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Variants of theorems of Schur, Baer and Hall

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Abstract If a group G is 'restricted' modulo its hypercentre, then to what extent does G have an equally restricted normal subgroup L with G/L hypercentral? We consider these questions where restricted means finite-π, Chernikov, locally finite-π, polycyclic or polycyclic-by-finite.

Keywords Hypercentre · Hypocentre · Central series

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1 Introduction

For any group G denote its centre by $\zeta_1(G)$ and its hypercentre by $\zeta(G)$. If t is a positive integer, say $t = \Pi_{\text{primes } p}\, p^{e(p)}$, let $e(t)$ denote the maximum of the $e(p)$ (so $e(1) = 0$) and $h(t)$ the sum of all the $e(p)$. Set $a(t) = [e(t)/2] + 1$, where $[r]$ denotes the integer part of a real number r, and set $b(t) = t^{(e(t)+1)/2}$. Obviously $b(t) \leq t^{a(t)}$. The following variant of theorems of Schur and Baer was essentially proved by de Falco et al. [2].

Theorem A (cf. [2,5]) *Let G be a group with $G/\zeta(G)$ finite of order t. Then G has a normal subgroup L with G/L hypercentral and with L of finite order dividing $t^{a(t)+1}$ and at most $b(t)t$.*

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de Falco et al. [2] gives no specific bounds. The later paper [5] by Kurdachenko et al. contains two proofs of Theorem A, one shorter with no bounds and one with just a bound for $|L|$ slightly larger than $b(t)t$.

There are variants of the classical Schur and Baer theorems where finite is replaced by notions like Chernikov, polycyclic or locally finite, see [7], especially Page 115. Here we consider corresponding questions in the context of Theorem A. The following is the main result of this paper.

Theorem B *Let G be a group with $G/\zeta(G)$ a Chernikov group. Then G has a normal Chernikov subgroup L with G/L hypercentral.*

A minor variation to our proof of Theorem B gives yet another short proof of Theorem A. In fact we prove Theorem A with rather better bounds than those stated above, but with bounds less briefly explained. Let Z be a central subgroup of a group G of finite index dividing t. Then (Schur's theorem) the order $|G'|$ of the derived subgroup of G is finite and in fact boundedly so (e.g. the easy proof of [11] 1.18, yields that $log_t|G'| \leq (t-1)^2 + 1$. Given t define the integers $c(t)$ and $d(t)$ as follows: $c(t)$ is the least integer such that for any G and Z as above (but with fixed t), $|G'|$ divides $t^{c(t)}$ and $d(t)$ is the least integer with $|G'| \leq d(t)$. Notice that if s divides t, then $c(s) \leq c(t)$ and $d(s) \leq d(t)$. By Theorem 1 of [12] we have $c(t) \leq [e(t)/2]+1$ and $d(t) \leq t^{(e(t)+1)/2}$. Hence Theorem A follows from the following.

Theorem C *Let G be a group with $Z = \zeta(G)$ of finite index in G dividing t. Then G has a normal subgroup L with G/L hypercentral and with L of finite order dividing $t^{c(t)+1}$ and at most $d(t)t$.*

Wiegold [13] has a different type of bound for $d(t)$. Assume $t > 1$, let q be the least integer to divide t and set $t' = log_q t$ and $t'' = [t']$; clearly $e(t) \leq h(t) \leq t'' \leq t' \leq t$. Then Wiegold proves that $d(t) \leq t^{(t'-1)/2}$. In fact one can do a little better than this (see [12] Theorems 2 and 3), namely that $c(t) \leq [t''/2]$ and $d(t) \leq t^{(t''-1)/2}$ unless $t = p^e q$ or pq^e with $p > q$ primes and $e \geq 1$ when $c(t) \leq [t''/2] + 1$. Further if $t = pq^e$ with $e \geq 2$ or if $t = p^e q$ with $p^e > q^{e+1}$, then $d(t) \leq t^{(t''-1)/2}$. With the exceptional $t = p^e q$ (e.g. $t = 6$) we have of course Wiegold's bound $d(t) \leq t^{(t'-1)/2}$.

The obvious analogues of Theorem B, with Chernikov replaced by polycyclic or polycyclic-by-finite, are false, see Example 1 below. We do however have the following easier result.

Theorem D *Let G be a group with $G/\zeta(G)$ a locally finite π-group for some set π of primes (e.g. π the set of all primes). Then G has a locally finite, normal π-subgroup L with G/L hypercentral.*

Casolo, Dardano and Rinauro in their recent paper [1] prove the corresponding result to Theorem A in the context of Hall's theorem. Specifically they prove the following.

Theorem E (see [1] Theorem A) *Let L be a finite normal subgroup of the group G such that G/L is hypercentral. Then the index $(G{:}\zeta(G))$ is finite and divides $|Aut\, L|.|\zeta_1(L)|$.*

Simple examples show that the corresponding statements are false with finite replaced by Chernikov, polycyclic, polycyclic-by-finite, or locally finite, see Examples 2, 3 and 4 below. Theorem E is a very easy consequence of our final theorem.

Theorem F *Let A be a finite abelian normal subgroup of the group G and let H be a normal subgroup of G in $C_G(A)$ and containing A. Suppose every finite image of $G/C_G(H)$ is nilpotent. Then $(H/A) \cap \zeta(G/A) = A(H \cap \zeta(G))/A$; that is, if ϕ denotes the natural projection of G onto G/A, then $H\phi \cap \zeta(G\phi) = (H \cap \zeta(G))\phi$.*

Whenever we have $A \leq K \leq H$ note that $(K/A) \cap \zeta(G/A) = A(K \cap \zeta(G))/A$. To derive Theorem E from Theorem F, set $H = C_G(L)$ and $A = H \cap L$. Clearly H/A is G-isomorphic to $HL/L \leq G/L$, which is hypercentral. Consequently $H/A \leq \zeta(G/A)$. Also $L \leq C_G(H)$, so $G/C_G(H)$ is hypercentral. Then Theorem F implies that $H \leq A.\zeta(G)$. Clearly $(G : H)$ divides $|Aut L|$. Therefore $(G : \zeta(G))$ divides $|Aut L|.|A|$.

2 Proof of the Theorems

Lemma 1 *Let V be a finite elementary abelian p-group and G a nilpotent subgroup of $Aut V$. Then as G-module $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$, where for each i the G-composition factors of V_i are all G-isomorphic. In particular if V as G-module has a non-trivial factor centralized by G, then V has a non-zero element fixed by G and a non-trivial image centralized by G.*

To obtain such a decomposition of V, see [10] 7.15. Note that the hypothesis there that the field F is algebraically closed is only used to ensure that the Jordan decomposition of each g in G takes place in $GL(n, F)$. If $g \in G$ has finite order, then trivially g_u and g_d lie in $\langle g \rangle$, so here, as H is finite, we can dispense with the algebraic closure hypothesis. (Actually it suffices just to have F perfect, e.g. see [9] 3.1.6, which of course automatically covers the $F = GF(p)$ case.)

Remark Suppose G is a nilpotent group and V a finite G-module such that $V = [V, G]$. If q is prime, Lemma 1 shows that V/qV has no trivial G-composition factors. Thus nor does any G-image of V/qV; in particular nor does any $q^i V/q^{i+1} V$. Applying this for every q dividing the order of V shows that V itself has no trivial G-composition factors.

As well as $\zeta(G) = \cup_{w \geq 0} \zeta_w(G)$, the hypercentre of G we consider $\gamma G = \cap_{w \geq 0} \gamma^{w+1} G$, the hypocentre G; here w runs over the ordinals, $\{\zeta_w(G)\}$ is the upper central series of G and $\{\gamma^{w+1}(G)\}$ is the lower central series of G. Let $k \geq 0$ and $t \geq 1$ be integers. If $(G : \zeta_k(G)) = t$, then clearly $\zeta(G) = \zeta_{k+e(t)}(G)$. Also by Baer's theorem $|\gamma^{k+1} G|$ is finite (see [8] 14.5.1), so $G/\gamma G$ is nilpotent. Then $G/\gamma G.\zeta_k(G)$ is nilpotent of order dividing t, so $G/\gamma G$ is nilpotent of class at most $k + e(t)$ and $\gamma G = \gamma^{k+e(t)+1} G$. Suppose instead that $|\gamma^{k+1} G| = d$. Clearly then $\gamma G = \gamma^{k+e(d)+1}(G)$. Also the upper central series of G intersected with $\gamma^{k+1} G$ has length at most $e(d)$ and $\zeta(G)/(\zeta(G) \cap \gamma^{k+1} G)$ embeds into $G/\gamma^{k+1} G$ as a G-group and hence has G-central height at most k. Therefore $\zeta(G!) = \zeta_{k+e(d)}(G)$. The above might seem rather pedantic, but one needs to be slightly careful in dealing with γG for infinite groups G. We use these remarks below.

We now start on the proofs of Theorem B and, indirectly, Theorem C. Thus below G denotes a group with $G/\zeta(G)$ a Chernikov group. Set $Z = \zeta(G)$ and $\Gamma = \gamma G$.

Suppose first that G is finite and that $(G : Z)$ divides t. Now $G/C_G(Z)$ stabilizes the upper central series of G and hence is nilpotent. Therefore $\Gamma \leq C_G(Z)$, $\Gamma \cap Z \leq \zeta_1(\Gamma)$, $(\Gamma : \Gamma \cap Z)$ divides t and $|\Gamma'|$ divides $t^{c(t)}$ and is at most $d(t)$. Set $V = \Gamma/\Gamma'$. Clearly Γ centralizes V, so $G/C_G(V)$ is nilpotent. By the Remarks above V has no trivial G-composition factors, so $\Gamma \cap Z \leq \Gamma'$. Thus $(\Gamma : \Gamma')$ divides t. Therefore $|\Gamma|$ divides $t^{c(t)+1}$ and is at most $d(t)t$.

Now suppose that G is finitely generated. Again we have $t = (G : Z)$ finite. Also G is polycyclic-by-finite, so there exists an integer k with $\zeta_k(G) = Z$. By Baer's theorem $\gamma^{k+1}G$ is finite, so $\Gamma \leq \gamma^{k+1}G$ is finite and G/Γ is nilpotent. Now G is residually finite. Hence there is a normal subgroup N of G of finite index with $\Gamma \cap N = \langle 1 \rangle$. Clearly $ZN/N \leq \zeta(G/N)$ and $\Gamma \cong \Gamma N/N = \gamma(G/N)$. By the finite case we have that $|\Gamma|$ divides $t^{c(t)+1}$ and is at most $d(t)t$. Also by the finite case we have $[\Gamma, Z] \leq \Gamma \cap N = \langle 1 \rangle$ and $\Gamma \cap Z \leq \Gamma \cap \Gamma' N = \Gamma'$.

The Proof of Theorem B. Suppose $X \leq Y$ are finitely generated subgroups of G. Clearly $\gamma X \leq \gamma Y$, so $L = \cup_X \gamma X$ is a normal subgroup of G. By the finitely generated case above we have that $[\gamma X, X \cap Z] = \langle 1 \rangle$ and $\gamma X \cap Z \leq (\gamma X)'$; further $X/\gamma X$ is nilpotent. If $x \in L$ and $z \in Z$, there exists an X with $x \in \gamma X$ and $z \in X \cap Z$. Then $[x, z] = 1$ and hence $[L, Z] = \langle 1 \rangle$. Also

$$L \cap Z = \cup_X (\gamma X \cap Z) \leq \cup_X (\gamma X)' = L'.$$

Now G/L is locally nilpotent since each $X/\gamma X$ is nilpotent and locally nilpotent Chernikov groups are hypercentral. Hence G/LZ and G/L are hypercentral. Further $L/(L \cap Z)$ is Chernikov and $L \cap Z \leq \zeta_1(L)$. Therefore L' is Chernikov by Polovickii's theorem (see [7] 4.23). Consequently L is Chernikov. The proof is complete. □

The Proof of Theorem C. Here we have $(G : Z)$ dividing t. Let X be a finitely generated subgroup of G with $XZ = G$. By the finitely generated case we have that $X/\gamma X$ is nilpotent and that $|\gamma X|$ divides $t^{c(t)+1}$ and is at most $d(t)t$. Choose X so that $|\gamma X|$ is maximal. If Y is any finitely generated subgroup of G containing X, then $\gamma X \leq \gamma Y$ since $[\gamma X, X] = \gamma X$. By the maximal choice of X we have $\gamma X = \gamma Y$. This is for all such Y and consequently $L = \gamma X$ is normal in G. If ψ is the natural map of G onto G/L, then $X\psi$ is nilpotent and $G\psi = X\psi.Z\psi$. Consequently $G\psi$ is hypercentral. The proof is complete. □

Comments on the above proofs. Notice that in general, unlike the finitely generated case, in Theorem C we cannot prove that γG is finite; just consider the infinite locally dihedral 2-group. However, since $L = \gamma X = [\gamma X, X]$, so $L = [L, G] \leq \gamma G$ and $G/\gamma G$ is hypercentral. Further L is actually the hypercentral residual of G and in particular L is fully invariant in G.

A similar remark applies to Theorem B. If $A = \oplus_{i \geq 1} \langle a_i \rangle$ is free abelian of infinite rank and $x \in Aut A$ is given by $a_i x = a_{i-1} + a_i$ for all i (with $a_0 = 0$), then the split extension G of A by $\langle x\rangle$ is hypercentral and yet $\gamma G = A$ is not Chernikov. Suppose $\alpha = ch(G)$, the central height of G, and $\beta = ch(G/L)$. Assuming $(G : Z) = t$, if $e = e(t)$, then $\beta \leq \alpha + e$. On the other hand if $|L| = d$ and if $f = e(d)$, then $\alpha \leq f + \beta$, so if either of α and β is infinite, they both are and $\alpha \leq \beta \leq \alpha + e$.

Let $k \geq 0$ and $t \geq 1$ be integers. Suppose G is a group with $(G : \zeta_k(G))$ finite. By Baer's theorem $\gamma^{k+1}G$ is finite. More precisely there exists an integer-valued function $\tau(k, t)$ such that if $(G : \zeta_k(G))$ divides t, then $|\gamma^{k+1}G|$ divides $t^{\tau(k,t)}$. For set $\tau(0, t) = 1$ and (via Schur's theorem) set $\tau(1, t) = c(t)$. Suppose $k \geq 2$ and $|\gamma^k(G).\zeta_1(G)/\zeta_1(G)|$ divides s. Then by [8] 14.5.2, or more precisely by its proof, $|\gamma^{k+1}G|$ divides $(st)^{2\tau(1,st)+1}$. Thus we can define $\tau(k, t)$ inductively on k by setting

$$\tau(k, t) = (\tau(k-1, t) + 1)(2\tau(1, st) + 1) \text{ for } s = t^{\tau(k-1,t)}.$$

The above implies (cf. [1] Proposition 3) that if $(G : \zeta_k(G)) = t$, then the order of $\gamma^{2k+1}G$ divides a power of t whose exponent is bounded by a function of t only, namely it divides $\max\{t^{c(t)+1}, t^{\tau(2.c(t),t)}\}$. For as we saw above $G/\gamma G$ is nilpotent of class at most $k+e(t)$, so if $k \geq e(t)$, then $|\gamma^{2k+1}G| = |\gamma G|$, which divides $t^{c(t)+1}$ and if $k \leq e(t)$, then $|\gamma^{2k+1}G|$ divides $t^{\tau(2.e(t),t)}$, since $\tau(k, t)$ is an increasing function of k.

Lemma 2 *Let G be a π-torsion-free group for π some set of primes. If $G/\zeta(G)$ is a locally finite π-group, then $G = \zeta(G)$.*

Proof If X is a finitely generated subgroup of G, then X is nilpotent-by-finite, $\zeta(X) = \zeta_k(X)$ for some finite k and $X/\zeta_k(X)$ is a finite π-group. Hence $\gamma^{k+1}(X)$ is also a finite π-group (e.g. [7] Page 115 or use the above). But G is π-torsion-free; therefore $\gamma^{k+1}(X) = \langle 1\rangle$ and so G is locally nilpotent. But then $\zeta(G)$ is π-isolated in G (see [4] 4.8b). Therefore $\zeta(G) = G$. (Alternatively, if T is the maximal periodic normal subgroup of G, then T is a π'-group, so $T \leq \zeta(G)$ and $\zeta(G/T)$ is isolated in G/T by [6] 2.3.9i); thus again $\zeta(G) = G$.) □

The Proof of Theorem D. Let $X \leq Y$ be finitely generated subgroups of G. Then $X/\zeta(X)$ is a finite π-group, so by Theorem C there exists a finite normal π-subgroup L_X of G with X/L_X hypercentral and hence nilpotent. Clearly we may choose L_X so that X/L_X is π-torsion-free. Then $L_X = X \cap L_Y$. Set $L = \cup_X L_X$. Then L is a locally finite, normal π-subgroup of G with G/L π-torsion-free and locally nilpotent. By the lemma above G/L is hypercentral.

Example 1 If $G/\zeta(G)$ is polycyclic, there is no need for G to be (polycyclic-by-finite)-by-hypercentral.

Let A be a divisible abelian 2-group of rank 2. Then $Aut A \cong GL(2, \mathbb{Z}_2)$. Let $H = \langle x, h\rangle \leq GL(2, \mathbb{Z}_2)$; here $x \neq 1$ permutes the standard basis of $(\mathbb{Z}_2)^{(2)}$ and $h = diag(k, k^{-1})$ where $k \in 1 + 2\mathbb{Z}_2 \leq \mathbb{Z}_2$ has infinite order. Set $A_i = \{a \in A : |a| \leq 2^i\}$ for $i = 0, 1, 2, \ldots$. Then $[A_i, h] \leq A_{i-1}$ for all $i > 0$; also $A_i^x = A_i$ and $[A_i, {}_2x] \leq A_{i-1}$. Further H is infinite dihedral, so $\zeta_1(H) = \langle 1\rangle$.

Let $G = HA$ be the split extension of A by H. Then $\zeta(G) = A$ and $G/\zeta(G) \cong H$ is polycyclic. Suppose T is any polycyclic-by-finite normal subgroup of G. Then

$T \cap A \leq A_i$ for some i. If m is a positive integer with $h^m \in T$, then h^m stabilizes the series $\langle 1 \rangle < A_1 < A_2 < \cdots < A_i < A$ and hence h^{mn} centralizes A for some $n \geq 1$ (e.g. [11] 1.21), contradicting h of infinite order. Consequently $H \cap T = \langle 1 \rangle$ and so G/T cannot be hypercentral.

The Proof of Theorem F. Define K by $K/A = (H/A) \cap \zeta(G/A)$. We induct on the exponent e of A. Suppose first that $e = p$, a prime and that $\zeta(G) = \langle 1 \rangle$. If $K > A$, then there exists $k \in K \backslash A$ with $kA \in \zeta_1(G/A)$. Then $V = \langle k \rangle A$ is abelian and normal in G and clearly $[v^p, g] = [v, g]^p = 1$ for all v in V and g in G. Also $V \leq H$, so $G/C_G(V)$ is an image of $G/C_G(H)$ and hence is nilpotent. It follows that $V \cap \zeta_1(G) \neq \langle 1 \rangle$, either because $V^p \neq \langle 1 \rangle$ or by the Remark above, contradicting the assumption that $\zeta(G) = \langle 1 \rangle$. Thus in this case $K = A$. Applying this to $G/\zeta(G)$ yields that if A is elementary abelian, then

$$K/A \leq (H/A) \cap A.\zeta(G)/A = A(H \cap \zeta(G))/A \leq K/A.$$

Now suppose that p is just some prime dividing e and set $B = A^p$. By the case above

$$(H/A) \cap \zeta(G/A) = (A/B)((H/B) \cap \zeta(G/B))/(A/B).$$

Also by induction on e we have $(H/B) \cap \zeta(G/B) = B(H \cap \zeta(G))/B$. Therefore

$$(H/A) \cap \zeta(G/A) = (A/B)(B(H \cap \zeta(G))/B)/(A/B) = A(H \cap \zeta(G))/A.$$

The proof is complete. □

Let L be a finite group of order d. Then any series of subgroups of L has length at most $h(d)$, the minimal number of generators of L is at most $h(d)$ and $Aut L$ has order at most $d^{h(d)}$. For example, appying this to Theorem E yields that $(G : \zeta(G))$ is at most $d^{h(d)+1}$.

Assume $k \geq 0$ and $d > 1$ are integers and suppose G is a group with $L = \gamma^{k+1}G$ of order d. Then from Theorem 2 of [3] we have $(G : \zeta_{2k}(G)) \leq d^s$, where $s = r^k + h(d)$ and r is the rank of $Aut L$. Note that r is bounded by a function of d only; for example $r \leq h(d)^2$. Also $\zeta(G) = \zeta_{k+e(d)}(G)$, see Remarks above; consequently $(G : \zeta_{k+e(k)}(G)) \leq d^{h(d)+1}$. Thus if $k \geq e(d)$, then $(G : \zeta_{2k}(G)) \leq d^{h(d)+1}$ and if $k < e(d)$, then $(G : \zeta_{2k}(G)) \leq d^s$ for $s = r^k + h(d) \leq r^{e(d)} + h(d) \leq h(d)^{2.e(d)} + h(d) = u(d)$ say. We have proved the following (cf. [1] Corollary A′). If $|\gamma^{k+1}G| = d$, then $(G : \zeta_{2k}(G)) \leq d^{u(d)}$ for u as above, a function of d only.

Unlike the previous case we need not have that $(G : \zeta_{2k}(G))$ divides a power of d, for if $G = Sym(3)$ and $k = 1$, then $d = 3$ and $(G : \zeta_{2k}(G)) = 6$.

For the analogues of Theorem E the results are negative.

Example 2 If G is (infinite cyclic)-by-hypercentral, then $G/\zeta(G)$ need not be polycyclic-by-finite.

Let $A = \mathbb{Z}$, $B = \mathbb{Z}[1/2]$, g the automorphism $b \mapsto -b$ of B and G the split extension $\langle g \rangle B$. Then A is infinite cyclic and normal in G and G/A is hypercentral, being an infinite locally dihedral 2-group. Finally if $x \in G \backslash B$, then x acts fixed-point freely on B, so $\langle 1 \rangle = \zeta_1(G) = \zeta(G)$. Clearly G is not polycyclic-by-finite.

Example 3 If G is (locally finite)-by-hypercentral, then $G/\zeta(G)$ need not be periodic.

Let G be the wreath product of a cyclic group of prime order p by an infinite cyclic group. Then G' is an elementary abelian p-group and yet $\zeta(G) = \zeta_1(G) = \langle 1 \rangle$.

Example 4 If G is Chernikov-by-hypercentral, then $G/\zeta(G)$ need not be Chernikov or even periodic.

Let G be the split extension of the Prüfer p-group P for the odd prime p by the infinite cyclic group $\langle ab \rangle$, where a is the inversion automorphism of P and b is an automorphism of P of infinite order that stabilizes the (only) composition series of P. Then $G' = P$ and so is Chernikov, but $\zeta(G) = \langle 1 \rangle$, so $G/\zeta(G)$ is not even periodic.

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