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Some properties of autocentral automorphisms of a group

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Abstract Let G be a group, $Aut(G)$ and $L(G)$ denote the full automorphisms group and absolute centre of G, respectively. The automorphism $\alpha \in Aut(G)$ is called autocentral if $g^{-1}\alpha(g) \in L(G)$, for all $g \in G$. In the present paper, we investigate the properties of such automorphisms.

Keywords Central automorphism · Absolute centre · Autocentral automorphism

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1 Introduction

Let G be a group and g_1, g_2 be elements of G, then $g_1^{g_2} = g_2^{-1}g_1g_2$ and $[g_1, g_2] = g_1^{-1}g_1^{g_2} = g_1^{-1}g_1^{\varphi_{g_2}}$ denote the *conjugate* of g_1 by g_2 and the *commutator* of g_1 and g_2, respectively, where φ_{g_2} is the inner automorphism of G. As in Hegarty [4], if $\alpha \in Aut(G)$ and $g \in G$ then the *autocommutator* of g and α is defined to be $[g, \alpha] = g^{-1}g^{\alpha} = g^{-1}\alpha(g)$.

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Now, using the above notation, one may define

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \text{ for all } \alpha \in Aut(G)\},$$
$$K(G) = \langle\, [g, \alpha] \mid g \in G\,,\ \alpha \in Aut(G)\,\rangle,$$

which are called the *absolute centre* and the *autocommutator subgroup* of G, respectively [8,4]. Autocommutator subgroup and absolute centre are already studied in [2,6].

Particularly, if α runs in the set of all inner automorphisms of G, $Inn(G)$, then the set of all elements $g \in G$ such that $[g, \alpha] = 1$ is the centre, $Z(G)$ of G, and the subgroup generated by the set of all commutators $[g, \alpha]$ with $g \in G$ is the derived subgroup G' of G. One can easily check that the absolute centre is a characteristic subgroup contained in the centre of G and $K(G)$ is a characteristic subgroup of G containing the derived subgroup. The automorphism $\alpha \in Aut(G)$ is said to be a central automorphism, if $[g, \alpha] = g^{-1}g^{\alpha} \in Z(G)$, for all $g \in G$. The set of all such automorphisms is denoted by $Aut_c(G)$, which is a normal subgroup of Aut(G) (see [3] for more detail).

Now, we call $\alpha \in Aut(G)$ to be *autocentral automorphism*, when $[g, \alpha] = g^{-1}g^{\alpha} \in L(G)$, for all $g \in G$. In [4], Hegarty denotes the set of all such autocentral automorphisms by

$$Var(G) = \{\alpha \in Aut(G) \mid [g, \alpha] \in L(G), \text{ for all } g \in G\}.$$

Clearly, $Var(G)$ is a normal subgroup of $Aut(G)$ contained in $Aut_c(G)$. In [4], it is shown that if $G/L(G)$ is a finite group, then $Var(G)$ is finite and so $Aut(G)$ and $K(G)$ are finite.

The properties of $Aut_c(G)$ are studied by many authors, see for instance [1,3,5]. In the present article, we study some properties of autocentral automorphisms of a given group G. We recall that in [3], Franciosi et al. show that if $Z(G)$ is torsion-free and $Z(G)/G' \cap Z(G)$ is torsion, then $Aut_c(G)$ acts trivially on $Z(G)$ and it is an abelian torsion-free group. They also prove that $Aut_c(G)$ is trivial, when $Z(G)$ is torsion-free and G/G' is torsion. In [5], Jamali et al. proved that $Aut_c(G) \cong Hom(G/G', Z(G))$, if G is a finite group and $Z(G) \leq G'$. Also Adney and Yen [1] have established a one-to-one correspondence between $Aut_c(G)$ and $Hom(G, Z(G))$, provided G is a purely non-abelian finite group, that is a non-abelian group with no nontrivial abelian direct factor.

We prove the following results, which give the behavior of the autocentral subgroup, $Var(G)$, of an arbitrary group G.

Theorem A *Let G be a group with $L(G)$ torsion-free and assume $E(G) = [G, C_{Aut(G)}(Var(G))]$. Then*

(a) *$Var(G)$ is torsion-free, and*
(b) *if $G/E(G)$ is torsion, then $Var(G) = \langle 1 \rangle$.*

Theorem B *Let G be a group such that $L(G)$ is contained in $E(G)$, then*

$$Var(G) \cong Hom(G/E(G), L(G)).$$

Theorem C *Let G be a purely non-abelian finite group, then*

$$Var(G) \cong Hom(G, L(G)).$$

One notes that, in the above theorems if the inner automorphisms to be the full automorphisms group of G or $L(G)$ to be the whole centre of G then we obtain the results mentioned above in [3,5,1].

2 Autocentral automorphisms of a group

Let G and H be any groups, then $Hom(G, H)$ denotes the set of all homomorphisms from G into H. Clearly, if H is an abelian group then $Hom(G, H)$ forms a group under the following operation $(fg)(x) = f(x)g(x)$, for all $f, g \in Hom(G, H)$ and $x \in G$.

Now, using the notation of the previous section, we have the following useful result.

Proposition 1 *If G is a group, $L(G)$ and $Var(G)$ are as defined before, then $Var(G) \cong Hom(G/L(G), L(G))$. In particular, $Var(G)$ is an abelian group.*

Proof Define the following map

$$\begin{aligned} \psi : Var(G) &\longrightarrow Hom(G/L(G), L(G)) \\ \alpha &\longmapsto \alpha^* \end{aligned}$$

so that $\alpha^* : G/L(G) \longrightarrow L(G)$ and given by $\alpha^*(gL(G)) = g^{-1}\alpha(g)$, for all $g \in G$. The definition of $L(G)$ implies that, every automorphism of G acts trivially on $L(G)$. It follows that α^* is a well-defined homomorphism of $G/L(G)$ into $L(G)$. Clearly, ψ is well-defined and it is a homomorphism, since for all $\alpha_1, \alpha_2 \in Var(G)$ and $g \in G$, we have

$$\begin{aligned} (\alpha_1\alpha_2)^*(gL(G)) &= g^{-1}\alpha_1\alpha_2(g) = g^{-1}\alpha_1(\alpha_2(g)) \\ &= g^{-1}\alpha_1(gg^{-1}\alpha_2(g)) = g^{-1}\alpha_1(g)\,\alpha_1(g^{-1}\alpha_2(g)) \\ &= g^{-1}\alpha_1(g)\;\; g^{-1}\alpha_2(g) = \alpha_1^*(gL(G))\,\alpha_2^*(gL(G)). \end{aligned}$$

Now, if $\alpha_1^* = \alpha_2^*$ then $g^{-1}\alpha_1(g) = g^{-1}\alpha_2(g)$, for all $g \in G$, which implies that $\alpha_1 = \alpha_2$ and hence ψ is one-to-one. The homomorphism ψ is also surjective, for let $\beta \in Hom(G/L(G), L(G))$, then we define the map

$$\begin{aligned} \alpha : G &\longrightarrow G \\ g &\longmapsto g\beta(gL(G)). \end{aligned}$$

If we show that α is in $Var(G)$, then it is evident that $\alpha^* = \beta$, which shows that ψ is an isomorphism and the result holds. On the other hand, as $L(G)$ is contained in the centre of G and it is abelian, it follows that $Hom(G/L(G), L(G))$ is also abelian and so is $Var(G)$. Now, it remains to show that $\alpha \in Var(G)$. The properties of α being well-defined and homomorphism follow easily. The homomorphism α is injective, for if $g \in Ker\alpha$ then $1 = \alpha(g) = g\beta(gL(G))$ and so $g^{-1} = \beta(gL(G)) \in L(G)$. Thus $gL(G) = 1_{G/L(G)}$ and so $\beta(gL(G))$ is identity, which follows that $1 = \alpha(g) = g$. The map α is also surjective. Note that $Im\beta \subseteq Im\alpha$, since if $l \in Im\beta$ then for some $g \in G$, $\beta(gL(G)) = l \in L(G)$. Now $\alpha(l) = l\beta(lL(G)) = l$ and so $l \in Im\alpha$. Using the latter containment, for every element $g \in G$, we have $g = \alpha(g)(\beta(gL(G)))^{-1} \in Im\alpha$, which implies that G is contained in $Im\alpha$ and hence gives the surjectivity of α. Therefore, α is an automorphism and the proof is completed. □

The following corollaries may be interested in their own right.

Corollary 1 *Let G be a finite p-group, then so is* $Var(G)$.

Proof By the assumption, the subgroup $L(G)$ and hence $Hom(G/L(G), L(G))$ are finite p-groups. Now, the result follows by using Proposition 1. □

Corollary 2 *Let G be a finite group. Then*

(i) *if* $(|\frac{G}{L(G)}|, |L(G)|) = 1$, *then* $Var(G) = \langle 1 \rangle$;
(ii) *if* $\frac{G}{L(G)}$ *is abelian and* $Var(G)$ *is trivial, then* $(|\frac{G}{L(G)}|, |L(G)|) = 1$.

Proof (i) It is obvious, by Proposition 1.
(ii) Proposition 1 implies that $Hom(G/L(G), L(G)) = \langle 1 \rangle$. Using the fundamental theorem of finitely generated abelian groups and the property that $Hom(C_m, C_n) \cong C_{(m,n)}$ we obtain $(|\frac{G}{L(G)}|, |L(G)|) = 1$. □

Remark 1 If G and H are finite groups and H is abelian, then the assumption $Hom(G, H) = 1$ does not imply that $(|G|, |H|) = 1$. For a counter example, take the alternating group A_5, then $A_5{}' = A_5$ and $Hom(A_5{}', C_3) = \langle 1 \rangle$, but $(60, 3) \neq 1$. Therefore, the assumption of $G/L(G)$ being abelian in part (ii) of Corollary 2 can not be removed.

3 Proofs of the main theorems

To prove our main theorems we need to construct the subgroup $E(G)$ of the given group G, which is contained in $K(G)$ and its elements are fixed under each autocentral automorphism of G. Let

$$C_{Aut(G)}(Var(G)) = \{\alpha \in Aut(G)|\ \alpha\beta = \beta\alpha, \quad \forall \beta \in Var(G)\}$$

be the centralizer of $Var(G)$ in $Aut(G)$ and put $E(G) = [G, C_{Aut(G)}(Var(G))]$. One can easily see that the subgroup $E(G)$ is characteristic in G and containing the derived subgroup $G' = [G, Inn(G)]$. Note that, $L(G)$ is contained in $Z(G)$ and every central automorphism commutes with each inner automorphism of G, and so $Inn(G) \leq C_{Aut(G)}(Var(G))$.

The following lemma gives the important property of $E(G)$, while $K(G)$ does not carry over such a property.

Lemma 1 *If G is an arbitrary group, then $Var(G)$ acts trivially on the subgroup $E(G)$ of G.*

Proof Take an automorphism $\alpha \in Var(G)$ then $x^{-1}\alpha(x) \in L(G)$, for all $x \in G$ and so $\alpha(x) = xl$, for some $l \in L(G)$. Now, let $\beta \in C_{Aut(G)}(Var(G))$ then using the property of β we have

$$\begin{aligned}\alpha([x,\beta]) &= \alpha(x^{-1}\beta(x)) = \alpha(x)^{-1}\alpha\beta(x) = l^{-1}x^{-1}\beta\alpha(x)\\ &= l^{-1}x^{-1}\beta(xl) = l^{-1}x^{-1}\beta(x)\beta(l) = x^{-1}\beta(x) = [x,\beta],\end{aligned}$$

which gives the result. □

Now, we are ready to prove Theorem A.

Proof of Theorem A (a) Clearly $Hom(G/L(G), L(G))$ is abelian, since $L(G)$ is abelian. By the assumption, $L(G)$ is a torsion-free group and hence Proposition 1 gives the result.

(b) We show that $\alpha(x) = x$, for any $\alpha \in Var(G)$ and all $x \in G$. Since $G/E(G)$ is torsion, $x^n \in E(G)$, for some $n \in \mathbb{N}$. By Lemma 1, we have $\alpha(x)^n = \alpha(x^n) = x^n$ and so $x^{-n}\alpha(x)^n = 1$ and as $x^{-1}\alpha(x) \in L(G)$ we have $(x^{-1}\alpha(x))^n = 1$. By the assumption, $L(G)$ is torsion-free and hence $x^{-1}\alpha(x) = 1$ that is $\alpha(x) = x$, for all $\alpha \in Var(G)$ and $x \in G$. Thus $Var(G) = \langle 1\rangle$. □

The following example guaranties the assumption of Theorem A, i.e. there exists an infinite group with torsion-free absolute centre.

Example 1 Let G be a group with the following presentation

$$G = \langle x, y \mid x^{2^n-1} = 1,\ x^y = x^2\rangle,$$

where $n \geq 2$. Clearly the group G is the semidirect product of $\mathbb{Z}_{2^n-1}$ by $\mathbb{Z}$, via the automorphism of order n, given by $x \longmapsto x^2$ and $y \longmapsto y$. It is obvious that all elements of G have the form $x^i y^j$ such that $1 \leq i \leq 2^n - 1$ and $j \in \mathbb{Z}$. Clearly, $Z(G) = \langle y^n\rangle$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_{2^n-1} \rtimes \mathbb{Z}_n$ a group of order $n(2^n - 1)$.

Now, any automorphism of G must take x to a power of x and y to $x^i y^j$ for some $1 \leq i \leq 2^n - 1$ and $j \in \{-1, 1\}$, but $j = -1$ does not satisfy the relators, so all the automorphisms of G must be of the form $x \longmapsto x^i$ and $y \longmapsto x^k y$ for some $1 \leq i, k \leq 2^n - 1$, for which $(i, 2^n - 1) = 1$. Then every automorphism must take y^n to $(x^k y)^n = y^n$, and hence all the automorphisms must fix $Z(G) = \langle y^n\rangle$, point-wise. Therefore $L(G) = Z(G)$ and so $L(G)$ is torsion-free.

Remark 2 Using Theorem A in the above example, it follows that $Var(G)$ is torsion-free. On the other hand, the main theorem of Hegarty in [4] implies that $Var(G)$ is finite, therefore $Var(G) = Aut_c(G) = \langle 1\rangle$. One notes that the converse of part (b) in Theorem A is not true in general, because $E(G) = K(G) = \langle x\rangle$ and so $G/E(G)$ can not be torsion.

Proof of Theorem B We first show that $Var(G) \cong Hom(\frac{G}{E(G)L(G)}, L(G))$. Consider the map

$$\begin{aligned}\varphi : Var(G) &\longrightarrow Hom(\frac{G}{E(G)L(G)}, L(G))\\ \alpha &\longmapsto \overline{\alpha},\end{aligned}$$

such that $\overline{\alpha} : \frac{G}{E(G)L(G)} \longrightarrow L(G)$ given by $\overline{\alpha}(gE(G)L(G)) = g^{-1}\alpha(g)$, for all $g \in G$. Clearly, $\overline{\alpha}$ is a well-defined homomorphism, since for all g_1 and g_2 in G, if $g_1E(G)L(G) = g_2E(G)L(G)$ then ${g_1}^{-1}g_2 \in E(G)L(G)$. By the definition of $L(G)$ and Lemma 1, $\alpha({g_1}^{-1}g_2) = {g_1}^{-1}g_2$ and so ${g_1}^{-1}\alpha(g_1) = {g_2}^{-1}\alpha(g_2)$. On the other hand, $\overline{\alpha}$ is a homomorphism, for

$$\begin{aligned}\overline{\alpha}(g_1E(G)L(G)g_2E(G)L(G)) &= \overline{\alpha}(g_1g_2E(G)L(G))\\ &= (g_1g_2)^{-1}\alpha(g_1g_2)\\ &= {g_2}^{-1}{g_1}^{-1}\alpha(g_1)\alpha(g_2)\\ &= {g_1}^{-1}\alpha(g_1)\,{g_2}^{-1}\alpha(g_2)\\ &= \overline{\alpha}(g_1E(G)L(G))\,\overline{\alpha}(g_2E(G)L(G)).\end{aligned}$$

Clearly, the map φ is a well-defined monomorphism. It is also surjective, since for every $\beta \in Hom(\frac{G}{E(G)L(G)}, L(G))$, consider the map

$$\begin{aligned}\alpha : G &\longrightarrow G\\ g &\longmapsto g\,\beta(gE(G)L(G)).\end{aligned}$$

We show that α is in $Var(G)$. Clearly, α a well-defined homomorphism and it is also an injective map, for if $k \in Ker\alpha$ then $1 = \alpha(k) = k\,\beta(kE(G)L(G))$ and so $k^{-1} = \beta(k\,E(G)L(G)) \in L(G) \leq E(G)L(G)$, which implies that $1 = \alpha(k) = k$ and so $Ker\alpha = \langle 1\rangle$. To prove that α is surjective, we show that $Im\beta \subseteq Im\alpha$. If $l \in Im\beta$, then $\beta(gE(G)L(G)) = l \in L(G)$, for some $g \in G$. Since $L(G) \leq E(G)L(G)$, we have $\alpha(l) = l\,\beta(lE(G)L(G)) = l$ and hence $l \in Im\alpha$. Now, for any element $g \in G$, $g = \alpha(g)\,\beta(gE(G)L(G))^{-1} \in Im\alpha$, which implies that $G = Im\alpha$, and hence α is surjective. Therefore, α is an automorphism in $Var(G)$ and so $\overline{\alpha} = \beta$, which shows that φ is an automorphism. Using the isomorphism φ, it is obvious that if $L(G) \leq E(G)$, then $Var(G) \cong Hom(G/E(G), L(G))$, which completes the proof. □

The following result gives a description of the centralizer of the centre of G in $Var(G)$.

Proposition 2 *Let G be a group, then*

$$C_{Var(G)}(Z(G)) \cong Hom\left(\frac{G}{E(G)Z(G)}, L(G)\right).$$

Proof Consider the map

$$\begin{aligned}\sigma : C_{Var(G)}(Z(G)) &\longrightarrow Hom\left(\frac{G}{E(G)Z(G)}, L(G)\right)\\ f &\longmapsto \sigma_f,\end{aligned}$$

such that

$$\begin{aligned}\sigma_f : \frac{G}{E(G)Z(G)} &\longrightarrow L(G)\\ xE(G)Z(G) &\longmapsto x^{-1}f(x).\end{aligned}$$

Clearly by Lemma 1, the autocentral automorphism f fixes every element of $E(G)$ and since $f \in C_{Var(G)}(Z(G))$, it also fixes every element of the centre of G, which guarantees the well-defined property of σ_f. The rest of the proof is similar to the proof of Theorem B. □

Proof of Theorem C We consider the map

$$\begin{aligned}\sigma : Var(G) &\longrightarrow Hom(G, L(G))\\ f &\longmapsto \sigma_f,\end{aligned}$$

so that

$$\begin{aligned}\sigma_f : G &\longrightarrow L(G)\\ g &\longmapsto g^{-1}f(g).\end{aligned}$$

Clearly, σ_f is a well-defined homomorphism and hence $\sigma_f \in Hom(G, L(G))$. One can easily check that the map σ is well-defined and monomorphism. It is surjective, for if $h \in Hom(G, L(G))$ then the map $f : G \longrightarrow G$ given by $f(g) = gh(g)$ is an endomorphism of G, and also $g^{-1}f(g) = h(g) \in L(G) \leq Z(G)$, which implies that f is a central endomorphism and hence f is a normal endomorphism, that is f commutes with every inner automorphism of G [7]. Clearly, the finite group G satisfies the maximal and minimal conditions properties for its normal subgroups. Now, since f is a normal endomorphism, it implies that Imf is normal in G. It is easy to see that f^n is also a normal endomorphism and hence Imf^n is a normal subgroup of G, for all $n \geq 1$. Thus the following two series terminate.

$$\begin{aligned}Kerf \leq Kerf^2 \leq \cdots,\\ Imf \geq Imf^2 \geq \cdots.\end{aligned}$$

Hence, there exists $r \in \mathbb{N}$ such that

$$\begin{aligned}Kerf^r &= Kerf^{r+1} = \cdots = I,\\ Imf^r &= Imf^{r+1} = \cdots = K.\end{aligned}$$

Now, we show that $G = IK$. For every element $g \in G$, $f^r(g) \in Imf^r = Imf^{2r}$ and so $f^r(g) = f^{2r}(h)$, for some $h \in G$. Therefore $f^r(g) = f^r(f^r(h))$ which implies that $(f^r(h))^{-1}g \in Kerf^r = K$, that is $g \in IK$ and hence $G = IK$. Clearly, $I \cap K = \langle 1 \rangle$ and therefore $G = I \times K$, that is to say

$$G = Kerf^r \times Imf^r.$$

Clearly, $Kerf^r$ is abelian, as $Kerf^n \leq Z(G)$, for all $n \in \mathbb{N}$. By the assumption, G is purely non-abelian and hence $Kerf^r = \langle 1 \rangle$, which implies that $Kerf = \langle 1 \rangle$. This shows that f is injective and $Imf^r = G$, which follows that $Imf = G$, that is f is surjective and hence $f \in Var(G)$. Clearly $\sigma(f) = \sigma_f = h$, which implies that σ is surjective and so

$$Var(G) \cong Hom(G, L(G)).$$

□

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