Ricerche mat. (2017) 66:415–424
DOI 10.1007/s11587-016-0309-5

On an anisotropic problem with singular nonlinearity having variable exponent

Sofiane El-Hadi Miri[1]

Received: 9 May 2016 / Revised: 5 August 2016 / Published online: 13 October 2016

Abstract We consider the following anisotropic problem, with singular nonlinearity having a variable exponent

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i-2} \partial_i u \right] = \frac{f}{u^{\gamma(x)}} & in\ \Omega, \\ u = 0 & on\ \Omega, \\ u \geq 0 & in\ \Omega; \end{cases}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$ and $\gamma(x) > 0$ is a smooth function, having a convenient behavior near $\partial\Omega$. f is assumed to be a non negative function belonging to a suitable Lebesgue space $L^m(\Omega)$. We will also assume without loss of generality that $2 \leq p_1 \leq p_2 \leq \cdots \leq p_N$. Using approximation techniques, we obtain existence and regularity of positive solutions to the considered problem.

Keywords Anisotropic problem · Singular nonlinearity · Approximation

Mathematics Subject Classification 135J65 · 35B65 · 35J70

✉ Sofiane El-Hadi Miri
mirisofiane@yahoo.fr

[1] Laboratoire d'Analyse Non Linéaire et Mathématiques Appliquées, Université de Tlemcen, BP 119, 13000 Tlemcen, Algeria

1 Introduction

We consider in this paper, the following problem

$$\begin{cases} -Lu = \frac{f}{u^{\gamma(x)}} & in\ \Omega, \\ u = 0 & on\ \Omega, \\ u \geq 0 & in\ \Omega, \end{cases} \tag{1}$$

where

$$Lu = \sum_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i - 2} \partial_i u \right],$$

$\gamma(x) > 0$ is assumed to be a regular function, say for example $\gamma(x) \in C(\overline{\Omega})$, and Ω is a bounded regular domain in $\mathbb{R}^N$. We will assume without loss of generality that $2 \leq p_1 \leq p_2 \leq \cdots \leq p_N$ and that f is a non negative function belonging to a suitable Lebesgue space $L^m(\Omega)$.

When the differential operator is assumed to be semilinear, and $\gamma(x) = \gamma$, Boccardo and Orsina in their leading work [2], obtained existence and regularity of the solution, and this was generalized to the case of the p-laplacian in [7], and to the the case of the anisotropic operator L in [14].

In the very recent work [3] the authors consider a singular semilinear elliptic problem with variable exponent $\gamma(x)$, they obtained existence and regularity of the solution, under some conditions on the behavior of the function $\gamma(x)$ near the boundary of Ω.

There exists a huge literature, devoted to the study of the anisotropic operator L, as it has many applications in fluid dynamics, and physical phenomena with anisotropic diffusion, we cite for example [8–11], and the references therein.

When a singular nonlinearity is considered in interaction with different types of differential operators as the laplacian or the p-laplcian, we invite the reader to see the works [1,4–6,10,12,15,16,18].

Problem (1) is associated to the following anisotropic Sobolev spaces

$$W^{1,(p_i)}(\Omega) = \left\{ v \in W^{1,1}(\Omega)\, ;\, \partial_i v \in L^{p_i}(\Omega) \right\}$$

and

$$W_0^{1,(p_i)}(\Omega) = W^{1,(p_i)}(\Omega) \cap W_0^{1,1}(\Omega)$$

endowed by the usual norm

$$\|v\|_{W_0^{1,(p_i)}(\Omega)} = \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}.$$

Definition 1.1 We will say that $u \in W_0^{1,(p_i)}(\Omega)$ is an "energy" solution to (1) if and only if

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi = \int_{\Omega} \frac{f\varphi}{u^{\gamma(x)}} \qquad \forall \varphi \in C_0^1(\Omega),$$

and we will say that u is a "weak" solution to (1) if $\partial_i u^{p_i-1} \in L^1(\Omega)$, $\frac{f}{u^{\gamma(x)}} \in L^1_{loc}(\Omega)$, and one has the identity

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi = \int_{\Omega} \frac{f\varphi}{u^{\gamma(x)}} \qquad \forall \varphi \in C_0^1(\Omega).$$

We will also very often use the following indices

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}$$

and

$$\overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}, \quad p_\infty = \max\left\{p_N, \overline{p}^*\right\}$$

The following Theorem states some anisotropic Sobolev type inequalities, for more details we refer to the early works [13,17,20].

Theorem 1.2 *There exists a positive constant C, depending only on Ω, such that for every $v \in W_0^{1,(p_i)}(\Omega)$, we have*

$$\|v\|_{L^{\overline{p}^*}(\Omega)}^{p_N} \leq C \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}^{p_i}, \tag{2}$$

$$\|v\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}^{\frac{1}{N}} \quad \forall r \in \left[1, \overline{p}^*\right] \tag{3}$$

and $\forall v \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$, $\overline{p} < N$

$$\left(\int_{\Omega} |v|^r\right)^{\frac{N}{\overline{p}}-1} \leq C \prod_{i=1}^{N} \left(\int_{\Omega} |\partial_i v|^{p_i} |v|^{t_i p_i}\right)^{\frac{1}{p_i}}, \tag{4}$$

for every r and t_j chosen in such a way to have

$$\begin{cases} \frac{1}{r} = \frac{\gamma(x)_i(N-1)-1+\frac{1}{p_i}}{t_i+1} \\ \sum_{i=1}^{N} \gamma(x)_i = 1. \end{cases}$$

In the whole paper, C will denote a constant that may change from line to line.

2 Approximation problems

All the results obtained in this section, are direct consequences of the ones presented in [2,14], but for the reader convenience we present them in details.

Let us first consider the following approximation problems

$$\begin{cases} -Lu_n = \frac{f_n}{\left(u_n+\frac{1}{n}\right)^{\gamma(x)}} & in\ \Omega, \\ u_n = 0 & on\ \Omega, \\ u_n \geq 0 & in\ \Omega, \end{cases} \tag{5}$$

where $f_n = T_n(f)$.

Recalling that

$$T_n(s) = \begin{cases} n\frac{s}{|s|} & if\ \ |s| > n \\ s & if\ \ |s| \leq n \end{cases}$$

Lemma 2.1 *The problem (5) has a solution in $W_0^{1,(p_i)}(\Omega)$.*

Proof We will follow the same reasoning as in [2].

Fix $n \in \mathbb{N}$, and let $v \in L_{\overline{p}^*}(\Omega)$. Consider the equation

$$-Lw = \frac{f_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}}, \tag{6}$$

it is clear that the previous problem has a unique solution whenever the right hand side belongs to $L^s(\Omega)$ with $s \geq p'_\infty$ see for instance [8,9]. Denoting $w = S(v)$,

using w as test function in (6), we obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i w|^{p_i} = \int_\Omega \frac{wf_n}{\left(|v| + \frac{1}{n}\right)^{\gamma(x)}} \leq n^{\gamma(x)+1} \int_\Omega |w|$$

by Sobolev inequality (2),

$$\|w\|_{L^{\overline{p}^*}(\Omega)}^{p_N} \leq C \sum_{i=1}^{N} \int_\Omega |\partial_i w|^{p_i},$$

by Hölder inequality

$$\int_{\Omega} |w| \leq \left(\int_{\Omega} |w|^{\overline{p}^*} \right)^{\frac{1}{\overline{p}^*}}.$$

Hence

$$\|w\|_{L^{\overline{p}^*}(\Omega)}^{p_N} \leq C n^{\gamma(x)+1} \|w\|_{L^{\overline{p}^*}(\Omega)},$$

and then

$$\|w\|_{L^{\overline{p}^*}(\Omega)} \leq C' \left(n^{\gamma(x)+1}\right)^{\frac{1}{p_N-1}} = R_N,$$

which means that the ball of radius R_N in $L^{\overline{p}^*}(\Omega)$ is invariant by S, and so by Sobolev embedding and Schauder's fixed point theorem we conclude that the approximation problem (5) has a solution in $W_0^{1,(p_i)}(\Omega)$, for every fixed n.

□

Lemma 2.2 *The sequence $\{u_n\}_n$ is increasing with respect to n.*

Proof We recall that $f_n = T_n(f)$ and so $0 \leq f_n \leq f_{n+1}$

$$-Lu_n = \frac{f_n}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \leq \frac{f_{n+1}}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}}$$

as

$$-Lu_{n+1} = \frac{f_{n+1}}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}}$$

and so one has that

$$\begin{aligned} -Lu_n + Lu_{n+1} &\leq f_{n+1} \left[\frac{1}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}} - \frac{1}{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}} \right] \\ &\leq f_{n+1} \left[\frac{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)} - \left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)} \left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}} \right] \end{aligned}$$

using $(u_n - u_{n+1})^+$ as test function in the last inequality, the right hand side gives

$$f_{n+1} \left[\frac{\left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)} - \left(u_n + \frac{1}{n+1}\right)^{\gamma(x)}}{\left(u_n + \frac{1}{n+1}\right)^{\gamma(x)} \left(u_{n+1} + \frac{1}{n+1}\right)^{\gamma(x)}} \right] (u_n - u_{n+1})^+ \leq 0.$$

Now, taking into account the problems associated to u_n and to u_{n+1}, it follows that

$$\int_\Omega (-Lu_n + Lu_{n+1})\,(u_n - u_{n+1})^+ \leq 0.$$

Thus

$$\sum_{i=1}^N \int_\Omega \left(|\partial_i u_n|^{p_i-2}\,\partial_i u_n - |\partial_i u_{n+1}|^{p_i-2}\,\partial_i u_{n+1}\right)\partial_i\,(u_n - u_{n+1})^+ \leq 0.$$

Integrating over the subset of Ω where $u_n \geq u_{n+1}$ and using the following inequality for $p_i \geq 2$

$$C_0\,|\partial_i\,(u_n - u_{n+1})|^{p_i} \leq \left(|\partial_i u_n|^{p_i-2}\,\partial_i u_n - |\partial_i u_{n+1}|^{p_i-2}\,\partial_i u_{n+1}\right)\partial_i\,(u_n - u_{n+1})$$

we reach that

$$\sum_{i=1}^N \int_\Omega \left|\partial_i\,(u_n - u_{n+1})^+\right|^{p_i} \leq 0.$$

Hence

$$u_n \leq u_{n+1},$$

which allows us to conclude that $\{u_n\}_n$ is increasing with respect to n.

□

Remark 2.3 We limit ourselves to the case $p_i \geq 2$ because (at our knowledge), the operator L verify a strong maximal principle only in the case $p_i \geq 2$ see for instance [8], maximal principle that will be necessary in the sequel.

Lemma 2.4 *For all $n \in \mathbb{N}$, u_n the solution to the approximation problem* (5)*, is such that $u_n \in L^\infty(\Omega)$ and for all $K \subset\subset \Omega$, $u_n \geq C_K > 0$.*

Proof By some modifications in the theory of Leray-Lions operators theory one can show the existence of solution to

$$-Lu_1 = \frac{f_1}{(u_1 + 1)^{\gamma(x)}}$$

and so

$$-Lu_1 = \frac{f_1}{(\|u_1\|_\infty + 1)^{\gamma(x)}} \geq 0$$

the strong maximum principle, and the monotonicity of $\{u_n\}_n$ give that $u_n \geq C_K > 0$. The $L^\infty(\Omega)$ estimate of $\{u_n\}_n$, is a direct consequence of Stampachia result [19], as done in [2]. □

3 Passage to the limit

For fixed δ, let $\Omega_\delta = \{x \in \Omega, dist(x, \partial\Omega) < \delta\}$

Theorem 3.1 *Let* $s = \dfrac{N\overline{p}}{N(\overline{p}-1)+\overline{p}}$ *and* $f \in L^s(\Omega)$, *assume that there exists a* $\delta > 0$ *such that* $\gamma(x) \le 1$ *in* Ω_δ, *then the sequence* $\{u_n\}_n$ *of solutions to* (5), *is bounded in* $W_0^{1,(p_i)}(\Omega)$.

Proof Put $\omega_\delta = \Omega \backslash \overline{\Omega_\delta}$, by the previous results we know that $u_n \ge C_{\omega_\delta} > 0$. Now using u_n as test function in (5) we obtain

$$\begin{aligned}
\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} &= \int_\Omega \frac{f_n(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} u_n \\
&= \int_{\overline{\Omega_\delta}} \frac{f_n(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} u_n + \int_{\omega_\delta} \frac{f_n(x)}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} u_n \\
&\le \int_{\overline{\Omega_\delta}} f(x) u_n^{1-\gamma(x)} + \int_{\omega_\delta} \frac{f(x)}{C_{\omega_\delta}^{\gamma(x)}} u_n \\
&\le \int_{\overline{\Omega_\delta} \cap \{u_n \le 1\}} f(x) + \int_{\overline{\Omega_\delta} \cap \{u_n \ge 1\}} f(x) u_n + \int_{\omega_\delta} \frac{f(x)}{C_{\omega_\delta}^{\gamma(x)}} u_n \\
&\le \|f\|_{L^1(\Omega)} + \left(1 + \left\| C_{\omega_\delta}^{-\gamma(x)} \right\|_{L^\infty(\Omega)}\right) \int_\Omega f(x) u_n
\end{aligned}$$

Using Hölder and Sobolev inequalities, we then obtain

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} \le \|f\|_{L^1(\Omega)} + C\left(1 + \left\| C_{\omega_\delta}^{-\gamma(x)} \right\|_{L^\infty(\Omega)}\right) \|f\|_{L^s(\Omega)} \left[\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i}\right]^{\frac{1}{p_N}}$$

which implies that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_n|^{p_i} \le C$$

where C is a constant independant of n. □

Theorem 3.2 *Let* $s = \dfrac{N\overline{p}}{N(\overline{p}-1)+\overline{p}}$ *and* $f \in L^s(\Omega)$, *assume that there exists a* $\delta > 0$ *such that* $\gamma(x) \le 1$ *in* Ω_δ, *then problem* (1) *posses a solution* $u \in W_0^{1,(p_i)}(\Omega)$.

Proof By the previous proposition $\{u_n\}_n$ is bounded in $W_0^{1,(p_i)}(\Omega)$, thus (up to a subsequence) $\{u_n\}_n$ converges weakly to some u in $W_0^{1,(p_i)}(\Omega)$. On the other hand,

$\{u_n\}_n$ converges strongly in $L^{\theta}(\Omega)$ for $\theta < \overline{p}^*$, thus $\{u_n\}_n$ converges to u almost everywhere in Ω. So one has that for every $\varphi \in C_0^1(\Omega)$

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_{\Omega} |\partial_i u_n|^{p_i-2} \partial_i u_n \partial_i \varphi = \sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi$$

By the fact that

$$0 \le \left| \frac{f_n(x)\varphi}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} \right| \le \left\| \varphi C_{\omega}^{-\gamma(x)} \right\|_{L^{\infty}(\Omega)} f(x)$$

for every $\varphi \in C_0^1(\Omega)$, whenever $\varphi \neq 0$ and on the set where $u_n \ge C_{\omega}$, ω being the support of φ; the dominated Lebesgue's theorem permits us to conclude that

$$\lim_{n\to+\infty} \int_{\Omega} \frac{f_n(x)\varphi}{\left(u_n + \frac{1}{n}\right)^{\gamma(x)}} = \int_{\Omega} \frac{f(x)\varphi}{u^{\gamma(x)}}$$

by the sequel, the limit u of the sequence $\{u_n\}_n$ verify

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i-2} \partial_i u \partial_i \varphi = \int_{\Omega} \frac{f(x)\varphi}{u^{\gamma(x)}}.$$

□

Theorem 3.3 *Assume that for some $\gamma^* > 1$ and some $\delta > 0$ we have $\|\gamma\|_{L^{\infty}(\Omega)} \le \gamma^*$. Provided that $f \in L^s(\Omega)$ with $s = \dfrac{N(\gamma^* - 1 + \overline{p})}{N(\overline{p} - 1) + \overline{p}\gamma^*}$, problem (1) has a solution u in $L^{\alpha}(\Omega)$ with $\alpha = \dfrac{N(\gamma^* - 1 + \overline{p})}{(N - \overline{p})}$, belonging to $W_{loc}^{1,(p_i)}(\Omega)$.*

Proof Let us use $u_n^{\gamma^*}$ as test function in (5), so we obtain for every $i = 1, 2, ..., N$

$$\begin{aligned}
\int_{\Omega} |\partial_i u_n|^{p_i} u_n^{\gamma^*-1} &\le \int_{\overline{\Omega_{\delta}}} f(x) u_n^{\gamma^*-\gamma(x)} + \int_{\omega_{\delta}} \frac{f(x)}{C_{\omega_{\delta}}^{\gamma(x)}} u_n^{\gamma^*} \\
&\le \|f\|_{L^1(\Omega)} + \left(1 + \left\| C_{\omega_{\delta}}^{-\gamma(x)} \right\|_{L^{\infty}(\Omega)}\right) \int_{\Omega} f(x) u_n^{\gamma^*} \\
&\le \|f\|_{L^1(\Omega)} + \left(1 + \left\| C_{\omega_{\delta}}^{-\gamma(x)} \right\|_{L^{\infty}(\Omega)}\right) \left(\int_{\Omega} f^s(x)\right)^{\frac{1}{s}} \left(\int_{\Omega} u_n^{\gamma^*\beta}\right)^{\frac{1}{\beta}}
\end{aligned}$$

with $\beta = \dfrac{N(\gamma^* - 1 + \overline{p})}{(N - \overline{p})\gamma^*}$, and so

$$\int_{\Omega} |\partial_i u_n|^{p_i} u_n^{\gamma^*-1} \le C_1 + C_2 \left(\int_{\Omega} u_n^{\gamma^*\beta}\right)^{\frac{1}{\beta}},$$

thus

$$\left(\int_\Omega |\partial_i u_n|^{p_i}\, u_n^{\gamma^*-1}\right)^{\frac{1}{p_i}} \leq \left(C_1 + C_2\left(\int_\Omega u_n^{\gamma^*\beta}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{p_i}}$$

which implies that

$$\prod_{i=1}^{N}\left(\int_\Omega |\partial_i u_n|^{p_i}\, u_n^{\gamma^*-1}\right)^{\frac{1}{p_i}} \leq \left(C_1 + C_2\left(\int_\Omega u_n^{\gamma^*\beta}\right)^{\frac{1}{\beta}}\right)^{\sum_{i=1}^{N}\frac{1}{p_i}} = \left(C_1 + C_2\left(\int_\Omega u_n^{\alpha}\right)^{\frac{1}{\beta}}\right)^{\frac{N}{\overline{p}}}$$

with the following choice of exponents

$$\begin{cases} t_i p_i = \gamma^* - 1 \\ r = \alpha = \dfrac{N\,(\gamma^* - 1 + \overline{p})}{(N - \overline{p})} \\ \frac{1}{r} = \frac{\gamma_i (N-1) - 1 + \frac{1}{p_i}}{t_i + 1} \end{cases}$$

Sobolev inequality (4) gives

$$\left(\int_\Omega u_n^{\alpha}\right)^{\frac{N}{\overline{p}}-1} \leq \left(C_1 + C_2\left(\int_\Omega u_n^{\alpha}\right)^{\frac{1}{\beta}}\right)^{\frac{N}{\overline{p}}}$$

and so

$$\left(\int_\Omega u_n^{\alpha}\right)^{1-\frac{\overline{p}}{N}} \leq C_1 + C_2\left(\int_\Omega u_n^{\alpha}\right)^{\frac{1}{\beta}}$$

by the fact that

$$\frac{1}{\beta} < 1 - \frac{\overline{p}}{N}$$

we conclude that $\{u_n\}_n$ is bounded in $L^{\alpha}(\Omega)$ with $\alpha = \dfrac{N\,(\gamma^* - 1 + \overline{p})}{(N - \overline{p})}$ and by the monotone convergence theorem, $\{u_n\}_n$ converges strongly to $u \in L^{\alpha}(\Omega)$.

On the other side using $u_n^{\gamma^*}$ as test function in (5) we get

$$\sum_{i=1}^{N}\int_\Omega |\partial_i u_n|^{p_i}\, u_n^{\gamma^*-1} \leq C$$

by strong maximum principle we have for every compact $K \subset\subset \Omega$

$$C_K^{\gamma^*-1} \sum_{i=1}^{N} \int_{\Omega} |\partial_i u_n| \leq C$$

thus we obtain weak convergence of $\{u_n\}_n$ to u in $W_{loc}^{1,(p_i)}(\Omega)$.

To complete the proof, we follow the same steps as in the previous Proposition. □

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