Ricerche mat. (2017) 66:629–644
DOI 10.1007/s11587-017-0317-0

Uniqueness of renormalized solutions for a class of parabolic equations

Ahmed Aberqi[1] · **Jaouad Bennouna**[2] · **Mohamed Hammoumi**[2]

Received: 31 May 2016 / Revised: 23 January 2017 / Published online: 8 February 2017

Abstract In this paper, we prove uniqueness of renormalized solution for a class of doubly nonlinear parabolic problems.

$$\begin{cases} \dfrac{\partial e^{\beta u}}{\partial t} - \triangle_p u + \operatorname{div}(c(x,t)|u|^{\gamma-1}u) + d(x,t)|\nabla u|^{\delta-1} = f - \operatorname{div}(F) & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ e^{\beta(u(x,0))} = e^{\beta(u_0(x))} & \text{in } \Omega. \end{cases} \tag{1}$$

Keywords Dirichlet problem · Non coercive problems · Uniqueness results · Renormalized solution

Mathematics Subject Classification Primary 47A15; Secondary 46A32 · 47D20

The original version of this article was revised: the first name of the second author was incorrect. Now, it has been corrected.

✉ Ahmed Aberqi
aberqi_ahmed@yahoo.fr

Jaouad Bennouna
jbennouna@hotmail.com

Mohamed Hammoumi
hammoumi.mohamed09@gmail.com

1 Department of Electric and Computer Engineering, Laboratory LISA, National School of Applied Sciences, University of Fez, Fez, Morocco

2 Department of Mathematics, Laboratory LAMA, Faculty of Sciences Dhar El Mahraz, University of Fez, B.P 1796 Atlas Fez, Morocco

1 Introduction

In the present paper, we establish the uniqueness for renormalized solutions for a class of doubly nonlinear parabolic equations, whose prototype:

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \triangle_p u + \operatorname{div}(c(x,t)|u|^{\gamma-1}u) + d(x,t)|\nabla u|^{\delta-1} = f - \operatorname{div}(F) & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ b(u(x,0)) = b(u_0(x)) & \text{in } \Omega. \end{cases} \tag{2}$$

In the problem (2) the framework is the following: Ω is a bounded open set of $\mathbb{R}^N$, with Lipschitz-continuous boundary $\partial\Omega$, $N \geq 2$, $T > 0$, while the data f, F and $b(u_0)$ are respectively in $L^1(Q_T)$, $(L^{p'}(Q_T))^N$, and $L^1(\Omega)$. b is strictly increasing $C^1(\mathbb{R})$-function, unbounded of s. $\triangle_p u$ is the p-Laplace operator, $Q_T = \Omega \times (0,T)$, data $c(.,.)$, $d(.,.)$, γ, δ, will be defined later, see Sect. 2 (8–11).

Starting with the paper [9] the authors proved an existence result of a weak solutions for the non coercive problem 3 in the stationary case ($b_t(u) = 0$) using the symmetrization method. More later Di Nardo et al. [10] have shown the existence of renormalized solution for the parabolic version, more precisely in the linear case ($b(u) = u$), and the uniqueness for such solution in the paper [6], Aberqi et al. [1,2] have proved the existence of a renormalized solutions for 3 with more general parabolic terms ($b_t(x,u)$).

In the present work we prove the uniqueness or such solution, under some local control of Lipschitz coefficient (see Theorem 3.1).

In general, the concept of the weak solution is not sufficient to determine the solution physically observed due to the lack of uniqueness of the solution. It appears necessary to select among all the physically weak solutions feasible solution. The renormalized solutions allowed to have results of existence and uniqueness for certain equations that are not accessible within the solutions in the sense of distributions see the counter example given by Serrin [14] in the linear case, and Bénilan et al. [5] in the case of p-Laplacian operator. To overcome this difficulties we work in the framework of renormalized solutions (see 14–18), this notion was introduced by Diperna and Lions [12] in their study of Boltzmann equations, see also [4,8].

The paper is concerned with giving a careful account on both existence and uniqueness of renormalized solution, we want to stress that, while the existence result follows a rather standard approximation argument see [1] due to the nonlinearity $b(u)$ and non coercive terms $c(x,t)|u|^{\gamma-1}u$ and $d(x,t)|\nabla u|^{\delta-1}$ and the measure $f - \operatorname{div}(F)$.

In order to perform the uniqueness, the paper is planned in the following way. Section 2 is devoted to specify the assumptions on b, a, H, f, and $b(u_0)$ and to give the definition of a renormalized solution of 3, and we prove some technical Lemmas whose play a crucial role to prove the uniqueness results. In Sect. 3, we prove that there exists a unique renormalized solution see Theorem 3.1.

2 Basic assumptions and definitions

In this section, we recall the definition of renormalized solutions to the following nonlinear parabolic problem:

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \operatorname{div}\big(a(x,t,u,\nabla u) - \phi(x,t,u)\big) + H(x,t,\nabla u) = f - \operatorname{div}(F) & \text{in} \quad Q_T \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ b(u(x,0)) = b(u_0(x)) & \text{in } \Omega, \end{cases} \tag{3}$$

where Q_T is the cylinder $\Omega \times (0,T)$, Ω is a bounded open set of $\mathbb{R}^N$, with $N \geq 2$, $T > 0$, $p > 1$.

Throughout this paper, we assume the following assumptions hold true.

- $b : \mathbb{R} \to \mathbb{R}$ is strictly increasing C^1-function, $b(0) = 0$,

$$\text{and } 0 < b_0 \leq b'(s) \leq b_1 \ \forall s \quad \text{and with} \quad b_1 < \left(\frac{2}{\alpha}\right)^{\frac{p-1}{2}} \tag{4}$$

  where α is a strictly real number defined below in (6).
- $a : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that there is $\alpha > 0$, and for $k > 0$, there exists $\nu_k > 0$ and a function $h_k \in L^{p'}(Q_T)$ such that, $\forall |s| \leq k, \forall (\xi, \eta) \in \mathbb{R}^{2N}$, for a.e. $(x,t) \in Q_T$,

$$|a(x,t,s,\xi)| \leq \ \big(h_k(x,t) + |s|^{p-1} + |\xi|^{p-1}\big) \tag{5}$$

$$a(x,t,s,\xi)\xi \geq \alpha|\xi|^p, \tag{6}$$

$$(a(x,t,s,\xi) - a(x,t,s,\eta)\cdot(\xi - \eta) > 0, \ \xi \neq \eta, \tag{7}$$

- $\phi : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}^N$ be a Carathéodory function such that

$$|\phi(x,t,s)| \leq c(x,t)|b(s)|^\gamma, \tag{8}$$

$$c(.,.) \in (L^r(Q_T))^N, \quad r = \frac{p+N}{p-1}, \quad \text{and} \quad \gamma = \frac{(N+2)(p-1)}{N+p}, \tag{9}$$

- $H : \Omega \times (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that

$$|H(x,t,\xi)| \leq d(x,t)|\xi|^\delta, \tag{10}$$

$$\text{with } \delta = \frac{p + N(p-1)}{N+2}, \ d(.,.) \text{ belonging a suitable Lorentz space } L^{N+2,1}(Q_T), \tag{11}$$

Moreover we assume that the source terms

$$f \in L^1(Q_T), \ F \in (L^{p'}(Q_T))^N, \tag{12}$$

$$u_0 \in L^1(\Omega). \tag{13}$$

Definition 2.1 A measurable function u defined on Q_T is called a renormalized solution of (3) if:

$$b(u) \in L^{\infty}((0,T), L^1(\Omega)). \tag{14}$$

$$T_k(b(u)) \in L^p((0,T), W_0^{1,p}(\Omega)), \quad \text{for any } k > 0, \tag{15}$$

$$\lim_{m\to+\infty} \frac{1}{m} \int_{\{(x,t)\in Q_T : |b(u(x,t))|\le m\}} a(x,t,u,\nabla u)\nabla u \, dx\, dt = 0, \tag{16}$$

and if, for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support, we have in the sense of distributions

$$\frac{\partial S(b(u))}{\partial t} - div\Big(S'(b(u))\big(a(x,t,u,\nabla u) - \phi(x,t,u)\big)\Big) + S''(b(u))\big(a(x,t,u,\nabla u) -\phi(x,t,u)\big)\nabla b(u) + H(x,t,\nabla u)S'(b(u)) = f S'(b(u)) + div(S'(b(u))F) -S''(b(u))F\nabla b(u), \tag{17}$$

and

$$S(b(u))(t=0) = S(b(u_0)) \quad \text{in} \quad \Omega. \tag{18}$$

Remark 2.1 Note that conditions (4), (14), and (15) allow to define ∇u and $\nabla b(u)$ almost everywhere in Q_T.

Remark 2.2 Note that for a renormalized solution, due to (15), each term in (17) has a meaning in $L^1(Q_T) + L^{p'}((0,T); W^{-1,p'}(\Omega))$. Indeed, since $|T_k(b(u))| \le k$, we can choose k such that $supp(S') \in [-k,k]$. Then the properties of S, the functions S' and S'' are bounded in $\mathbb{R}$. We have $S(b(u)) = S(T_k(b(u))) \in L^p((0,T); W_0^{1,p}(\Omega))$ and $\frac{\partial S(b(u))}{\partial t} \in D'(Q_T)$. The term $S'(b(u))a(x,t,u,\nabla u)$ identifies with $S'(T_k(b(u)))a(x,t,u,\nabla b^{-1}(T_k(b(v))))$ a.e. in Q_T, where $v = b(u)$ and $u = b^{-1}(T_k(b(v)))$ in $\{|b(u)| \le k\}$, by (4) and (5) we have

$$\Big|S'(T_K(b(u)))a(x,t,u,\nabla u)\Big| \le \|S'\|_{L^\infty(\mathbb{R})}\Big[h_k(x,t)+|u|^{p-1}+|\nabla b^{-1}(T_K(v))|^{p-1}\Big] \le \|S'\|_{L^\infty(\mathbb{R})}\Big[h_k(x,t) + |u|^{p-1} + \frac{1}{b_0^{p-1}}|\nabla T_K(v)|^{p-1}\Big] \text{a.e. in } Q_T, \tag{19}$$

Using (6, 17) it follows that $S'(b(u))a(x,t,u,\nabla u) \in (L^{p'}(Q_T))^N$. In view of (4, 6, 9, 10, 15, 19), we obtain:

$$\begin{cases} S''(u)a(x,t,u,\nabla u)\nabla b(u) & \text{in } L^1(Q_T) \\ S''(u)\phi(x,t,u)\nabla b(u) & \text{in } L^1(Q_T) \\ S'(T_k(b(u)))\phi(x,t,u) & \text{in } L^1(Q_T) \\ S'(b(u))H(x,t,\nabla u) & \text{in } L^1(Q_T) \\ S'(u)f & \text{in } L^1(Q_T), \end{cases} \tag{20}$$

Consequently, we have $\dfrac{\partial S(b(u))}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q_T)$ and $S(b(u)) \in L^p(0,T,W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$. Which implies that $S(b(u))$ belongs to $C([0,T];L^1(\Omega))$ so the initial condition (18) makes sense.

The existence theorem of renormalized solution of (3):

Theorem 2.1 *Under assumptions* (4)–(13) *there exists at least a renormalized solution u of problem* (3).

Proof of Theorem 2.1 The existence theorem of renormalized solution of (3) is proved in ([10]) in the linear case (b(u) = u) and by author in ([1,2]). □

Remark 2.3 To prove the uniqueness result for the problem (3), due to, the presence of a general monotone operator $A(u) = -div(a(x,t,u,\nabla u)$, and the non-linearity $b(u)$, a standard approach does not feasible. To overcome this difficulty we draw upon the idea included in ([6]), for which we recall some basic results that will be a key point.

Lemma 2.1 *(see* [7]*) Let v be a function in $W_0^{1,p}(\Omega) \cap L^2(\Omega)$ with $p \geq 1$. Then there exists a positive constant C, depending on N, p, such that*

$$\| v \|_{L^\gamma(\Omega)} \leq C \| \nabla v \|^{\theta}_{(L^p(\Omega))^N} \| v \|^{1-\theta}_{L^2(\Omega)}$$

for every θ and γ satisfying

$$0 \leq \theta \leq 1, \quad 1 \leq \gamma \leq +\infty, \quad \frac{1}{\gamma} = \theta\left(\frac{1}{p} - \frac{1}{N}\right) + \frac{1-\theta}{2}.$$

An immediate consequence of the previous result:

Corollary 2.1 *Let $v \in L^p((0,T);W^{1,p}(\Omega)) \cap L^\infty((0,T);L^2(\Omega))$, with $p \geq 1$. Then $v \in L^\sigma(\Omega)$ with $\sigma = p(\frac{N+2}{N})$ and*

$$\int_{Q_T} |v|^\sigma dxdt \leq C \| v \|^{\frac{2p}{N}}_{L^\infty(0,T,L^2(\Omega))} \int_{Q_T} |\nabla v|^p dxdt.$$

Lemma 2.2 *Let ω be an open subset of $\mathbb{R}^N$, $N \geq 1$, $F \in L^p(\Omega)$, and $\overline{u} : \Omega \to [0,+\infty]$ and $\overline{v} : \Omega \to [0,+\infty]$ be two measurable functions. Then there exists a sequence n_j(related for simplicity as n) of real numbers such that*

$$\underline{\lim}_{n\to+\infty} \overline{\lim}_{\delta\to 0} \left[\frac{1}{\delta}\int_{n-\delta\leq|\overline{u}|\leq n+\delta} |F|^p\,dx\,dt + \frac{1}{\delta}\int_{n-\delta\leq|\overline{v}|\leq n+\delta} |F|^p\,dx\,dt\right] = 0$$

Proof see [6], Lemma 6. □

Lemma 2.3 *Under the assumptions* (4)–(13)*, any renormalized solution u of* (3) *satisfies the following estimate for any $n \geq 1$ and any $0 < \delta < 1$*

$$\frac{1}{\delta}\int_{\{n-\delta\leq|b(u)|\leq n+\delta\}} a(x,t,u,\nabla u)\nabla b(u)\,dx\,dt \leq \epsilon(n,\delta),$$

with $\lim_{n\to+\infty}\lim_{\delta\to 0}\epsilon(n,\delta)=0.$

Proof Using the same proof (Lemma 5, p. 356, [6]), adding the analysis of the two lower order terms ϕ and H, and taking into account the non-linearity $b(u)$. □

Let $S_n \in W^{1,\infty}(\mathbb{R})$be the function defined by

$$\begin{cases} S_n(0)=0 & \\ S_n'(r)=1 & \text{for } |r|\leq n \\ S_n'(r)=n+1-|r| & \text{for } n\leq |r|\leq n+1 \\ S_n'(r)=0 & \text{for} |r|\geq n+1. \end{cases} \tag{21}$$

Since $supp S_n' \subset [-n-2, n+2]$, by setting $S=S_{n+1}$, $\forall n>0$ in (17), we have in the sense of distributions

$$\begin{aligned} &\frac{\partial S_{n+1}(b(u))}{\partial t} - div\Big(S_{n+1}'(b(u))\big(a(x,t,u,\nabla u)-\phi(x,t,u)\big)\Big) \\ &\quad + S_{n+1}''(b(u))\big(a(x,t,u,\nabla u)-\phi(x,t,u)\big)\nabla b(u) + H(x,t,\nabla u)S_{n+1}'(b(u)) \\ &= f S_{n+1}'(b(u)) + div(S_{n+1}'(b(u))F) - S_{n+1}''(b(u))F\nabla b(u), \end{aligned} \tag{22}$$

For a real numbers $n>0$ and $0<\delta<1$, using the admissible test function

$$R_n^\delta(r) = \frac{1}{\delta}(T_{n+\delta}(r) - T_{n-\delta}(r)) \tag{23}$$

in (22), we get

$$\begin{aligned} &\int_0^T \Big\langle \frac{\partial S_{n+1}(b(u))}{\partial t}, R_n^\delta(b(u))\Big\rangle dt + \int_{Q_T} a(x,t,u,\nabla u)S_{n+1}'(b(u))\nabla R_n^\delta(b(u))\,dx\,dt \\ &\quad + \int_{Q_T} S_{n+1}''(b(u))a(x,t,u,\nabla u)\nabla b(u)R_n^\delta(b(u))\,dx\,dt \\ &\quad - \int_{Q_T} \phi(x,t,u)S_{n+1}'(b(u))\nabla R_n^\delta(b(u))\,dx\,dt \\ &\quad - \int_{Q_T} S_{n+1}''(b(u))\phi(x,t,u)\nabla b(u)R_n^\delta(b(u))\,dx\,dt \\ &\quad + \int_{Q_T} H(x,t,\nabla u)S_{n+1}'(b(u))R_n^\delta(b(u))\,dx\,dt \\ &= \int_{Q_T} f_n S_{n+1}'(b(u))R_n^\delta(b(u))\,dx\,dt \end{aligned}$$

$$+\int_{Q_T} S'_{n+1}(b(u))F\nabla R_n^\delta(b(u))\,dx\,dt$$
$$+\int_{Q_T} S''_{n+1}(b(u))F\nabla b(u)R_n^\delta(b(u))\,dx\,dt. \tag{24}$$

Remark that $R_n^\delta(b(u)) = R_n^\delta(S_{n+1}(b(u)))$ as soon as $0 < \delta < 1$, and then we have

$$\int_0^T \Big\langle\frac{\partial S_{n+1}(b(u))}{\partial t}, R_n^\delta(S_{n+1}(b(u)))\Big\rangle\,dt = \int_0^T \Big\langle\frac{\partial S_{n+1}(b(u))}{\partial t}, R_n^\delta(b(u))\Big\rangle\,dt$$
$$= \int_\Omega\int_0^T \frac{\partial \widetilde{R}_n^\delta(S_{n+1}(b(u)))}{\partial t}\,dt\,dx$$
$$= \int_\Omega \widetilde{R}_n^\delta(S_{n+1}(b(u)))(T)\,dx - \int_\Omega \widetilde{R}_n^\delta(S_{n+1}(b(u)))(t=0)\,dx$$
$$\geq -\int_\Omega \widetilde{R}_n^\delta(S_{n+1}(b(u)))(t=0)\,dx$$
$$\geq -\int_\Omega |b(u_0)|\chi_{|b(u_0)|\geq n-1}\,dx \tag{25}$$

where $\widetilde{R}_n^\delta(s) = \displaystyle\int_0^{S_{n+1}(b(u_0))} R_n^\delta(s)\,ds$ for any $n > 1$ and any $0 < \delta < 1$.
The definitions (21) and (23) permit to obtain from (24) and (25) that for any $n > 1$ and any $0 < \delta < 1$,

$$\frac{1}{\delta}\int_{\{n-\delta\leq|b(u)|\leq n+\delta\}} a(x,t,u,\nabla u)\nabla b(u)\,dx\,dt$$
$$\leq 2\int_{\{n+1\leq|b(u)|\leq n+2\}} a(x,t,u,\nabla u)\nabla b(u)\,dx\,dt$$
$$+\frac{1}{\delta}\int_{\{n-\delta\leq|b(u)|\leq n+\delta\}} c(x,t)|b(u)|^\gamma\,\nabla b(u)\,dx\,dt$$
$$+2\int_{\{n+1\leq|b(u)|\leq n+2\}} c(x,t)|b(u)|^\gamma\,\nabla b(u)\,dx\,dt$$
$$+2\int_{\{|b(u)|>n-1\}} d(x,t)|\nabla u|^\delta\,dx\,dt + 2\int_{\{|b(u)|>n-1\}} |f|\,dx\,dt$$
$$+2\int_{\{n+1\leq|b(u)|\leq n+2\}} |F||\nabla b(u)|\,dx\,dt$$
$$+\frac{1}{\delta}\int_{\{n-\delta\leq|b(u)|\leq n+\delta\}} |F||\nabla b(u)|\,dx\,dt + \int_\Omega |b(u_0)|\chi_{|b(u_0)|\geq n-1}\,dx. \tag{26}$$

⋆ **Estimates for the first lower order:** Note that the terms involving $\phi(x,t,u)$ in (22) not equal to 0 for any $n > 0$, and any $\delta > 0$ (as in equation 24 in [6]). By (8,9), (21, 24) and using Hölder inequality, Gagliardo–Nirenberg (see Corollary 2.1) together with Young inequality yields to

$$
\begin{aligned}
&\int_{Q_T} \phi(x,t,u) S'_{n+1}(b(u)) \nabla R_n^{\delta}(b(u))\, dx\, dt \\
&\quad \le \frac{1}{\delta} \int_{\{n-\delta \le |b(u)| \le n+\delta\}} c(x,t) |b(u)|^{\gamma} \nabla b(u)\, dx\, dtn \\
&\quad \le \frac{1}{\delta} \Big(\int_{\{n-\delta \le |b(u)| \le n+\delta\}} (c(x,t)|b(u)|^{\gamma})^{p'}\, dx\, dt \Big)^{\frac{1}{p'}} \\
&\qquad \times \Big(\int_{\{n-\delta \le |b(u)| \le n+\delta\}} (|\nabla b(u)|)^{p}\, dx\, dt \Big)^{\frac{1}{p}} \\
&\quad \le \frac{1}{\delta} \Big(\int_{\{n-\delta \le |b(u)| \le n+\delta\}} c^{r}(x,t)\, dx\, dt \Big)^{\frac{1}{r}} \\
&\qquad \times \Big(\int_{\{n-\delta \le |b(u)| \le n+\delta\}} |b(u)|^{\frac{(N+2))}{N}})\, dx\, dt \Big)^{\frac{N+1}{N+p}} \\
&\qquad \times \Big(\int_{\{n-\delta \le |b(u)| \le n+\delta\}} (|\nabla b(u)|)^{p}\, dx\, dt \Big)^{\frac{1}{p}} \\
&\quad \le \frac{1}{\delta} C ||c||_{L^r(Q_T \cap \{n-\delta \le |b(u)| \le n+\delta\})} \frac{N+1}{N+p} \int_{\{n-\delta \le |b(u)| \le n+\delta\}} |b(u)|^{p}\, dx\, dt \\
&\qquad + \frac{1}{\delta} C ||c||_{L^r(Q_T \cap \{n-\delta \le |b(u)| \le n+\delta\})}
\end{aligned} \tag{27}
$$

The coercive character (6) of a and choosing the norm of $c(x,t)$ small enough, we get

$$
\begin{aligned}
&\frac{1}{\delta} \int_{\{n-\delta \le |b(u)| \le n+\delta\}} c(x,t) |b(u)|^{\gamma} |\nabla b(u)|\, dx\, dt \\
&\quad \le \frac{1}{4\delta} \int_{\{n-\delta \le |b(u)| \le n+\delta\}} a(x,t,u,\nabla u) \nabla b(u)\, dx\, dt + \epsilon(\delta).
\end{aligned} \tag{28}
$$

in the same way

$$
\begin{aligned}
&\int_{\{n+1 \le |b(u)| \le n+2\}} c(x,t) |b(u)|^{\gamma} |\nabla b(u)|\, dx\, dt \\
&\quad \le C_1 \int_{\{n+1 \le |b(u)| \le n+2\}} a(x,t,u,\nabla u) \nabla b(u)\, dx\, dt.
\end{aligned} \tag{29}
$$

⋆ **Estimates for the second lower order:** By Hölder inequality (in Lorentz space), we have

$$
\begin{aligned}
&\int_{\{|b(u)| > n-1\}} d(x,t) |\nabla u|^{\delta}\, dx\, dt \\
&\quad \le ||d||_{L^{N+2,1}(Q \cap \{|b(u)| > n-1\})} |||\nabla u|^{\delta}||_{L^{\frac{N+2}{N+1},\infty}(Q \cap \{|b(u)| > n-1\})}
\end{aligned} \tag{30}
$$

and by using (Lemma A.1, see ([10]) in Appendix) we have

$$|||\nabla u|^{\frac{N(p-1)+p}{N+2}}||_{L^{\frac{N+2}{N+1},\infty}} = sup_{k>0}k\Big(maes\{(x,t) : |\nabla u|^{\frac{N(p-1)+p}{N+2}} > k\}\Big)^{\frac{N+2}{N+1}} \leq C_4$$

Finally by using Young inequality and the coercivity of a for the sixth term of the right hand (26), we obtain from (25) to (30) that for any $n > 1$ and any $0 < \delta < 1$

$$\begin{aligned}
&\frac{1}{\delta}\int_{\{n-\delta\leq|b(u)|\leq n+\delta\}} a(x,t,u,\nabla u)\nabla b(u)\,dx\,dt \leq \frac{C_3}{\delta}\int_{\{n-\delta\leq|b(u)|\leq n+\delta\}} |F|^{p'}\,dx\,dt\\
&\quad +C_5\int_{\{n+1\leq|b(u)|\leq n+2\}} a(x,t,u,\nabla u)\nabla b(u)\,dx\,dt\\
&\quad +C_4||d||_{L^{N+2,1}(Q\cap\{|b(u)|>n-1\})} + C_6||F||_{L^{p'}(Q_T)}\Big(\int_{\{n+1\leq|b(u)|\leq n+2\}} |\nabla u|^p\Big)^{\frac{1}{p}}\\
&\quad +2\int_{\{|b(u)|>n-1\}} |f|\,dx\,dt + \int_{\Omega} |b(u_0)|\chi_{|b(u_0)|\geq n-1}\,dx.
\end{aligned}$$

Since $f \in L^1(Q_T)$, $a(x,t,u,\nabla u)\nabla b(u) \in L^1(Q_T)$ and conditions (14), (16) we have

$$\lim_{n\to+\infty}\lim_{\delta\to+\infty}\frac{1}{\delta}\int_{\{n-\delta\leq|b(u)|\leq n+\delta\}} a(x,t,u,\nabla u)\nabla b(u)\,dx\,dt = 0$$

so that Lemma 2.3 is established.

3 Uniqueness of renormalized solution

In this section, we assume a local control of Lipschitz coefficients to prove the following uniqueness theorem

Theorem 3.1 *Assume that assumptions (4)–(13) hold true and moreover that for any compact set D of $\mathbb{R}$, there exists $L_D \in L^{p'}(Q_T)$ and $\rho_D > 0$ such that $\forall s, \overline{s} \in D$*

$$|a(x,t,s,\xi) - a(x,t,\overline{s},\xi)| \leq \Big(L_D(x,t) + \rho_D|\xi|^{p-1}\Big)|s-\overline{s}| \tag{31}$$

$$|\phi(x,t,s) - \phi(x,t,\overline{s})| \leq L_D(x,t)|s-\overline{s}| \tag{32}$$

$$|b'(s) - b'(\overline{s})| \leq \beta_D|s-\overline{s}| \tag{33}$$

for almost every $(x,t) \in Q_T$ and for every $\xi \in \mathbb{R}^N$. Then the problem (3) has a unique renormalized solution.

For the sake of shortness, we denote by $\{|u| \leq k\}$ (resp. $\{|u| < k\}$) the measurable subset $\{(x,t) \in Q_T; |u(x,t)| \leq k\}$ (resp. $\{(x,t) \in Q_T; |u(x,t)| < k\}$.) Moreover the explicit dependence in x and t of the functions a, ϕ and H will be omitted so that $a(x,t,u,\nabla u) = a(u,\nabla u)$, $\phi(x,t,u) = \phi(u)$.

Proof of Theorem 3.1 Let u and v be two renormalized solutions of (3) for the same data f and F and initial condition $b(u_0)$. We define a smooth approximation of T_n by T_n^σ and

$$(T_n^\sigma)'(r) = \begin{cases} 1 & \text{for } |r| \le n \\ \frac{n+\sigma-|r|}{\sigma} & \text{for } n \le |r| \le n+\sigma \\ 0 & \text{for} |r| \ge n+\sigma. \end{cases} \quad (34)$$

□

Using $\frac{1}{k}T_k(T_n^\sigma(b(u)) - T_n^\sigma(b(v)))$ as test function in the difference of Eq. (17) for u and v in which we take $S = T_n^\sigma$, we obtain

$$\frac{1}{k}\int_0^T \Big\langle \frac{\partial (T_n^\sigma(b(u)) - T_n^\sigma(b(v)))}{\partial t}; T_k(T_n^\sigma(u_1) - T_n^\sigma(v_1))\Big\rangle dt \\ + I_{1,n}^\sigma + I_{2,n}^\sigma = I_{3,n}^\sigma + I_{4,n}^\sigma + I_{5,n}^\sigma + I_{6,n}^\sigma + I_{7,n}^\sigma, \quad (35)$$

where

$$I_{1,n}^\sigma = \frac{1}{k}\int_{Q_T} \Big[(T_n^\sigma)'(b(u))a(u,\nabla u) - (T_n^\sigma)'(b(v))a(v,\nabla v)\Big]\nabla T_k(T_n^\sigma(b(u)) \\ - T_n^\sigma(b(v)))\,dx\,dt,$$

$$I_{2,n}^\sigma = \frac{1}{k}\int_{Q_T} \Big[(T_n^\sigma)''(b(u))a(u,\nabla u)\nabla b(u) - (T_n^\sigma)''(b(v))a(v,\nabla v)\nabla b(v)\Big]T_k(T_n^\sigma(b(u)) \\ - T_n^\sigma(b(v)))\,dx\,dt,$$

$$I_{3,n}^\sigma = \frac{1}{k}\int_{Q_T} \Big[(T_n^\sigma)'(b(u))\phi(u) - (T_n^\sigma)'(b(v))\phi(v)\Big]\nabla T_k(T_n^\sigma(b(u)) - T_n^\sigma(b(v)))\,dx\,dt,$$

$$I_{4,n}^\sigma = \frac{1}{k}\int_{Q_T} \Big[(T_n^\sigma)''(b(u))\phi(u)\nabla b(u) - (T_n^\sigma)''(b(v))\phi(v)\nabla b(v)\Big]T_k(T_n^\sigma(b(u)) \\ - T_n^\sigma(b(v)))\,dx\,dt,$$

$$I_{5,n}^\sigma = \frac{1}{k}\int_{Q_T} f\Big[(T_n^\sigma)'(b(u)) - (T_n^\sigma)'(b(v))\Big]T_k(T_n^\sigma(b(u)) - T_n^\sigma(b(v)))\,dx\,dt,$$

$$I_{6,n}^\sigma = \frac{1}{k}\int_{Q_T} \Big[(T_n^\sigma)'(b(u)) - (T_n^\sigma)'(b(v))\Big]F\nabla T_k(T_n^\sigma(b(u)) - T_n^\sigma(b(v)))\,dx\,dt,$$

$$I_{7,n}^\sigma = \frac{1}{k}\int_{Q_T} \Big[T_k(T_n^\sigma(b(u)) - T_n^\sigma(b(v)))F\nabla\big((T_n^\sigma)'(b(u)) - (T_n^\sigma)'(b(v))\big)\,dx\,dt.$$

For any $k > 0, n > 0, \sigma > 0$. Now we will pass to the limit of each term of (35) when σ and k tends to 0, and n tend to $+\infty$.

- For the first term in the right-hand sid of (35), upon of Lemma 2.4 ([3]), and due to

  $T_n^\sigma(b(u))(t=0) = T_n^\sigma(b(v))(t=0) = T_n^\sigma(b(u_0))$ a.e. in Ω, we have

$$\lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}\int_0^T \Big\langle \frac{\partial (T_n^\sigma(b(u)) - T_n^\sigma(b(v)))}{\partial t}; T_k(T_n^\sigma(b(u)) - T_n^\sigma(b(v)))\Big\rangle dt$$

$$= \lim_{k\to 0}\lim_{\sigma\to 0}\frac{1}{k}\int_{Q_T}\overline{T_k}(T_n^\sigma(b(u)) - T_n^\sigma(b(v)))\,dt\,dx$$
$$= \int_{Q_T}|T_n(b(u)) - T_n(b(v))|\,dt\,dx$$

where again $\overline{T_k}(z) = \int_0^z T_k(r)dr$. We deduce from the above equality that for almost any $t \in (0,T)$

$$\lim_{n\to+\infty}\lim_{k\to 0}\lim_{\sigma\to 0}\int_0^T \Big\langle \frac{\partial(T_n^\sigma(b(u)) - T_n^\sigma(b(v)))}{\partial t}; T_k(T_n^\sigma(b(u)) - T_n^\sigma(b(v)))\Big\rangle dt$$
$$= \int_{Q_T}|b(u) - b(v)|dtdx \tag{36}$$

- For a fixed $n > 0$, we studied the behavior of $I_{1,n}^\sigma$ when σ and k tends to 0:

We have, when $\sigma \to 0$, $T_n^\sigma)'(r) \to \chi_{|r|\le n}$ a.e. in Q_T and strongly in $L^q(Q_T)$ for any $q < +\infty$ and $T_n^\sigma)(r) \to T_n(r)$ a.e. in Q_T and strongly in $L^p(Q_T)$. Since supp $(T_n^\sigma)' \subset [-n-\sigma, n+\sigma]$ we have

$$\lim_{\sigma\to 0} I_{1,n}^\sigma = \frac{1}{k}\int_{Q_T}\Big[\chi_{\{|b(u)|\le n\}}a(u,\nabla u) - \chi_{\{|b(v)|\le n\}}a(v,\nabla v)\Big]\nabla T_k(T_n(b(u))$$
$$-T_n(b(v)))dxdt = I_{1,n}$$

Rewritten $I_{1,n}$ as $I_{1,n} = I_{11} + I_{12} + I_{13} + I_{14} + I_{15}$, where

$$I_{11} = \frac{1}{k}\int_{\{|b(u)-b(v)|\le k, |b(u)|\le n, |b(v)|\le n\}}\Big(a(u,\nabla u) - a(u,\nabla v)\Big)(\nabla u - \nabla v)b'(u)dxdt$$
$$I_{12} = \frac{1}{k}\int_{\{|b(u)-b(v)|\le k, |b(u)|\le n, |b(v)|\le n\}}\Big(a(u,\nabla v) - a(v,\nabla v)\Big)(\nabla u - \nabla v)b'(u)dxdt$$
$$I_{13} = \frac{1}{k}\int_{\{|b(u)-b(v)|\le k, |b(u)|\le n, |b(v)|\le n\}}\Big(a(u,\nabla u) - a(v,\nabla v)\Big)\nabla v(b'(u) - b'(v))dxdt$$
$$I_{14} = \frac{1}{k}\int_{\{|T_n(b(u))-T_n(b(v))|\le k, |b(u)|>n, |b(v)|\le n\}}a(v,\nabla v)\nabla b(v)dxdt$$
$$I_{15} = \frac{1}{k}\int_{\{|T_n(b(u))-T_n(b(v))|\le k, |b(u)|\le n, |b(v)|>n\}}a(u,\nabla u)\nabla b(u)dxdt$$

We use the monotonicity of $a(s,\xi)$ with respect to ξ and $b'(s) > 0$ for all $s \in \mathbb{R}$, we obtain

$$I_{11} \ge 0. \tag{37}$$

It remains to prove that I_{12} goes to zero as k goes to zero. Indeed using the local Lipschitz condition (31) and (35) on a we get

$$|I_{12}| \leq \frac{b_1}{k}\int_{Q_T} \nu(x)\chi_{\{|b(u)-b(v)|\leq k\}}|u-v|\Big(L_D(x,t)+\rho_D|\nabla v|^{p-1}\Big)(|\nabla u|+|\nabla v|)dxdt$$
$$\leq \frac{b_1}{b_0}\int_{\{|b(u)-b(v)|\leq k\}}\Big(L_D(x,t)+\rho_D|\nabla v|^{p-1}\Big)(|\nabla u|+|\nabla v|)dxdt$$

Due to regularity of u, v, and L_D we have

$$\Big(L_D(x,t)+\rho_D|\nabla v|^{p-1}\Big)(|\nabla u|+|\nabla v|)\in L^1(Q_T).$$

Since $\chi_{\{|b(u)-b(v)|\leq k\}}$ tends to zero almost everywhere in Q_T as k goes to zero, the Lebesgue dominated convergence allows us to conclude that, for all $n\geq 1$:

$$\limsup_{k\to 0} I_{12}=0. \tag{38}$$

We denote by C_n the compact subset $[b^{-1}(-n-1), b^{-1}(n+1)]$, and remark that due to the local Lipschitz character of b',there exists a positive real number β_n such that

$$|b'(r_1)-b'(r_2)|\leq \beta_n|r_1-r_2|,$$

for all r_1 and r_2 lying in C_n. Using now (4) again leads to

$$|b'(r_1)-b'(r_2)|\leq \frac{\beta_n}{b_0}|b(r_1)-b(r_2)| \tag{39}$$

for all r_1 and r_2 lying in C_n, then $|b'(u)-b'(v)|\leq \frac{k\beta_n}{b_0}$ on $\{|b(u)-b(v)|\leq k, |b(u)|\leq n, |b(v)|\leq n\}$, and in view (4) we obtain

$$|I_{13}|\leq \frac{\beta_n}{b_0}\int_{Q_T}|a(u,\nabla u)-a(v,\nabla v)||\nabla v|\chi_{\{|b(u)-b(v)|\leq k, b(u)\neq b(v), |b(u)|\leq n, |b(v)|\leq n\}}dxdt$$

Due to regularity of u, v, ∇u and ∇v we have

$$|a(T_n(u),\nabla T_n(u))-a(T_n(v),\nabla T_n(v))||\nabla T_n(v)|\in L^1(Q_T).$$

Since $\chi_{\{|b(u)-b(v)|\leq k, b(u)\neq b(v), |b(u)|\leq n, |b(v)|\leq n\}}$ tends to zero almost everywhere in Q_T as k goes to zero, the Lebesgue dominated convergence allows us to conclude that, for all $n\geq 1$:

$$\limsup_{k\to 0} I_{13}=0. \tag{40}$$

In view of the definition of T_n, we have

$$I_{14}=\frac{1}{k}\int_{\{|T_n(b(u))-T_n(b(v))|\leq k, |b(u)|>n, |b(v)|\leq n\}} a(v,\nabla v)\nabla b(v)dxdt$$

It is possible to obtain

$$|I_{14}| = \frac{1}{k}\int_{\{n-k<|b(v)|\leq k\}} a(v,\nabla v)\nabla b(v)dxdt$$

Using (35), (6) we conclude that

$$\liminf_{n\to+\infty}\limsup_{k\to 0} I_{14} \geq 0 \tag{41}$$

Similarly to the argument of limit I_{14}, we conclude

$$\liminf_{n\to+\infty}\limsup_{k\to 0} I_{15} \geq 0 \tag{42}$$

We obtain from (37) to (42) that

$$\liminf_{n\to+\infty}\limsup_{k\to 0} I_1 \geq 0$$

then

$$\liminf_{n\to+\infty}\limsup_{k\to 0}\lim_{\sigma\to 0} I_1^\sigma \geq 0 \tag{43}$$

The limit of $I_{2,n}^\sigma$: In view of the definition of T_n^σ it is possible to obtain

$$\begin{aligned}|I_{2,n}^\sigma| &\leq \int_{Q_T}\frac{1}{\sigma}\Big[\chi_{\{n\leq|b(u)|\leq n+\sigma\}}a(u,\nabla u)\nabla b(u)+\chi_{\{n\leq|b(v)|\leq n+\sigma\}}a(v,\nabla v)\nabla b(v)\Big]dxdt\\ &\leq \frac{1}{\sigma}\int_{\{n\leq|b(u)|\leq n+\sigma\}} a(u,\nabla u)\nabla b(u)dxdt+\frac{1}{\sigma}\int_{\{n\leq|b(v)|\leq n+\sigma\}} a(v,\nabla v)\nabla b(v)dxdt\end{aligned} \tag{44}$$

Using (44) and the estimates of Lemma 2.3, then we obtain

$$\liminf_{n\to+\infty}\limsup_{\sigma\to 0} I_{3,n}^\sigma = 0 \tag{45}$$

The limit of $I_{3,n}^\sigma$: We first write that for almost any $t\in(0,T)$

$$\begin{aligned}\limsup_{\sigma\to 0}|I_{3,n}^\sigma| &= \Big|\frac{1}{k}\int_{Q_T}[\chi_{\{|b(u)|\leq n\}}\phi(u)-\chi_{\{|b(v)|\leq n\}}\phi(v)]\nabla T_k(T_n(b(u))\\ &\quad -T_n(b(v)))\,dx\,dt\Big|\\ &\leq I_{31}+I_{32}+I_{33},\end{aligned}$$

where

$$I_{31} = \frac{1}{k}\int_{Q_T}\chi_{\{|b(u)|\leq n \text{ and } |b(v)|>n\}}|\phi(u)||\nabla T_k(b(u)-nsign(b(v)))|dxdt$$

$$I_{32} = \frac{1}{k}\int_{Q_T}\chi_{\{|b(v)|\leq n \text{ and } |b(u)|>n\}}|\phi(v)||\nabla T_k(b(v)-nsign(b(u)))|dxdt$$

and

$$I_{33} = \frac{1}{k}\int_{Q_T} \chi_{\{|b(u)|\le n \text{ and } |b(v)|\le n\}}|\phi(u)-\phi(v)||\nabla T_k(b(u)-b(v))|dxdt$$

We estimate I_{31} and I_{32} by (8) we obtain

$$\begin{aligned} I_{31} &\le \frac{1}{k}\int_{Q_T} \chi_{\{|b(u)|\le n \text{ and } |b(v)|>n\}}\chi_{\{|b(u)-nsign(b(v))|\le k\}}|\phi(u)||\nabla b(u)|dxdt \\ &\le \frac{1}{k}\int_{\{n-k\le|b(u)|\le n\}} |\phi(u)||\nabla b(u)|dxdt \end{aligned} \tag{46}$$

and similarly

$$I_{32} \le \frac{1}{k}\int_{\{n-k\le|b(v)|\le n\}} |\phi(v)||\nabla b(v)|dxdt. \tag{47}$$

Applying Lemma 2.2 in (46) and (47), we obtain:

$$\lim_{k\to 0} I_{31} = \lim_{k\to 0} I_{32} = 0, \quad \text{for any} \quad n > 0. \tag{48}$$

Finally, since the function ϕ is locally Lipschitz continuous, we have for some positive L_D element of $L^{p'}(Q_T)$

$$\begin{aligned} I_{33} &= \frac{1}{k}\int_{Q_T} \chi_{\{|b(u)|\le n \text{ and } |b(v)|\le n\}}|\phi(u)-\phi(v)||\nabla T_k(T_n(b(u))-T_n(b(v)))|dxdt \\ &\le \frac{1}{k}\int_{Q_T} \chi_{\{|T_n(b(u))-T_n(b(v))|\le k\}}L_D(x,t)\nu(x)|u-v||\nabla T_k(T_n(b(u)) \\ &\quad -T_n(b(v)))|dxdt \end{aligned}$$

by (4) we obtain

$$\begin{aligned} I_{33} &\le \frac{1}{k}\int_{Q_T} \chi_{\{|T_n(b(u))-T_n(b(v))|\le k\}}\frac{1}{b_0}L_D(x,t)|T_n(b(u))-T_n(b(v))||\nabla T_k(T_n(b(u)) \\ &\quad -T_n(b(v)))|dxdt \\ &\le \frac{1}{b_0}\int_{\{|T_n(b(u))-T_n(b(v))|\le k\}} \chi_{\{|T_n(b(u))-T_n(b(v))|\le k\}}L_D(x,t)(|\nabla T_n(b(u))| \\ &\quad +|\nabla T_n(b(v))|)dxdt. \end{aligned}$$

Since L_C belongs to $L^{p'}(Q_T)$ and due to (15) the function $L_D(x,t)(|\nabla T_n(b(u))| + |\nabla T_n(b(v))|)$ belongs to $L^1(Q_T)$. Using $\chi_{\{|T_n(b(u))-T_n(b(v))|\le k\}}$ tends to 0 almost everywhere in Q_T as k goes to 0 and is bounded by 1, the Lebesgue dominated convergence theorem leads to

$$\lim_{k\to 0} I_{33} = 0, \quad \text{for any} \quad n > 0. \tag{49}$$

Using (48) and (49) we obtain:

$$\liminf_{n\to+\infty} \limsup_{k\to 0} I_{3,n}^{\sigma} = 0 \tag{50}$$

The limit of $I_{4,n}^{\sigma}$: We have for any σ and $k > 0$

$$|I_{4,n}^{\sigma}| \leq \frac{1}{\sigma}\left[\int_{\{n<|b(u)|<n+\sigma\}} |\phi(u)||\nabla b(u)|dxdt + \int_{\{n<|b(v)|<n+\sigma\}} |\phi(v)||\nabla b(v)|dxdt\right] dxdt$$

Using the Lemma 2.2, we get

$$\liminf_{n\to+\infty} \limsup_{\sigma\to 0} I_{4,n}^{\sigma} = 0 \tag{51}$$

The limit of $I_{5,n}^{\sigma}$: Using Lebesgue's theorem, the definition of T_n^{σ}, it is possible to conclude that

$$\begin{aligned}\lim_{n\to+\infty} \lim_{\sigma\to 0} |I_{5,n}^{\sigma}| &\leq \lim_{n\to+\infty} \frac{1}{k}\int_{Q_T} |f||\chi_{\{|b(u)|\leq n\}} - \chi_{\{|b(v)|\leq n\}}||T_k(T_n(b(u)) \\ &\quad -T_n(b(v)))|\, dx\, dt \\ &\leq \lim_{n\to+\infty}\left(\int_{\{|b(u)|\geq n\}} |f|\, dx\, dt + \int_{\{|b(v)|\geq n\}} |f|\, dx\, dt\right) = 0.\end{aligned}$$

Then

$$\liminf_{n\to+\infty} \lim_{\sigma\to 0} J_{5,n}^{\sigma} = 0 \tag{52}$$

The limit of $I_{6,n}^{\sigma}$: We have for almost every $t \in (0, T)$

$$\begin{aligned}\limsup_{\sigma\to 0} |I_{6,n}^{\sigma}| &= \left|\frac{1}{k}\int_{Q_T} [\chi_{\{|b(u)|\leq n\}} - \chi_{\{|b(v)|\leq n\}}]F\nabla T_k(T_n(b(u)) - T_n(b(v)))\, dx\, dt\right| \\ &\leq I_{61} + I_{62},\end{aligned}$$

where

$$\begin{aligned}I_{61} &= \frac{1}{k}\int_{Q_T} \chi_{\{|b(u)|\leq n \text{ and } |b(v)|>n\}}|F||\nabla T_k(b(u) - nsign(b(v)))|dxdt \\ I_{62} &= \frac{1}{k}\int_{Q_T} \chi_{\{|b(v)|\leq n \text{ and } |b(u)|>n\}}|F||\nabla T_k(b(v) - nsign(b(u)))|dxdt \\ I_{61} &\leq \frac{1}{k}\int_{Q_T} \chi_{\{|b(u)|\leq n \text{ and } |b(v)|>n\}}\chi_{\{|b(u)-nsign(b(v))|\leq k\}}|F||\nabla b(u)|dxdt \\ &\leq \frac{1}{k}\int_{\{n-k\leq|b(u)|\leq n\}} |F||\nabla b(u)|dxdt\end{aligned} \tag{53}$$

and similarly

$$I_{62} \leq \frac{1}{k} \int_{\{n-k \leq |b(v)| \leq n\}} |F||\nabla b(v)|dxdt. \tag{54}$$

Applying Lemma 2.2 in (53) and (54), we obtain:

$$\lim_{k \to 0} I_{61} = \lim_{k \to 0} I_{62} = 0, \quad \text{for any} \quad n > 0. \tag{55}$$

The limit of $I^{\sigma}_{7,n}$: We have for any σ and $k > 0$

$$|I^{\sigma}_{7,n}| \leq \frac{1}{\sigma} \Big[\int_{\{n<|b(u)|<n+\sigma\}} |F||\nabla b(u)|dxdt + \int_{\{n<|b(v)|<n+\sigma\}} |F||\nabla b(v)|dxdt \Big] dxdt$$

Using the Lemma 2.2, we get

$$\liminf_{n \to +\infty} \limsup_{\sigma \to 0} I^{\sigma}_{7,n} = 0 \tag{56}$$

The proof of Theorem 3.1 is complete.

References

1. Aberqi, A., Bennouna, J., Mekkour, M., Redwane, H.: Renormalized solution for a nonlinear parabolic equation with lower order terms. Aust. J. Math. Anal. Appl. **10**(1,15), 1–15 (2013)
2. Aberqi, A., Bennouna, J., Redwane, H.: A nonlinear parabolic problems with lower order terms and measure data. Thai J. Math. **14**(1), 115–130 (2016)
3. Boccardo, L., Murat, F., Puel, J.-P.: Existence of bounded solutions for nonlinear elliptic unilateral problems. Ann. Mat. Pura Appl. **152**, 183–196 (1988)
4. Boccardo, L., Giachetti, D., Diaz, J.-I., Murat., F.: Existence and regularity of renormalized solutions of some elliptic problems involving derivatives of nonlinear terms. J. Differ. Equ. **106**, 215–237 (1993)
5. Bénilan, Ph, Boccardo, L., Gallouët, Th, Gariepy, R., Pierre, M., Vázquez, J.L.: An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Sc. Norm. Super. Pisa Cl. Sci. **22**(4), 241–273 (1995)
6. Blanchard, D., Murat, F., Redwane, H.: Existence and uniqueness of a renormalized solution for fairly general class of non linear parabolic problems. J. Difer. Equ. **177**, 331–374 (2001)
7. Brézis, H., Browder, F.E.: Strongly nonlinear parabolic initial-boundary value problems. Proc. Natl. Acad. Sci. USA **76**, 38–40 (1976)
8. Dal Maso, G., Murat, F., Orsina, L., Prignet, A.: Renormalized solutions of elliptic equations with general measure data. Ann. Sc. Norm. Super. Pisa Cl. Sci. **28**(4), 741–808 (1999)
9. Del Vecchio, T., Posteraro, M.R.: An existence result for elliptic equations with measure data. Adv. Differ. Equ. **1**, 899–917 (1996)
10. Di Nardo, R., Féo, F., Guibé, O.: Existence result for nonlinear parabolic equations with lower order terms. Anal. Appl. **9**(2), 161–186 (2011)
11. Di Nardo, R., Perrotta, A.: Uniquness results for nonlinear elliptic problems ith two lowers order terms. Bull. Sci. Math. **137**, 107–128 (2013)
12. Diperna, R.-J., Lions, P.-L.: On the cauchy problem for the Boltzmann equations: global existence and weak stability. Ann. Math. **130**, 285–366 (1989)
13. Landes, R.: On the existence of weak solutions for quasilinear parabolic initial-boundary problems. Proc. R. Soc. Edinb. Sect. **A89**, 321–366 (1981)
14. Serrin, J.: Pathological solution of elliptic differential equations. Ann. Sc. Norm. Super. Pisa Cl. Sci. **18**, 385–387 (1964)
15. Simon, J.: Compact set in the space $L^p(0, T, B)$. Ann. Mat. Pura Appl. **146**, 65–96 (1987)