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Intermediate weighted spaces and domains of semi-groups

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Abstract Interpolation results of Lions (Ann Scuola Norm Sup Pisa t 13:389–403, 1959), Lions (Math Scand 9:147–177, 1961), Lions and Peetre (Publ Math IHS 19:5–68, 1964) are extended to embed domains of semi-groups into some weighted spaces studied in Artola (Bolletino UMI 5(9):125–158, 2012), Artola (Bolletino UMI, in press, 2016). Hardy's inequality for weighted spaces (see Bolletino UMI 5(9):125–158, 2012), being necessary for the treatment, the weights are required to belong to the Hardy class $\mathcal{H}(p)$, $(1 \leq p \leq +\infty)$ defined in Artola (Bolletino UMI 5(9):125–158, 2012.

Keywords Interpolation · Weighted spaces · Hardy's inequality and Hardy's class $\mathcal{H}(p)$ · Semi-groups

Mathematics Subject Classification 46 Functional analysis · 47 Operator theory

1 Introduction

The paper describes the construction of some intermediate weighted spaces between the domain $D(\Lambda)$ of an unbounded operator Λ, which is the infinitesimal generator of a convenient semi-group $G(t), 0 \leq t < \infty$ and the space A that is a space Σ satisfying

$$D(\Lambda) \subset \Sigma \subset A$$

where $\subset$ means algebraic and topological imbedding.

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It is of interest for applications to construct intermediate space which have the *interpolation property*: when $(D(\Lambda_1), \Sigma_1, A_1)$ is another set analogous to $(D(\Lambda), \Sigma, A)$, then a linear mapping π from A_1 into A such that π *restricted to* $D(\Lambda_1)$ is continuous from $D(\Lambda_1)$ *into* $D(\Lambda)$ (say $\pi \in \mathcal{L}(D(\Lambda_1), D(\Lambda))$), then π is continuous from Σ_1 *into* Σ (i.e. $\pi \in \mathcal{L}(\Sigma_1, \Sigma)$.

The usual procedure to prove that an intermediate space has the interpolation property is, either by a direct proof, or by showing that the space can be identified with another space which has the property. That second method is employed to embed domains of semi-groups into some weighted spaces studied in [2,4] but adapted here to a semi-group framework where the weight $c(t)$ belongs to the Hardy class $\mathcal{H}(p)$, $1 \leq p \leq \infty$, that is: $c(t) \in \mathcal{H}(p)$ if and only if the Hardy operator $\phi \longrightarrow \mathcal{H}(\phi)$ is continuous from $L^p_c(R^+; A)$ into itself, where $\mathcal{H}(\phi)(t) = t^{-1} \int_0^t \phi(\tau) d\tau$ and $L^p_c(R^+; A) = \{\phi; c\phi \in L^p(R^+; A)\}$.

A similar problem solved in [11], deals with weights $c(t) = t^\alpha$, $\alpha + 1/p \in (0, 1)$, so that c belongs to $\mathcal{H}(p)$ and $1/c$ to $\mathcal{H}(p')$, $1/p + 1/p' = 1$, in that case.

Section 2 extends all results of [11,12], to weighted spaces with weights c or $1/c$ in Hardy's class. Section 2.1, introduces a semi-group satisfying a *SG-condition* and an intermediate weighted space Σ between $D(\Lambda)$ and A which extends that of [11]. Then in Sect. 2.2, because $c \in \mathcal{H}(p)$, we show that Σ can be identified with a trace space belonging to the class of weighted spaces studied mainly in [2]. In Sect. 2.3, we recall that trace spaces of [2] have the interpolation property, so that we obtain the result for Σ.

In Sect. 2.4, duality results are extended on introducing the weight $1/c$ which lead to a problem posed by Lions in [12] which either is in general partly solved, or solved in a particular case (see Proposition 2.4).

Section 3 is mainly devoted to spaces studied in [4,5], called "intermediate mean weighted spaces" which extend those of [13] called "espaces de moyenne". Generally they are not trace spaces but have the interpolation property. Thus we obtain some results for certain intermediate spaces again between $D(\Lambda)$ and A.

In Sect. 4, we consider as in [11] the case of n commutative semi-groups which is very important for applications to PDE and leads to a class of weighted Besov or Sobolev spaces probably not well known when equipped with that weights type.

Finally, on using a main result of Lions on distribution semi-groups,[1] we generalise some results of [13] for intermediate mean weighted spaces between $D(\Lambda^m)$ $(m \geq 1)$ and A and also between $D(\Lambda^{j+1})$, $D(\Lambda^j)$ on adapting a theorem established in [4] to extend the reiteration theorem of Lions–Peetre. Furthermore, a formal example is given that involves certain weighted Besov spaces that are probably new.

2 An intermediate space between $D(\Lambda)$ and A

If X, Y are Banach spaces, we denote by $\mathcal{L}(X, Y)$ resp. ($\mathcal{L}(X)$ if $X = Y$), the space of linear mappings from X *into* Y.

[1] See [13, pp. 53–54].

Let A be a real or complex Banach space with norm denoted by $|.|$. Consider an unbounded, closed, operator Λ, with domain $D(\Lambda)$ dense in A. Provided with the norm $|a|_{D(\Lambda)} = |a| + |\Lambda(a)|$, $D(\Lambda)$ is a Banach space. When $\pi \in \mathcal{L}(A)$, we denote the norm of π by $\|\pi\|$.

We assume that Λ is the infinitesimal generator of a semi-group $t \longrightarrow G(t)$ which is continuous and uniformly bounded on $R^+ =]0, +\infty[$. More explicitly this means that *SG-condition*:

The function $t \longrightarrow G(t)$ is strongly continuous from $t \geq 0$ to A, with $G(0)a = a$ and there is a constant M such that $\|G(t)\| \leq M < \infty$.

To construct an intermediate space between $D(\Lambda)$ and A we begin with a simple space connected with a trace space.

2.1 The space $\Sigma(p,c;D(\Lambda),A)$

Let p, $1 \leq p < \infty$ and c: $t \longrightarrow c(t)$, $R^+ = (0, +\infty) \longrightarrow R^+$, be a locally integrable function satisfying

$$\forall t_0 > 0, \quad (i)\ c \in L(0, t_0; R^+), \quad (ii)\ 1/c \in L^{p'}(0, t_0; R^+), \tag{2.1}$$

and

$$\frac{c}{t} \in L^p(t_0, +\infty; R^+). \tag{2.2}$$

Moreover, we assume that,[2]

$$\sup_{t>0} \left(\int_t^{+\infty} \left[\frac{c(\tau)}{\tau} \right]^p d\tau \right)^{1/p} \left(\int_0^t \frac{d\tau}{[c(\tau)]^{p'}} \right)^{1/p'} < \infty. \tag{2.3}$$

Remark 2.1 (1) Condition (2.3) characterizes the weights c for which the Hardy operator:

$$f \longrightarrow \mathcal{H}(f)(t) = \frac{1}{t} \int_0^t f(\tau) d\tau$$

is continuous from $L_c^P(A)$ into itself when $cf \in L^p(A) = L^p(R^+; A)$, which is a Banach space when equipped with the natural norm $|f|_{L^p(A)} = \left(\int_{R^+} |f(\tau)|_A^p \right)^{1/p}$ (usual definition holds for $p = \infty$).

In what follows we shall write $c \in \mathcal{H}(p)$ to say that c satisfies (2.1) and (2.3)

(2) A weight c which is a non-increasing function,obviously belongs to $\mathcal{H}(p)$ for all p, and if ϕ is a non-increasing function then $\phi\ c \in \mathcal{H}(p)$ when $c \in \mathcal{H}(p)$.

(3) It is of interest to notice that (2.2) is only a necessary condition for $c \in \mathcal{H}(p)$ (see [2]).

[2] See [2].

We define the space $\Sigma(p, c; D(\Lambda), A)$ as a subspace of A such that: $a \in A$, with

$$\frac{c(t)}{t}(G(t)a - a) \in L^p(A). \tag{2.4}$$

$\Sigma(p, c; D(\Lambda), A)$ becomes a Banach space, When equipped with the norm

$$|a|_\Sigma = |a| + \left(\int_{R^+}\left[\frac{c(\tau)}{\tau}\right]^p |G(\tau)a - a|^p\right)^{1/p}. \tag{2.5}$$

Proposition 2.2 *Assume that* (2.1i) *and* (2.2) *hold, and let G satisfy the "SG-condition". Then the space* $\Sigma(p, c; D(\Lambda), A)$ *is an intermediate space between* $D(\Lambda)$ *and* A. *That is*

$$D(\Lambda) \subset \Sigma(p, c; D(\Lambda), A) \subset A \tag{2.6}$$

where $\subset$ *means algebraic and topological inclusion.*

Proof Naturally we have to check only the first inclusion.

From semi-group theory[3] *for* $a \in D(\Lambda)$, the function $t \longrightarrow G(t)a$ is continuous and derivable, the derivative being $G(t)\Lambda a$, so that

$$G(t)a - a = \int_0^t G(\tau)\Lambda a d\tau$$

and consequently

$$|G(t)a - a| \leq Mt\,|\Lambda a|\,;$$

for a *fixed* T_0, we then deduce from (2.1i)

$$\frac{c(t)}{t}(G(t)a - a) \in L^p(0, T_0; R^+)$$

with

$$\int_0^{T_0}\left[\frac{c(t)}{t}\right]^p |G(t)a - a|^p\,dt \leq M^p\,|\Lambda a|^p \int_0^{T_0} c(t)^p dt.$$

On the other hand, we have

$$\frac{c(t)}{t}|G(t)a - a| \leq \frac{c(t)}{t}(M + 1)\,|a|\,,$$

and from (2.2)

$$\frac{c(t)}{t}(G(t)a - a) \in L^p(T_0, +\infty; R^+).$$

[3] We refer to [7].

This complete the proof for the first inclusion algebraically and also topologically, because there is a positive constant K, such that

$$|a|_{\Sigma} \leq K(|a| + |\Lambda a|).$$

We next show that the space $\Sigma(p, c; D(\Lambda), A)$ can be identified with a trace space.

2.2 The space $\Sigma(p, c; D(\Lambda), A)$ as a trace space

We define, as a particular case of spaces studied in [2]:

$$\begin{aligned} W(p, c; D(\Lambda), A) &= W^{(1)}(p, c, D(\Lambda); p, c, A) \\ &= \{u;\ cu \in L^p(D(\Lambda), cDu \in L^p(A)\}, \end{aligned}$$

where $W(p, c; D(\Lambda), A)$ is a Banach space when provided with the natural norm.

The last condition on Du must be understood as follows: u is differentiable in the sense of distributions on R^+ with values in A, Du being locally integrable, so that the product with c takes a sense.

Now since Du is locally integrable, u is absolutely continuous and hence continuous. Then we can consider that u is continuously differentiable with values in A and $u(t)$ is defined for $t\ \in]0, +\infty[$. Therefore when $\lim_{t \longrightarrow +0} u(t) = a$ *in A exists*, we shall say that u has a *trace* $u(0) = a$ *at* $t = 0$.

For the existence of traces ($\neq$0), we have from [2]:

Proposition 2.3 *Conditions (2.1) are necessary and sufficient for the existence of a trace $a \neq 0$.*

Assuming that conditions (2.1) hold, let $T(p, c; D(\Lambda), A) = T_0^1(p, c, D(\Lambda); p, c, A)$ the space spanned in A by $u(o) = a$ when u spans the space $W(p, c; D(\Lambda), A)$. The space $T(p, c; D(\Lambda), A)$ is a Banach space, When equipped with the norm:

$$|a|_{T(p,c;D(\Lambda),A)} = \inf_{u(0)=a} |u|_{W(p,c;D(\Lambda),A)}\,. \tag{2.7}$$

The space $T(p, c; D(\Lambda), A)$ is called a trace space (here of order 0).

Remark 2.4 The definition of the norm of $T(p, c; D(\Lambda), A)$ shows that the space can be interpreted as the quotient space $W(p, c; D(\Lambda), A)/W_0(p, c; D(\Lambda), A)$ where

$$W_0(p, c; D(\Lambda), A) = \{u;\ u \in W(p, c; D(\Lambda), A),\ u(0) = 0\}.$$

From (2.1), the space of traces $T(p, c; D(\Lambda), A)$ exists and we have:

Theorem 2.5 *Assume that Λ is the infinitesimal generator of a continuous and uniformly bounded semi-group $t \longrightarrow G(t)$. Given p with $1 \leq p \leq \infty$, let $c \in \mathcal{H}(p)$. Then the linear mapping*

$$u \longrightarrow u(o)\ is\ continuous\ from\ W(p, c; D(\Lambda), A)\ onto\ \Sigma(p, c; D(\Lambda), A).$$

Proof of Theorem 2.5 The proof involves two steps:

Step 1: $u \longrightarrow u(0)$ *is continuous from* $W(p, c, \Lambda)$ *into* $\Sigma(p, c; D(\Lambda), A)$.

Since $u \in W(p, c; D(\Lambda), A),\ \forall t > 0,\ u(t) \in D(\Lambda)$, we can write

$$\frac{du}{dt} - \Lambda u = f \tag{2.8}$$

and from the definition of $W(p, c; D(\Lambda), A)$, we deduce that

$$cf \in L^p(A). \tag{2.9}$$

Therefore[4] the method of Cauchy to solve (2.8) gives

$$u(t) = G(t)u(0) + \int_0^t G(t-\sigma) f(\sigma) d\sigma,$$

and when we set $u(0) = a$, we have

$$u(t) - a = G(t)a - a + \int_0^t G(t-\sigma) f(\sigma) d\sigma$$

and

$$G(t)a - a = \int_0^t \frac{du}{dt}(\sigma) d\sigma - \int_0^t G(t-\sigma) f(\sigma) d\sigma.$$

On setting $\frac{du}{dt} = u'$, we get

$$\frac{c(t)}{t} |G(t)a - a| \leq I_1(t) + I_2(t)$$

where

$$I_1(t) = c(t)\mathcal{H}\left(|u'|\right)(t),$$
$$I_2 = \frac{c(t)}{t} \int_0^t \|G(t-\sigma\| \, |f(\sigma)| \, d\sigma.$$

As $c \in \mathcal{H}(p)$ and because $cu' \in L^p(A)$ the weighted Hardy inequality gives

$$\int_{R^+} [I_1(t)]^p dt \leq \kappa \int_{R^+} \left|c(t)u'(t)\right|^p dt, \quad \kappa = \text{constant}.$$

For I_2, because $\forall t \geq 0$, $\|G(t)\| \leq M < \infty$, and from (2.9), the procedure is analogous to that for I_1 and Hardy's inequality gives the required estimate. We deduce that $a \in \Sigma(p, c; D(\Lambda), A)$. Thus the result for the step 1 is proved.

[4] See for example [7].

Step 2: *The mapping* $u \longrightarrow u(0)$ *is onto.*

Assume $a \in \Sigma(p, c; D(\Lambda), A)$. We have to check that a function $u \in W(p, c; D(\Lambda), A)$ can be found such that $u(0) = a$.

On introducing[5]

$$v(t) = t^{-1} \int_0^t G(\sigma) a d\sigma = \mathcal{H}(G(\sigma)a). \tag{2.10}$$

We define a continuous function from $t \geq 0$ into A with

$$v(0) = a. \tag{2.11}$$

Now we use

Lemma 2.6 *For any* $t > 0$

$$\int_0^t G(\tau) dt \in \mathcal{L}(A; D(\Lambda))$$

and

$$\Lambda \int_0^t G(\tau) d\tau = G(t) - I, \quad I = \text{identity}.$$

Proof: Obviously when $a \in D(\Lambda)$, we obtain

$$\Lambda \int_0^t G(\tau) a d\tau = \int_0^t \Lambda G(\tau) a d\tau = \int_0^t \frac{d}{dt} (G(\tau)a] d\tau = G(t)a - a.$$

$D(\Lambda)$ being dense in A, and the operator Λ being closed, that is true for every $a \in A$. The lemma is proved.

Therefore, $v(t) \in D(\Lambda)$, and

$$\Lambda v(t) = t^{-1} \left(G(t)a - a \right).$$

Consequently, we deduce

$$c(t) \Lambda v(t) = \frac{c(t)}{t} \left(G(t)a - a \right)$$

belongs to $L^p(A)$, because $a \in \Sigma(p, c.D(\Lambda), A)$.

Consider now the derivative $v' = dv/dt$. From (2.10) we have

$$tv' + v = G(t)a$$

[5] We follow an adaptation of a Gagliardo method given by Lions [11] using the weights $t^{\alpha}, \alpha + 1/p \in (0, 1)$ which are in $\mathcal{H}(p)$.

so that

$$v'(t) = t^{-1}\left(G(t)a - a\right) + t^{-1}a - t^{-2}\int_0^t G(\tau)d\tau,$$

and finally

$$v' = \Lambda v - w, \tag{2.12}$$

where

$$w(t) = t^{-2}\int_0^t (G(\tau)a - a)d\tau.$$

In order to check that

$$cv' \in L^p(A), \tag{2.13}$$

and because $c\Lambda v \in L^p(A)$ it is sufficient to obtain the result for w. Since

$$|w(t)| \le t^{-1}\int_0^t t^{-1}\left|G(\tau)a - a\right| d\tau \le t^{-1}\int_0^t \tau^{-1}\left|G(\tau)a - a\right| d\tau,$$

in conjunction with the definition of $\Sigma(p, c; D(\Lambda), A)$ and on applying the weighted Hardy inequality, we obtain $cw \in L^p(A)$.

Hence (2.13) holds and Theorem 2.5 is proved, if we consider the function u defined by $u(t) = \Phi(t)v(t)$, where the function Φ is once continuously differentiable on $t \ge 0$ *and vanishes for t sufficiently large*, and is such that $\Phi(0) = 1$. Thus $u \in W(p, c; D(\lambda), A)$ and $u(0) = a$.

2.3 Interpolation property

From [1], we know that any trace theorem gives an interpolation result and we have proved in [2] that, generally, the traces of weighted spaces (of the Hardy's class) have the interpolation property.

Let $(A_1, \Lambda_1, W_1(p_1, c_1; D(\Lambda_1), A_1)T_1)$, be a set analogous to (A, Λ, W, T). Therefore

$$\pi \in \mathcal{L}(D(\Lambda), D(\Lambda_1)) \cap \mathcal{L}(A, A_1)),$$

implies

$$\pi \in \mathcal{L}(T(p, c, D(\Lambda), A), T_1(p_1, c_1, D(\Lambda_1), A_1).$$

Thus, since $\Sigma(p, c, D(\Lambda), A) = T(p, c, D(\Lambda), A)$, we have

Theorem 2.7 *Given* $\{(A, \Lambda, W, \Sigma), (A_1, \Lambda_1, W_1, \Sigma_1)\}$*, we assume* $\pi \in \mathcal{L}(D(\Lambda), D(\Lambda_1)) \cap \mathcal{L}(A, A_1)$*. Then*

$$\pi \in \mathcal{L}(\Sigma(p, c; D(\Lambda), A), \Sigma_1(p_1, c_1; D(\Lambda_1), A_1)).$$

Remark 2.8 Define $\hat{c}(t) = t^{-1/p}c(t)$ with $c \in \mathcal{H}(p)$, then also $\hat{c} \in \mathcal{H}(p)$. Therefore, when we consider spaces constructed with c replaced by $\hat{c}$, the previous result still holds.

If we denote by $L^p_*(A)$ the space $L^p(A)$ equipped with the Haar measure on R^*, we have the equivalence:

$$\hat{c}u \in L^p(A) \simeq cu \in L^p_*(A).$$

2.4 Duality

Since $D(\Lambda)$ is dense in A, we may identify A', the dual (or antidual) of A, as a subspace of $D(\Lambda)'$ and

$$A' \subset D(\Lambda)', \ (with\ injective\ mapping).$$

Indeed if i is the injective mapping of $D(\Lambda)$ *into* A, then its range is dense in A. This implies that the adjoint i^* is a continuous injective mapping from A' *into* $D(\Lambda)'$, whose the range is dense.

Assume

$$A\ is\ a\ reflexive\ Banach\ space. \tag{2.14}$$

From Proposition 3.1, and Theorem 4.7 of [2], we obtain

$$\begin{aligned} [w(p, c.D(\Lambda), A)]' &= W(p', c^{-1}, A'; p', c^{-1},\ D(\Lambda)') \\ &= W(p', c^{-1}; A', D(\Lambda)'), \end{aligned} \tag{2.15}$$

$$\begin{aligned} [T(p, c; D(\Lambda), A)]' &= T_0^1(p', c^{-1}, A'; p', c^{-1}, D(\Lambda)') \\ &= T(p', c^{-1}; A', D(\Lambda)'). \end{aligned} \tag{2.16}$$

where $1/p + 1/p\prime = 1$.

Remark 2.9 For the moment, we note that only the condition (2.1)′ is needed for existence of the trace space $T_0^1(p', c^{-1}, A', p', c^- D(\Lambda)')$, that is

$$(2.1)' :\ \forall t_0 > 0,\ \int_0^{t_0} \frac{d\tau}{[c(\tau)]^{p'}} < +\infty,\ \int_0^{t_0} [c(\tau)]^p d\tau < +\infty,$$

which is (globally) (2.1).

To study the dual of the space $\Sigma(p, c; D(\Lambda), A)$, we observe that $G(t) \in \mathcal{L}(D(\Lambda), D(\Lambda))$, the norm of $G(t)$ in that space being majorized by M and $G(t)$ is defined as a semi group into $D(\Lambda)$.

If $\tilde{G}(t)$ denotes the adjoint of $G(t)$ in $\mathcal{L}(D(\Lambda), D(\Lambda))$, we have:

$$\tilde{G}(t) \in \mathcal{L}(D(\Lambda)', D(\Lambda)'), \tag{2.17}$$

$\tilde{G}(t)$ being a semi-group on satisfying the *SG-condition* in $D(\Lambda)'$.

Let Λ^* be the adjoint of Λ with domain $D(\Lambda^*) = A'$ which is an element of $\mathcal{L}(A', D(\Lambda)')$.

To prove that $[\Sigma(p, c; D(\Lambda), A)]' = \Sigma'(p', c^{-1}; A', D(\Lambda)')$, equipped with the norm

$$|a'|_{\Sigma'} = |a'|_{D(\Lambda)'} + \left(\int_{R^+} [(tc(t)^{-1}]^{p'} \left|\tilde{G}(t)a' - a'\right|^{p'}_{D(\Lambda)'} dt\right)^{1/p'} < \infty, \tag{2.18}$$

is an intermediate space between A' and $D(\Lambda)'$, we need to assume:

$$(2.2)' : \ \forall t_0 > 0, \int_{t_0}^{+\infty} \frac{dt}{[tc(t)]^{p'}} < \infty,$$

which together with (2.1ii) gives the result by the same method that used for Proposition 2.2.

To prove that

$$\Sigma'(p', c^{-1}; A', D(\Lambda)') \equiv T(p', c^{-1}; A', D(\Lambda)') \tag{2.19}$$

we need to assume moreover that

$$c^{-1} \in \mathcal{H}(p'). \tag{2.20}$$

Thus we have obtained.

Theorem 2.10 *Let $G(t)$ be a semi-group with infinitesimal generator Λ, satisfying the SG-condition in the reflexive Banach space A.*

Let $\tilde{G}(t)$ be the adjoint semi-group in $D(\Lambda)$ and for p with $1 < p < \infty$, we assume $c \in \mathcal{H}(p)$ and $c^{-1} \in \mathcal{H}(p')$.

Then the dual space of $\Sigma(p, c.D(\Lambda), A)$ is algebraically and topologically equivalent to the space $T(p', c^{-1}; A', D(\Lambda)')$ equipped with the norm of $T_0^1(p', c^{-1}, A'; p', c^{-1}, D(\Lambda)')$ that is

$$|a'|_{T(p',c^{-1};A',D(\Lambda)')} = \underset{u(o)=a'}{Inf} |u|_{W(p',c^{-1};A',D(\Lambda)')} . \tag{2.21}$$

On an other hand, from [2], every continuous linear form L on $\Sigma(p, c; D(\Lambda), A)$ may be written (with non uniqueness).

$$L(a) =< a', a > + \int_0^{+\infty} \frac{c(t)}{t} < f(t), G(t)a - a > dt,$$

where $<, >$ denotes the bracket in the duality $< A', A >$, with

$$a' \in A', \quad f \in L^{p'}(R^+; A'), \quad a \in \Sigma(p, c; D(\Lambda), A).$$

We note that, *a.e. on t*, one has

$$< f(t), G(t)a - a >=< \tilde{G}(t) f(t) - f(t), a >,$$

and we deduce

Lemma 2.11 *Let $f \in L^{p'}(R^+; A\prime)$. Then the function*

$$g(t) = \frac{c(t)}{t}(\tilde{G}(t) f(t) - f(t))$$

belongs to $L^1(R^+; D(\Lambda)\prime)$.

The lemma is the dual result to Proposition 2.2.

Proof (1) First we have that $g \in L^1(1, \infty; A\prime)$ because

$$|g(t)|_{A\prime} \leq (M + 1)t^{-1}c(t) \, |f(t)|_{A\prime},$$

and from (2.2), the result follows by Hölder's inequality.

(2) Now, we check that $g \in L^1(0, 1.D(\Lambda)\prime)$.
When $a\prime \in A\prime$, it follows that

$$\begin{aligned} \left|\tilde{G}(t)a\prime - a\prime\right|_{D(\Lambda)\prime} &\leq \sup_{a\in D(\Lambda)} \frac{\left|< \tilde{G}(t)a\prime - a\prime, a >\right|}{|a|_{D(\Lambda)}} \\ &= \mathop{sup}_{a\in D(\Lambda)} \frac{|< a\prime, G(t)a - a >|}{|a|_{D(\Lambda)}} \\ &\leq Mc(t) \, |a\prime|_{A\prime} \end{aligned}$$

so that

$$|g(t)|_{D(\Lambda)\prime} \leq Mc(t) \, |f(t)|_{A\prime},$$

and from (2.1i), the result follows by Hölder inequality.

Taking in account (2.17–2.19) and the Lemma 2.11, we can finally give an equivalent version of Theorem 2.10.

Theorem 2.12 *Let $G(t)$ semi-group with infinitesimal generator Λ, satisfying the SG-condition in the reflexive Banach space A.*

Let $1 < p > +\infty$, and assume $c \in \mathcal{H}(p)$, $c^{-1} \in \mathcal{H}(p')$, $1/p + 1/p' = 1$. Then we can write every element $a \in \Sigma(p, c; D(\Lambda), A)$ as

$$a = a_0 + \int_0^{+\infty} (G(t) f(t) - f(t)) \, \frac{dt}{tc(t)}, \tag{2.22}$$

where

$$f \in L^p(R^+; \; D(\Lambda)), \quad a_0 \in D(\Lambda). \tag{2.23}$$

The representation (2.21, 2.22) is non-unique.

2.4.1 A problem of Lions

In [12] the following problem is considered:

Problem P:

Let Φ be the set

$$\Phi = \{\phi;\ \phi \geq 0,\ on\ R^+, \quad \phi \in L^p(1, +\infty), \quad t\phi \in L^p(0, 1)\}, \tag{2.24}$$

and denote by $S_\phi(p, D(\Lambda), A)$ the space of $a \in A$ such that

$$\phi(t)\,(G(t)a - a) \in L^p(R^+; A).$$

Equipped with the norm

$$|a|_{S_\phi} = |a| + \left(\int_0^{+\infty} \phi(t)^p\, |G(t)a - a|^p\, dt\right)^{1/p} \tag{2.25}$$

$S_\phi(p, D(\Lambda), A)$ becomes a Banach space.

From (2.23), we have

$$D(\Lambda) \subset S_\phi(p, D(\lambda), A) \subset A.$$

Then a question is: *what are the functions $\phi \in \Phi$ such that the space $S_\phi(p, D(\Lambda), A)$ is an interpolation space?*

A partial answer is given here:

Indeed if we define $\phi = \frac{c(t)}{t}$, *with* $c \in \mathcal{H}(p)$, then thanks to (2.1i) and (2.3), $\phi \in \{\Phi\}$ and from Sect. 2.2, the space $S_\phi(p, D(\Lambda), A)$ is identified with the trace space $T(p, c; D(\Lambda), A)$ which is an interpolation space.

Then we have

$$\tilde{\Phi} = \{\phi \in \Phi;\ tc \in \mathcal{H}(p)\} \subset \Phi. \tag{2.26}$$

Now a dual problem is

Problem P:*

Let Ψ be the set

$$\Psi = \{\psi;\ \psi \geq 0, on\ \ R^+, \quad \psi \in L^{p'}(1, +\infty), \quad t\psi \in L^{p'}(0, 1)\}. \tag{2.27}$$

Then, for every $f \in L^p(R^+, D(\Lambda))$ the function

$$t \longrightarrow \psi(t)\,(G(t)f(t) - f(t))$$

belongs to $L^1(R^+; A)$ and we can consider the mapping

$$\{a_0, f\} \longrightarrow a = a_0 + \int_{R^+} \psi(t)(G(t)f(t) - f(t))dt$$

from $D(\Lambda) \times L^p(R^+; D(\Lambda))$ *into* A.

Denote by $S_\psi(p, D(\Lambda), A)$ the range of the mapping (2.26) equipped with the norm

$$|a|_{S_\psi} = Inf\left\{|a_0|_{D(\Lambda)} + \left(\int_0^{+\infty} |f(t)|^p_{D(\Lambda)}\, dt\right)^{1/p}\right\} \tag{2.28}$$

for every

$$a_0 + \int_0^{+\infty} \psi(t)\,(G(t)f(t) - f(t)dt)\,dt = a.$$

Then $S_\psi(p, D(\Lambda), A)$ is a Banach space which is also an intermediate space:

$$D(\Lambda) \subset S_\psi(p, D(\Lambda), A) \subset A.$$

Therefore the problem is: *"what are the functions ψ satisfying (2.27) such that $S_\psi(p, D(\Lambda), A)$ is an interpolation space?"*

From Theorem 2.12 an answer is given when $\psi(t) = \frac{1}{tc(t)}$, $1/c \in \mathcal{H}(p\prime)$. Thus on setting

$$\tilde{\Psi} = \{\psi \in \Psi; \quad t\psi \in \mathcal{H}(p\prime)\},$$

we obtain[6]

$$\tilde{\Psi} \subset \Psi.$$

Remark 2.13 (1) In [10,11] the case with weights t^α is studied under the assumption $\theta = \alpha + 1/p \in]0, 1[$ which implies both that $c \in \mathcal{H}(p)$ and $\frac{1}{c} \in \mathcal{H}(p\prime)$ so that the associated functions ϕ, ψ are $\phi(t) = t^{\alpha-1}$, $\psi(t) = t^{-(\alpha+1)}$.

(2) If we assume that the function ϕ (resp. ψ) is such that $log(t\phi)$ $(resp.\ \log(t\psi))$ is of finite order distinct from -1, with respect to $\log t$ when $t \longrightarrow 0$ *or* $t \longrightarrow \infty$, $(condition\mathcal{A})$, and since we can naturally assume that $[t\phi]^{-1} \in L^{p'}(0, t_0)$, $(resp.\ [t\psi]^{-1} \in L^p(0, t_0))$ for any $t_0 > 0$, then from a result of Bourbaki [6], we can check that

$$\int_0^t \phi(\tau)^p d\tau \simeq k_0 t\phi^p(t) \text{ and } \int_0^t \frac{d\tau}{[\tau\phi(\tau)]^{p'}} \simeq k_1 \frac{t^{1-p\prime}}{\phi^{p\prime}(t)},\ as\ t \longrightarrow +0\ or\ t \longrightarrow \infty,$$

for constants k_i $(i = 0, 1)$.

Consequently (2.3) holds for $c(t) = t\phi(t)$, and we obtain $c \in \mathcal{H}(p)$. An analogous result holds for $c^*(t) = t\psi(t)$ and $c^* \in \mathcal{H}(p\prime)$.

From remark 2.13(2), we have

Proposition 2.14 *Assume that Condition A of Remark 2.13, holds. Then the solution of Problem P (resp. of Problem P*) is given by $\phi(t) = \frac{c(t)}{t}$, with $c \in \mathcal{H}(p)$, (resp. by $\psi(t) = \frac{c^*(t)}{t}$, with $c^* \in \mathcal{H}(p\prime)$, and we can take $c^*(t) = 1/c(t)$.*

[6] As for (2.25) the imbedding is only algebraically.

3 Intermediate weighted mean spaces

In [2] we have defined the space $\hat{\Sigma}_\theta = \Sigma(p, \theta, \hat{c}, D(\Lambda); p, \theta - 1, \hat{c}, A)$, where $\theta \in (0, 1)$, (and $\hat{c}$ as in Remark 2.8), by

$$a \in \hat{\Sigma}_\theta \; if \; we \; can \; find \; a_i(t), \quad (i = 0, 1) \; with$$
$$a = a_0(t) + a_1(t), \quad t^\theta \hat{c} a_0 \in L^p(D(\Lambda)), \; t^{\theta-1}\hat{c} a_1 \in L^P(A). \tag{3.1}$$

Equipped with the norm

$$|a|_{\hat{\Sigma}_\theta} = \inf_{a(t)+a_1(t)=a} (\left|\hat{c}a_0\right|_{L^p(D(\Lambda))}, \left|\hat{c}a_1\right|_{L^p(A)}). \tag{3.2}$$

$\hat{\Sigma}_\theta$ becomes a Banach space.

Since properties of spaces named “Espaces de moyennes” by Lions–Peetre extend to the weighted space Σ_θ, we call these spaces “weighted mean spaces”.

One has

Theorem 3.1 *The space $\hat{\Sigma}_\theta, \theta \in (0, 1)$ is an intermediate space between $D(\Lambda)$ and A having the interpolation property.*

Proof We have only to prove the interpolation property.

To do that we use a result (of Lemma 4.3 and Remark 4.4 from [2]) that gives

$$|a|_{\Sigma_\theta =} \underset{a_0(t)+a_1(t)=a}{Inf} \left|t^\theta \hat{c} a_0\right|^{1-\theta}_{L^p(D(\Lambda)} \left|t^{\theta-1}\hat{c} a_1\right|^\theta . \tag{3.3}$$

Consider, another set (A_1, Λ_1) like (A, Λ) and let $\pi \in \mathcal{L}(A, A_1)$ (with norm ω_0 in that space) which restricted on $D(\Lambda)$ *belongs to* $\mathcal{L}(D(\Lambda), D(\Lambda_1))$, (with norm ω_1) then we have to prove that

$$\pi \in \mathcal{L}(\Sigma(p, \theta, \hat{c}, D(\Lambda), p, \theta - 1, \hat{c}, A), \Sigma(p, \theta, \hat{c}, D(\Lambda_1); p\theta - 1, \hat{c}, A_1).$$

Then if we denote for the moment Σ^i_θ, $i = 0, 1$, the space Σ_θ by (A, Λ) (*resp.* (A_1, Λ_1)), we have when $a \in \Sigma^0_\theta$, *that* $\pi a = \pi a_0 + \pi a_1$, and (obviously) with the notations used in inequality (3.3) :

$$\begin{aligned}|\pi a|_{\Sigma^1_\theta} &\leq \left|t^\theta \hat{c}\pi a_0\right|^{1-\theta}_{L^p(D(\Lambda_1))} \left|t^{\theta-1}\hat{c}\pi a_1\right|_{L^p(A_1)} \\ &\leq \omega_0^{1-\theta}\omega_1^\theta \left|t^\theta \hat{c} a_0\right|_{L^p(D(\Lambda)} \left|t^{\theta-1}\hat{c} a_1\right|_{L^p(A)}\end{aligned}$$

and, again with (3.3), we obtain

$$|\pi a|_{\Sigma^1_\theta} \leq \omega_0^{1-\theta}\omega_1^\theta \, |a|_{\Sigma^0_\theta},$$

and the theorem is proved.

If we denote by ω the norm of $\pi \in \mathcal{L}(\Sigma_\theta^0, \Sigma_\theta^1)$, this shows that

$$\omega \leq \omega_0^{1-\theta}\omega_1^{\theta}.$$

From the comparison with traces spaces made in [2], we obtain

Theorem 3.2 *When $\theta = 1/p$, $1 \leq p < +\infty$, it follows that*

$$\Sigma_{1/p} = \Sigma(p, 1/p, \hat{c}, D(\Lambda); p, -1/p\prime, \hat{c}, A) \equiv T(p, c; D(\Lambda), A) \tag{3.4}$$

with equivalent norms.

Proof See [2] and (Sect. 5, Theorem 5.2).

Another definition of the space Σ_θ is given in [2]: Consider a function $t \longrightarrow v(t)$, defined on $R^+$with values in $D(\Lambda)$, and assume that v belongs to the space

$$V = \{v; v(t) \in D(\Lambda)\ a.e.,\ t^\theta \hat{c}v \in L^p(D(\Lambda), t^{\theta-1}\hat{c}v \in L^p(A)\}$$

which is a Banach Space when equipped with the natural norm.

Then when

$$t^{-\theta}[\hat{c}(t)]^{-1} \in \Psi \tag{3.5}$$

the integral $\int_0^{+\infty} \frac{v(t)}{t}dt$ exists in A.

Proposition 3.3 *Consider the space spanned by $a = \int_0^{+\infty} \frac{v(t)}{t}dt$. When v spans the space V, which is a Banach space provided with the norm*

$$\underset{\int_0^{+\infty} \frac{v(t)}{t}dt = a}{Inf} \quad [max\ (\left|t^\theta \hat{c}v\right|_{L^p(D(\Lambda))}, \left|t^{\theta-1}\hat{c}v\right|_{L^p(A)})], \tag{3.6}$$

then that space can be identified with the space Σ_θ.

Proof (see [2, Theorem 4.2]).

Remark 3.4 The condition (3.5) means that

$$\int_0^1 \frac{t^{-\theta p\prime}}{[\hat{c}(t)]^{p\prime}}dt < \infty, \int_1^{+\infty} \frac{t^{-\theta p\prime}}{[t\hat{c}(t)]^{p\prime}} < \infty,$$

we know that is implied by the condition $t^{-\theta}[\hat{c}(t)]^{-1} \in \mathcal{H}(p\prime)$ which is satisfied when $[\hat{c}(t)]^{-1} \in \mathcal{H}(p\prime)$.

Henceforth, we assume that the last conditon holds.

We may introduce a variant for the Space Σ of section2: Let the space Σ_θ, $\theta \in (0, 1)$,be the subspace of A such that

$$\hat{c}(t)t^{-(1-\theta)}(G(t)a - a) \in L^p(A). \tag{3.7}$$

Equipped with the norm

$$|a|_{\Sigma_\theta} = |a| + [\int_0^{+\infty} t^{-(1-\theta)p} \hat{c}^p(t)\,|G(t)a - a|^p\,dt]^{1/p}, \tag{3.8}$$

Σ_θ is a Banach space.

Remark 3.5 Observe that when $G(t)$ is relaced by $e^{-kt}G(t),\ k > 0$, Λ becomes $\Lambda + kI$ with the same domain $D(\Lambda)$, and consequently

$$\|G(t)\| \leq Me^{-kt},\ k > 0 \tag{3.9}$$

without change of the space $\hat{\Sigma}_\theta$. Moreover, the space Σ_θ is not changed when we assume that (3.7) holds.

Thus we can assume now that (3.7) holds and since a space does not change when we replace its norm with an equivalent norm,

we can take $\|a\|_1 = \underset{t\geq 0}{Sup}\,|G(t)a|$which is equivalent to $|a|$ (note that $|a| \leq \|a\|_1 \leq M\,|a|$ and the new norm of $G(t)$ is $\underset{t\geq 0}{Sup}\,\frac{\|G(t)a\|_1}{\|a\|_1} \leq 1$.)

Then we may assume that $G(t)$ has a norm ≤ 1, and eventually, if necessary, we can change again $G(t)\ in\ G(t)e^{-kt}$ to have $\|G(t)\| \leq e^{-kt}$, $k > 0$. Thus we will assume in what follows that

$$(I - G(t))^{-1}\ exists\ for\ t \geq 0. \tag{3.10}$$

Another variant for the space $\hat{\Sigma}_\theta$, is the space S_θ such that a function v can be found with values $a.e.\ in\ D(\Lambda)$ for which $a = \int_0^{+\infty} \frac{v(t)}{t} dt\ in\ A$, and

$$t^\theta \hat{c}(t)(I - G(t))^{-1}\Lambda v(t) \in L^p(A). \tag{3.11}$$

Then S_θ become a Banach space equipped with the norm

$$|a|_{S_\theta=} |a| + [\int_0^{+\infty} [t^\theta \hat{c}(t)]^p\,|(I - G(t)\Lambda v(t)|^p\,dt]^{1/p}. \tag{3.12}$$

Therefore the main result of Sect 3 is

Theorem 3.6 *Assume that $G(t)$ satisfies the SG-condition and (3.13)*, $1 \leq p \leq \infty$, *and also that $\hat{c} \in \mathcal{H}(p)$, $[\hat{c}]^{-1} \in \mathcal{H}(p\prime)$.*

Then the following conditions are equivalent :

(1) - $a \in \hat{\Sigma}_\theta = \Sigma(p, \theta, \hat{c}, D(\Lambda); p, \theta - 1, \hat{c}, A)$
(2) - $a \in \Sigma_\theta$.
(3) - $a \in S_\theta$.

An immediate deduction is that $\hat{\Sigma}_\theta = \Sigma_\theta = S_\theta$ with equivalent norms. Note that the conditions upon the weight may be relaxed.

Proof I) (1)$\Longrightarrow$(2).

From the definition when $a \in \hat{\Sigma}_\theta$ we have together with (3.1) that $a = a_0(t) + a_1(t)$ *a.e.* But we know that $|(I - G(t))a| \leq t\,|\Lambda a|$, for every $a \in D(\Lambda)$, and therefore

$$|(I - G(t)a_0(t)| \leq t\,|\Lambda a_0(t)|\,.$$

Since $\|I - G(t)\| \leq 2$, consequently

$$|(I - G(t))a| \leq t\,|\Lambda a_0(t)| + 2\,|a_1(t)|\,,$$

and the result follows from (3.1).

II) (3)$\Longrightarrow$(1).

Let v satisfy (3.11). We must to check that

$$t^\theta \hat{c} v \in L^p(D(\Lambda)), \quad t^{\theta-1}\hat{c} v \in L^p(A) \tag{3.13}$$

which, from remark 3.5, implies that (1) holds.

We can write

$$\Lambda v(t) = (I - G(t))\,(I - G(t))^{-1}\,\Lambda v(t)$$

and as $\|I - G(t)\| \leq 2$, the first condition (3.13) is obtained from (3.11).

Then, since

$$v(t) = (I - G(t))\,(I - G(t))^{-1}\,v(t)$$

we conclude that

$$|v(t)|_A \leq t\left|(I - G(t))^{-1}\,\Lambda v(t)\right|_A$$

and the second condition of (3.13) is satisfied from (3.11).

III) (2) $\Longrightarrow$(3).

Assume given $a \in \Sigma_\theta$, and define v by

$$v(t) = \frac{k}{t}(I - G(t))^2\Lambda^{-1}a \tag{3.14}$$

where k is a constant to be chosen.

Now $\hat{c}(t)t^\theta\,(I - G(t))^{-1}\,\Lambda v(t) = k\hat{c}(t)t^{-(1-\theta)}\,(I - G(t))\,a \in L^p(A)$, because $a \in \Sigma_\theta$. Thus v satisfies (3.11), which implies that (3.13) is satisfied so that the integral $I(v) = \int_0^{+\infty} \frac{v(t)}{t} dt$ exists.

It remains to prove that

$$I(v) = \underset{\epsilon \longrightarrow +0}{Lim}\, k \int_\epsilon^{+\infty} t^{-2}\,(I - G(t))^2\,\Lambda^{-1} a dt = \Lambda[\Lambda^{-1}a] = a. \tag{3.15}$$

with a convenient choice of k found with the help of Lemmas (1-1) and (1-2) ([13, p. 53]) for $\alpha = 1$, $\mu = 2$. Then (3.15) is true when $k^{-1} = -\int_0^{+\infty} t^{-2}(1-e^{-1})^2 dt$ (see also in Sect. 5).

4 Commutative semi-groups

One consider ν *unbounded operators* $\Lambda_i,\ i = 1, 2 \ldots, \nu$ with domain $D(\Lambda_i)$ *dense in* A, every Λ_i being the infinitesimal generator of a semi-group $G_i(t)$, satisfying the *SG-condition*.

Moreover, assume that

$$\forall i, j, \ and\ \forall (s,t) \geq 0,\ G_i(s)G_j(t) = G_j(t)G_i(s). \tag{4.1}$$

This implies that $G_j(t) \in \mathcal{L}(D(\Lambda_i), D(\Lambda_j))$ and consequently that

$$\forall a \in D(\Lambda_i),\quad \Lambda_i G_j(t)a = G_j(t)\Lambda_i a. \tag{4.2}$$

Definition 4.1 We denote by $W(p, c, \Lambda_1, \ldots, \Lambda_\nu)$ the space of functions u satisfying

$$cu \in L^p(R^+; D(\Lambda_i)),\quad i = 1, \ldots, \nu, \tag{4.3}$$

$$c\frac{du}{dt} \in L^p(R^+; A), \tag{4.4}$$

with $1 < p \leq \infty,\ c \in \mathcal{H}(p)$.

Provided with the norm

$$\left(\int_0^\infty c^p(t)[|u(t)|^p + \sum_{i=1}^{\nu} |\Lambda_i u(t)|^p + |u\prime(t)|^p]dt\right)^{1/p},$$

the space $W(p, c, \Lambda_1, \ldots, \Lambda_\nu)$ is a Banach space.

Definition 4.2 Denote by $\Sigma(p, c, \Lambda_1, \ldots, \Lambda_\nu)$ the space

$$a \in A,\ t^{-1}c(t)\,(G_i(t)a - a)) \in L^p(R^+; A),\ i = 1, \ldots, \nu. \tag{4.5}$$

Equipped with the norm

$$|a| + \sum_{i=1}^{\nu}\left(\int_0^{+\infty} [t^{-1}c(t)]^p\,|G_i(t)a - a|^p\,dt\right)^{1/p},$$

$\Sigma(p, c, \Lambda_1, \ldots, \Lambda_\nu)$ becomes a Banach space.

Denote by $D(\Lambda_1, \ldots, \Lambda_\nu)$, *the space of* $a \in D(\Lambda_i),\ for\ i = 1, \ldots, \nu$, which is a Banach space when provided with the norm

$$|a|_A + \sum_{i=1}^{\nu} |\Lambda_i a|_A.$$

We have

$$D(\Lambda_1, \ldots, \Lambda_\nu) \subset \Sigma(p, c, \Lambda_1, \ldots, \Lambda_\nu) \subset A,$$

and we want to show that the space $\Sigma(p, c, \Lambda_1, \ldots, \Lambda_\nu)$ can be identified with the trace space $T(p, c, \Lambda_1, \ldots, \Lambda_\nu)$ of $W(p, c, \Lambda_1, \ldots, \Lambda_\nu)$.

For this purpose we have

Theorem 4.3 *Assume that every* Λ_i *satisfies the SG-condition and that (4.1) holds. Moreover assume* $c \in \mathcal{H}(p),\ 1 \leq p \leq \infty$. *Then the linear mapping*

$$u \longrightarrow u(0)\ is\ continuous\ from\ W(p, c, \Lambda_1, \ldots, \Lambda_\nu)\ onto\ \Sigma(p, c, \Lambda_1, \ldots, \Lambda_\nu).$$

Proof On recalling the first step in the proof for the Theorem 2.5, we have only to prove that the mapping is "$onto''$.

Let $a \in \Sigma(p, c, \Lambda_1, \ldots, \Lambda_\nu)$. To construct a function $u \in W(p, c, \Lambda_1, \ldots, \Lambda_\nu)$, satisfying $u(o) = a$, we let

$$v_i(t) = t^{-1} \int_0^t G_i(\tau) d\tau,$$

and define[7]

$$v(t) = v_1(t) v_2(t) \ldots v_\nu(t) a.$$

In what follows, to fix the ideas, it is sufficient to consider only the case when $\nu = 2$.

Since $v_i(t)\ commutes\ with\ G_i(t)$, Lemma 2.6 implies,

$$\Lambda_i v_i(t) = t^{-1} \left(G_i(t) - I \right)$$

and therefore $\|v_i(t)\| \leq M$, follows from $\|G_i(t)\| \leq M_i$.

Then

$$\Lambda_1 v(t) = t^{-1} \left(G_1(t) a - a \right) V_2(t)$$

[7] This is an idea of Gagliardo [9].

and

$$\|\Lambda_1 v(t)\| \leq M_2 t^{-1} \left| G_1(t)a - a \right|.$$

So that $c\Lambda_1 v \in L^p(A)$, and we deduce

$$c\Lambda_i v(t) \in L^p(A), \quad \forall i \equiv 1, 2, \ldots, \nu. \tag{4.6}$$

Now to consider $v\prime(t)$:

$$v\prime(t) = w_1(t) + w_2(t)$$

with

$$\begin{aligned} w_1(t) &= v\prime_1(t) v_2(t) a, \\ w_2(t) &= v_1(t) v\prime_2(t) a. \end{aligned} \tag{4.7}$$

To prove that

$$cv\prime \in L^p(A), \tag{4.8}$$

it is sufficient to prove

$$cw_1 \in L^p(A). \tag{4.9}$$

The second step in the proof of Theorem 2.2, gives

$$v\prime_i(t) = \Lambda_i v_i(t) - t^{-2} \int_0^t (G_i(\tau) - I)\, d\tau,$$

and thus

$$\begin{aligned} w_1 = w_1^1 + w_1^2, \ w_1^1(t) &= \Lambda_1 v_1(t) v_2(t) a, \\ w_1^2(t) = t^{-2} \int_0^t & (G_1(\tau) - I)\, v_2(\tau) a d\tau. \end{aligned}$$

But (4.6), implies $cw_1^1 \in L^p(A)$, and it remains only to estimate w_1^2. Since $\|v_2(t)\| < M_2$, we have

$$\left| w_1^2(t) \right| \leq M_2 t^{-1} \int_0^t t^{-1} \left| G(\tau)a - a \right| d\tau \leq M_2 t^{-1} \int_0^t \tau^{-1} \left| G(\tau)a - a \right| d\tau,$$

so that, from the definition of the space $\Sigma(p, c, \Lambda_1, \ldots, \Lambda_\nu)$ and from the weighted Hardy inequality, we obtain $cw_1^2 \in L^p(A)$. Thus (4.8) is proved.

Finally as in the proof of Theorem 2.2, we could define $u \in W(p, c, \Lambda_1, \ldots, \Lambda_\nu)$ by $u(t) = \Phi(t)v(t)$ with $u(0) = a$.

Example 4.4 We consider the space $A = L^q(R^n)$. Let $x = (x_1, \ldots, x_n) \in R^n$ and denote by

$$\Lambda_i = \partial/\partial x_i$$

the infinitesimal generator of the semi-group defined by

$$G_i(t) f(x) = f(x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n).$$

The hypothesis of Theorem 4.3 is satisfied, and the space

$$D(\Lambda_1, \ldots, \Lambda_n) = W^{1,q}(R^n) = \{f;\ f \in L^q(R^n),\ \partial f/\partial x_i \in L^q(R^n),\ i = 1, \ldots, n\}$$

is the usual Sobolev space of order 1 (see [16]).

Let $\Omega = \{x, t;\ t \geq 0) \subset R^n_x \times R_t$, and assume that $f \in L^p_t[L^p_x(\Omega_t)] \simeq \int_0^\infty \left(\int_{R^n_x} |f(x,t)|^q dx\right)^{p/q} dt$. Then "$u \in W(p, c, \Lambda_1, \ldots, \Lambda_n)$" is equivalent to

$$cu,\ c(\partial u/\partial x_i) \in L^p_t[L^p_x(R^n)]\ i = 1, \ldots, n,\ \ c(\partial u/\partial t) \in L^p_t[L^p_x(R^n)], \tag{4.10}$$

and Theorem 4.3 shows that the mapping $u \longrightarrow u(x, o)$ is continuous from the space of u satisfying (4.10) onto the functions f such that

$$f \in L^q(R^n_x) \tag{4.11}$$

$$\int_0^\infty [t^{-1}c(t)]^p \left(\int_{R^n_x} |f(x_1, \ldots, x_i + t, \ldots, x_n) - f(x)|^q\, dx\right)^{p/q} < \infty, \quad \forall i = 1, \ldots, n. \tag{4.12}$$

For $p = q$, $c(t) \equiv 1$, the result is given in Gagliardo [9] where the case $p = 1$ is solved. For $c(t) = t^\alpha$, $\alpha + 1/p \in]0, 1[$ (where $t^\alpha \in \mathcal{H}(p)$ *and* $t^{-\alpha} \in \mathcal{H}(p\prime)$). See, for example, the discussion by: Lions [12], Peetre [16] and also Slobodetskii [17], and Vacherin [20] ($p = q = 2,\ \alpha \in (0, 1/2)$). Nevertheless, as far as the present author is aware, there are few, if any, results for weights spaces belonging generally to $\mathcal{H}(p)$.

5 Intermediate mean spaces between $D(\Lambda^m)$ and A

Let Λ, $G(t)$, satisfying the *SG-condition*. For $m \in N,\ m \geq 1$ we denote by $D(\Lambda^m)$, the space of $a \in D(\Lambda)$, such that $\Lambda a \in D(\Lambda), \ldots,\ \Lambda^{m-1}a \in D(\Lambda)$, which equipped with the graph norm:

$$\sum_{i=0}^{m} \left|\Lambda^i a\right|,$$

is a Banach space.

The main Lemma of Lions ([13, pp. 53–54]) uses the notion of distribution semi-groups, and is reproduced as lemma below.

Lemma 5.1 *Let* $\alpha \in N,\ \alpha \geq 1$ *be given. Consider* $\mu \in N,\ \mu \geq \alpha + 1$, *and put* $k_{\alpha,\mu} = \int_0^\infty t^{-(\alpha+1)} \left(1 - e^{-t}\right)^\mu dt$.

On the other hand let $a \in A$ *such that*

$$\underset{\epsilon \longrightarrow +0}{Lim} \int_\epsilon^\infty t^{-(\alpha+1)} (I - G(t)^\mu \cdot adt,\ exists.$$

Therefore $a \in D(\Lambda^\alpha)$ *and*

$$\underset{\epsilon \longrightarrow +0}{Lim} \frac{(-1)^\alpha}{k_{\alpha,\mu}} \int_\epsilon^\infty t^{-(\alpha+1)} (I - G(t))^\mu \cdot adt = \Lambda^\alpha(a). \tag{5.1}$$

Now we return to Theorem 3.6 where we change $D(\Lambda)$ in $D(\Lambda^m)$ and θ in $m\theta$. We have

Theorem 5.2 *Assume that* $G(t)$ *satisfies the SG-condition, and (3.13) with* $1 \leq p \leq \infty$, *and also that* $\hat{c} \in \mathcal{H}(p),\ [\hat{c}]^{-1} \in \mathcal{H}(p\prime)$.

Then the following conditions are equivalent:

(1) $a \in \hat{\Sigma}(p, \theta m, , \hat{c}, D(\Lambda^m); p, (\theta - 1)m, \hat{c}, A)$,

(2) $a \in \Sigma_{\theta m}$,

(3) $a \in S_{\theta m}$.

We have

$$\Sigma_{\theta m} = \{a \in A,\ t^{-(1-\theta)m} \hat{c}\, (I - G(t))^m\, a \in L^p(A)\}, \tag{5.2}$$

and $S_{\theta m}$ is the space of $a \in A$, such that a function v can be found with values *a.e. in* $D(\Lambda^m)$ such that

$$\int_0^\infty \frac{v(t)}{t} dt = a \ in\ A,\ and\ t^{m\theta} \hat{c}\, (I - G(t))^{-m}\, \Lambda^m v(t) \in L^p(A). \tag{5.3}$$

Proof First we use the homogeneity result of [2], which states

$$\hat{\Sigma}(p, \theta m, \hat{c}, D(\Lambda^m); p, (\theta - 1)m, \hat{c}, A) = \hat{\Sigma} p, \theta,\ \hat{c}, D(\Lambda^m); p, \theta - 1), \hat{c}, A).$$

Then the same proof as that for Theorem 3.6 (upto some obvious modifications) shows (1)$\Longrightarrow$(2) and (3)$\Longrightarrow$(1). To prove that (2)$\Longrightarrow$(3), define $v(t) = kt^{-m}(I - G(t))^{2m} \Lambda^{-m} a$, with $k = (-1)^m / K_{2m,m}$ (notation of the Lemma 5.1). The proof is analogous to that of Theorem 3.6 on using the Lemma 5.1.

Now when

$$(1 - \theta)m = \alpha, \tag{5.4}$$

Theorem 5.2 gives

$$D(\Lambda^\alpha) = \hat{\Sigma}(p, \theta, \hat{c}, D(\Lambda^m); p, \theta - 1, \hat{c}, A), \tag{5.5}$$

with equivalent norm.

Let $X_i = \hat{\Sigma}(p, \theta_i, \hat{c}, D(\Lambda^m); p, \theta_i - 1, \hat{c}, A),\ i = 0, 1$. From the reiteration result for "mean weighted spaces" (see [2, Proposition 6.7]), we have

$$\hat{\Sigma}(p, \xi_0, \hat{c}, D(\Lambda^m); p, \xi_1, \hat{c}, A) = \hat{\Sigma}(p, \eta_0, \hat{c}, X_0; p, \eta_1, \hat{c}, X_1), \tag{5.6}$$

where

$$\theta = \frac{\xi_0}{\xi_0 - \xi_1},\ \theta_0 < \theta < \theta_1,\ \eta_i = (\xi_0 - \xi_1)(\theta - \theta_i), \quad i = 0, 1. \tag{5.7}$$

On choosing $(1-\theta_0)m = j + 1$ (resp. $(1-\theta_1)m = j$) to give $X_0 = D(\Lambda^{j+1})$ (resp. $X_1 = D(\Lambda^j)$) and on taking $\xi_0 = \theta$, (resp. $\xi_1 = \theta - 1$) in (5.6) we can claim

Theorem 5.3 *Assume,* $(1-\theta)m = j + \eta,\ 0 < j \in N,\ 0 < \eta < 1$*, then*

$$\hat{\Sigma}(p, \theta, \hat{c}, D(\Lambda^m); p, \theta - 1, \hat{c}, A) = \hat{\Sigma}(p, 1-\eta, \hat{c}, D(\Lambda^{j+1}); p, -\eta, \hat{c}, D(\Lambda^j)), \tag{5.8}$$

with equivalent norms.

Moreover, Λ *being an isomorphism from* $D(\Lambda)$ *onto A,from (3.10), we obtain*

$$\begin{aligned}&\hat{\Sigma}(p, \theta, \hat{c}, D(\Lambda^m); p, \theta - 1, \hat{c}, A)\\ &\quad = \{a; a \in D(\Lambda^j),\ \Lambda^j(a) \in \hat{\Sigma}(p, 1-\eta, \hat{c}, D(\Lambda); p, -\eta, \hat{c}, A),\end{aligned} \tag{5.9}$$

with equivalent norms.

For completeness, we must consider now the case where $(1-\theta)m \in N$. Let $(1-\theta)m = j + 1$, and define $\hat{\Sigma}(p, \theta, \hat{c}, D(\Lambda^m); p, \theta - 1, \hat{c}, A)$ as an intermediate space between $D(\Lambda^{j+2})$ and $D(\Lambda^j)$. By choosing $(1-\theta_0)m = j+2$ (resp. $(1-\theta_1)m = j$, the reiteration theorem gives

$$\hat{\Sigma}(p, \theta, \hat{c}, D(\Lambda^m); p, \theta - 1, \hat{c}, A) = \hat{\Sigma}(p, \eta_0, \hat{c}, D(\Lambda^{j+2});\ \ p, \eta_1, \hat{c}, D(\Lambda^j)),$$

where

$$\eta_0 = \theta - \theta_0, \quad \eta_1 = \theta - \theta_1.$$

Since $m\eta_0 = 1,\ m\eta_1 = -1$, we therefore have

$$\hat{\Sigma}(p, \theta, \hat{c}, D(\Lambda^m);\ \ p, \theta - 1, \hat{c}, A = \hat{\Sigma}(p, \frac{1}{2}, \hat{c}, D(\Lambda^{j+2}); p, -\frac{1}{2}, \hat{c}, D(\Lambda^j))$$

and as Λ^j is an isomorphism from $\hat{\Sigma}(p, \frac{1}{2}, \hat{c}, D(\Lambda^{j+2}), p, -\frac{1}{2}, \hat{c}, D(\Lambda^j))$ onto the space

$$\hat{\Sigma}(p, \frac{1}{2}, \hat{c}, D(\Lambda^2);\ \ p, -\frac{1}{2}, \hat{c}, A),$$

we have proved:

Theorem 5.4 *Assume* $(1-\theta)m \in N$. *The choice* $(1-\theta)m = j = 1$, *leads to the relation*

$$\hat{\Sigma}(p, \theta, \hat{c}, D(\Lambda^m); p, \theta - 1, \hat{c}, A)$$
$$= \{a; a \in D(\Lambda^j), \Lambda^j(a) \in \hat{\Sigma}(p, \frac{1}{2}, \hat{c}, D(\Lambda^2);\ p, -\frac{1}{2}, \hat{c}, A)\}.$$

Moreover the norms

$$|a|_{\hat{\Sigma}(p,\theta,\hat{c},D(\Lambda^m);p,\theta-1,\hat{c},A)} \quad \textit{and} \quad |a|_{D(\Lambda^j)} + \left|\Lambda^j a\right|_{\hat{\Sigma}(p,\frac{1}{2},\hat{c},D(\Lambda^2);p,-\frac{1}{2},\hat{c},A)}$$

are equivalent.

Example 5.5 We can consider as in Example 4.4, the case of $A = L^q(R^n)$, $1 < q < \infty$ along with the operators

$$\Lambda_i = \partial/\partial x_i, \quad i = 1, \ldots, n$$

which are infinitesimal generators of

$$G_i(t)f(x) = f(x_1, \ldots, x_{i-1}, x_i + t, x_{i+1}, \ldots, x_n).$$

In consequence

$$D(\Lambda_i) = \{v;\ v \in L^q(R^n), \quad \frac{\partial v}{\partial x_i} \in L^q(R^n)\}$$

and $D(\Lambda) = D(\Lambda_1, \ldots, \Lambda_n) = W^{1,q}(R^n)$.

Then denote by $D(\Lambda^m)$, the space of $v \in A$, with $\Lambda^{\alpha_1}v, \ldots, \Lambda_n^{\alpha_n}v \in L^q(R^n)$, $\forall(\alpha_1, \ldots, \alpha_n)$, such that $\alpha_1 + \cdots + \alpha_n \leq m$, and $\alpha_i \in N$, $\alpha_i \geq 0$. This gives

$$D(\Lambda^m) = W^{m,q}(R^n) = \{v; D^\alpha v \in L^q(R^n), |\alpha| \leq m\} \tag{5.10}$$

which is a Sobolev space provided with the usual norm

$$\sum_{|\alpha| \leq m} \left|D^\alpha v\right|_{L^p(R^n)}.$$

If we consider the case where $c \equiv 1$ (called the unweighted case) and put

$$\hat{\Sigma}(p, \theta, t^{-1/p}, W^{m,q}(R^n); p, \theta - 1, t^{-1/p}, L^q(R^n)) = B^{(1-\theta)m,q}(R^n) \tag{5.11}$$

with $(1-\theta)m = j + \eta$, $j \in N$, $0 < \eta < 1$, then [13] implies

$$a \in B^{(1-\theta)m,q}(R^n)$$

which means

(i) $v \in W^{j,q}(R^n)$,
(ii) $\forall\alpha, |\alpha| = j, \ and \quad \forall i, \ t^{-(1/p+\eta)}\left(G_i(t)D^\alpha v - D^\alpha v\right) \in L_t^p(L_x^q(R^n)$.

The spaces $B^{(1-\theta)m,q}(R^n)$ are Besov spaces (see [18]).
In our case when we put for instance $m = 1$, Theorem 3.6 yields

$$\hat{\Sigma}(p, \theta, \hat{c}, W^{1,q}(R^n); \ p, \theta - 1, \hat{c}, L^q(R^n) = B_c^{(1-\theta)m,q}(R^n), \tag{5.12}$$

where formally $a \in B_c^{1-\theta,q}(R^n)$ means:

(i) $a \in W^{1,q}(R^n)$
(ii) $\forall i \in \{1, \ldots, n\}, \ t^{-(1-\theta)}\hat{c}\,(a - G_i(t)a) \in L_t^p[L_x^q(R^n)$.

Thus the interpretation of applications in terms of intermediate mean spaces leads to the introduction of some weighted types of Besov spaces whose the properties are unknown at the least to the present author.

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