,0} = 0$  for  $h \geq 1$ , we have the following corollary.

**Corollary 1** Given  $a \in B_R$ , we have

$$a_{2n} = (2n)! \sum_{k=1}^n \binom{\frac{1}{2}}{k} \sum_{\substack{i_1+i_2+\dots+i_{n-k+1}=2k \\ i_1+2i_2+\dots+(n-k+1)i_{n-k+1}=n+k}} \times (2k)! \prod_{j=1}^{n-k+1} \frac{1}{i_j! ((2j-1)!)^{i_j}} \prod_{j=1}^{n-k+1} a_{2j-1}.$$

On the other hand, it is also possible to determine the sequences  $a \in B_R$ , with  $a_2 \in R^*$  and such that  $a_2$  is a square in  $R$ , by fixing the terms  $a_{2k}, k = 1, 2, \dots$  and finding the terms with odd index as functions of the ones with even index.

**Theorem 4** Given  $a \in B_R$  such that  $a_2 \in R^*$  and  $x^2 = a_2$  is solvable in  $R$ , we have

$$a_{2n+1} = (2n+1)! \sum_{k=0}^n a_2^{\frac{1}{2}-k} \binom{\frac{1}{2}}{k} B_{n,k}(x_1, \dots, x_{n-k+1}), \quad \forall n \geq 0,$$

where  $x_i = \frac{1}{(2i+2)!} \sum_{k=0}^{i+1} \binom{2i+2}{k} a_{2k} a_{2(n-k+1)}$ , and  $a_2^{\frac{1}{2}} \in R$  is a solution of  $x^2 = a_2$ .

*Proof* Let  $A(t)$  be the e.g.f. of  $a$ , with the same notation used in the proof of Theorem 3, we have  $(D(t))^2 = (P(t))^2 - 1$ , where  $D(t)$  and  $P(t)$  as in (4) and (3), respectively. Since the product  $P(t) \cdot P(t)$  is equal to

$$(P(t))^2 = 1 + a_2 t^2 + \sum_{n=2}^{+\infty} \left( \sum_{k=0}^n \binom{2n}{2k} a_{2k} a_{2n-2k} \right) \frac{t^{2n}}{(2n)!},$$

we find

$$\begin{aligned} D(t) &= t a_2^{\frac{1}{2}} \left( 1 + \sum_{n=2}^{+\infty} \frac{a_2^{-1}}{(2n)!} \left( \sum_{k=0}^n \binom{2n}{2k} a_{2k} a_{2n-2k} \right) t^{2n-2} \right)^{\frac{1}{2}} \\ &= t a_2^{\frac{1}{2}} \left( 1 + \sum_{n=1}^{+\infty} \frac{a_2^{-1}}{(2n+2)!} \left( \sum_{k=0}^{n+1} \binom{2n+2}{2k} a_{2k} a_{2n-2k+2} \right) t^{2n} \right)^{\frac{1}{2}} \end{aligned}$$

Then, considering the formal Maclaurin series expansion of  $(1+X)^{\frac{1}{2}}$  and by Definition 1, we obtain

$$D(t) = \sum_{n=0}^{+\infty} \left( \sum_{k=0}^n a_2^{1/2-k} \binom{\frac{1}{2}}{k} B_{n,k}(x_1, \dots, x_{n-k+1}) \right) t^{2n+1}.$$

Now the thesis follows by a comparison of the corresponding coefficients of the odd powers of  $t$  in the expansion (4) of  $D(t)$ . □

*Remark 3* When  $R = \mathbb{Z}$ ,  $B_{\mathbb{Z}}$  contains many well-known and important integer sequences. We mention here some of them as interesting examples.

We have seen that the Euler zigzag numbers belong to  $B_{\mathbb{Z}}$ . They are listed in OEIS [16] as A000111. Thus, all the sequences having as e.g.f. a power of  $\sec(t) + \tan(t)$  are in  $B_{\mathbb{Z}}$ .

For instance the sequence A001250 in OEIS, whose  $n$ -th element is the number of alternating permutations of order  $n$ , has e.g.f.  $(\sec(t) + \tan(t))^2$ .

Moreover, the sequence A000667, which is the Boustrophedon transform of all-1's sequence, has e.g.f.  $e^t(\sec(t) + \tan(t))$  and belongs to  $B_{\mathbb{Z}}$ .

Another sequence in  $B_{\mathbb{Z}}$  is A000831, with e.g.f.  $\frac{1+\tan(t)}{1-\tan(t)}$ .

The sequences A006229 and A002017 also belong  $B_{\mathbb{Z}}$  since they have exponential generating functions of the shape  $\exp(f(t))$ , with  $f(t)$  odd function. Indeed, they have e.g.f.  $e^{\tan(t)}$  and  $e^{\sin(t)}$ , respectively.

Thanks to Theorem 3 and Corollary 1, we have new interesting identities connecting many sequences in OEIS. Furthermore, it is quite surprising that all these (very different) sequences satisfy the same limiting conditions.

In the following, we will introduce a new transform of sequences that arises from the study of  $B_R$ , which will allow us to consider  $U_R$  as a dynamic ultrametric space. Given  $a, b = a^{-1} \in U_R$ , we know that

$$\sum_{h=0}^n \binom{n}{h} a_h b_{n-h} = 0, \quad \forall n \geq 1,$$

from which it follows that

$$a_n = -\sum_{h=0}^{n-1} \binom{n}{h} a_h b_{n-h}, \quad b_n = -\sum_{h=0}^{n-1} \binom{n}{h} b_h a_{n-h}.$$

If  $a \in B_R$ , i.e.  $A(-t) = A(t)^{-1}$ , then, for all  $n \geq 2$ , we have

$$-\sum_{h=1}^{n-1} \binom{n}{h} (-1)^h a_h a_{n-h} = \begin{cases} 0 & \text{if } n \text{ odd} \\ 2a_n & \text{if } n \text{ even} \end{cases} \tag{5}$$

Thus it is natural to define the following transform.

**Definition 3** The *autoconvolution* transform  $\mathcal{A}$  maps a sequence  $a \in H_R$  into a sequence  $b = \mathcal{A}(a) \in H_R$ , where

$$\begin{cases} b_0 = a_0 \\ b_{2n+1} = a_{2n+1}, \quad \forall n \geq 0 \\ b_{2n} = -\frac{1}{2} \sum_{h=1}^{2n-1} \binom{2n}{h} (-1)^h a_h a_{2n-h}, \quad \forall n \geq 1 \end{cases}.$$

The following proposition is a straightforward consequence.

**Proposition 3** Given any  $a \in U_R$ , we have  $a \in B_R \Leftrightarrow \mathcal{A}(a) = a$ .

*Proof* If  $a \in B_R$ , we have  $b = \mathcal{A}(a) = a$ . Indeed, from Definition 3, we find  $b_0 = a_0 = 1$ ,  $b_{2n+1} = a_{2n+1}$  and, thanks to relation (5),  $b_{2n} = a_{2n}$ . On the other hand, when  $a \in U_R$  and  $\mathcal{A}(a) = a$ , if we consider the e.g.f.  $A(t)$  of  $a$ , we obtain

$$A(-t)A(t) = \sum_{m=0}^{+\infty} c_m \frac{t^m}{m!}, \quad c_m = \sum_{h=0}^m \binom{m}{h} (-1)^h a_h a_{m-h},$$

where  $c_0 = a_0^2 = 1$  and for all  $n \geq 1$ . Hence

$$c_{2n-1} = \sum_{h=0}^{2n-1} \binom{2n-1}{h} (-1)^h a_h a_{2n-1-h} = 0,$$

and from Definition 3

$$c_{2n} = \sum_{h=0}^{2n} \binom{2n}{h} (-1)^h a_h a_{2n-h} = 2a_{2n} + \sum_{h=1}^{2n-1} \binom{2n}{h} (-1)^h a_h a_{2n-h} = 0.$$

Thus  $A(-t)A(t) = 1$ , i. e.  $a \in B_R$ . □

Finally, we introduce another transform strictly related to  $\mathcal{A}$ .

**Definition 4** The transform  $\mathcal{U}$  maps a sequence  $a \in H_R$  into a sequence  $\mathcal{U}(a) = b \in H_R$  as follows:

$$\begin{cases} b_0 = a_0 \\ b_{2n+1} = a_{2n+1}, \quad \forall n \geq 0 \\ b_{2n} = (2n)! \sum_{k=0}^n \binom{\frac{1}{2}}{k} B_{n+k, 2k}(x_1, \dots, x_{n-k+1}), \quad \forall n \geq 1 \end{cases},$$

where  $x_i = \frac{a_{2i-1}}{(2i-1)!}$ .

**Proposition 4** Given any sequence  $a \in U_R$ , we have  $\mathcal{U}(a) \in B_R$ , moreover a sequence  $a \in U_R$  is in  $B_R$  if and only if  $a = \mathcal{U}(a)$ .

*Proof* Let us consider the e.g.f.  $B(t)$  of  $b = \mathcal{U}(a)$ . From Definition 4 and since  $a \in U_R$  we have

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{b_{2n}}{(2n)!} t^{2n} &= \sum_{n=0}^{+\infty} \left( \sum_{k=0}^n \binom{\frac{1}{2}}{k} B_{n+k, 2k}(x_1, \dots, x_{n-k+1}) \right) t^{2n} \\ &= \sum_{k=0}^{+\infty} \binom{\frac{1}{2}}{k} \sum_{n=k}^{+\infty} B_{n+k, 2k}(x_1, \dots, x_{n-k+1}) t^{2n} \end{aligned}$$

where, for all  $i \geq 1, x_i = \frac{a_{2i-1}}{(2i-1)!} = \frac{b_{2i-1}}{(2i-1)!}$ . Thus if we set  $m - k = n$  and we take into account the formal Maclaurin power series of  $(1 + X)^{\frac{1}{2}}$  and the definition of partial ordinary Bell polynomials, as in the proof of Theorem 3, we finally find the relation

$$\sum_{n=0}^{+\infty} \frac{b_{2n}}{(2n)!} t^{2n} = \left( 1 + \left( \sum_{n=1}^{+\infty} \frac{b_{2n-1}}{(2n-1)!} t^{2n-1} \right)^2 \right)^{\frac{1}{2}}$$

or, equivalently,

$$\begin{aligned} 1 &= \left( \sum_{n=0}^{+\infty} \frac{b_{2n}}{(2n)!} t^{2n} \right)^2 - \left( \sum_{n=1}^{+\infty} \frac{b_{2n-1}}{(2n-1)!} t^{2n-1} \right)^2 \\ &= \left( \sum_{n=0}^{+\infty} \frac{b_{2n}}{(2n)!} t^{2n} + \sum_{n=1}^{+\infty} \frac{b_{2n-1}}{(2n-1)!} t^{2n-1} \right) \left( \sum_{n=0}^{+\infty} \frac{b_{2n}}{(2n)!} t^{2n} - \sum_{n=1}^{+\infty} \frac{b_{2n-1}}{(2n-1)!} t^{2n-1} \right) \\ &= B(t)B(-t), \end{aligned}$$

i.e.  $b = \mathcal{U}(a) \in B_R$ . Therefore if  $a \in U_R$  and  $a = b = \mathcal{U}(a)$  we have  $a \in B_R$ , and, from the results of Theorem 3, if  $a \in B_R$  we can find  $a = \mathcal{U}(a)$ .  $\square$

**Proposition 5** *Given  $a \in H_R$ , with e.g.f.  $A(t)$ , then  $\mathcal{U}(a)$  has e.g.f.*

$$U(t) = \left( 1 + \left( \frac{A(t) - A(-t)}{2} \right)^2 \right)^{\frac{1}{2}} + \frac{A(t) - A(-t)}{2}.$$

*Proof* We can write  $A(t) = P(t) + D(t)$ , where  $P(t)$  and  $D(t)$  as in (3) and in (4) respectively. We have  $A(-t) = P(t) - D(t)$  and consequently  $D(t) = \frac{A(t) - A(-t)}{2}$ . The terms in odd places of  $\mathcal{U}(a)$  have e.g.f.  $D(t)$ . By Theorem 3, the terms in the even places of  $\mathcal{U}(a)$  have e.g.f.  $P(t) = (1 + (D(t))^2)^{\frac{1}{2}}$ , since  $\mathcal{U}(a) \in B_R$ . Thus, we have

$$U(t) = \left( 1 + (D(t))^2 \right)^{\frac{1}{2}} + D(t). \quad \square$$

Given  $a, b \in H_R$ , let us define

$$\delta(a, b) := 2^{-k},$$

if  $a_i = b_i$ , for any  $0 \leq i \leq k - 1$  and  $a_k \neq b_k$ . It is well-known that  $\delta$  is an ultrametric in  $H_R$ . Indeed,

- $\delta(a, b) = 0 \Leftrightarrow a = b$ ,
- $\delta(a, b) = \delta(b, a)$ ,
- $\delta(a, c) \leq \max(\delta(a, b), \delta(b, c))$ ,

for any  $a, b, c \in H_R$ . Thus,  $(H_R, \delta)$  is an ultrametric space.

Let us recall that we denote  $H_R^{(n)}$  the ring whose elements are sequences of elements of  $R$  with length  $n$ . Similarly,  $U_R^{(n)}$  and  $B_R^{(n)}$  are the subgroups of  $H_R^{(n)*}$  corresponding to the subgroups  $U_R$  and  $B_R$  of  $H_R^*$ , respectively.

**Theorem 5** *Given any  $a \in U_R$ , we have*

$$\delta(\mathcal{A}^n(a), \mathcal{U}(a)) \leq \frac{1}{2^{2(n+1)}},$$

where  $\mathcal{A}^n = \underbrace{\mathcal{A} \circ \dots \circ \mathcal{A}}_{n\text{-times}}$ .

*Proof* We prove the thesis by induction.

Let us denote  $a' = \mathcal{U}(a)$  and  $b = \mathcal{A}(a)$ . It is straightforward to check that

$$a' = (1, a_1, a_1^2, a_3, \dots), \quad b = (1, a_1, a_1^2, a_3, \dots).$$

Thus,  $a'$  and  $b$  coincide at least in the first 4 terms, i.e.,

$$\delta(\mathcal{A}(a), \mathcal{U}(a)) \leq \frac{1}{2^4}.$$

Now, let us suppose that given  $b = \mathcal{A}^n(a)$ , we have  $\delta(\mathcal{A}^n(a), \mathcal{U}(a)) \leq \frac{1}{2^{2(n+1)}}$ , i.e.  $b_i = a'_i$  for all  $i \leq 2n + 1$  and consider  $c = \mathcal{A}(b)$ . Since  $a' \in B_R$ , we remember that for all  $n \geq 2$  we have

$$-\sum_{h=1}^{n-1} \binom{n}{h} (-1)^h a'_h a'_{n-h} = \begin{cases} 0 & \text{if } n \text{ odd} \\ 2a'_n & \text{if } n \text{ even} \end{cases}.$$

Thus, by Definition 3, we obtain  $c_i = a'_i$  for all  $i \leq 2n + 3$ , since  $(b_0, \dots, b_{2n+1}) = (a'_0, \dots, a'_{2n+1}) \in B_R^{(2n+2)}$  by inductive hypothesis. Hence, we have proved that

$$\delta(\mathcal{A}^{n+1}(a), \mathcal{U}(a)) \leq \frac{1}{2^{2(n+2)}}.$$

□

As a consequence of Theorem 5, we can observe that  $\mathcal{A}$  can be considered as an approximation of  $\mathcal{U}$ . Indeed, given a sequence  $a \in U_R$ , sequences  $\mathcal{A}^n(a)$  have more elements equal to elements of  $\mathcal{U}(a)$  for increasing values of  $n$ .

*Example 1* Given  $a = (a_0, a_1, a_2, a_3, a_4, a_5) \in U_R^{(6)}$ , then

$$\mathcal{U}(a) = (1, a_1, a_1^2, a_3, 4a_1a_3 - 3a_1^4, a_5)$$

and

$$\mathcal{A}(a) = (1, a_1, a_1^2, a_3, 4a_1a_3 - 3a_2^2, a_5).$$

Considering  $\mathcal{A}^2$ , we obtain

$$\mathcal{A}^2(a) = (1, a_1, a_1^2, a_3, 4a_1a_3 - 3a_1^4, a_5) = \mathcal{U}(a).$$

In other words, given any sequence  $a \in U_R^{(6)}$ ,  $\mathcal{A}^2(a) = \mathcal{U}(a) \in B_R^{(6)}$ , i.e., in  $U_R^{(6)}$  the transforms  $\mathcal{A}^2$  and  $\mathcal{U}$  are identical.

From Theorem 5 easily follows the next corollary.

**Corollary 2** *Given any  $a \in U_R^{(2^n)}$ , we have*

$$\mathcal{U}(a) = \mathcal{A}^{n-1}(a).$$

Moreover, for any  $a \in U_R$ , we have

$$\mathcal{U}(a) = \lim_{n \rightarrow +\infty} \mathcal{A}^n(a).$$

Clearly, if two sequences  $a, b \in H_R$  coincide in the first  $k$  terms, then  $\mathcal{A}(a)$  and  $\mathcal{A}(b)$  coincide at least in the first  $k$  terms. Thus, we have the following proposition.

**Proposition 6** *Given any  $a, b \in H_R$ , then*

$$\delta(\mathcal{A}(a), \mathcal{A}(b)) \leq \delta(a, b).$$

By the previous proposition, we have that  $\mathcal{A}$  is a contraction mapping on the ultrametric space  $(H_R, \delta)$ . As a first interesting consequence, we can observe that  $\mathcal{A}$  is a continuous function. Moreover, we have that the ultrametric group  $(U_R, \star, \delta)$  with the contraction mapping  $\mathcal{A}$  is an ultrametric dynamic space, where the set of fixed points is the subgroup  $B_R$ . In this way, we have found a very interesting example of ultrametric dynamic space. Ultrametric dynamics are very studied in several fields, see [15] for a good reference about dynamics on ultrametric spaces.

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