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A note on autocentral automorphisms of groups

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Abstract The absolute centre $L(G)$ of a group G is the subgroup of all elements fixed by every automorphism of G, and an automorphism of G is autocentral if it acts trivially on the factor group $G/L(G)$. Autocentral automorphisms have been introduced by Moghaddam and Safa (Ricerche Mat 59:257–264, 2010). The aim of this paper is to obtain new informations on the behaviour of autocentral automorphisms of a group. We also consider the relations between the group of autocentral automorphisms and that of class preserving automorphisms of a group.

Keywords Absolute centre · Autocentral automorphism · Class preserving automorphism

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1 Introduction

Let G be a group. If g is an element of G and α is an automorphism of G, the element

$$[g, \alpha] = g^{-1}g^{\alpha} = g^{-1}\alpha(g)$$

is the *autocommutator* of g and α. Of course, if α is the inner automorphism determined by an element a of G, then the autocommutator $[g, \alpha]$ coincides with the ordinary commutator $[g, a]$. The set $L(G)$, consisting of all elements g of G fixed by every automorphism of G is a central characteristic subgroup of G, which is called the *absolute centre* (or the *autocentre*) of G. Then

$$L(G) = \{g \in G \mid [g, \alpha] = 1 \; \forall \alpha \in \mathrm{Aut}(G)\},$$

where $\mathrm{Aut}(G)$ is the group of all automorphisms of G. Moreover, the subgroup $K(G)$ generated by all autocommutators $[g, \alpha]$ (where $g \in G$ and $\alpha \in \mathrm{Aut}(G)$) is the *autocommutator subgroup* of G. Observe that if in the above considerations the full automorphism group $\mathrm{Aut}(G)$ is replaced by the group $\mathrm{Inn}(G)$ of all inner automorphisms of G, then we obtain the usual definitions of centre and commutator subgroup.

The absolute centre and the autocommutator subgroup have been introduced by Hegarty [1], who proved in particular that if the absolute centre $L(G)$ of a group G has finite index, then both the autocommutator subgroup $K(G)$ and the automorphism group $\mathrm{Aut}(G)$ are finite, a result that must be compared with the celebrated theorem of Schur [4] on the finiteness of the commutator subgroup of a central-by-finite group.

Let G be a group, and let N be a characteristic subgroup of G. We shall denote by $\mathrm{Aut}_N(G)$ the normal subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms of G inducing the identity on the factor group G/N, i.e.

$$\mathrm{Aut}_N(G) = \{\alpha \in \mathrm{Aut}(G) \mid [g, \alpha] \in N \; \forall g \in G\}.$$

In particular, if we choose $N = Z(G)$, then $\mathrm{Aut}_{Z(G)}(G)$ is precisely the group $\mathrm{Aut}_c(G)$ of all central automorphisms of G. If $N = L(G)$, the group $\mathrm{Aut}_{L(G)}(G)$ will be denoted here by $\mathrm{Aut}_L(G)$; the elements of $\mathrm{Aut}_L(G)$ are called *autocentral automorphisms* of G (see also [2], where this concept has been introduced). Our main result on the group of autocentral automorphisms is the following.

Theorem *Let G and $\bar{G}$ be groups such that $L(G) \simeq L(\bar{G})$ and $G/L(G) \simeq \bar{G}/L(\bar{G})$. Then the groups of autocentral automorphisms $\mathrm{Aut}_L(G)$ and $\mathrm{Aut}_L(\bar{G})$ are isomorphic.*

Recall also that an automorphism α of a group G is called *class preserving* if the image x^{α} belongs to the conjugacy class x^G, for all $x \in G$. The set $\mathrm{Aut}^c(G)$ of all class preserving automorphisms of G is a normal subgroup of $\mathrm{Aut}(G)$, which of course contains the inner automorphism group $\mathrm{Inn}(G)$. Class preserving automorphisms were introduced by Yadav [5], who investigated conditions under which two (finite) groups have isomorphic class preserving automorphism groups. The relations between the groups $\mathrm{Aut}_L(G)$ and $\mathrm{Aut}^c(G)$ are considered in the last part of the paper.

Most of our notation is standard and can be found in [3].

2 Statements and proofs

Let G be a group. If x is an element of G and α is an automorphism of G, the autocommutator $[x, \alpha]$ is of course an ordinary commutator in the holomorph group of G. Thus the following lemma is just an application of the usual commutator laws; it gives rules that can be used in the study of autocommutators.

Lemma 1 *Let x and y be elements of a group G and $\alpha, \beta \in Aut(G)$. Then the following identities hold:*

(a) $[xy, \alpha] = [x, \alpha]^y [y, \alpha]$;
(b) $[x, \alpha^{-1}] = ([x, \alpha]^{-1})^{\alpha^{-1}}$;
(c) $[x^{-1}, \alpha] = ([x, \alpha]^{-1})^{x^{-1}}$;
(d) $[x, \alpha\beta] = [x, \beta][x, \alpha]^{\beta} = [x, \beta][x, \alpha][x, \alpha, \beta]$;
(e) $[x, \alpha]^{\beta} = [x^{\beta}, \alpha^{\beta}]$;
(f) $[x, \alpha^{-1}, \beta]^{\alpha}[\alpha, \beta^{-1}, x]^{\beta}[\beta, x^{-1}, \alpha]^{x} = 1$.

Our main theorem is a special case of the following result.

Theorem 1 *Let G and H be groups, and let M and N be characteristic subgroups of G and H, respectively, such that $M \leq L(G)$ and $N \leq L(H)$. If $G/M \simeq H/N$ and $M \simeq N$, then the groups $Aut_M(G)$ and $Aut_N(H)$ are isomorphic.*

Proof Let Let $\varphi : G/M \longrightarrow H/N$ and $\psi : M \longrightarrow N$ be isomorphisms. Let α be any element of the group $\mathrm{Aut}_M(G)$. Clearly, for each element h of H there exists an element g_h of G such that $\varphi(g_h M) = hN$ and $h \in N$ if and only if $g_h \in M$. If g is any other element of G such that $\varphi(gM) = hN$, the product $g_h g^{-1}$ belongs to M and hence

$$g_h^{\alpha}(g^{-1})^{\alpha} = (g_h g^{-1})^{\alpha} = g_h g^{-1},$$

because $M \leq L(G)$. Then

$$[g, \alpha] = g^{-1} g^{\alpha} = g_h^{-1} g_h^{\alpha} = [g_h, \alpha].$$

As the autocommutator $[g_h, \alpha]$ belongs to M, the above equality allows us to define a new map

$$f_{\alpha} : H \longrightarrow H$$

by putting

$$f_{\alpha}(h) = h\psi([g_h, \alpha])$$

for each element h of H. Observe here that if h belongs to N, then g_h lies in M, so that $[g_h, \alpha] = 1$ and hence $f_{\alpha}(h) = h$, i.e. the restriction of f_{α} to N is the identity map.

Let h_1 and h_2 be arbitrary elements of H. Clearly,

$$\varphi(g_{h_1}g_{h_2}M) = \varphi(g_{h_1}M)\varphi(g_{h_2}M) = h_1h_2N,$$

and so $[g_{h_1}g_{h_2}, \alpha] = [g_{h_1h_2}, \alpha]$. Since M is contained in $Z(G)$ and N lies in $Z(H)$, it follows that

$$\begin{aligned} f_\alpha(h_1)f_\alpha(h_2) &= h_1\psi([g_{h_1}, \alpha])h_2\psi([g_{h_2}, \alpha]) = h_1h_2\psi([g_{h_1}, \alpha][g_{h_2}, \alpha]) \\ &= h_1h_2\psi([g_{h_1}g_{h_2}, \alpha]) = h_1h_2\psi([g_{h_1h_2}, \alpha]) = f_\alpha(h_1h_2). \end{aligned}$$

Therefore f_α is a homomorphism.

Let k be an element of the kernel of f_α. Then

$$1 = f_\alpha(k) = h\psi([g_k, \alpha]),$$

so that $k = \psi([g_k, \alpha])^{-1}$ belongs to N, and hence $k = f_\alpha(k) = 1$. Therefore the homomorphism f_α is injective. Moreover, if h is any element of H, we have

$$\begin{aligned} f_\alpha\big(h\psi([g_h, \alpha]^{-1}\big) &= f_\alpha(h)\psi([g_h, \alpha]^{-1}) \\ &= h\psi([g_h, \alpha])\psi([g_h, \alpha]^{-1}) = h. \end{aligned}$$

It follows that f_α is also surjective, and hence it is an automorphism of H. Observe also that $[h, f_\alpha] = h^{-1}f_\alpha(h) = \psi([g_h, \alpha]$ belongs to N for each element h of H, and so the automorphism f_α belongs to the group $\mathrm{Aut}_N(H)$.

Let α and β be elements of $\mathrm{Aut}_M(G)$, and let h be any element of H. Then

$$\begin{aligned} (f_\alpha f_\beta)(h) &= f_\beta\big(f_\alpha(h)\big) = f_\beta\big(h\psi([g_h, \alpha])\big) \\ &= f_\beta(h)f_\beta\big(\psi([g_h, \alpha])\big) = h\psi([g_h, \beta])\psi([g_h, \alpha]) \\ &= h\psi([g_h, \beta][g_h, \alpha]) = h\psi([g_h, \beta][g_h, \alpha]^\beta) \\ &= h\psi([g_h, \alpha\beta]) = f_{\alpha\beta}(h). \end{aligned}$$

Therefore $f_\alpha f_\beta = f_{\alpha\beta}$, and hence the map

$$\tau : \alpha \in \mathrm{Aut}_M(G) \longmapsto f_\alpha \in \mathrm{Aut}_N(H)$$

is a group homomorphism.

Consider now the inverse isomorphisms

$$\varphi^{-1} : H/N \longrightarrow G/M$$

and

$$\psi^{-1} : N \longrightarrow M.$$

The above method applied to φ^{-1} and ψ^{-1} allows to construct, for each element γ of $\mathrm{Aut}_N(H)$, an automorphism f_γ of G which belongs to $\mathrm{Aut}_M(G)$, and the map

$$\omega : \gamma \in \mathrm{Aut}_N(H) \longmapsto f_\gamma \in \mathrm{Aut}_M(G)$$

is a homomorphism. It is easy to prove that $\omega \circ \tau$ is the identity map of $\mathrm{Aut}_M(G)$ and $\tau \circ \omega$ is the identity map of $\mathrm{Aut}_N(H)$. Therefore τ is an isomorphism and $\mathrm{Aut}_M(G) \simeq \mathrm{Aut}_N(H)$. □

It was remarked in the introduction that if a is any element of a group G and α is the inner automorphism of G determined by a, the autocommutator $[g, \alpha]$ coincides with the ordinary commutator $[g, a]$ for each element g of G. It follows that if the inner automorphism α is autocentral, then the subgroup $[G, a]$ is contained in $L(G)$, i.e. the coset $aL(G)$ belongs to the centre of $G/L(G)$. In particular, if the group $\mathrm{Aut}^c(G)$ of all class preserving automorphisms is contained in $\mathrm{Aut}_L(G)$, we obtain that the commutator subgroup G' lies in the absolute centre $L(G)$ of G.

Our second main result shows that if G is any finite group in which the commutator subgroup and the absolute centre coincide, then $\mathrm{Aut}^c(G) = \mathrm{Aut}_L(G)$.

Theorem 2 *Let G be a finite group such that $G' = L(G)$. Then $Aut^c(G) \simeq Hom\big(G/G', G'\big)$ and $Aut^c(G) = Aut_L(G)$.*

Proof Let α be any class preserving automorphism of G. Clearly, $[xu, \alpha] = [x, \alpha]$ for all elements x of G and u of $G' = L(G)$, and so the map

$$f_\alpha : xG' \in G/G' \longmapsto [x, \alpha] \in G'$$

can be considered. As $G' = L(G) \leq Z(G)$, we have

$$f_\alpha(xyG') = [xy, \alpha] = [x, \alpha][y, \alpha] = f_\alpha(xG') f_\alpha(yG')$$

for all elements x and y of G, and hence f_α is a homomorphism. Observe also that, if α and β are two class preserving automorphisms of G, and x is any element of G, then

$$f_{\alpha\beta}(xG') = [x, \alpha\beta] = [x, \beta][x, \alpha]^\beta = [x, \alpha][x, \beta] = (f_\alpha + f_\beta)(x).$$

Therefore the map

$$\psi : \alpha \in \mathrm{Aut}^c(G) \longmapsto f_\alpha \in \mathrm{Hom}(G/G', G')$$

is a homomorphism, which is promptly seen to be injective.

Conversely, if f is any homomorphism of G/G' into G', consider the map

$$\alpha_f : G \longrightarrow G,$$

defined by putting $\alpha_f(x) = xf(xG')$ for each element x of G. It is clear that f is a homomorphism. If x is an element of G such that $\alpha_f(x) = 1$, then $x = f(xG')^{-1}$ belongs to G', and so $x = 1$. Therefore α_f is injective, and hence it is an automorphism of the finite group G. Moreover, α_f acts trivially on G/G', and so it is an autocentral automorphism of G, because $G' = L(G)$. Finally, we have

$$\psi(\alpha_f) = f_{\alpha_f} = f,$$

so that ψ is an isomorphism, and the groups $\mathrm{Aut}^c(G)$ and $\mathrm{Hom}(G/G', G')$ are isomorphic.

On the other hand, $\mathrm{Aut}_L(G)$ is naturally isomorphic to the homomorphism group $\mathrm{Hom}\big(G/L(G), L(G)\big)$ (see also [2], Proposition 1), and hence in our case we obtain

$$\mathrm{Aut}^c(G) \simeq \mathrm{Aut}_L(G).$$

As $\mathrm{Aut}^c(G)$ acts trivially on G/G', and $G' = L(G)$, it follows that all class preserving automorphisms are autocentral, so that $\mathrm{Aut}^c(G) = \mathrm{Aut}_L(G)$, and the proof is complete. □

Observe finally that part of the statement of Theorem 2 can be generalized to certain types of infinite groups. Recall that a group G is *cohopfian* if it not isomorphic to any of its proper subgroups, i.e. if every injective endomorphism of G is an automorphism; for instance, every Černikov group is obviously cohopfian. The argument of the above proof can be used to show that if G is any cohopfian group such that $G' = L(G)$, then the groups $\mathrm{Aut}^c(G)$ and $\mathrm{Aut}_L(G)$ are isomorphic.

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