Ricerche mat. (2011) 60:263–297
DOI 10.1007/s11587-011-0109-x

Ground states for Schrödinger–Poisson type systems

Giusi Vaira

Received: 18 October 2010 / Revised: 29 December 2010 / Published online: 19 February 2011

Abstract In this paper we consider the following elliptic system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + u + \lambda K(x)\phi u = a(x)|u|^{p-1}u & x \in \mathbb{R}^3 \\ -\Delta \phi = K(x)u^2 & x \in \mathbb{R}^3 \end{cases}$$

where λ is a real parameter, $p \in (1, 5)$ if $\lambda < 0$ while $p \in (3, 5)$ if $\lambda > 0$ and $K(x), a(x)$ are non-negative real functions defined on $\mathbb{R}^3$. Assuming that $\lim_{|x|\to+\infty} K(x) = K_\infty > 0$ and $\lim_{|x|\to+\infty} a(x) = a_\infty > 0$ and satisfying suitable assumptions, but not requiring any symmetry property on them, we prove the existence of positive ground states, namely the existence of positive solutions with minimal energy.

Keywords Non-autonomous Schrödinger–Poisson system · Lack of compactness · Variational methods

Mathematics Subject Classification (2000) 35J05 · 35J10 · 35J50 · 35J60

1 Introduction and main results

The basis of the mathematical formalism of Quantum Mechanics lies in the fact that any state of a particle in the 3-dimensional space can be described, at a given moment, by a definite (in general complex) function ψ of the coordinates x: $|\psi|^2 dx$ is the

Communicated by V. Coti-Zelata.

G. Vaira (✉)
SISSA, Via Bonomea, 265, 34014 Trieste, Italy
e-mail: vaira@sissa.it

probability that the coordinates of the particle associated to ψ will find their values in the element dx. The function ψ is called the *wave function* of the system. The sum of the probability of all possible values of the coordinates must, by definition, be equal to unity:

$$\int_{\mathbb{R}^3} |\psi|^2 \, dx = 1.$$

This equation is what is called the *normalization equation* for the wave function. The central problem of the theory is to know the wave equation, i.e. the equation of propagation of the wave ψ. For example the behavior of a single particle of mass $m > 0$ can be described by the linear Schrödinger equation:

$$\imath\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + Q(x)\psi, \quad x \in \mathbb{R}^3, \ t \in \mathbb{R} \tag{1.1}$$

where $\hbar$ is the Planck constant and $Q : \mathbb{R}^3 \to \mathbb{R}$ is the time independent potential of the particle at the position $x \in \mathbb{R}^3$.

Differently, in the presence of many particles one can try to simulates the effects of the mutual interactions by introducing a nonlinear term. Then one is led to a nonlinear equation of the form

$$\imath\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + Q(x)\psi - |\psi|^{p-1}\psi, \quad x \in \mathbb{R}^3, \ t \in \mathbb{R}. \tag{1.2}$$

with $p > 1$. Let us suppose that the particle moves in its own gravitational field where the field is generated by the particles probability density via classical Newtons field equation. Then the potential Q is given (up to constants) by

$$Q(x) = -\int_{\mathbb{R}^3} \frac{1}{|x-y|}|\psi|^2 \, dy,$$

namely Q is the solution of the Poisson equation

$$\Delta Q = |\psi|^2.$$

If we look for standing waves, namely waves of the form

$$\psi(x,t) = u(x)e^{i\omega t}, \quad \omega > 0, \ x \in \mathbb{R}^3, \ t \in \mathbb{R}, \tag{1.3}$$

then the system that we are deal with is given by

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + \omega\hbar u - Qu = |u|^{p-1}u, & x \in \mathbb{R}^3 \\ -\Delta Q = u^2. & x \in \mathbb{R}^3 \end{cases} \tag{1.4}$$

If, instead, in (1.2) we suppose that ψ is a charged wave, in order to find standing waves of the form (1.3) in equilibrium with a purely electrostatic field, the system that we deal with (see f.i. [4,5]) is given by

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + V(x)u + \phi u = |u|^{p-1}u & x \in \mathbb{R}^3 \\ -\Delta\phi = u^2 & x \in \mathbb{R}^3 \end{cases} \tag{1.5}$$

where $V(x) := Q(x) + \hbar\omega$.

Jointing systems (1.4) and (1.5) we are concerned with the existence of positive solutions for the following generalized nonlinear system in $\mathbb{R}^3$

$$\begin{cases} -\Delta u + u + \lambda K(x)\phi u = a(x)|u|^{p-1}u & x \in \mathbb{R}^3 \\ -\Delta\phi = K(x)u^2 & x \in \mathbb{R}^3 \end{cases} \tag{P_λ}$$

where λ is a real parameter. We remark that if $\lambda < 0$, respectively, $\lambda > 0$, we obtain a generalization of system (1.4), resp. a generalization of system (1.5). The case $\lambda = 0$ is not an interesting one and then we do not consider it.

Similar problems have been widely investigated, but many researches mainly concern either the autonomous case or, in the non autonomous case, the search of the so-called semi-classical states. We refer the reader interested in a detailed bibliography to the survey paper [1].

All these works deal with systems like (P_λ) with the nonlinearity $f(s) = s^p$ with p subcritical. Recently, in [2] the author considers the case of a nonlinearity which satisfies the general hypotheses introduced by Berestycki and Lions and, by using concentration and compactness argument he proves the existence of a non trivial non-radial solution for (P_λ) with $\lambda > 0$ and $K = a \equiv 1$.

Instead, in [9] the author considers (P_λ) with $\lambda > 0$ and $K = a \equiv 1$ but in a bounded domain Ω of $\mathbb{R}^3$ and he proves that if p is near the critical Sobolev exponent the number of positive solutions is greater than the Lusternik-Schnirelmann category of Ω.

As we will see later (Sect. 2), the second equation of (P_λ) has a unique positive solution $\phi_u \in D^{1,2}(\mathbb{R}^3)$.

Hence the system (P_λ) can be easily transformed into a single equation. Indeed, substituting ϕ_u into the first equation of (P_λ) we have to study the equivalent problem

$$-\Delta u + u + \lambda K(x)\phi_u u = a(x)|u|^{p-1}u. \tag{P'_λ}$$

The problem (P'_λ) is variational in nature, that is its solutions are the critical points of the C^2 functional $I_\lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined as follows

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1}\, dx. \tag{1.6}$$

In the sequel we always assume that $a(x)$ and $K(x)$ verify, respectively

$$\text{(a1)}\quad \lim_{|x|\to+\infty} a(x) = a_\infty > 0,\ \alpha(x) := a(x) - a_\infty \in L^{\frac{6}{5-p}}(\mathbb{R}^3);$$

$$\mathcal{A} := \inf_{\mathbb{R}^3} a(x) > 0;$$

$$\text{(K1)}\quad \lim_{|x|\to+\infty} K(x) = K_\infty > 0,\ \eta(x) := K(x) - K_\infty \in L^2(\mathbb{R}^3);$$

$$\mathcal{K} := \inf_{\mathbb{R}^3} K(x) > 0.$$

In [7] the authors consider the case in which $K_\infty \equiv 0$ and ground and bound states for the problem (P'_λ) with $\lambda > 0$ has been found.

Since any symmetry assumption on $K(x)$ and $a(x)$ is done, one has to face various difficulties. Here we have to distinguish the different cases that can arise.

If $\lambda > 0$ then the competing effect of the nonlocal term with the nonlinear term gives rise to very different situations as p varies in the interval $(1, 5)$.

If $\lambda < 0$ the nonlocal term and the nonlinear term have both an attractive effect, then no problems with the various $p \in (1, 5)$ appear.

In our research we get $p \in (3, 5)$ if $\lambda > 0$ while $p \in (1, 5)$ if $\lambda < 0$.

However, in any case, the lack of compactness of the embedding of $H^1(\mathbb{R}^3)$ in $L^q(\mathbb{R}^3)$, $q \in (2, 6)$, prevents from using the variational techniques at least in a standard way. Hence a basic step in the study of (P'_λ) is a careful investigation of the behavior of the Palais-Smale sequences for the functional I_λ.

In [7], since $K_\infty \equiv 0$, it is proved that the only obstacle to the compactness are the solutions of the *problem at infinity* $-\Delta u + u = a_\infty |u|^{p-1}u$, which has a unique radial positive ground state with an exponential decay to zero at infinity. These facts permit to deduce not only that the compactness condition is recovered below a certain threshold, but also that, above the first level in which the Palais-Smale condition fails, some other energy interval exists where the compactness hold. For (P_λ) the corresponding problem at infinity turns out to be the system

$$\begin{cases} -\Delta u + u + \lambda K_\infty \phi u = a_\infty |u|^{p-1}u & x \in \mathbb{R}^3 \\ -\Delta \phi = K_\infty u^2. & x \in \mathbb{R}^3 \end{cases} \qquad (\mathrm{P}^\infty_\lambda)$$

Again, in Sect. 5, we study the behavior of the Palais-Smale sequences and we prove that the “bad” levels for the compactness can be located by the energy of the solutions of (P^∞_λ), but in striking contrast with the scalar case, very few is known on the ground states of (P^∞_λ). This might depend on the fact that the study of (P^∞_λ) requires some work far from trivial.

In the present paper we deal with the existence of positive ground states for (P_λ), with $\lambda \in \mathbb{R}^3 \backslash \{0\}$.

In Sect. 4, we prove that if $\lambda < 0$, then there exists a positive, radial ground state of (P^∞_λ), while if $\lambda > 0$, we prove the existence of a positive ground state solution for (P^∞_λ) but nothing is known about its radial symmetry.

In order to find critical levels of I_λ, we need to look into the geometry of the functional. The study is carried out considering I_λ constrained on its Nehari manifold $\mathcal{N}_\lambda$, where I_λ turns out to be bounded from below. The analysis of I_λ on $\mathcal{N}_\lambda$ highlights the different features of (P_λ) according to the sign of $\eta(x)$ and of $\alpha(x)$. Actually, let be $\lambda < 0$. As we shall see in Sect. 6.1, if we assume either

(H_1) $K(x) \geq K_\infty$; $a(x) \geq a_\infty$ for all $x \in \mathbb{R}^3$ and $a(x) - a_\infty > 0$ on a positive measure set;

or

(H_2) $K(x) \leq K_\infty$; $a(x) \geq a_\infty$ for all $x \in \mathbb{R}^3$ and $a(x) - a_\infty > 0$ on a positive measure set;

or

(H_3) $K(x) \geq K_\infty$; $a(x) \leq a_\infty$ for all $x \in \mathbb{R}^3$ and $K(x) - K_\infty > 0$ on a positive measure set;

then the problem can be faced by a minimization argument obtaining sufficient conditions to have a ground state solution (see Theorems 6.1, 6.2, 6.3).

Instead, if $\lambda > 0$, we will prove, in Sect. 6.2, that under the assumptions (H_1) or (H_2) or

(H_4) $K(x) \leq K_\infty$; $a(x) \leq a_\infty$ for all $x \in \mathbb{R}^3$ and $a_\infty - a(x) > 0$ on a positive measure set;

the problem (P_λ) admits a ground state solution (see Theorems 6.4, 6.5, 6.6).

2 Notations and preliminaries

Hereafter we use the following notation:

- $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with the standard scalar product and norm

$$(u, v) = \int_{\mathbb{R}^3} [\nabla u \nabla v + uv] dx; \quad \|u\|^2 = \int_{\mathbb{R}^3} \left[|\nabla u|^2 + u^2 \right] dx.$$

- $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

- H^{-1} denotes the dual space of $H^1(\mathbb{R}^3)$.
- $L^q(\Omega)$, $1 \leq q \leq +\infty$, $\Omega \subseteq \mathbb{R}^3$, denotes a Lebesgue space, the norm in L^q is denoted by $|u|_{q,\Omega}$ when Ω is a proper subset of $\mathbb{R}^3$, by $|\cdot|_p$ when $\Omega = \mathbb{R}^3$.

- For any $\rho > 0$ and for any $z \in \mathbb{R}^3$, $B_\rho(z)$ denotes the ball of radius ρ centered at z.
- C, C', C_i are various positive constants.
- S_q is the best Sobolev constant for the embedding of $H^1(\mathbb{R}^3)$ in $L^q(\mathbb{R}^3)$, $q \in (2, 6)$, that is

$$S_q = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|}{|u|_q}.$$

- $\bar{S}$ is the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$, that is

$$\bar{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|_{D^{1,2}}}{|u|_6}.$$

It is easy to see that (P_λ) can be reduced into a single equation with a nonlocal term. Actually, considering for all $u \in H^1(\mathbb{R}^3)$ the linear functional L_u defined in $D^{1,2}(\mathbb{R}^3)$ by

$$L_u(v) = \int_{\mathbb{R}^3} K(x) u^2 v \, dx,$$

the Hölder inequality and the Sobolev inequality imply

$$\begin{aligned} |L_u(v)| &= \left| \int_{\mathbb{R}^3} (K(x) - K_\infty) u^2 v \, dx + \int_{\mathbb{R}^3} K_\infty u^2 v \, dx \right| \\ &\leq \int_{\mathbb{R}^3} |\eta(x)| u^2 |v| \, dx + \int_{\mathbb{R}^3} K_\infty u^2 |v| \, dx \\ &\leq |v|_6 \left(\int_{\mathbb{R}^3} |\eta(x)|^{6/5} u^{12/5} \, dx \right)^{5/6} + K_\infty |v|_6 |u^2|_{6/5} \\ &\leq \bar{S}^{-1} \left[S_6^{-2} |\eta|_2 + K_\infty S_{12/5}^{-2} \right] \|u\|^2 \|v\|_{D^{1,2}}. \end{aligned} \tag{2.1}$$

Hence, by the Lax-Milgram theorem, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla \phi_u \nabla v dx = \int_{\mathbb{R}^3} K(x) u^2 v dx \quad \forall\, v \in D^{1,2}(\mathbb{R}^3) \tag{2.2}$$

that is ϕ_u is a weak solution of $-\Delta\phi_u = K(x)u^2$. Moreover, since K is positive, $\phi_u > 0$ when $u \neq 0$ and the following representation formula holds

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} u^2(y)dy = \frac{1}{|x|} * Ku^2. \tag{2.3}$$

By using (2.1) and Sobolev inequality we obtain

$$\|\phi_u\|_{D^{1,2}} = \|L_u\|_{\mathcal{L}(D^{1,2},\mathbb{R})} \leq M_1 \cdot \|u\|^2; \tag{2.4}$$

where

$$M_1 := \bar{S}^{-1}\left[S_6^{-2}|\eta|_2 + K_\infty S_{12/5}^{-2}\right].$$

By using again Hölder and Sobolev inequalities, we find

$$\begin{aligned} \int_{\mathbb{R}^3} K(x)\phi_u u^2\, dx &= \int_{\mathbb{R}^3} \eta(x)\phi_u u^2\, dx + \int_{\mathbb{R}^3} K_\infty \phi_u u^2\, dx \\ &\leq |\phi_u|_6 \left(|\eta u^2|_{6/5} + K_\infty |u|_{12/5}^2\right) \\ &\leq M_1^2 \|u\|^4. \end{aligned} \tag{2.5}$$

In the same way one can prove the existence of a unique positive $\tilde{\phi}_u \in D^{1,2}(\mathbb{R}^3)$ and the existence of a unique positive $\bar{\phi}_u \in D^{1,2}(\mathbb{R}^3)$ which are, respectively, solutions of

$$\text{(a)} \quad -\Delta\tilde{\phi}_u = K_\infty u^2; \qquad \text{(b)} \quad -\Delta\bar{\phi}_u = \eta(x)u^2. \tag{2.6}$$

Reasoning as before we find

$$\text{(a)} \quad \|\tilde{\phi}_u\|_{D^{1,2}} \leq M_2\|u\|^2; \qquad \text{(b)} \quad \|\bar{\phi}_u\|_{D^{1,2}} \leq M_3\|u\|^2 \tag{2.7}$$

with $M_2 := K_\infty \bar{S}^{-1} \cdot S_{12/5}^{-2}$ and $M_3 := \bar{S}^{-1} \cdot S_6^{-2}|\eta|_2$.
Furthermore

$$\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_u u^2\, dx \leq M_2^2\|u\|^4 \tag{2.8}$$

and

$$\int_{\mathbb{R}^3} \eta(x)\bar{\phi}_u u^2\, dx \leq M_3^2\|u\|^4. \tag{2.9}$$

Let us now define the operators

$$\Phi, \bar{\Phi}, \tilde{\Phi} : H^1(\mathbb{R}^3) \longrightarrow D^{1,2}(\mathbb{R}^3)$$

as

$$\Phi[u] = \phi_u, \quad \bar{\Phi}[u] = \bar{\phi}_u, \quad \tilde{\Phi}[u] = \tilde{\phi}_u.$$

In the following lemma we summarize some properties of $\Phi, \bar{\Phi}, \tilde{\Phi}$ useful to study our problem.

Lemma 2.1 (1) $\Phi, \bar{\Phi}, \tilde{\Phi}$ *are continuous;*
(2) $\Phi, \bar{\Phi}, \tilde{\Phi}$ *map bounded sets into bounded sets;*
(3) *If* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$ *then*
  (i) $\Phi[u_n] \rightharpoonup \Phi[u]$ *in* $D^{1,2}(\mathbb{R}^3)$*;*
  (ii) $\bar{\Phi}[u_n] \rightharpoonup \bar{\Phi}[u]$ *in* $D^{1,2}(\mathbb{R}^3)$*;*
  (iii) $\tilde{\Phi}[u_n] \rightharpoonup \tilde{\Phi}[u]$ *in* $D^{1,2}(\mathbb{R}^3)$*;*
(4) $\Phi[tu] = t^2\Phi[u]$, $\bar{\Phi}[tu] = t^2\bar{\Phi}[u]$ *and* $\tilde{\Phi}[tu] = t^2\tilde{\Phi}[u]$ *for all* $t \in \mathbb{R}$.

Proof (1) The continuity can be proved in the same way as done in Lemma 2.1-(1) of [7].
(2) It is a straight consequence of (2.4), (2.7)-(a) and of (2.7)-(b).
(3) Let $(u_n)_n \subset H^1(\mathbb{R}^3)$ be such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. Then u_n is bounded in $H^1(\mathbb{R}^3)$ and in $L^6(\mathbb{R}^3)$ and, by the previous point, $\Phi[u_n]$, $\bar{\Phi}[u_n]$ and $\tilde{\Phi}[u_n]$ are bounded too. Therefore, up to a subsequence,
  (a) $\bar{\Phi}[u_n] \rightharpoonup \bar{\phi}$ in $D^{1,2}(\mathbb{R}^3)$;
  (b) $\tilde{\Phi}[u_n] \rightharpoonup \tilde{\phi}$ in $D^{1,2}(\mathbb{R}^3)$.
  (c) $\Phi[u_n] \rightharpoonup \phi$ in $D^{1,2}(\mathbb{R}^3)$;
  Claim 1: $\bar{\phi} \equiv \bar{\Phi}[u]$.
  By (a), for any $v \in D^{1,2}(\mathbb{R}^3)$, we get

$$(\bar{\Phi}[u_n], v)_{D^{1,2}} \to (\bar{\phi}, v)_{D^{1,2}}. \tag{2.10}$$

Let us prove that, for all $v \in D^{1,2}(\mathbb{R}^3)$, as $n \to +\infty$

$$\left(\bar{\Phi}[u_n], v\right)_{D^{1,2}} = \int\limits_{\mathbb{R}^3} \eta(x)u_n^2 v\,dx \to \int\limits_{\mathbb{R}^3} \eta(x)u^2 v\,dx = (\bar{\Phi}[u], v)_{D^{1,2}} \tag{2.11}$$

This relation with (2.10) and the uniqueness of the solution of $-\Delta\phi = \eta(x)u^2$ will imply the claim.

Being $\eta \in L^2(\mathbb{R}^3)$, to any $\epsilon > 0$, there corresponds $\rho \equiv \rho(\epsilon) > 0$ such that

$$|\eta|_{2,\mathbb{R}^3\setminus B_\rho(0)} < \epsilon.$$

Then, by using the boundedness of the sequence $(u_n)_n$, we deduce

$$\int_{\mathbb{R}^3\setminus B_\rho(0)} \eta(x)v\left(u_n^2-u^2\right)\,dx \le C(v)\epsilon, \quad n\in\mathbb{N}. \tag{2.12}$$

On the other hand $\eta v \in L^{3/2}(\mathbb{R}^3)$. Moreover, easily follows that $z_n := u_n^2 - u^2$ is bounded in $L^3(B_\rho(0))$ and so (see [12]) $z_n \rightharpoonup 0$ in $L^3(B_\rho(0))$. Hence, for any $\epsilon > 0$, we have

$$\left|\int_{B_\rho(0)} \eta(x)vz_n\,dx\right| \le \hat{C}(v)\epsilon \tag{2.13}$$

for large n. Then (2.12) and (2.13) and the arbitrary choice of ϵ give (2.11).
Claim 2: $\tilde{\phi} \equiv \tilde{\Phi}[u]$.

Actually, let be $\rho \in C_0^\infty(\mathbb{R}^3)$ and let $\Omega = \operatorname{supp}\rho$. The case of a general $v \in D^{1,2}(\mathbb{R}^3)$ follows by a density argument.

(c) implies

$$\int_{\mathbb{R}^3} \nabla\tilde{\phi}\nabla\rho\,dx \leftarrow \int_{\mathbb{R}^3} \nabla\tilde{\Phi}[u_n]\nabla\rho\,dx = \int_{\mathbb{R}^3} K_\infty u_n^2\rho\,dx$$
$$= \int_{\mathbb{R}^3\setminus\Omega} K_\infty u_n^2\rho\,dx + \int_{\Omega} K_\infty u_n^2\rho\,dx = \int_{\Omega} K_\infty u_n^2\rho\,dx$$

Now since $u_n \to u$ in $L^2_{loc}(\mathbb{R}^3)$ then

$$\int_{\Omega} K_\infty u_n^2\rho\,dx \to \int_{\Omega} K_\infty u^2\rho\,dx = \int_{\mathbb{R}^3} K_\infty u^2\rho\,dx.$$

Hence by the uniqueness of the solution of $-\Delta\tilde{\phi} = K_\infty u^2$ the claim follows.
Claim 3: $\phi \equiv \Phi[u]$.

Indeed, for all $v \in D^{1,2}(\mathbb{R}^3)$, collecting the previous results we find

$$(\phi, v)_{D^{1,2}} \leftarrow \int_{\mathbb{R}^3} \nabla\Phi[u_n]\nabla v\,dx = \int_{\mathbb{R}^3} K(x)u_n^2 v\,dx = \int_{\mathbb{R}^3} \eta(x)u_n^2 v\,dx$$
$$+ \int_{\mathbb{R}^3} K_\infty u_n^2 v\,dx = (\bar{\Phi}[u_n], v)_{D^{1,2}} + (\tilde{\Phi}[u_n], v)_{D^{1,2}}$$
$$\to (\bar{\Phi}[u], v)_{D^{1,2}} + (\tilde{\Phi}[u], v)_{D^{1,2}}$$
$$= \int_{\mathbb{R}^3} \eta(x)u^2 v\,dx + \int_{\mathbb{R}^3} K_\infty u^2 v\,dx = (\Phi[u], v)_{D^{1,2}}.$$

Then, by the uniqueness of the solution, the claim follows.

(4) A direct computation gives the assertion. □

In the following lemma we establish a characterization of the weak convergence for the Poisson term that is useful in the sequel. The proof can be made in a similar way as in [8].

Lemma 2.2 *Let us define the operator*

$$\bar{T} : \left[H^1(\mathbb{R}^3)\right]^4 \to \mathbb{R}$$

such that for all $(u, v, w, z) \in \left[H^1(\mathbb{R}^3)\right]^4$:

$$\bar{T}(u, v, w, z) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\eta(x)\eta(y)}{|x-y|} u(x)v(x)w(y)z(y)\, dx\, dy.$$

Then for all $(u_n)_n, (v_n)_n, (w_n)_n \subset H^1(\mathbb{R}^3)$ *such that* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$, $v_n \rightharpoonup v$ *in* $H^1(\mathbb{R}^3)$, $w_n \rightharpoonup w$ *in* $H^1(\mathbb{R}^3)$ *and for all* $z \in H^1(\mathbb{R}^3)$ *we have*

$$\bar{T}(u_n, v_n, w_n, z) \to \bar{T}(u, v, w, z).$$

Remark 1 Lemma 2.2 can be proved also for the maps

$$\tilde{T}, T : \left[H^1(\mathbb{R}^3)\right]^4 \to \mathbb{R}$$

such that for all $(u, v, w, z) \in \left[H^1(\mathbb{R}^3)\right]^4$

$$\tilde{T}(u, v, w, z) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K_\infty K_\infty}{|x-y|} u(x)v(x)w(y)z(y)\, dx\, dy$$

and

$$T(u, v, w, z) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\eta(x) K_\infty}{|x-y|} u(x)v(x)w(y)z(y)\, dx\, dy.$$

3 Variational setting

In this section we describe the variational framework for the study of the critical points of the functional I_λ define in (1.6).

It is convenient to consider I_λ restricted to a natural constraint, the Nehari manifold, that contains all the critical points of I_λ and on which I_λ turns out to be bounded from below. We set

$$\mathcal{N}_\lambda := \left\{u \in H^1(\mathbb{R}^3)\backslash\{0\} : G_\lambda(u) = 0\right\}$$

where

$$G_\lambda(u) = I'_\lambda(u)[u] = \|u\|^2 + \lambda \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} a(x)|u|^{p+1} dx.$$

We remark that there holds

$$I_{\lambda_{|\mathcal{N}_\lambda}}(u) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|^2 + \lambda\left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\, dx \quad (3.1)$$

$$= \frac{1}{4}\|u\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} a(x)|u|^{p+1}\, dx \quad (3.2)$$

$$= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} a(x)|u|^{p+1}\, dx - \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\, dx \quad (3.3)$$

Next lemma contains the statement of the main properties of $\mathcal{N}_\lambda$. The proof can be made in a similar way as in [7].

Lemma 3.1 (1) *$\mathcal{N}_\lambda$ is a C^1 regular manifold diffeomorphic to the sphere of $H^1(\mathbb{R}^3)$;*
(2) *I_λ is bounded from below on $\mathcal{N}_\lambda$ by a positive constant;*
(3) *u is a free critical point of I_λ if and only if u is a critical point of I_λ constrained on $\mathcal{N}_\lambda$.*

We set

$$m_\lambda := \inf\{I_\lambda(u) : u \in \mathcal{N}_\lambda\}.$$

By (2) of Lemma 3.1 it turns out that m_λ is a positive number.

From (1) of Lemma 3.1 it follows that to any $u \in H^1(\mathbb{R}^3)$ there corresponds a (unique) $t(u) > 0$, called the *projection* of u on $\mathcal{N}_\lambda$, such that

$$I_\lambda(t(u)u) = \max_{t\geq 0} I_\lambda(tu). \quad (3.4)$$

4 The problem at infinity

Since $K(x) \xrightarrow{|x|\to\infty} K_\infty$ and $a(x) \xrightarrow{|x|\to\infty} a_\infty$, it can be possible to prove that the problem at infinity related to (P'_λ) turns out to be the following problem

$$-\Delta u + u + \lambda K_\infty \tilde{\phi}_u u = a_\infty |u|^{p-1} u. \qquad (\mathrm{P}'^{\infty}_{\lambda})$$

The solutions of (P'^{∞}_λ) are the critical points of the functional $\mathcal{J}_\lambda \in C^2(H^1(\mathbb{R}^3), \mathbb{R})$ defined as

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_u u^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a_\infty |u|^{p+1}\, dx.$$

Let

$$\mathcal{M}_\lambda := \left\{ u \in H^1(\mathbb{R}^3)\backslash\{0\} : H_\lambda(u) = 0 \right\},$$

where

$$H_\lambda(u) = \mathcal{J}'_\lambda(u)[u] = \|u\|^2 + \lambda \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_u u^2 dx - \int_{\mathbb{R}^3} a_\infty |u|^{p+1}\, dx,$$

the Nehari manifold related to $\mathcal{J}_\lambda$ and set

$$c_\lambda := \inf\{\mathcal{J}_\lambda(u) : u \in \mathcal{M}_\lambda\}.$$

It is easy to prove that (1)-(2)-(3) of Lemma 3.1 hold for $\mathcal{M}_\lambda$. Hence c_λ is a positive number.

Moreover to any $u \in H^1(\mathbb{R}^3)$ there corresponds a (unique) $\xi(u) > 0$ called the *projection* of u on $\mathcal{M}_\lambda$ such that

$$\mathcal{J}_\lambda(\xi(u)u) = \max_{\xi \geq 0} \mathcal{J}_\lambda(\xi u). \qquad (4.1)$$

In the sequel we find a positive ground state for the problem (P'^{∞}_λ).

4.1 The case $\lambda < 0$

Without loss of generality, let us assume $\lambda = -1$ and let $\bar{\mathcal{J}} := \mathcal{J}_{-1}$, $\bar{\mathcal{M}} := \mathcal{M}_{-1}$ and $\bar{c} := c_{-1}$. The aim of this section is to find a positive ground state solution for the problem

$$-\Delta u + u - K_\infty \tilde{\phi}_u u = a_\infty |u|^{p-1} u, \quad u \in H^1(\mathbb{R}^3), \qquad ((\mathrm{SN})_\infty)$$

A first remark is in order.

Remark 2 Let ϕ be the weak solution in $\mathbb{R}^3$ of the Poisson equation

$$-\Delta\phi = f.$$

We denote by f^* the spherically symmetric rearrangement of f, that is the function whose level sets $\{x \in \mathbb{R}^3 : f^*(x) > t\} = \{x \in \mathbb{R}^3 : [f(x)] > t\}^*$, and by v the weak solution of the problem

$$-\Delta v = f^*.$$

Then by Theorem 1 of [10] it follows that

$$\int_{\mathbb{R}^3} |\nabla v|^q \, dx \geq \int_{\mathbb{R}^3} |\nabla \phi|^q \, dx \tag{4.1.1}$$

for all $0 < q \leq 2$.

Let now be $\phi = \tilde{\phi}_u$ and $f = K_\infty u^2$. By (4.1.1) with $q = 2$ it follows that

$$\|\tilde{\phi}_u\|^2_{D^{1,2}} \leq \|\tilde{\phi}_{u^*}\|^2_{D^{1,2}}.$$

Hence, since $\tilde{\phi}_u$ solves (2.6)-(a) and $\tilde{\phi}_{u^*}$ solves $-\Delta\tilde{\phi}_{u^*} = K_\infty(u^*)^2$, we find

$$\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_u u^2 \, dx = \|\tilde{\phi}_u\|^2_{D^{1,2}} \leq \|\tilde{\phi}_{u^*}\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u^*} (u^*)^2 \, dx. \tag{4.1.2}$$

Proposition 4.1 *The problem* (SN)$_\infty$ *has a positive radial ground state* $\bar{w} \in \bar{\mathcal{M}}$ *such that* $\bar{\mathcal{J}}(\bar{w}) = \bar{c}$.

Proof Let $(u_n)_n$, $u_n \in \bar{\mathcal{M}}$ be such that $\bar{\mathcal{J}}(u_n) \to \bar{c}$. Let $t_n > 0$ such that $t_n|u_n| \in \bar{\mathcal{M}}$. Then

$$\begin{aligned}\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u_n} u_n^2 \, dx + \int_{\mathbb{R}^3} a_\infty |u_n|^{p+1} \, dx &= \|u_n\|^2 \\ &= t_n^2 \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u_n} u_n^2 \, dx + t_n^{p-1} \int_{\mathbb{R}^3} a_\infty |u_n|^{p+1} \, dx.\end{aligned}$$

Hence

$$(1 - t_n^2) \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u_n} u_n^2 \, dx + \left(1 - t_n^{p-1}\right) \int_{\mathbb{R}^3} a_\infty |u_n|^{p+1} \, dx = 0. \tag{4.1.3}$$

The equality (4.1.3) implies $t_n = 1$, $n \in \mathbb{N}$. Therefore, we can assume $u_n \geq 0$.

We denote by u_n^* the Schwartz symmetric function associated to u_n and let $t_n^* > 0$ be such that $t_n^* u_n^* \in \bar{\mathcal{M}}$. It is well known that $\|u_n^*\|^2 \le \|u_n\|^2$ and $|u_n^*|_{p+1} = |u_n|_{p+1}$. Since $t_n^* u_n^* \in \bar{\mathcal{M}}$ and $u_n \in \bar{\mathcal{M}}$ then it follows that, by using also (4.1.2),

$$\begin{aligned}
0 &= \left(t_n^*\right)^2 \|u_n^*\|^2 - \left(t_n^*\right)^4 \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u_n^*} (u_n^*)^2 \, dx - \left(t_n^*\right)^{p+1} \int_{\mathbb{R}^3} a_\infty |u_n^*|^{p+1} \, dx \\
&\le \left(t_n^*\right)^2 \|u_n\|^2 - (t_n^*)^4 \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u_n} u_n^2 \, dx - \left(t_n^*\right)^{p+1} \int_{\mathbb{R}^3} a_\infty |u_n|^{p+1} \, dx \\
&= \left(t_n^*\right)^2 \left(\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u_n} u_n^2 \, dx + \int_{\mathbb{R}^3} a_\infty |u_n|^{p+1} \, dx \right) \\
&\quad - \left(t_n^*\right)^4 \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u_n} u_n^2 \, dx - \left(t_n^*\right)^{p+1} \int_{\mathbb{R}^3} a_\infty |u_n|^{p+1} \, dx
\end{aligned}$$

that is

$$\left(1 - \left(t_n^*\right)^2\right) \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u_n} u_n^2 \, dx + \left(1 - \left(t_n^*\right)^{p-1}\right) \int_{\mathbb{R}^3} a_\infty |u_n|^{p+1} \, dx \ge 0$$

and this implies $t_n^* \le 1$. Hence $\bar{\mathcal{J}}(u_n^*) \le \bar{\mathcal{J}}(u_n)$. Therefore we can also suppose that u_n is radial. Since H_r^1 is compactly embedded into $L^{p+1}(\mathbb{R}^3)$, from standard arguments it follows that $\bar{c}$ is achieved at some $\bar{w} \in \bar{\mathcal{M}}$ which is non-negative and radial. Since $\bar{w} \in \bar{\mathcal{M}}$ then $\bar{w} \ne 0$. By continuity and by the uniqueness of the limit we obtain also $\bar{\mathcal{J}}(\bar{w}) = \bar{c}$, completing the proof. □

4.2 The case $\lambda > 0$

Without loss of generality we can assume $\lambda = 1$. Our aim is to find a positive ground state solution of the problem

$$-\Delta u + u + K_\infty \tilde{\phi}_u u = a_\infty |u|^{p-1} u. \qquad ((\mathrm{SP})_\infty)$$

However a minimization argument on the Nehari manifold $\mathcal{M}_1$ is more complicated than the case in which $\lambda < 0$. This is due to the fact that, by Remark 2, it is easy to see that we cannot deduce the existence of a ground state solution for the problem $(\mathrm{SP})_\infty$ simply passing to a radial minimizing sequence.

Then we have to analyze the compactness situation for the limiting problem $(\mathrm{SP})_\infty$.

Lemma 4.1 *Let $(u_n)_n$ be a bounded Palais-Smale sequence of $\mathcal{J}_1$, namely*

$$\begin{aligned}
&\text{(a)}\ \ \mathcal{J}_1(u_n)\ \textit{is bounded};\\
&\text{(b)}\ \ \nabla \mathcal{J}_1(u_n) \to 0\ \textit{strongly in}\ H^1(\mathbb{R}^3).
\end{aligned} \qquad (4.2.1)$$

Then replacing $(u_n)_n$*, if necessary, with a subsequence, there exist a solution* $\bar{u}$ *of* $(\mathrm{SP})_\infty$*, a number* $k \in \mathbb{N} \cup \{0\}$*,* k *functions* $u^1, \ldots, u^k$ *of* $H^1(\mathbb{R}^3)$ *and* k *sequences of points* (y_n^j)*,* $y_n^j \in \mathbb{R}^3, 0 \leq j \leq k$ *such that*

$$\begin{aligned} &\text{(i) } |y_n^j| \to +\infty,\ |y_n^j - y_n^i| \to +\infty \textit{ if } i \neq j,\ n \to +\infty; \\ &\text{(ii) } u_n - \textstyle\sum_{j=1}^k u^j(\cdot - y_n^j) \longrightarrow \bar{u}, \textit{ in } H^1(\mathbb{R}^3); \\ &\text{(iii) } \mathcal{J}_1(u_n) \to \mathcal{J}_1(\bar{u}) + \textstyle\sum_{j=1}^k \mathcal{J}_1(u^j); \\ &\text{(iv) } u^j \textit{ are non trivial weak solutions of } (\mathrm{SP})_\infty. \end{aligned} \tag{4.2.2}$$

Proof Let $(u_n)_n$ be a bounded Palais-Smale sequence. Then there exists $\bar{u}$ such that, up to a subsequence,

$$\begin{aligned} u_n \rightharpoonup \bar{u} \quad & \text{in } H^1(\mathbb{R}^3) \text{ and in } L^{p+1}(\mathbb{R}^3) \\ u_n(x) \to \bar{u}(x) \quad & \text{a.e. on } \mathbb{R}^3. \end{aligned}$$

Furthermore, taking into account (3)-(iii) of Lemma 2.1, we deduce that $\nabla \mathcal{J}_1(\bar{u}) = 0$, that is $\bar{u}$ is a weak solution $(\mathrm{SP})_\infty$. Let us define $z_{n,1} := u_n - \bar{u}$. Then $z_{n,1}$ goes weakly to zero in $H^1(\mathbb{R}^3)$ but not strongly.

A direct computation shows that

$$\|u_n\|^2 = \|z_{n,1}\|^2 + \|\bar{u}\|^2 + o(1). \tag{4.2.3}$$

Moreover, according to the Brezis-Lieb Lemma [6] we deduce

$$|u_n|_{p+1}^{p+1} = |\bar{u}|_{p+1}^{p+1} + |z_{n,1}|_{p+1}^{p+1} + o(1). \tag{4.2.4}$$

Now, by using Lemma 2.2 for $\tilde{T}$ (see also Remark 1) we find

$$\begin{aligned} \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u_n} u_n^2\, dx &= \tilde{T}(u_n, u_n, u_n, u_n) \\ &= \tilde{T}(u_n, u_n, u_n, z_{n,1}) + \tilde{T}(u_n, u_n, u_n, \bar{u}) \\ &= \tilde{T}(u_n, u_n, u_n, z_{n,1}) + \tilde{T}(\bar{u}, \bar{u}, \bar{u}, \bar{u}) + o(1) \\ &= \tilde{T}(z_{n,1}, z_{n,1}, z_{n,1}, z_{n,1}) + \tilde{T}(\bar{u}, \bar{u}, \bar{u}, \bar{u}) + o(1) \\ &= \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{z_{n,1}} z_{n,1}^2\, dx + \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{\bar{u}} \bar{u}^2\, dx + o(1) \end{aligned} \tag{4.2.5}$$

Hence by using (4.2.3), (4.2.4) and (4.2.5) we find

$$\begin{aligned}\mathcal{J}_1(u_n) &= \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\int\limits_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u_n} u_n^2\, dx - \frac{1}{p+1}\int\limits_{\mathbb{R}^3} a_\infty |u_n|^{p+1}\, dx \\ &= \frac{1}{2}\|z_{n,1}\|^2 + \frac{1}{2}\|\bar{u}\|^2 + \frac{1}{4}\int\limits_{\mathbb{R}^3} K_\infty \tilde{\phi}_{z_{n,1}} z_{n,1}^2\, dx \\ &\quad + \frac{1}{4}\int\limits_{\mathbb{R}^3} K_\infty \tilde{\phi}_{\bar{u}} \bar{u}^2\, dx - \frac{1}{p+1}\int\limits_{\mathbb{R}^3} a_\infty |z_{n,1}|^{p+1}\, dx \\ &\quad - \frac{1}{p+1}\int\limits_{\mathbb{R}^3} a_\infty |\bar{u}|^{p+1}\, dx + o(1) \\ &= \mathcal{J}_1(\bar{u}) + \mathcal{J}_1(z_{n,1}) + o(1). \end{aligned} \tag{4.2.6}$$

Moreover, for all $h \in H^1(\mathbb{R}^3)$, since by Lemma 8.1 of [11]

$$|u_n|^{p-1}u_n = |\bar{u}|^{p-1}\bar{u} + |z_{n,1}|^{p-1}z_{n,1} + o(1), \text{ in } H^{-1} \tag{4.2.7}$$

and since by Lemma 2.2

$$\tilde{T}(z_{n,1}, z_{n,1}, z_{n,1}, h) = o(1),$$

then, reasoning as before, we can prove

$$\begin{aligned} o(1) = (\nabla \mathcal{J}_1(u_n), h) &= \big(\nabla \mathcal{J}_1(z_{n,1}), h\big) + (\nabla \mathcal{J}_1(\bar{u}), h) + o(1) \\ &= \big(\nabla \mathcal{J}_1(z_{n,1}), h\big) + o(1) \end{aligned}$$

so that

$$\nabla \mathcal{J}_1(z_{n,1}) = o(1). \tag{4.2.8}$$

Set

$$\delta := \limsup_{n\to+\infty} \left(\sup_{y\in\mathbb{R}^3} \int\limits_{B_1(y)} |z_{n,1}|^{p+1}\, dx \right).$$

It is easy to see that $\delta > 0$. Actually, if $\delta = 0$ would be true, then by Lemma 1.21 of [11], $z_{n,1} \to 0$ in $L^{p+1}(\mathbb{R}^3)$ would hold, contradicting the fact that u_n does not converge strongly to $\bar{u}$ in $L^{p+1}(\mathbb{R}^3)$. Then we may assume the existence of $y_n^1 \subset \mathbb{R}^3$, such that

$$\int\limits_{B_1(y_n^1)} |z_{n,1}|^{p+1}\, dx > \frac{\delta}{2}.$$

Let us consider $z_{n,1}(\cdot + y_n^1)$. We can assume $z_{n,1}(\cdot + y_n^1) \rightharpoonup u^1$ in $H^1(\mathbb{R}^3)$ and so $z_{n,1}(x + y_n^1) \to u^1(x)$ a.e. on $\mathbb{R}^3$. Since

$$\int_{B_1(0)} |z_{n,1}(x + y_n^1)|^{p+1}\, dx > \frac{\delta}{2},$$

then, by Rellich Theorem, it follows

$$\int_{B_1(0)} |u^1(x)|^{p+1}\, dx > \frac{\delta}{2}.$$

Hence $u^1 \neq 0$. However since $z_{n,1}$ goes weakly to zero in $H^1(\mathbb{R}^3)$ then (y_n^1) must be unbounded and, up to a subsequence, we may assume that $|y_n^1| \to +\infty$. Furthermore (4.2.8) implies $\nabla \mathcal{J}_1(u^1) = 0$.

Finally, let us set

$$z_{n,2}(x) = z_{n,1}(x) - u^1\big(x - y_n^1\big).$$

Then, using (4.2.3), (4.2.4) and, again, by Brezis-Lieb Lemma we have

$$\begin{aligned} \|z_{n,2}\|^2 &= \|u_n\|^2 - \|\bar{u}\|^2 - \|u^1\|^2 + o(1) \\ |z_{n,2}|_{p+1}^{p+1} &= |u_n|_{p+1}^{p+1} - |\bar{u}|_{p+1}^{p+1} - |u^1|_{p+1}^{p+1} + o(1). \end{aligned}$$

Moreover, by using again Lemma 2.2

$$\begin{aligned} \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{z_{n,2}} (z_{n,2})^2\, dx &= \tilde{T}(z_{n,2}, z_{n,2}, z_{n,2}, z_{n,2}) = \tilde{T}(z_{n,1}, z_{n,1}, z_{n,1}, z_{n,1}) \\ &\quad - \tilde{T}(u^1, u^1, u^1, u^1) + o(1) \\ &= \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{z_{n,1}} (z_{n,1})^2\, dx - \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u^1} (u^1)^2\, dx + o(1). \end{aligned}$$

Then we get

$$\mathcal{J}_1(z_{n,2}) = \mathcal{J}_1(z_{n,1}) - \mathcal{J}_1\big(u^1\big) + o(1),$$

hence, by using (4.2.6), we obtain

$$\mathcal{J}_1(u_n) = \mathcal{J}_1(\bar{u}) + \mathcal{J}_1\big(u^1\big) + \mathcal{J}_1\left(z_{n,2}\right) + o(1).$$

As before one can prove that

$$\nabla \mathcal{J}_1(z_{n,2}) = o(1) \quad \text{in } H^1(\mathbb{R}^3).$$

Now, if $z_{n,2} \to 0$ in $H^1(\mathbb{R}^3)$ we are done. Otherwise $z_{n,2} \rightharpoonup 0$ and not strongly and we repeat the argument. By iterating this procedure we obtain sequences of points $y_n^j \in \mathbb{R}^3$ such that $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ if $i \neq j$ as $n \to +\infty$ and a sequence of functions $z_{n,j}(x) = z_{n,j-1}(x) - u^{j-1}(x - y_n^{j-1})$ with $j \geq 2$ such that

$$z_{n,j}(x + y_n^j) \rightharpoonup u^j(x) \quad \text{in } H^1(\mathbb{R}^3) \quad \nabla \mathcal{J}_1(u^j) = 0$$

and

$$\mathcal{J}_1(u_n) = \mathcal{J}_1(\bar{u}) + \sum_{j=1}^{k} \mathcal{J}_1(u^j) + \mathcal{J}_1(z_{n,k}) + o(1)$$

Then, since $\mathcal{J}_1(u^j) \geq c_1$ for all j and $\mathcal{J}_1(u_n)$ is bounded, the iteration must stop at some finite index k. □

Proposition 4.2 *c_1 is achieved by some positive $w \in \mathcal{M}_1$ such that $\mathcal{J}_1(w) = c_1$.*

Proof Let $(u_n)_n, u_n \in \mathcal{M}_1$, a minimizing sequence for $\mathcal{J}_1$, that is $\mathcal{J}_1(u_n) \to c_1$ as $n \to +\infty$. Let $t_n > 0$ such that $t_n|u_n| \in \mathcal{M}_1$. Then since also $u_n \in \mathcal{M}_1$ we get

$$\left(t_n^2 - t_n^{p+1}\right) \|u_n\|^2 + \left(t_n^4 - t_n^{p+1}\right) \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u_n} u_n^2 \, dx = 0$$

and this implies $t_n = 1, n \in \mathbb{N}$. Hence we can consider $u_n \geq 0$. Moreover $(u_n)_n$ is bounded. Indeed, by using (3.1) and the fact that $p \in (3, 5)$ we get

$$\mathcal{J}_1(u_n) \geq \left(\frac{1}{2} - \frac{1}{p+1}\right) \|u_n\|^2$$

from which it follows that $(u_n)_n$ is bounded since $\mathcal{J}_1(u_n)$ it is.

By the Ekeland variational principle there exists $(\tilde{u}_n)_n, \tilde{u}_n \in \mathcal{M}_1$ such that

(a) $\mathcal{J}_1(\tilde{u}_n) \to c_1$ as $n \to +\infty$;
(b) $\nabla \mathcal{J}_{1_{|\mathcal{M}_1}}(\tilde{u_n}) \to 0$ strongly in $H^1(\mathbb{R}^3)$ as $n \to +\infty$;
(c) $\|u_n - \tilde{u}_n\| \to 0$ as $n \to +\infty$.

We prove that $\nabla \mathcal{J}_1(\tilde{u}_n) \to 0$ as $n \to +\infty$. Indeed,

$$o(1) = \nabla \mathcal{J}_{1_{|\mathcal{M}_1}}(\tilde{u_n}) = \nabla \mathcal{J}_1(\tilde{u_n}) - \sigma_n \nabla H_1(\tilde{u}_n) \tag{4.2.9}$$

for some $\sigma_n \in \mathbb{R}$. Then, since $\tilde{u}_n \in \mathcal{M}_1$, taking the scalar product with $\tilde{u}_n$, we find

$$o(1) = (\nabla \mathcal{J}_1(\tilde{u}_n), \tilde{u}_n) - \sigma_n (\nabla H_1(\tilde{u}_n), \tilde{u}_n).$$

Thus we obtain

$$\sigma_n(\nabla H_1(\tilde{u}_n), \tilde{u}_n) \to 0.$$

But an easy computation shows that $(\nabla H_1(\tilde{u}_n), \tilde{u}_n) < -C < 0$. Hence $\sigma_n \to 0$ as $n \to +\infty$. Furthermore, since $\nabla H_1(\tilde{u}_n)$ is bounded, then $\sigma_n \nabla H_1(\tilde{u}_n) \to 0$ as $n \to +\infty$. This implies $\nabla \mathcal{J}_1(\tilde{u_n}) \to 0$ in $H^1(\mathbb{R}^3)$ as $n \to +\infty$.

Since $\mathcal{J}_1''$ maps bounded sets onto bounded sets, then by the mean value theorem it follows that also $\nabla \mathcal{J}_1(u_n) \to 0$ in $H^1(\mathbb{R}^3)$ as $n \to +\infty$. Then $(u_n)_n$ is a bounded Palais-Smale sequence at l! evel c_1. Hence we can apply Lemma 4.1 to $(u_n)_n$. Since $\mathcal{J}_1(u_n) \to c_1$, if $\bar{u} \neq 0$, then

$$c_1 = \mathcal{J}_1(\bar{u}) + \sum_{j=1}^{k} \mathcal{J}_1(u^j) \geq (k+1)c_1$$

and this implies $k = 0$. Hence u_n converges strongly to $\bar{u}$ in $H^1(\mathbb{R}^3)$.

If, instead, $\bar{u} = 0$, then

$$c_1 = \mathcal{J}_1(\bar{u}) + \sum_{j=1}^{k} \mathcal{J}_1(u^j) \geq kc_1$$

and this implies $k = 1$ and, up to translation, $u_n \to u^1$ in $H^1(\mathbb{R}^3)$.

In any case c_1 is achieved by some non-negative w. Furthermore since $\|u_n\| \geq C > 0$ then, by the strong convergence, also $w \neq 0$ and this would imply $w \in \mathcal{M}_1$. Furthermore, by continuity and by the uniqueness of the limit, we get $\mathcal{J}_1(w) = c_1$. □

5 A compactness lemma

In this section we deal with the behavior of the Palais-Smale sequences of I_λ where now λ can be positive or negative. This study will be basic to our search of critical points of I_λ.

Lemma 5.1 *Let $(u_n)_n$ be a (PS) sequence of I_λ constrained on $\mathcal{N}_\lambda$, i.e. $u_n \in \mathcal{N}_\lambda$ and*

$$\begin{aligned} &\text{(a)}\ I_\lambda(u_n) \text{ is bounded};\\ &\text{(b)}\ \nabla I_{\lambda|_{\mathcal{N}_\lambda}}(u_n) \to 0 \text{ strongly in } H^1(\mathbb{R}^3). \end{aligned} \tag{5.1}$$

Then replacing $(u_n)_n$, if necessary, with a subsequence, there exist a solution u of (P'_λ), a number $k \in \mathbb{N} \cup \{0\}$, k functions $u^1, \ldots, u^k$ of $H^1(\mathbb{R}^3)$ and k sequences of points (y_n^j), $y_n^j \in \mathbb{R}^3$, $0 \leq j \leq k$ such that

$$\begin{aligned} &\text{(i)}\ |y_n^j| \to +\infty,\ \ |y_n^j - y_n^i| \to +\infty \text{ if } i \neq j,\ \ n \to +\infty;\\ &\text{(ii)}\ u_n - \textstyle\sum_{j=1}^k u^j(\cdot - y_n^j) \longrightarrow \bar{u},\ \text{ in } H^1(\mathbb{R}^3);\\ &\text{(iii)}\ I_\lambda(u_n) \to I_\lambda(\bar{u}) + \textstyle\sum_{j=1}^k \mathcal{J}_\lambda(u^j);\\ &\text{(iv)}\ u^j \text{ are non trivial weak solutions of } (\mathrm{P}'^{\infty}_\lambda). \end{aligned} \tag{5.2}$$

Moreover we agree that in the case $k = 0$, the above holds without u^j.

Proof We first observe that $(u_n)_n$ is bounded. Indeed, if $\lambda > 0$ or $\lambda < 0$ with $p < 3$, then by (3.1)

$$I_\lambda(u_n) \geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_n\|^2.$$

If $\lambda < 0$ and $p \geq 3$ then

$$I_\lambda(u_n) \geq \frac{1}{4}\|u_n\|^2.$$

In both cases, being $I_\lambda(u_n)$ bounded, $(u_n)_n$ is bounded too.

In a similar way as in the proof of Proposition 4.2, one can prove that

$$\nabla I_\lambda(u_n) \to 0 \quad \text{in } H^1(\mathbb{R}^3). \tag{5.3}$$

Since u_n is bounded in $H^1(\mathbb{R}^3)$, there exists $\bar{u} \in H^1(\mathbb{R}^3)$ such that, up to a subsequence,

$$u_n \rightharpoonup \bar{u} \quad \text{in } H^1(\mathbb{R}^3) \text{ and in } L^{p+1}(\mathbb{R}^3)$$
$$u_n(x) \to \bar{u}(x) \quad \text{a.e. on } \mathbb{R}^3.$$

Furthermore, taking into account (3) of Lemma 2.1, we deduce that $\nabla I_\lambda(\bar{u}) = 0$, that is $\bar{u}$ is a weak solution of (P'_λ).

If $u_n \to \bar{u}$ in $H^1(\mathbb{R}^3)$, we are done. So we can assume that $(u_n)_n$ does not converge strongly to $\bar{u}$ in $H^1(\mathbb{R}^3)$. Set

$$z_n^1(x) = u_n(x) - \bar{u}(x).$$

Obviously, $z_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, but not strongly. As in the proof of Lemma 4.1 one can show that (4.2.3)–(4.2.4) and (4.2.7) hold. Moreover, by using Lemma A.2 of [3] we infer

$$\alpha(x)|z_n^1|^{p-1}z_n^1 \to 0 \quad \text{in } H^{-1}. \tag{5.4}$$

Furthermore, an easy computation shows that

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2\,dx = &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\eta(x)\eta(y)}{|x-y|}u_n^2(x)u_n^2(y)\,dx\,dy\\ &+2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\eta(x)K_\infty}{|x-y|}u_n^2(x)u_n^2(y)\,dx\,dy\\ &+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K_\infty K_\infty}{|x-y|}u_n^2(x)u_n^2(y)\,dx\,dy.\end{aligned}$$

Hence, by using Lemma 2.2, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2\,dx &= \bar{T}(u_n,u_n,u_n,u_n)+2T(u_n,u_n,u_n,u_n)+\tilde{T}(u_n,u_n,u_n,u_n)\\ &= \bar{T}\left(z_n^1,z_n^1,z_n^1,z_n^1\right)+2T\left(z_n^1,z_n^1,z_n^1,z_n^1\right)+\tilde{T}\left(z_n^1,z_n^1,z_n^1,z_n^1\right)\\ &\quad +\bar{T}(\bar{u},\bar{u},\bar{u},\bar{u})+2T(\bar{u},\bar{u},\bar{u},\bar{u})+\tilde{T}(\bar{u},\bar{u},\bar{u},\bar{u})+o(1)\\ &= \int_{\mathbb{R}^3} K(x)\phi_{z_n^1}\left(z_n^1\right)^2 dx+\int_{\mathbb{R}^3} K(x)\phi_{\bar{u}}\bar{u}^2\,dx+o(1).\end{aligned}$$

Claim:

$$\int_{\mathbb{R}^3} K(x)\phi_{z_n^1}\left(z_n^1\right)^2 dx=\int_{\mathbb{R}^3} K_\infty\tilde{\phi}_{z_n^1}\left(z_n^1\right)^2 dx+o(1). \tag{5.5}$$

Indeed:

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} K(x)\phi_{z_n^1}\left(z_n^1\right)^2 dx-\int_{\mathbb{R}^3} K_\infty\tilde{\phi}_{z_n^1}\left(z_n^1\right)^2 dx\right|\\ &\quad\le \underbrace{\int_{\mathbb{R}^3}|\eta(x)|\bar{\phi}_{z_n^1}\left(z_n^1\right)^2 dx}_{(I)}+2\underbrace{\int_{\mathbb{R}^3}|\eta(x)|\tilde{\phi}_{z_n^1}\left(z_n^1\right)^2 dx}_{(II)}.\end{aligned}$$

Since $\eta\in L^2(\mathbb{R}^3)$, we get $\eta^{6/5}\in L^{5/3}(\mathbb{R}^3)$. Now, we observe that $(z_n^1)_n$ is bounded in $H^1(\mathbb{R}^3)$ hence in $L^6(\mathbb{R}^3)$. Then $((z_n^1)^{12/5})_n$ is bounded in $L^{5/2}(\mathbb{R}^3)$ and so (see [12]) to any any choice of $\rho>0$ we get $(z_n^1)^{12/5}\rightharpoonup 0$ in $L^{5/2}(B_\rho(0))$. Hence, for any $\epsilon>0$, we get

$$\int_{B_\rho(0)}\eta(x)^{6/5}\left(z_n^1\right)^{12/5} dx<\epsilon.$$

Moreover, by (2) of Lemma 2.1 $\bar{\phi}_{z_n^1}$ and $\tilde{\phi}_{z_n^1}$ are bounded too and to any $\epsilon>0$ there exists $\bar{\rho}\equiv\bar{\rho}(\epsilon)>0$ such that

$$|\eta|_{2,\mathbb{R}^3\setminus B_\rho(0)}<\epsilon,\quad \forall\,\rho\ge\bar{\rho}.$$

Hence

$$
\begin{aligned}
(I) &\le \bar{S}^{-1}\|\bar{\phi}_{z_n^1}\|_{D^{1,2}}\left(\int\limits_{\mathbb{R}^3}|\eta(x)|^{6/5}\left(z_n^1\right)^{12/5}dx\right)^{5/6}\\
&\le C\left(\int\limits_{B_\rho(0)}|\eta(x)|^{6/5}\left(z_n^1\right)^{12/5}dx+\int\limits_{\mathbb{R}^3\setminus B_\rho(0)}|\eta(x)|^{6/5}\left(z_n^1\right)^{12/5}dx\right)^{5/6}\\
&\le C\left(\epsilon+|\eta|^{6/5}_{2,\mathbb{R}^3\setminus B_\rho(0)}|z_n^1|^{12/5}_{6,\mathbb{R}^3\setminus B_\rho(0)}\right)^{5/6}\\
&\le \tilde{C}\left(\epsilon+\epsilon^{6/5}\|z_n^1\|^{12/5}\right)^{5/6}
\end{aligned}
$$

from which it follows $(I)=o(1)$. By similar arguments one can also show that $(II)=o(1)$ and the claim is proved. Hence

$$
\int\limits_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2\,dx=\int\limits_{\mathbb{R}^3}K(x)\phi_{\bar{u}}\bar{u}^2\,dx+\int\limits_{\mathbb{R}^3}K_\infty\tilde{\phi}_{z_n^1}\left(z_n^1\right)^2dx+o(1). \tag{5.6}
$$

Argue as before it is possible to verify that for any $h\in H^1(\mathbb{R}^3)$ the following holds:

$$
\int\limits_{\mathbb{R}^3}K(x)\phi_{u_n}u_nh\,dx=\int\limits_{\mathbb{R}^3}K(x)\phi_{\bar{u}}\bar{u}h\,dx+\int\limits_{\mathbb{R}^3}K_\infty\tilde{\phi}_{z_n^1}z_n^1h\,dx+o(1). \tag{5.7}
$$

Therefore, (4.2.3), (4.2.4), (5.4), (4.2.7) together with (5.6) and (5.7), respectively, allow to obtain

$$
\begin{aligned}
I_\lambda(u_n)&=\frac{1}{2}\|u_n\|^2+\lambda\frac{1}{4}\int\limits_{\mathbb{R}^3}K(x)\phi_{u_n}u_n^2(x)dx-\frac{1}{p+1}\int\limits_{\mathbb{R}^3}a(x)|u_n|^{p+1}dx\\
&=\frac{1}{2}\|z_n^1\|^2+\frac{1}{2}\|\bar{u}\|^2+\lambda\frac{1}{4}\int\limits_{\mathbb{R}^3}K(x)\phi_{\bar{u}}\bar{u}^2(x)dx+\lambda\frac{1}{4}\int\limits_{\mathbb{R}^3}K_\infty\tilde{\phi}_{z_n^1}\left(z_n^1\right)^2dx\\
&\quad-\frac{1}{p+1}\int\limits_{\mathbb{R}^3}a(x)|\bar{u}|^{p+1}dx-\frac{1}{p+1}\int\limits_{\mathbb{R}^3}|z_n^1|^{p+1}dx+o(1)\\
&=I_\lambda(\bar{u})+\mathcal{J}_\lambda\left(z_n^1\right)+o(1).
\end{aligned} \tag{5.8}
$$

and, for all $h \in H^1(\mathbb{R}^3)$,

$$\begin{aligned} o(1) = (\nabla I_\lambda(u_n), h) &= (u_n, h) + \int_{\mathbb{R}^3} \left[\lambda K(x)\phi_{u_n} u_n h - a(x)|u_n|^{p-1} u_n h\right] dx \\ &= (\bar{u}, h) + \int_{\mathbb{R}^3} \left[\lambda K(x)\phi_{\bar{u}} \bar{u} h - a(x)|\bar{u}|^{p-1}\bar{u} h\right] dx \\ &\quad + (z_n^1, h) + \int_{\mathbb{R}^3} \left[\lambda K_\infty \tilde{\phi}_{z_n^1} z_n^1 h - |z_n^1|^{p-1} z_n^1 h\right] dx + o(1) \\ &= \left(\nabla I_\lambda(\bar{u}), h\right) + \left(\nabla \mathcal{J}_\lambda\left(z_n^1\right), h\right) + o(1) = \left(\nabla \mathcal{J}_\lambda\left(z_n^1\right), h\right) + o(1) \end{aligned}$$

so that

$$\nabla \mathcal{J}_\lambda\left(z_n^1\right) = o(1) \quad \text{in } H^1(\mathbb{R}^3). \tag{5.9}$$

Furthermore

$$\begin{aligned} 0 = (\nabla I_\lambda(u_n), u_n) &= (\nabla I_\lambda(\bar{u}), \bar{u}) + \left(\nabla \mathcal{J}_\lambda\left(z_n^1\right), z_n^1\right) + o(1) \\ &= \left(\nabla \mathcal{J}_\lambda\left(z_n^1\right), z_n^1\right) + o(1). \end{aligned}$$

Setting

$$\delta := \limsup_{n \to +\infty} \left(\sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |z_n^1|^{p+1} dx \right),$$

we have, as in Lemma 4.1, $\delta > 0$. Then we may assume the existence of $y_n^1 \subset \mathbb{R}^3$, such that

$$\int_{B_1(y_n^1)} |z_n^1|^{p+1} dx > \frac{\delta}{2}.$$

Let us now consider $z_n^1(\cdot + y_n^1)$. We may assume that $z_n^1(\cdot + y_n^1) \rightharpoonup u^1$ in $H^1(\mathbb{R}^3)$ and, then, $z_n^1(\cdot + y_n^1)(x) \to u^1(x)$ a.e. on $\mathbb{R}^3$. Since

$$\int_{B_1(0)} |z_n^1\left(x + y_n^1\right)|^{p+1} dx > \frac{\delta}{2},$$

from the Rellich theorem it follows that

$$\int_{B_1(0)} |u^1(x)|^{p+1} dx > \frac{\delta}{2}$$

and, thus, $u^1 \neq 0$. But, since $z_n^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, (y_n^1) must be unbounded and, up to a subsequence, we can assume that $|y_n^1| \to +\infty$. Furthermore (5.9) implies $\nabla \mathcal{J}_\lambda(u^1) = 0$.

Finally, let us set

$$z_n^2(x) = z_n^1(x) - u^1\left(x - y_n^1\right).$$

Then, using (4.2.3), (4.2.4) and, again, the Brezis-Lieb Lemma we have

$$\begin{aligned} \|z_n^2\|^2 &= \|u_n\|^2 - \|\bar{u}\|^2 - \|u^1\|^2 + o(1) \\ |z_n^2|_{p+1}^{p+1} &= |u_n|_{p+1}^{p+1} - |\bar{u}|_{p+1}^{p+1} - |u^1|_{p+1}^{p+1} + o(1). \end{aligned}$$

Moreover

$$\begin{aligned} \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{z_n^2}\left(z_n^2\right)^2 dx &= \tilde{T}\left(z_n^2, z_n^2, z_n^2, z_n^2\right) = \tilde{T}\left(z_n^1, z_n^1, z_n^1, z_n^1\right) \\ &\quad - \tilde{T}\left(u^1, u^1, u^1, u^1\right) + o(1) \\ &= \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{z_n^1}\left(z_n^1\right)^2 dx - \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{u^1}\left(u^1\right)^2 dx + o(1). \end{aligned}$$

This implies

$$\mathcal{J}_\lambda\left(z_n^2\right) = \mathcal{J}_\lambda\left(z_n^1\right) - \mathcal{J}_\lambda\left(u^1\right) + o(1),$$

hence, by using (5.8), we obtain

$$I_\lambda(u_n) = I_\lambda(\bar{u}) + \mathcal{J}_\lambda\left(z_n^1\right) + o(1) = I_\lambda(\bar{u}) + \mathcal{J}_\lambda\left(u^1\right) + \mathcal{J}_\lambda\left(z_n^2\right) + o(1).$$

As before one can prove that

$$\nabla \mathcal{J}_\lambda\left(z_n^2\right) = o(1) \quad \text{in } H^1\left(\mathbb{R}^3\right).$$

Now, if $z_n^2 \to 0$ in $H^1(\mathbb{R}^3)$ we are done. Otherwise $z_n^2 \rightharpoonup 0$ and not strongly and we repeat the argument. By iterating this procedure we obtain sequences of points

$y_n^j \in \mathbb{R}^3$ such that $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ if $i \neq j$ as $n \to +\infty$ and a sequence of functions $z_n^j(x) = z_n^{j-1}(x) - u^{j-1}(x - y_n^{j-1})$ with $j \geq 2$ such that

$$z_n^j\left(x + y_n^j\right) \rightharpoonup u^j(x) \quad \text{in } H^1\left(\mathbb{R}^3\right) \; \nabla \mathcal{J}_\lambda\left(u^j\right) = 0$$

and

$$I_\lambda\left(u_n\right) = I_\lambda(\bar{u}) + \sum_{j=1}^{k} \mathcal{J}_\lambda\left(u^j\right) + \mathcal{J}_\lambda\left(z_n^k\right) + o(1)$$

Then, since $\mathcal{J}_\lambda(u^j) \geq c_\lambda$ for all j and $I_\lambda(u_n)$ is bounded, the iteration must stop at some finite index k. □

We say that $(u_n)_n$, $u_n \in \mathcal{N}_\lambda$, is a $(PS)_d$- sequence if $I_\lambda(u_n) \to d$ and (5.1)-(b) holds.

Corollary 5.1 *Let $(u_n)_n$ be a $(PS)_d-$sequence. Then $(u_n)_n$ is relatively compact for all $d \in (0, c_\lambda)$. Moreover, if $I_\lambda(u_n) \to c_\lambda$, then either $(u_n)_n$ is relatively compact or the statement of Lemma 5.1 holds with $k = 1$.*

Proof Let consider a $(PS)_d-$sequence $(u_n)_n$ and apply to it Lemma 5.1, taking into account that $\mathcal{J}_\lambda(u^j) \geq c_\lambda$, for all j. When $I_\lambda(u_n) \to d < c_\lambda$ (5.2)(iii) gives $k = 0$, and, then, $u_n \to \bar{u}$ in $H^1(\mathbb{R}^3)$. When $I_\lambda(u_n) \to c_\lambda$, if u_n is not compact then (5.2)(iii) implies $k = 1$ and $\bar{u} = 0$. □

6 Existence of ground states for (P_λ)

In the following we will find ground states solutions for (P_λ). Here we have to distinguish the case in which λ is positive or negative.

6.1 The case $\lambda < 0$

Since $\lambda < 0$, without loss of generality we can take $\lambda = -1$. For simplicity, we set $\bar{I} := I_{-1}$, $\bar{\mathcal{N}} = \mathcal{N}_{-1}$ and $\bar{m} := m_{-1}$.

Next theorem provides a sufficient condition to solve the problem (P_λ) by using a minimization argument. In particular, if (H_1) holds, then, roughly speaking, the energy of a solution of (P_λ) cannot overcome the energy of a ground state of $(SN)_\infty$. Then a solution (ground state) of (P_λ) it is found without any other assumption.

Theorem 6.1 *Let (H_1) holds. Then there exist a positive ground state solution for* (P_λ).

Proof To prove the existence of a ground state solution for (P_λ) we just need to show that $\bar{m} < \bar{c}$. Indeed, by standard arguments and by using also Corollary 5.1, the theorem would follow.

Let $\bar{w} \in \bar{\mathcal{M}}$ such that $\bar{\mathcal{J}}(\bar{w}) = \bar{c}$ and let $t > 0$ such that $t\bar{w} \in \bar{\mathcal{N}}$. Then

$$\bar{m} \le \bar{I}(t\bar{w}) = \frac{t^2}{2}\|\bar{w}\|^2 - \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{\bar{w}}\bar{w}^2\,dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3} a(x)|\bar{w}|^{p+1}\,dx$$
$$\le \frac{t^2}{2}\|\bar{w}\|^2 - \frac{t^4}{4}\int_{\mathbb{R}^3} K_\infty\tilde{\phi}_{\bar{w}}\bar{w}^2\,dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3} a_\infty|\bar{w}|^{p+1}\,dx. \qquad (6.1.1)$$

Let us show that $t < 1$.

First of all we prove that $t \le 1$. Indeed since $\bar{w} \in \bar{\mathcal{M}}$ and $t\bar{w} \in \bar{\mathcal{N}}$ then

$$t^2\int_{\mathbb{R}^3} a_\infty|\bar{w}|^{p+1}\,dx + t^2\int_{\mathbb{R}^3} K_\infty\tilde{\phi}_{\bar{w}}\bar{w}^2\,dx = t^2\|\bar{w}\|^2$$
$$= t^{p+1}\int_{\mathbb{R}^3} a(x)|\bar{w}|^{p+1}\,dx + t^4\int_{\mathbb{R}^3} K(x)\phi_{\bar{w}}\bar{w}^2\,dx$$
$$\ge t^{p+1}\int_{\mathbb{R}^3} a_\infty|\bar{w}|^{p+1}\,dx + t^4\int_{\mathbb{R}^3} K_\infty\tilde{\phi}_{\bar{w}}\bar{w}^2\,dx.$$

Hence

$$\left(1 - t^{p-1}\right)\int_{\mathbb{R}^3} a_\infty|\bar{w}|^{p+1}\,dx + \left(1 - t^2\right)\int_{\mathbb{R}^3} K_\infty\tilde{\phi}_{\bar{w}}\bar{w}^2\,dx \ge 0$$

and this implies $t \le 1$. Furthermore $t \ne 1$. In fact, if $t = 1$ then

$$\int_{\mathbb{R}^3} a_\infty|\bar{w}|^{p+1}\,dx + \int_{\mathbb{R}^3} K_\infty\tilde{\phi}_{\bar{w}}\bar{w}^2\,dx = \|\bar{w}\|^2$$
$$= \int_{\mathbb{R}^3} a(x)|\bar{w}|^{p+1}\,dx + \int_{\mathbb{R}^3} K(x)\phi_{\bar{w}}\bar{w}^2\,dx.$$

Hence

$$\int_{\mathbb{R}^3} (a(x) - a_\infty)|\bar{w}|^{p+1}\,dx + \int_{\mathbb{R}^3} K(x)\phi_{\bar{w}}\bar{w}^2\,dx - \int_{\mathbb{R}^3} K_\infty\tilde{\phi}_{\bar{w}}\bar{w}^2\,dx = 0.$$

By (H_1) it follows the contradiction.

From (6.1.1) it follows

$$\bar{m} < \frac{1}{2}\|\bar{w}\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{\bar{w}} \bar{w}^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a_\infty |\bar{w}|^{p+1}\,dx$$
$$= \bar{\mathcal{J}}(\bar{w}) = \bar{c}.$$

□

Ground state solutions for (P_λ) can be found also under the assumptions (H_2) and (H_3), respectively. However since in these cases the energy of (P_λ) can exceed the energy of a ground state of (SN_∞), then we have to make a further assumption to insure that $\bar{m}$ is lower than $\bar{c}$. To do this, let us consider the problem

$$-\Delta u + u - K_\infty \tilde{\phi}_u u = a(x)|u|^{p-1}u. \tag{SN$_1$}$$

The solutions of (SN_1) are the critical points of the real functional $\bar{\mathcal{J}}_a$ defined on $H^1(\mathbb{R}^3)$ by

$$\bar{\mathcal{J}}_a(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_u u^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1}\,dx.$$

Let us define the Nehari manifold related to $\bar{\mathcal{J}}_a$

$$\bar{\mathcal{M}}_a := \left\{ u \in H^1(\mathbb{R}^3)\setminus\{0\} : \|u\|^2 = \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_u u^2\,dx + \int_{\mathbb{R}^3} a(x)|u|^{p+1}\,dx \right\},$$

and set

$$\bar{m}_a := \inf\left\{\mathcal{J}_a(u) : u \in \bar{\mathcal{M}}_a\right\}$$

which is a positive number. By using concentration-compactness arguments it is possible to show the following result:

Proposition 6.1 *If $a(x) \geq a_\infty$ with $a(x) - a_\infty > 0$ on a positive measure set, then there exists $\bar{w}_a \in \bar{\mathcal{M}}_a$ such that $\bar{\mathcal{J}}_a(\bar{w}_a) = \bar{m}_a$.*

At the same way we can consider the problem

$$-\Delta u + u - K(x)\phi_u u = a_\infty |u|^{p-1}u. \tag{SN$_2$}$$

The solutions of (SN_2) are the critical points of the real functional $\bar{\mathcal{J}}_K$ defined on $H^1(\mathbb{R}^3)$ by

$$\bar{\mathcal{J}}_K(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a_\infty |u|^{p+1}\,dx.$$

Let us define the Nehari manifold related to $\bar{\mathcal{J}}_K$

$$\bar{\mathcal{M}}_K := \left\{ u \in H^1(\mathbb{R}^3)\backslash\{0\} : \|u\|^2 = \int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx + \int_{\mathbb{R}^3} a_\infty |u|^{p+1}\,dx \right\},$$

and set

$$\bar{m}_K := \inf\left\{\bar{\mathcal{J}}_K(u) : u \in \bar{\mathcal{M}}_K\right\} > 0.$$

By using again concentration-compactness arguments it is possible to prove the following result:

Proposition 6.2 *If $K(x) \geq K_\infty$ with $K(x) - K_\infty > 0$ on a positive measure set, then there exists $\bar{w}_K \in \bar{\mathcal{M}}_K$ such that $\bar{\mathcal{J}}_K(\bar{w}_K) = \bar{m}_K$.*

Arguing as in Lemma 3.1-(1) it is possible to show that to any $u \in H^1(\mathbb{R}^3)$ there correspond a (unique) function $\xi u \in \bar{\mathcal{M}}_a$ and a (unique) function $\tau u \in \bar{\mathcal{M}}_K$ such that

$$\bar{\mathcal{J}}_a(\xi u) = \max_{t\geq 0} \bar{\mathcal{J}}_a(tu), \qquad \bar{\mathcal{J}}_K(\tau u) = \max_{t\geq 0} \bar{\mathcal{J}}_K(tu).$$

We are able now to prove the following results:

Theorem 6.2 *Let (H_2) holds. Moreover we assume*

$$\frac{K_\infty^2}{\mathcal{K}^2} < \left(\frac{\bar{c}}{\bar{m}_a}\right)^{\frac{2}{p+1}} - 1 \tag{6.1.2}$$

if $p \geq 3$ and

$$\frac{K_\infty^2}{\mathcal{K}^2} < \left(\frac{\bar{c}}{\bar{m}_a}\right)^{\frac{p-1}{4}} - 1 \tag{6.1.3}$$

if $p < 3$. Then there exists a positive ground state solution of (P_λ).

Proof Let us show that $\bar{m} < \bar{c}$. Indeed, standard arguments and Corollary 5.1 would imply the assertion of the theorem.

Let $\bar{w}_a \in \bar{\mathcal{M}}_a$ such that $\bar{\mathcal{J}}_a(\bar{w}_a) = \bar{m}_a$ and let $t > 0$ be such that $t\bar{w}_a \in \bar{\mathcal{N}}$.

Claim: $t \geq 1$.
Indeed, since $t\bar{w}_a \in \bar{\mathcal{N}}$, $\bar{w}_a \in \bar{\mathcal{M}}_a$ and (H_2) holds we find

$$\begin{aligned}\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{\bar{w}_a} \bar{w}_a^2\,dx + \int_{\mathbb{R}^3} a(x)|\bar{w}_a|^{p+1}\,dx &= \|\bar{w}_a\|^2 \\ &= t^2 \int_{\mathbb{R}^3} K(x)\phi_{\bar{w}_a} \bar{w}_a^2\,dx + t^{p-1}\int_{\mathbb{R}^3} a(x)|\bar{w}_a|^{p+1}\,dx \\ &\leq t^2 \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{\bar{w}_a} \bar{w}_a^2\,dx + t^{p-1}\int_{\mathbb{R}^3} a(x)|\bar{w}_a|^{p+1}\,dx.\end{aligned}$$

Hence

$$\left(1 - t^2\right)\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{\bar{w}_a} \bar{w}_a^2\,dx + \left(1 - t^{p-1}\right)\int_{\mathbb{R}^3} a(x)|\bar{w}_a|^{p+1}\,dx \leq 0$$

and the claim follows. Then

$$\begin{aligned}\bar{m} \leq \bar{I}(t\bar{w}_a) &= \frac{t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_{\bar{w}_a}\bar{w}_a^2\,dx + t^{p+1}\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} a(x)|\bar{w}_a|^{p+1}\,dx \\ &\leq t^{\max\{4,p+1\}}\left(\frac{1}{4}\int_{\mathbb{R}^3} K_\infty\tilde{\phi}_{\bar{w}_a}\bar{w}_a^2\,dx + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} a(x)|\bar{w}_a|^{p+1}\,dx\right) \\ &= t^{\max\{4,p+1\}}\bar{m}_a. \end{aligned} \tag{6.1.4}$$

Let us now estimate $t \geq 1$. We find

$$\begin{aligned}\|\bar{w}_a\|^2 &= t^2\int_{\mathbb{R}^3} K(x)\phi_{\bar{w}_a}\bar{w}_a^2\,dx + t^{p-1}\int_{\mathbb{R}^3} a(x)|\bar{w}_a|^{p+1}\,dx \\ &\geq t^{\min\{2,p-1\}}\left(\int_{\mathbb{R}^3} K(x)\phi_{\bar{w}_a}\bar{w}_a^2\,dx + \int_{\mathbb{R}^3} a(x)|\bar{w}_a|^{p+1}\,dx\right).\end{aligned}$$

Then, since $K(x)$, $a(x)$ are positive functions

$$\begin{aligned}t^{\min\{2,p-1\}} &\leq \frac{\|\bar{w}_a\|^2}{\int_{\mathbb{R}^3} K(x)\phi_{\bar{w}_a}\bar{w}_a^2\,dx + \int_{\mathbb{R}^3} a(x)|\bar{w}_a|^{p+1}\,dx} \\ &< \frac{\int_{\mathbb{R}^3} K_\infty\tilde{\phi}_{\bar{w}_a}\bar{w}_a^2\,dx}{\int_{\mathbb{R}^3} K(x)\phi_{\bar{w}_a}\bar{w}_a^2\,dx} + 1 \\ &< \frac{K_\infty^2}{\mathcal{K}^2} + 1.\end{aligned}$$

Substituting in (6.1.4) and by using (6.1.2)–(6.1.3) we find that $\bar{m} < \bar{c}$ completing the proof.

In a similar way one can prove also the following result: □

Theorem 6.3 *Let* (H_3) *holds. Moreover we assume*

$$\frac{a_\infty}{\mathcal{A}} < \left(\frac{\bar{c}}{\bar{m}_K}\right)^{\frac{2}{p+1}} - 1 \tag{6.1.5}$$

if $p \geq 3$ *and*

$$\frac{a_\infty}{\mathcal{A}} < \left(\frac{\bar{c}}{\bar{m}_K}\right)^{\frac{p-1}{4}} - 1 \tag{6.1.6}$$

if $p < 3$. *Then there exists a positive ground state solution of* (P_λ).

6.2 The case $\lambda > 0$

Without loss of generality we assume $\lambda = 1$. In this section we provide sufficient conditions to prove the existence of ground states solutions for (P_λ).

Theorem 6.4 *If* (H_2) *holds then* (P_λ) *admits a ground states solution.*

Proof To prove the theorem, we have just to show that $m_1 < c_1$. Let $w \in \mathcal{M}_1$ such that $\mathcal{J}_1(w) = c_1$ and let $t > 0$ such that $tw \in \mathcal{N}_1$.

Let us show that $t < 1$.

Since $tw \in \mathcal{N}_1$, $w \in \mathcal{M}_1$ and (H_1) holds then

$$\begin{aligned} t^{p+1}\|w\|^2 + t^{p+1}\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2\,dx &= t^{p+1}\int_{\mathbb{R}^3} a_\infty |w|^{p+1}\,dx \\ \leq t^{p+1}\int_{\mathbb{R}^3} a(x)|w|^{p+1}\,dx &= t^2\|w\|^2 + t^4\int_{\mathbb{R}^3} K(x)\phi_w w^2\,dx \\ \leq t^2\|w\|^2 + t^4\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2\,dx & \end{aligned}$$

from which it follows

$$\left(t^{p+1} - t^2\right)\|w\|^2 + \left(t^{p+1} - t^4\right)\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2\,dx \leq 0$$

and hence $t \leq 1$. Moreover $t \neq 1$. Indeed, by contradiction, if $t = 1$ then

$$\int_{\mathbb{R}^3} a_\infty |w|^{p+1}\, dx - \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2\, dx = \|w\|^2$$
$$= \int_{\mathbb{R}^3} a(x)|w|^{p+1}\, dx - \int_{\mathbb{R}^3} K(x)\phi_w w^2\, dx$$

and this implies

$$\int_{\mathbb{R}^3} (a_\infty - a(x))|w|^{p+1}\, dx - \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2\, dx + \int_{\mathbb{R}^3} K(x)\phi_w w^2\, dx = 0$$

and this is a contradiction since by (H_2)

$$-\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2\, dx + \int_{\mathbb{R}^3} K(x)\phi_w w^2\, dx \leq 0$$

and

$$\int_{\mathbb{R}^3} (a_\infty - a(x))|w|^{p+1}\, dx \leq 0$$

but not identically zero since $a(x) - a_\infty > 0$ on a positive measure set. Then

$$\begin{aligned} m_1 \leq I_1(tw) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|tw\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} K(x)\phi_{tw}(tw)^2\, dx \\ &\leq t^2\left(\frac{1}{2} - \frac{1}{p+1}\right)\|w\|^2 + t^4\left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2\, dx \\ &< \left(\frac{1}{2} - \frac{1}{p+1}\right)\|w\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_w w^2\, dx \\ &= \mathcal{J}_1(w) = c_1. \end{aligned}$$

□

Ground state solutions for (P_λ) can be found also under the assumptions (H_1) and (H_4) respectively. However since in these cases the energy of (P_λ) can exceed the energy of a ground state of (SP_∞), then we have to make a further assumption to insure that m_1 is lower than c_1. To do this, let us consider the problem

$$-\Delta u + u + K_\infty \tilde{\phi}_u u = a(x)|u|^{p-1}u. \qquad (\mathrm{SP}_1)$$

The solutions of (SP_1) are the critical points of the real functional I_a defined on $H^1(\mathbb{R}^3)$ by

$$I_a(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K_\infty \tilde{\phi}_u u^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a(x)|u|^{p+1}\,dx.$$

Let us define the Nehari manifold related to I_a

$$\mathcal{N}_a := \left\{ u \in H^1(\mathbb{R}^3)\setminus\{0\} : \|u\|^2 + \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_u u^2\,dx = \int_{\mathbb{R}^3} a(x)|u|^{p+1}\,dx \right\},$$

and set

$$m_a := \inf\{I_a(u) : u \in \mathcal{N}_a\}$$

which is a positive number. By using concentration-compactness arguments it is possible to show the following result:

Proposition 6.3 *If $a(x) \geq a_\infty$ with $a(x) - a_\infty > 0$ on a positive measure set, then there exists $w_a \in \mathcal{N}_a$ such that $I_a(w_a) = m_a$.*

At the same way we can consider the problem

$$-\Delta u + u + K(x)\phi_u u = a_\infty |u|^{p-1} u. \tag{SP$_2$}$$

The solutions of (SP_2) are the critical points of the real functional I_K defined on $H^1(\mathbb{R}^3)$ by

$$I_K(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3} a_\infty |u|^{p+1}\,dx.$$

Let us define the Nehari manifold related to I_K

$$\mathcal{N}_K := \left\{ u \in H^1(\mathbb{R}^3)\setminus\{0\} \,:\, \|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx = \int_{\mathbb{R}^3} a_\infty |u|^{p+1}\,dx \right\},$$

and set

$$m_K := \inf\{I_K(u) \,:\, u \in \mathcal{N}_K\} > 0.$$

By using again concentration-compactness arguments it is possible to prove the following result:

Proposition 6.4 *If $K(x) \le K_\infty$ with $K(x) - K_\infty > 0$ on a positive measure set, then there exists $w_K \in \mathcal{N}_K$ such that $I_K(w_K) = m_K$.*

Arguing as in Lemma 3.1-(1) it is possible to show that to any $u \in H^1(\mathbb{R}^3)$ there correspond a (unique) function $\xi u \in \mathcal{N}_a$ and a (unique) function $\tau u \in \mathcal{N}_K$ such that

$$I_a(\xi u) = \max_{t \ge 0} I_a(tu), \qquad I_K(\tau u) = \max_{t \ge 0} I_K(tu).$$

Then we are ready to prove the following results:

Theorem 6.5 *Let (H_1) holds. Moreover let us assume*

$$M_1^2 < \frac{p-1}{2(p+1)} \left[\left(\frac{\bar{c}}{m_a} \right)^{\frac{p-3}{p+1}} - 1 \right] \cdot \frac{1}{m_a}, \tag{6.2.1}$$

where

$$M_1 := \bar{S}^{-1} \left(S_6^{-2} |\eta|_2 + K_\infty S_{12/5}^{-2} \right).$$

Then (P_λ) *has a positive ground state solution.*

Proof In the following we show that $m_1 < c_1$. Then, standard arguments jointing with Corollary 5.1 give the desired assertion.

Let $w_a \in \mathcal{N}_a$ and let $t > 0$ be such that $t w_a \in \mathcal{N}_1$. We claim that $t \ge 1$. Indeed, since $w_a \in \mathcal{N}_a$ and $t w_a \in \mathcal{N}_1$ we find, by using also (H_1)

$$\begin{aligned} t^{p+1} \left(\|w_a\|^2 + \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{w_a} w_a^2 \, dx \right) &= t^{p+1} \int_{\mathbb{R}^3} a(x) |w_a|^{p+1} \, dx \\ &= t^2 \|w_a\|^2 + t^4 \int_{\mathbb{R}^3} K(x) \phi_{w_a} w_a^2 \, dx \\ &\ge t^2 \|w_a\|^2 + t^4 \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{w_a} w_a^2 \, dx \end{aligned}$$

Hence we find

$$\left(t^2 - t^{p+1} \right) \|w_a\|^2 + \left(t^4 - t^{p+1} \right) \int_{\mathbb{R}^3} K_\infty \tilde{\phi}_{w_a} w_a^2 \, dx \le 0$$

from which it follows the claim. We estimate $t \geq 1$.

$$
\begin{aligned}
0 &= t^2\|w_a\|^2 + t^4 \int\limits_{\mathbb{R}^3} K(x)\phi_{w_a} w_a^2\,dx - t^{p+1} \int\limits_{\mathbb{R}^3} a(x)|w_a|^{p+1}\,dx \\
&\leq t^4\|w_a\|^2 + t^4 \int\limits_{\mathbb{R}^3} K(x)\phi_{w_a} w_a^2\,dx - t^{p+1} \int\limits_{\mathbb{R}^3} a(x)|w_a|^{p+1}\,dx.
\end{aligned}
$$

Hence, since $K_\infty > 0$ we find

$$
t^{p-3} \leq \frac{\|w_a\|^2 + \int_{\mathbb{R}^3} K(x)\phi_{w_a} w_a^2\,dx}{\int_{\mathbb{R}^3} a(x)|w_a|^{p+1}\,dx} < 1 + \frac{\int_{\mathbb{R}^3} K(x)\phi_{w_a} w_a^2\,dx}{\|w_a\|^2}.
$$

Furthermore by using (2.5)

$$
\int\limits_{\mathbb{R}^3} K(x)\phi_{w_a} w_a^2\,dx \leq M_1^2 \cdot \|w_a\|^4
$$

where

$$
M_1 := \bar{S}^{-1}\left(|\eta|_2 \cdot S_6^{-2} + K_\infty S_{12/5}^{-2}\right).
$$

Since

$$
\begin{aligned}
m_a = I_a(w_a) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|w_a\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int\limits_{\mathbb{R}^3} K_\infty \tilde{\phi}_{w_a} w_a^2\,dx \\
&\geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\|w_a\|^2,
\end{aligned}
$$

then

$$
t < \left(1 + \frac{2(p+1)}{p-1} M_1^2 c_a\right)^{\frac{1}{p-3}}. \tag{6.2.2}
$$

Then, by using (H_1), the definition of m_a, the fact that $p \in (3, 5)$, (6.2.2) and (6.2.1) we find

$$
\begin{aligned}
m_1 \le I(tw_a) &= \frac{1}{4}t^2\|w_a\|^2 + t^{p+1}\left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} a(x)|w_a|^{p+1}\,dx \\
&\le t^{p+1}\left[\frac{1}{4}\|w_a\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3} a(x)|w_a|^{p+1}\,dx\right] \\
&\le t^{p+1}c_a \le \left(1 + \frac{2(p+1)}{p-1}M_1^2 c_a\right)^{\frac{p+1}{p-3}} c_a < c_1.
\end{aligned}
$$

□

In a similar way one can also prove the following result:

Theorem 6.6 *Let* (H_4) *holds. Moreover let us assume*

$$\frac{a_\infty}{\mathcal{A}} < \left(\frac{\bar{c}}{m_K}\right)^{\frac{p-3}{4}}. \tag{6.2.3}$$

Then (P_λ) *has a positive ground state solution.*

References

1. Ambrosetti, A.: On Schrödinger–Poisson systems. Milan J. Math. **76**, 257–274 (2008)
2. Azzollini, A.: Concentration and compactness in nonlinear Schrödinger–Poisson system with a general nonlinearity. J. Diff. Eqns. **249**(7), 1746–1765 (2010)
3. Bahri, A., Lions, P.L.: On the existence of a positive solution of semilinear elliptic equations in unbounded domains. Ann. Inst. Henri Poincaré **14**(3), 365–413 (1997)
4. Benci, V., Fortunato, D.: An eigenvalue problem for the Schrödinger–Maxwell equations. Top. Meth. Nonlinear Anal. **11**, 283–293 (1998)
5. Benci, V., Fortunato, D.: Solitary waves of the nonlinear Klein–Gordon equation coupled with Maxwell equations. Rev. Math. Phys. **14**, 409–420 (2002)
6. Brezis, H., Lieb, E.: A relation between pointwise convergence of functions and convergence of functionals. Proc. AMS **88**, 486–490 (1988)
7. Cerami, G., Vaira, G.: Positive solutions for some non autonomous Schrödinger–Poisson Systems. J. Diff. Eqns. **248**(3), 521–543 (2010)
8. Ianni, I., Ruiz, D.: Ground and bound states for a static Schrödinger–Poisson–Slater problem (preprint)
9. Siciliano, G.: Multiple solutions for Schrödinger–Poisson–Slater system. J. Math. Anal. Appl. **365**(1), 288–299 (2010)
10. Talenti, G.: Elliptic equations and rearrangements. Ann. Scuola Norm. Sup. Pisa **4**(3) (1976)
11. Willem, M.: Minimax Theorems, PNDEA, vol. 24. Birkhäuser, Basel (1996)
12. Willem, M.: Analyse Harmonique réelle. Hermann, Paris (1995)