

Strongly consistent multivariate conditional risk measures

Hannes Hoffmann¹ · Thilo Meyer-Brandis¹ · Gregor Svindland¹ 

Received: 14 October 2016 / Accepted: 28 December 2017 / Published online: 11 January 2018
© Springer-Verlag GmbH Germany, part of Springer Nature 2018

Abstract We consider families of strongly consistent multivariate conditional risk measures. We show that under strong consistency these families admit a decomposition into a conditional aggregation function and a univariate conditional risk measure as introduced Hoffmann et al. (Stoch Process Appl 126(7):2014–2037, 2016). Further, in analogy to the univariate case in Föllmer (Stat Risk Model 31(1):79–103, 2014), we prove that under law-invariance strong consistency implies that multivariate conditional risk measures are necessarily multivariate conditional certainty equivalents.

Keywords Multivariate risk measures · Strong consistency · Law-invariance · Conditional certainty equivalents · Systemic risk measures

JEL Classification G10 · G32

1 Introduction

Over the recent years the study of *multivariate risk measures*

$$\rho : L_d^\infty(\mathcal{F}) \rightarrow \mathbb{R}, \quad (1.1)$$

✉ Gregor Svindland
gregor.svindland@math.lmu.de

Hannes Hoffmann
hannes.hoffmann@math.lmu.de

Thilo Meyer-Brandis
meyer-brandis@math.lmu.de

¹ Department of Mathematics, University of Munich, Theresienstraße 39, 80333 Munich, Germany

that associate a risk level $\rho(X)$ to a d -dimensional vector $X = (X_1, \dots, X_d)$ of random risk factors at a given future time horizon T has increasingly gained importance. Here, $L_d^\infty(\mathcal{F})$ denotes the space of d -dimensional bounded random vectors on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

A natural extension of the static viewpoint of deterministic risk measurement in (1.1) is to consider *conditional risk measures* which allow for risk measurement under varying information. A conditional multivariate risk measure is a map

$$\rho_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G}), \quad (1.2)$$

that associates to a d -dimensional risk factor a \mathcal{G} -measurable bounded random variable, where $\mathcal{G} \subseteq \mathcal{F}$ is a sub- σ -algebra. We interpret $\rho_{\mathcal{G}}(X)$ as the risk of X given the information \mathcal{G} . In the present literature, conditional risk measures have mostly been studied within the framework of univariate *dynamic* risk measures, where one adjusts the risk measurement in response to the flow of information that is revealed when time elapses. For a good overview on univariate dynamic risk measures we refer the reader to Acciaio and Penner [1] or Tutsch [23]. One possible motivation to study conditional multivariate risk measures is thus the extension from univariate to multivariate dynamic risk measures, and to study the question of what happens to the risk of a system as new information arises in the course of time. In the context of multivariate risk measures, however, also a second interesting and important dimension of conditioning arises, besides dynamic conditioning: Risk measurement conditional on information in space in order to identify systemic relevant structures. In that case \mathcal{G} represents for example information on the state of a subsystem, and one is interested in questions of the type: How is the overall risk of the system affected, given that a subsystem is in distress? Or how is the risk of a single institution affected, given the entire system is in distress? In Föllmer [12] and Föllmer and Klüppelberg [13] the authors analyze such spatial conditioning in the context of univariate conditional risk measures, so-called spatial risk measures. Another field of application where these questions are important are systemic risk measures, which measure the risk of a financial network. For instance the systemic risk measures CoVaR of Adrian and Brunnermeier [3] or the systemic expected shortfall of Acharya et al. [2] can be considered to be examples of conditional multivariate risk measures. We refer to Hoffmann et al. [17] and the references therein for a thorough discussion about the relation to systemic risk measures.

When dealing with families of conditional risk measures, a frequently imposed requirement is that the conditional risk measurement behaves consistent in a certain way with respect to the flow of information. In particular, in the literature on univariate dynamic risk measures most often the so-called *strong consistency* is studied; c.f. Detlefsen and Scandolo [10], Cheridito et al. [7], Cheridito and Kupper [8], Kupper and Schachermayer [20], Penner [21]. Two univariate conditional risk measures $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{H}}$ with corresponding σ -algebras $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{F}$ are called strongly consistent if for all $X, Y \in L^\infty(\mathcal{F})$

$$\rho_{\mathcal{H}}(X) \leq \rho_{\mathcal{H}}(Y) \implies \rho_{\mathcal{G}}(X) \leq \rho_{\mathcal{G}}(Y), \quad (1.3)$$

i.e. strong consistency states that if Y is riskier than X given the information \mathcal{H} , then this risk preference also holds under less information.

The purpose of this paper is to study the concept of strong consistency for multivariate conditional risk measures. Note that the motivation and interpretation of strong consistency in (1.3) remains perfectly meaningful when extending to the multivariate case. In analogy to the univariate case we thus define strong consistency of two multivariate conditional risk

measures $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{H}}$ with $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{F}$ as in (1.3) for any d -dimensional risk vectors X and Y in $L_d^\infty(\mathcal{F})$. In the literature on dynamic consistency of univariate conditional risk measures a number of weaker consistency concepts have been suggested which could also be considered in the multivariate setting. For instance, one could consider acceptance consistency

$$\rho_{\mathcal{H}}(X) \leq \alpha \implies \rho_{\mathcal{G}}(X) \leq \alpha,$$

or rejection consistency

$$\rho_{\mathcal{H}}(X) \geq \alpha \implies \rho_{\mathcal{G}}(X) \geq \alpha,$$

where $\alpha \in \mathcal{G}$ denotes a risk level below (resp. above) which a risk X is considered to be acceptable (or is rejected, resp.) in every state, see Weber [24] or Tutsch [23]. This level is typically set to be 0 in the cash-additive case (cash-additivity is defined below). In this paper we only consider the stronger property of strong consistency given in (1.3). Our results, however, show how restrictive strong consistency is and may be interpreted as supporting weaker notions of consistency which are to be studied in future research.

As a first main result we prove that under some conditions the members of any family of strongly consistent multivariate conditional risk measures are necessarily of the following form:

$$\rho_{\mathcal{G}}(X) = \eta_{\mathcal{G}}(\Lambda_{\mathcal{G}}(X)), \tag{1.4}$$

where $\eta_{\mathcal{G}} : L^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$ is a univariate conditional risk measure, and $\Lambda_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F})$ is a (conditional) aggregation function. This subclass of multivariate conditional risk measures corresponds to the idea that we first aggregate the risk factors X and then evaluate the risk of the aggregated values. In fact many prominent examples of multivariate conditional risk measures are of type (1.4), for instance the Contagion Index of Cont et al. [9] or the SystRisk of Brunnermeier and Cheridito [5] from the systemic risk literature. Chen et al. [6] were the first to axiomatically describe this intuitive type of multivariate risk measures on a finite state space, and in Kromer et al. [18] this has been extended to general L^p -spaces, whereas the conditional framework was studied in Hoffmann et al. [17]. We also remark that in Kromer et al. [19] the authors study consistency of risk measures over time which can be decomposed as in (1.4). However, their definition of consistency differs from ours in (1.3) as they require consistency of the underlying univariate risk measure and the aggregation function in (1.4) simultaneously. We will comment more on that in Remark 4.10.

A requirement on the strongly consistent family of multivariate conditional risk measures we ask for here—which is automatically satisfied in the univariate case—is that it contains a terminal risk measure $\rho_{\mathcal{F}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F})$ under full information \mathcal{F} . Such a terminal risk measure is nothing but a statewise aggregation rule for the components of a risk $X \in L_d^\infty(\mathcal{F})$. In the univariate case, if $X \in L^\infty(\mathcal{F})$, there is of course no aggregation necessary. Indeed letting the terminal risk measure correspond to the identity mapping, i.e. $\rho_{\mathcal{F}} = -\text{id}$, we have that any univariate risk measure $\rho_{\mathcal{G}}$ with $\mathcal{G} \subseteq \mathcal{F}$ is strongly consistent with $\rho_{\mathcal{F}}$ by monotonicity, so the existence of a terminal risk measure which is strongly consistent with the other risk measures of the family is no further restriction. In the truly multivariate case, however, it is very natural that also under full information there is a rule for aggregating risk over the dimensions, and the risk measures in the family should be consistent with this terminal aggregation rule. Another requirement is that the aggregation of constants under $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{F}}$ basically coincide. In case these requirements are fulfilled, we show, as already mentioned, that the members of the family are necessarily of type (1.4). Indeed we show that by strong consistency the risk measures inherit a property called *risk-antitonicity* in Hoffmann et al. [17] from the terminal risk measure. This property is the essential axiom

behind allowing for a decomposition of type (1.4); see Theorem 3.9. A consequence of (1.4) and the stated requirements is that if the family of strongly consistent multivariate risk measures includes a deterministic risk measure $\rho_{\{\emptyset, \Omega\}}$, then all aggregations functions must necessarily be deterministic, and in fact monotone transformations of each other.

Along the path to this result we characterize strong consistency in terms of a tower property. It is well-known that for univariate conditional risk measures which are normalized on constants ($\eta_G(a) = -a$ for all $a \in L^\infty(\mathcal{G})$), strong consistency (1.3) is equivalent to the following tower property:

$$\rho_G(X) = \rho_G(-\rho_{\mathcal{H}}(X)) \text{ for all } X \in L^\infty(\mathcal{F}), \tag{1.5}$$

see e.g. Föllmer and Penner [14], Bion-Nadal [4], Tutsch [23], and the references therein. The recursive formulation (1.5) is often more useful than (1.3) when analyzing strong consistency. The formulation (1.5), however, cannot be extended in a straight forward manner to the multivariate case. Firstly, note that (1.5) is not even well-defined in the multivariate case since $\rho_{\mathcal{H}}(X)$ is not a d -dimensional random vector but a random number. Secondly, also in the univariate case the equivalence (1.3) \Leftrightarrow (1.5) only holds for risk measures that are normalized on constants, which in the *monetary* univariate case is implied up to a normalization by requiring that this class of risk measures satisfy cash-additivity ($\eta_G(X + a) = \eta_G(X) - a$). For multivariate risk measures there is neither a canonical extension of the concept of cash-additivity nor is it clear that such a property is desirable at all. One could of course think of generalized versions of cash-additivity in terms of eligible assets as in Farkas et al. [11] but this would have no effect on the problem that strong consistency cannot be characterized by (1.5). Nevertheless, we prove that there is a generalization of the recursive formulation (1.5) of strong consistency which holds for multivariate risk measures without any normalized on constants or cash-additivity assumption. Indeed, under some typical regularity assumptions, one of our first results is that two multivariate conditional risk measures ρ_G and $\rho_{\mathcal{H}}$ with $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{F}$ are strongly consistent if and only if for all $X \in L_d^\infty(\mathcal{F})$

$$\rho_G(X) = \rho_G(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X))\mathbf{1}_d), \tag{1.6}$$

where $\mathbf{1}_d$ is a d -dimensional vector with all entries equal to 1, and $f_{\rho_{\mathcal{H}}}^{-1}$ is the (well-defined) inverse of the function $f_{\rho_{\mathcal{H}}}$ associated to $\rho_{\mathcal{H}}$ given by

$$f_{\rho_{\mathcal{H}}} : L^\infty(\mathcal{H}) \rightarrow L^\infty(\mathcal{H}); \alpha \mapsto \rho_{\mathcal{H}}(\alpha\mathbf{1}_d). \tag{1.7}$$

The map $f_{\rho_{\mathcal{H}}}$ describes the risk of a system where each component is equipped with the same amount of (\mathcal{H} -constant) cash α . Note that if $\rho_{\mathcal{H}}$ is a univariate risk measure that is normalized on constants then $f_{\rho_{\mathcal{H}}} = -\text{id}$ is minus the identity map and (1.6) reduces to (1.5). We call a multivariate risk measure $\rho_{\mathcal{H}}$ normalized whenever $f_{\rho_{\mathcal{H}}} = -\text{id}$. In the multivariate normalized case (1.6) thus becomes

$$\rho_G(X) = \rho_G(-\rho_{\mathcal{H}}(X)\mathbf{1}_d).$$

Further, we remark that one can always "normalize" a given conditional risk measure $\rho_{\mathcal{H}}$ by putting

$$\bar{\rho}_{\mathcal{H}}(X) := -f_{\rho_{\mathcal{H}}}^{-1} \circ \rho_{\mathcal{H}}(X). \tag{1.8}$$

Then $\bar{\rho}_{\mathcal{H}}$ is a multivariate conditional risk measure with $f_{\bar{\rho}_{\mathcal{H}}} = -\text{id}$.

After studying strong consistency for general families of multivariate conditional risk measures, we move on to give a characterization of strongly consistent multivariate conditional risk measures which are also conditionally law-invariant. In contrast to before we do

not require consistency with respect to a risk measure under full information, but with respect to the initial risk measure given the trivial information $\{\emptyset, \Omega\}$. These studies were triggered by the results obtained in Föllmer [12] for univariate risk measures, where it is shown that the only family of univariate, strongly consistent, conditional, cash-additive, convex risk measures is the family of conditional entropic risk measures, i.e. conditional certainty equivalents of the form

$$\rho_{\mathcal{H}}(X) = -u^{-1}(\mathbb{E}_{\mathbb{P}}[u(X) | \mathcal{H}]), \quad X \in L^{\infty}(\mathcal{F}),$$

with deterministic utility function $u(x) = a + be^{\beta x}$ or $u(x) = a + bx$, where $a \in \mathbb{R}$ and $b, \beta > 0$ are constants. We also remark that Kupper and Schachermayer [20] showed this characterization for the case of dynamic risk measures by an alternative proof. In the multivariate case we will see that every strongly consistent family of multivariate conditionally law-invariant conditional risk measures consists of risk measures of type

$$\rho_{\mathcal{H}}(X) = f_{\rho_{\mathcal{H}}}(f_u^{-1}(\mathbb{E}_{\mathbb{P}}[u(X) | \mathcal{H}])), \quad X \in L_d^{\infty}(\mathcal{F}), \tag{1.9}$$

where $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is a multivariate utility function and $f_u(x) := u(x\mathbf{1}_d)$, $x \in \mathbb{R}$. In other words the normalized risk measure $\bar{\rho}_{\mathcal{H}}$ equals a multivariate conditional certainty equivalent $\bar{\rho}_{\mathcal{H}}(X) = -(f_u^{-1}(\mathbb{E}_{\mathbb{P}}[u(X) | \mathcal{H}])))$. For the study of univariate conditional certainty equivalents and their dynamic behavior we refer the interested reader to Frittelli and Maggis [16].

1.1 Structure of the paper

In Sect. 2 we introduce our notation and multivariate conditional risk measures. In Sects. 3 and 4 we prove our main results outlined above for two strongly consistent conditional risk measures, where the law-invariant case is studied in Sect. 4. Throughout Sect. 5 we extend these results to families of multivariate conditional risk measures. Auxiliary results and longer proofs are collected in the ‘‘Appendix’’.

2 Setup

Throughout this paper $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. For $d \in \mathbb{N}$ we denote by $L_d^{\infty}(\mathcal{F}) := \{X = (X_1, \dots, X_d) : X_i \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P}) \ \forall i\}$ the space of equivalence classes of \mathcal{F} -measurable, \mathbb{P} -almost surely (a.s.) bounded random vectors. It is a Banach space when equipped with the norm $\|X\|_{d,\infty} := \max_{i=1,\dots,d} \|X_i\|_{\infty}$ where $\|F\|_{\infty} := \text{esssup}|F|$ is the supremum norm for $F \in L^{\infty}(\Omega, \mathcal{F}, \mathbb{P})$. We will use the usual componentwise orderings on \mathbb{R}^d and $L_d^{\infty}(\mathcal{F})$, i.e. $x = (x_1, \dots, x_d) \geq y = (y_1, \dots, y_d)$ for $x, y \in \mathbb{R}^d$ if and only if $x_i \geq y_i$ for all $i = 1, \dots, d$, and similarly $X \geq Y$ if and only if $X_i \geq Y_i$ \mathbb{P} -a.s. for all $i = 1, \dots, d$. Note that, as usual, we identify random variables with the \mathbb{P} -a.s. equivalence classes they induce, and conversely any such equivalence class with an arbitrary representative, whenever this causes no confusion. $\mathbf{1}_d$ and $\mathbf{0}_d$ denote the d -dimensional vectors whose entries are all equal to 1 or all equal to 0, respectively.

Definition 2.1 Let $\mathcal{G} \subseteq \mathcal{F}$. A conditional risk measure (CRM) is a function

$$\rho_{\mathcal{G}} : L_d^{\infty}(\mathcal{F}) \rightarrow L^{\infty}(\mathcal{G}),$$

possessing the following properties:

- (i) There exists a position with zero risk, i.e. $0 \in \text{Im } \rho_{\mathcal{G}}$.
- (ii) **Strict Antitonicity:** $X \geq Y$ and $\mathbb{P}(X > Y) > 0$ implies $\rho_{\mathcal{G}}(X) \leq \rho_{\mathcal{G}}(Y)$ and $\mathbb{P}(\rho_{\mathcal{G}}(X) < \rho_{\mathcal{G}}(Y)) > 0$.
- (iii) **\mathcal{G} -Locality:** For all $A \in \mathcal{G}$ we have $\rho_{\mathcal{G}}(X \mathbb{1}_A + Y \mathbb{1}_{A^c}) = \rho_{\mathcal{G}}(X) \mathbb{1}_A + \rho_{\mathcal{G}}(Y) \mathbb{1}_{A^c}$.
- (iv) **Lebesgue property:** If $(X_n)_{n \in \mathbb{N}} \subset L_d^\infty(\mathcal{F})$ is a $\|\cdot\|_{d, \infty}$ -bounded sequence such that $X_n \rightarrow X$ \mathbb{P} -a.s., then

$$\rho_{\mathcal{G}}(X) = \lim_{n \rightarrow \infty} \rho_{\mathcal{G}}(X_n) \quad \mathbb{P}\text{-a.s.}$$

We remark that the properties in Definition 2.1 are standard in the literature on conditional risk measures. Note that strict antitonicity is sometimes also referred to as strong sensitivity in the literature. In order to stress the dimension we often use the term univariate conditional risk measure for a conditional risk measure as defined in Definition 2.1 with $d = 1$ and we typically denote it by $\eta_{\mathcal{G}}$. For $d > 1$ the risk measure $\rho_{\mathcal{G}}$ of Definition 2.1 is called multivariate conditional risk measure.

A standard assumption on univariate CRMs is *cash-additivity*, i.e. $\eta_{\mathcal{G}}(X + \alpha) = \eta_{\mathcal{G}}(X) - \alpha$ for all $\alpha \in L^\infty(\mathcal{G})$, which in particular implies that we postulate a certain behavior of the risk measure $\eta_{\mathcal{G}}$ on (\mathcal{G}) -constants $\alpha \in L^\infty(\mathcal{G})$ which turns out to be helpful in the study of consistency in dimension 1. Given that a multivariate analogue of cash-additivity is tricky to define and probably not reasonable to ask for, we do not require such a property. However, we will have to extract the behavior of a CRM on constants in the following way.

Definition 2.2 For every CRM $\rho_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$ we introduce the function

$$f_{\rho_{\mathcal{G}}} : L^\infty(\mathcal{G}) \rightarrow L^\infty(\mathcal{G}); \alpha \mapsto \rho_{\mathcal{G}}(\alpha \mathbf{1}_d)$$

and the corresponding inverse function

$$f_{\rho_{\mathcal{G}}}^{-1} : \text{Im } f_{\rho_{\mathcal{G}}} \rightarrow L^\infty(\mathcal{G}); \beta \mapsto \alpha \text{ such that } f_{\rho_{\mathcal{G}}}(\alpha) = \beta.$$

We call $\rho_{\mathcal{G}}$ *normalized on constants* if

$$f_{\rho_{\mathcal{G}}}(\alpha) = -\alpha \quad \text{for all } \alpha \in L^\infty(\mathcal{G}).$$

For any CRM $\rho_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$, let $\bar{\rho}_{\mathcal{G}} := -f_{\rho_{\mathcal{G}}}^{-1} \circ \rho_{\mathcal{G}}$ be the *normalized CRM* of $\rho_{\mathcal{G}}$.

In ‘‘Appendix A’’ we show that $f_{\rho_{\mathcal{G}}}^{-1}$ is well-defined. Also, it follows from Lemma A.1 and Lemma A.2 that $\bar{\rho}_{\mathcal{G}}$ is indeed a CRM which is normalized on constants.

Example 2.3 Consider a strictly increasing continuous function $u : \mathbb{R} \rightarrow \mathbb{R}$ such that $u(0) = 0$ and define

$$\rho_{\mathcal{G}}(X) := -\mathbb{E}_{\mathbb{P}} \left[u \left(\frac{1}{d} \sum_{i=1}^d X_i \right) \middle| \mathcal{G} \right], \quad X = (X_1, \dots, X_d) \in L_d^\infty(\mathcal{F}).$$

Then $\rho_{\mathcal{G}}$ is a CRM, and $f_{\rho_{\mathcal{G}}}(\alpha) = -u(\alpha)$, $\alpha \in L^\infty(\mathcal{G})$, whereas $f_{\rho_{\mathcal{G}}}^{-1}(\beta) = u^{-1}(-\beta)$, $\beta \in \text{Im } f_{\rho_{\mathcal{G}}}$. The corresponding normalized risk measure is the conditional certainty equivalent

$$\bar{\rho}_{\mathcal{G}}(X) = -u^{-1} \left(\mathbb{E}_{\mathbb{P}} \left[u \left(\frac{1}{d} \sum_{i=1}^d X_i \right) \middle| \mathcal{G} \right] \right), \quad X = (X_1, \dots, X_d) \in L_d^\infty(\mathcal{F}).$$

3 Strong consistency

In this section we study consistency of CRMs. We consider the most frequently used consistency condition for univariate risk measures in the literature which is known as strong consistency and extend it to the multivariate case. We refer to Detlefsen and Scandolo [10], Cheridito et al. [7], Cheridito and Kupper [8], Kupper and Schachermayer [20], and Penner [21] for more information on strong consistency of univariate risk measures. Kromer et al. [19] also study a kind of consistency for multivariate risk measures, however, as we will point out in Remark 4.10 below, their definition of consistency differs from our approach. For the remainder of this section we let \mathcal{G} and \mathcal{H} be two sub- σ -algebras of \mathcal{F} such that $\mathcal{G} \subseteq \mathcal{H}$, and let $\rho_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$ and $\rho_{\mathcal{H}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{H})$ be the corresponding CRMs.

Definition 3.1 (Strong consistency) The pair $\{\rho_{\mathcal{G}}, \rho_{\mathcal{H}}\}$ is called *strongly consistent* if

$$\rho_{\mathcal{H}}(X) \leq \rho_{\mathcal{H}}(Y) \Rightarrow \rho_{\mathcal{G}}(X) \leq \rho_{\mathcal{G}}(Y) \quad (X, Y \in L_d^\infty(\mathcal{F})). \tag{3.1}$$

Strong consistency states that if one risk is preferred to another risk in almost surely all states under more information, then this preference already holds under less information. Our first result shows that strong consistency can be equivalently defined by a recursive relation.

Lemma 3.2 *Equivalent are:*

- (i) $\{\rho_{\mathcal{G}}, \rho_{\mathcal{H}}\}$ is strongly consistent;
- (ii) For all $X \in L_d^\infty(\mathcal{F})$ it holds that

$$\rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X))\mathbf{1}_d\right),$$

where $f_{\rho_{\mathcal{H}}}^{-1}$ was defined in Definition 2.2.

Proof (i) \Rightarrow (ii): As for all $X \in L_d^\infty(\mathcal{F})$

$$\rho_{\mathcal{H}}(X) = \rho_{\mathcal{H}}\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X))\mathbf{1}_d\right),$$

it follows from strong consistency that

$$\rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X))\mathbf{1}_d\right).$$

(ii) \Rightarrow (i): Let $X, Y \in L_d^\infty(\mathcal{F})$ be such that $\rho_{\mathcal{H}}(X) \leq \rho_{\mathcal{H}}(Y)$. Then by antitonicity of $f_{\rho_{\mathcal{H}}}^{-1}$ (see Lemma A.1) and $\rho_{\mathcal{G}}$ it follows that

$$\rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X))\mathbf{1}_d\right) \leq \rho_{\mathcal{G}}\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(Y))\mathbf{1}_d\right) = \rho_{\mathcal{G}}(Y).$$

□

Remark 3.3 Let $\eta_{\mathcal{G}}$ and $\eta_{\mathcal{H}}$ be two univariate CRMs, where $\eta_{\mathcal{H}}$ is normalized on constants, i.e. $\eta_{\mathcal{H}}(\alpha) = -\alpha$ for all $\alpha \in L^\infty(\mathcal{H})$. Then $f_{\eta_{\mathcal{H}}}(\alpha) = f_{\eta_{\mathcal{H}}}^{-1}(\alpha) = -\alpha$ and thus strong consistency is equivalent to the tower property [see (1.5)]

$$\eta_{\mathcal{G}}(F) = \eta_{\mathcal{G}}(-\eta_{\mathcal{H}}(F)), \quad F \in L^\infty(\mathcal{F}).$$

Remark 3.4 If $\{\rho_{\mathcal{G}}, \rho_{\mathcal{H}}\}$ is strongly consistent so is the pair of normalized CRMs $\{\bar{\rho}_{\mathcal{G}}, \bar{\rho}_{\mathcal{H}}\}$ as defined in Definition 2.2 and vice versa, because $f_{\rho_{\mathcal{G}}}, f_{\rho_{\mathcal{H}}}$ and their inverse functions

are antitone (Lemma A.1). Since $f_{\bar{\rho}_G} = f_{\bar{\rho}_H} = -\text{id}$ strong consistency of the normalized CRMs is equivalent to

$$\bar{\rho}_G(F) = \bar{\rho}_G(-\bar{\rho}_H(F)\mathbf{1}_d), \quad F \in L^\infty(\mathcal{F}),$$

in analogy to Remark 3.3.

Lemma 3.5 *If $\{\rho_G, \rho_H\}$ is strongly consistent, then ρ_G uniquely determines the normalized CRM $\bar{\rho}_H = -f_{\rho_H}^{-1} \circ \rho_H$.*

Proof Suppose that there are two CRMs ρ_H^1 and ρ_H^2 which are strongly consistent with respect to ρ_G , i.e.

$$\rho_G\left(f_{\rho_H^1}^{-1}(\rho_H^1(X))\mathbf{1}_d\right) = \rho_G(X) = \rho_G\left(f_{\rho_H^2}^{-1}(\rho_H^2(X))\mathbf{1}_d\right), \quad X \in L_d^\infty(\mathcal{F}).$$

We will show that $f_{\rho_H^1}^{-1}(\rho_H^1(X)) = f_{\rho_H^2}^{-1}(\rho_H^2(X))$. Suppose that there exists an $X \in L_d^\infty(\mathcal{F})$ such that $A := \left\{f_{\rho_H^1}^{-1}(\rho_H^1(X)) > f_{\rho_H^2}^{-1}(\rho_H^2(X))\right\} \in \mathcal{H}$ has positive probability. Then, by the \mathcal{H} -locality of ρ_H^1 and ρ_H^2 and of $f_{\rho_H^1}^{-1}$ and $f_{\rho_H^2}^{-1}$ (Lemma A.1), we obtain

$$\begin{aligned} \rho_G(X\mathbb{1}_A) &= \rho_G\left(f_{\rho_H^1}^{-1}(\rho_H^1(X\mathbb{1}_A))\mathbf{1}_d\right) = \rho_G\left(f_{\rho_H^1}^{-1}(\rho_H^1(X))\mathbb{1}_A\mathbf{1}_d\right) \\ &\leq \rho_G\left(f_{\rho_H^2}^{-1}(\rho_H^2(X))\mathbb{1}_A\mathbf{1}_d\right) = \rho_G\left(f_{\rho_H^2}^{-1}(\rho_H^2(X\mathbb{1}_A))\mathbf{1}_d\right) \\ &= \rho_G(X\mathbb{1}_A). \end{aligned} \tag{3.2}$$

where the inequality (3.2) is strict with positive probability as ρ_G is strictly antitone, and hence we have a contradiction. Reverting the role of ρ_H^1 and ρ_H^2 in the definition of A proves the lemma. \square

In Hoffmann et al. [17] we studied under which conditions a (multivariate) conditional risk measure can be decomposed as in (1.4), i.e. into a conditional aggregation function and a univariate conditional risk measure. We will pursue showing that strong consistency of $\{\rho_G, \rho_{\mathcal{F}}\}$ is already sufficient to guarantee a decomposition (1.4) for both ρ_G and $\rho_{\mathcal{F}}$. To this end we need to clarify what we mean by a conditional aggregation function:

Definition 3.6 We call a function $\Lambda : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F})$ a *conditional aggregation function* if it fulfills the following properties:

Strict isotonicity: $X \geq Y$ and $\mathbb{P}(X > Y) > 0$ implies $\Lambda(X) \geq \Lambda(Y)$ and $\mathbb{P}(\Lambda(X) > \Lambda(Y)) > 0$.

\mathcal{F} -Locality: $\Lambda(X\mathbb{1}_A + Y\mathbb{1}_{A^c}) = \Lambda(X)\mathbb{1}_A + \Lambda(Y)\mathbb{1}_{A^c}$ for all $A \in \mathcal{F}$;

Lebesgue property: For any uniformly bounded sequence $(X_n)_{n \in \mathbb{N}}$ in $L_d^\infty(\mathcal{F})$ such that $X_n \rightarrow X$ \mathbb{P} -a.s., we have that

$$\Lambda(X) = \lim_{n \rightarrow \infty} \Lambda(X_n) \quad \mathbb{P}\text{-a.s.}$$

If in addition

$$\Lambda(L_d^\infty(\mathcal{J})) \subseteq L^\infty(\mathcal{J}) \text{ for all } \mathcal{H} \subseteq \mathcal{J} \subseteq \mathcal{F},$$

where $\mathcal{H} \subset \mathcal{F}$ we call Λ a \mathcal{H} -conditional aggregation function. Moreover, for any conditional aggregation function $\Lambda : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F})$ let

$$f_\Lambda : L^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F}); F \mapsto \Lambda(F\mathbf{1}_d)$$

and

$$f_{\Lambda}^{-1} : \text{Im } f_{\Lambda} \rightarrow L^{\infty}(\mathcal{F}); G \mapsto F \text{ such that } f_{\Lambda}(F) = G.$$

Remark 3.7 The name \mathcal{H} -conditional aggregation function refers to the fact that $\Lambda(x) \in L^{\infty}(\mathcal{H})$ for all $x \in \mathbb{R}^d$. Thus every conditional aggregation function is at least \mathcal{F} -conditional.

Before we state our decomposition result for strongly consistent CRMs in Theorem 3.9 we need the following definition.

Definition 3.8 A realization $\rho_{\mathcal{G}}(\cdot, \cdot)$ of a function $\rho_{\mathcal{G}} : L_d^{\infty}(\mathcal{F}) \rightarrow L^{\infty}(\mathcal{G})$ is a selection of one representative $\rho_{\mathcal{G}}(X, \cdot)$ of the equivalence class $\rho_{\mathcal{G}}(X) \in L^{\infty}(\mathcal{G})$ for each $X \in L_d^{\infty}(\mathcal{F})$. We call such a realization continuous whenever $\mathbb{R}^d \times \Omega \ni (x, \omega) \mapsto \rho_{\mathcal{G}}(x, \omega)$ is continuous in its first argument \mathbb{P} -a.s. Here we identify $x \in \mathbb{R}^d$ with the equivalence class in $L_d^{\infty}(\mathcal{F})$ determined by the corresponding constant random vector.

Theorem 3.9 Let $\rho_{\mathcal{G}} : L_d^{\infty}(\mathcal{F}) \rightarrow L^{\infty}(\mathcal{G})$ and $\rho_{\mathcal{F}} : L_d^{\infty}(\mathcal{F}) \rightarrow L^{\infty}(\mathcal{F})$ be CRMs such that $\{\rho_{\mathcal{G}}, \rho_{\mathcal{F}}\}$ is strongly consistent. Moreover, suppose that

$$\bar{\rho}_{\mathcal{G}}(x) = \bar{\rho}_{\mathcal{F}}(x) \text{ for all } x \in \mathbb{R}^d. \tag{3.3}$$

If $\rho_{\mathcal{G}}$ has a continuous realization $\rho_{\mathcal{G}}(\cdot, \cdot)$, then there exists a \mathcal{G} -conditional aggregation function $\Lambda_{\mathcal{G}} : L_d^{\infty}(\mathcal{F}) \rightarrow L^{\infty}(\mathcal{G})$ and a univariate CRM $\eta_{\mathcal{G}} : \text{Im } \Lambda_{\mathcal{G}} \rightarrow L^{\infty}(\mathcal{G})$ such that

$$\rho_{\mathcal{G}}(X) = \eta_{\mathcal{G}}(\Lambda_{\mathcal{G}}(X)) \text{ for all } X \in L_d^{\infty}(\mathcal{F}) \tag{3.4}$$

and

$$\eta_{\mathcal{G}}(\Lambda_{\mathcal{G}}(X)) = -\Lambda_{\mathcal{G}}(X) \text{ for all } X \in L_d^{\infty}(\mathcal{F}). \tag{3.5}$$

Let $\Lambda_{\mathcal{F}} := -\rho_{\mathcal{F}}$ and $\eta_{\mathcal{F}} := -\text{id}$ so that $\rho_{\mathcal{F}} = \eta_{\mathcal{F}} \circ \Lambda_{\mathcal{F}}$ for the conditional aggregation function $\Lambda_{\mathcal{F}}$ and the univariate CRM $\eta_{\mathcal{F}}$. Then $\Lambda_{\mathcal{F}}(\alpha) \in L^{\infty}(\mathcal{G})$ for all $\alpha \in L_d^{\infty}(\mathcal{G})$ and

$$\Lambda_{\mathcal{F}}(X) \leq \Lambda_{\mathcal{F}}(Y) \implies \Lambda_{\mathcal{G}}(X) \leq \Lambda_{\mathcal{G}}(Y) \quad (X, Y \in L_d^{\infty}(\mathcal{F})), \tag{3.6}$$

i.e. $\Lambda_{\mathcal{G}}$ and $\Lambda_{\mathcal{F}}$ are strongly consistent.

Conversely, suppose that the CRM $\rho_{\mathcal{G}} : L_d^{\infty}(\mathcal{F}) \rightarrow L^{\infty}(\mathcal{G})$ satisfies (3.4) and (3.5), then $\{\rho_{\mathcal{G}}, \rho_{\mathcal{F}}\}$ is strongly consistent and satisfies (3.3), where $\rho_{\mathcal{F}} := -\Lambda_{\mathcal{G}}$ is a CRM.

A proof of Theorem 3.9 is provided in ‘‘Appendix B’’. We remark that in Theorem 3.9 we require consistency of the pair $\{\rho_{\mathcal{G}}, \rho_{\mathcal{F}}\}$ where $\rho_{\mathcal{F}}$ is a CRM given the full information \mathcal{F} . Note that $\rho_{\mathcal{F}}$ is (apart from the sign) simply a conditional aggregation function as defined in Definition 3.6, so $\rho_{\mathcal{G}}$ is required to be consistent with some aggregation function under full information. This also explains $\Lambda_{\mathcal{F}}$. For $d = 1$ this consistency is automatically satisfied by monotonicity, the aggregation is simply the identity function, and clearly the assertion is trivial anyway. For higher dimensions, Theorem 3.9 states that if there exists an aggregation function which is consistent with $\rho_{\mathcal{G}}$, then $\rho_{\mathcal{G}}$ is automatically of type (3.4). Clearly, if we already know that (3.4) holds true, then $\rho_{\mathcal{G}}$ is consistent with $\rho_{\mathcal{F}} = -\Lambda_{\mathcal{G}}$.

Condition (3.3) is a strengthening of strong consistency in absence of any risk, that is for constants, since it states that in that case aggregation under $\rho_{\mathcal{G}}$ equals aggregation under full information $\rho_{\mathcal{F}}$ up to normalization with $f_{\rho_{\mathcal{G}}}^{-1}$ and $f_{\rho_{\mathcal{F}}}^{-1}$, respectively. Note that we may replace condition (3.3) by requiring $\bar{\rho}_{\mathcal{F}}(x) \in L^{\infty}(\mathcal{G})$ for all $x \in \mathbb{R}^d$. Indeed, in the setting of Theorem 3.9, assuming (3.3), the theorem shows that $\rho_{\mathcal{F}}(\alpha) = -\Lambda_{\mathcal{F}}(\alpha) \in L^{\infty}(\mathcal{G})$ for all

$\alpha \in L_d^\infty(\mathcal{G})$ and thus the same is true for $\bar{\rho}_{\mathcal{F}}$. Conversely, if $\bar{\rho}_{\mathcal{F}}(x) \in L^\infty(\mathcal{G})$, then by strong consistency we have that

$$\bar{\rho}_{\mathcal{G}}(x) = \bar{\rho}_{\mathcal{G}}(-\bar{\rho}_{\mathcal{F}}(x)\mathbf{1}_d) = \bar{\rho}_{\mathcal{F}}(x), \quad x \in \mathbb{R}^d,$$

which is condition (3.3). As regards the required existence of a continuous realization of $\rho_{\mathcal{G}}$, sufficient criteria are well-known, e.g. Kolmogorov’s criterion; see Theorem 2.1 in Revuz and Yor [22].

Remark 3.10 We show in Lemma A.3 that the inverse function $f_{\Lambda_{\mathcal{G}}}^{-1}$ of $f_{\Lambda_{\mathcal{G}}}$ is isotone and that $\Lambda_{\mathcal{G}}(X) = \Lambda_{\mathcal{G}}(f_{\Lambda_{\mathcal{G}}}^{-1}(\Lambda_{\mathcal{G}}(X))\mathbf{1}_d)$ for all $X \in L_d^\infty(\mathcal{F})$. Therefore it can be shown as in Lemma 3.2, that (3.6) is equivalent to

$$f_{\Lambda_{\mathcal{G}}}^{-1}(\Lambda_{\mathcal{G}}(X)) = f_{\Lambda_{\mathcal{F}}}^{-1}(\Lambda_{\mathcal{F}}(X)), \quad \text{for all } X \in L_d^\infty(\mathcal{F}).$$

Note that we cannot write the recursive form of the strong consistency of two CRMs $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{F}}$ as above, since $f_{\rho_{\mathcal{G}}}$ is only defined on $L^\infty(\mathcal{G})$ and not on $L^\infty(\mathcal{F})$ in contrast to $f_{\Lambda_{\mathcal{G}}}$.

The following Theorem 3.11 extends the results in Hoffmann et al. [17] for strong consistency based on our findings in Theorem 3.9.

Theorem 3.11 *If $\rho_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$ is a CRM with a continuous realization $\rho_{\mathcal{G}}(\cdot, \cdot)$, then the following three statements are equivalent*

- (i) $\rho_{\mathcal{G}}(\cdot, \cdot)$ is risk-antitone, that is $\rho_{\mathcal{G}}(X(\omega), \omega) \geq \rho_{\mathcal{G}}(Y(\omega), \omega)$ \mathbb{P} -a.s., implies $\rho_{\mathcal{G}}(X) \geq \rho_{\mathcal{G}}(Y)$;
- (ii) $\rho_{\mathcal{G}}$ is decomposable as in (3.4);
- (iii) $\rho_{\mathcal{G}}$ is strongly consistent with some aggregation function $\Lambda : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F})$, i.e. $\{\rho_{\mathcal{G}}, -\Lambda\}$ is strongly consistent, and $\bar{\rho}_{\mathcal{G}}(x) = -f_{\Lambda_{\mathcal{F}}}^{-1}(\Lambda_{\mathcal{F}}(x))$ for all $x \in \mathbb{R}^d$.

Proof The equivalence of (ii) and (iii) has been shown in Theorem 3.9 and that (i) implies (ii) follows from Proposition B.1. Finally, the proof of Theorem 3.9 shows that (iii) implies (i). □

Example 3.12 Recall Example 2.3. In that case $\bar{\rho}_{\mathcal{G}}(x) = -\frac{1}{d} \sum_{i=1}^d x_i$ for all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, so condition (3.3) is satisfied whenever $\rho_{\mathcal{F}}(X) = -l(\sum_{i=1}^d X_i)$ for some strictly increasing function $l : \mathbb{R} \rightarrow \mathbb{R}$, so in particular for $l = u$. Moreover, we may choose the realization of $\rho_{\mathcal{G}}$ such that $\rho_{\mathcal{G}}(x, \omega) = -u(\frac{1}{d} \sum_{i=1}^d x_i)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, which is obviously continuous in x . Replacing the sum by a conditional aggregation function, for instance $-\rho_{\mathcal{F}}(x) = \Lambda_{\mathcal{F}}(x) = \sum_{i=1}^d a_i x_i \mathbb{1}_A + \sum_{i=1}^d b_i x_i \mathbb{1}_{A^c}$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, where $a_i, b_i > 0$ are weights and $\mathbb{P}(A) > 0$, then indeed (3.3) requires that $A \in \mathcal{G}$.

4 Conditional law-invariance and strong consistency

As in the previous section, if not otherwise stated, throughout this section we let \mathcal{G} and \mathcal{H} be two sub- σ -algebras of \mathcal{F} such that $\mathcal{G} \subseteq \mathcal{H}$, and let $\rho_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$ and $\rho_{\mathcal{H}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{H})$ be the corresponding CRMs.

Definition 4.1 A CRM $\rho_{\mathcal{G}}$ is conditional law-invariant if $\rho_{\mathcal{G}}(X) = \rho_{\mathcal{G}}(Y)$ whenever the \mathcal{G} -conditional distributions $\mu_X(\cdot|\mathcal{G})$ and $\mu_Y(\cdot|\mathcal{G})$ of $X, Y \in L_d^\infty(\mathcal{F})$ are equal, i.e. if $\mathbb{P}(X \in A | \mathcal{G}) = \mathbb{P}(Y \in A | \mathcal{G})$ for all Borel sets $A \in \mathcal{B}(\mathbb{R}^d)$. In case $\mathcal{G} = \{\emptyset, \Omega\}$ is trivial, conditional law-invariance of $\rho_{\mathcal{G}}$ is also referred to as law-invariance.

In the law-invariant case we will often have to require a little more regularity of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

Definition 4.2 We say that $(\Omega, \mathcal{F}, \mathbb{P})$ is

- atomless**, if $(\Omega, \mathcal{F}, \mathbb{P})$ supports a random variable with continuous distribution;
- conditionally atomless given $\mathcal{H} \subset \mathcal{F}$** , if $(\Omega, \mathcal{F}, \mathbb{P})$ supports a random variable with continuous distribution which is independent of \mathcal{H} .

The next lemma shows that conditional law-invariance is passed from $\rho_{\mathcal{G}}$ (forward) to $\rho_{\mathcal{H}}$ by strong consistency. The proof is based on Föllmer [12].

Lemma 4.3 *If $\{\rho_{\mathcal{G}}, \rho_{\mathcal{H}}\}$ is strongly consistent and $\rho_{\mathcal{G}}$ is conditionally law-invariant, then $\rho_{\mathcal{H}}$ is also conditionally law-invariant.*

Proof Let $X, Y \in L^\infty(\mathcal{F})$ such that $\mu_X(\cdot|\mathcal{H}) = \mu_Y(\cdot|\mathcal{H})$ and let $A := \{\rho_{\mathcal{H}}(X) > \rho_{\mathcal{H}}(Y)\} \in \mathcal{H}$. Then the random variables $X \mathbb{1}_A$ and $Y \mathbb{1}_A$ have the same conditional distribution given \mathcal{G} . As $\rho_{\mathcal{G}}$ is conditionally law-invariant and strongly consistent with $\rho_{\mathcal{H}}$ we obtain

$$\begin{aligned} \rho_{\mathcal{G}}\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X) \mathbb{1}_A + \rho_{\mathcal{H}}(\mathbf{0}_d) \mathbb{1}_{A^c}) \mathbf{1}_d\right) &= \rho_{\mathcal{G}}(X \mathbb{1}_A) = \rho_{\mathcal{G}}(Y \mathbb{1}_A) \\ &= \rho_{\mathcal{G}}\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(Y) \mathbb{1}_A + \rho_{\mathcal{H}}(\mathbf{0}_d) \mathbb{1}_{A^c}) \mathbf{1}_d\right). \end{aligned}$$

On the other hand, by strict antitonicity of $\rho_{\mathcal{G}}$ and $f_{\rho_{\mathcal{H}}}^{-1}$

$$\rho_{\mathcal{G}}\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X) \mathbb{1}_A + \rho_{\mathcal{H}}(\mathbf{0}_d) \mathbb{1}_{A^c}) \mathbf{1}_d\right) \geq \rho_{\mathcal{G}}\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(Y) \mathbb{1}_A + \rho_{\mathcal{H}}(\mathbf{0}_d) \mathbb{1}_{A^c}) \mathbf{1}_d\right),$$

and the inequality is strict with positive probability if $\mathbb{P}(A) > 0$. Thus A must be a \mathbb{P} -nullset and interchanging X and Y in the definition of A shows that indeed $\rho_{\mathcal{H}}(X) = \rho_{\mathcal{H}}(Y)$. \square

While in Theorem 3.9 we had to require that the strongly consistent pair $\{\rho_{\mathcal{G}}, \rho_{\mathcal{H}}\}$ satisfies $\mathcal{H} = \mathcal{F}$, in this section we in some sense require the opposite extreme, namely that $\mathcal{G} = \{\emptyset, \Omega\}$ is trivial while $\mathcal{H} \subseteq \mathcal{F}$.

Assumption 1 For the rest of the section we assume that $\mathcal{G} = \{\emptyset, \Omega\}$. For simplicity we will write $\rho := \rho_{\mathcal{G}} = \rho_{\{\emptyset, \Omega\}}$.

In the following we extend the representation result of Föllmer [12] to multivariate CRMs.

Theorem 4.4 *Let $(\Omega, \mathcal{H}, \mathbb{P})$ be atomless and let $(\Omega, \mathcal{F}, \mathbb{P})$ be conditionally atomless given \mathcal{H} . Suppose that ρ is law-invariant. Then, $\{\rho, \rho_{\mathcal{H}}\}$ is strongly consistent if and only if ρ and $\rho_{\mathcal{H}}$ are of the form*

$$\rho(X) = g\left(f_u^{-1}(\mathbb{E}_{\mathbb{P}}[u(X)])\right) \quad \text{for all } X \in L_d^\infty(\mathcal{F}) \tag{4.1}$$

and

$$\rho_{\mathcal{H}}(X) = g_{\mathcal{H}}\left(f_u^{-1}(\mathbb{E}_{\mathbb{P}}[u(X) | \mathcal{H}])\right) \quad \text{for all } X \in L_d^\infty(\mathcal{F}) \tag{4.2}$$

where $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is strictly increasing and continuous, $f_u^{-1} : \text{Im } f_u \rightarrow \mathbb{R}$ is the inverse function of

$$f_u : \mathbb{R} \rightarrow \mathbb{R}; \quad x \mapsto u(x \mathbf{1}_d)$$

and $g : \mathbb{R} \rightarrow \mathbb{R}$ and $g_{\mathcal{H}} : L^\infty(\mathcal{H}) \rightarrow L^\infty(\mathcal{H})$ are strictly antitone, fulfill the Lebesgue property, $0 \in \text{Im } g \cap \text{Im } g_{\mathcal{H}}$, and $g_{\mathcal{H}}$ is \mathcal{H} -local.

In particular, for any CRM ρ of type (4.1) we have that $g = f_\rho$, and similarly for any CRM $\rho_{\mathcal{H}}$ of type (4.2) we have $g_{\mathcal{H}} = f_{\rho_{\mathcal{H}}}$, where f_ρ and $f_{\rho_{\mathcal{H}}}$ are defined in Definition 2.2.

The proof of Theorem 4.4 is provided in ‘‘Appendix C’’. Note that the common function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ appearing in (4.1) and (4.2) can be seen as a multivariate utility where u being strictly increasing means that $x, y \in \mathbb{R}^d$ with $x \geq y$ and $x \neq y$ implies $u(x) > u(y)$. So $f_u^{-1}(\mathbb{E}_{\mathbb{P}}[u(\cdot)])$ and $f_u^{-1}(\mathbb{E}_{\mathbb{P}}[u(\cdot) | \mathcal{H}])$ are (conditional) certainty equivalents – in the univariate case ($d = 1$) we clearly have $f_u^{-1} = u^{-1}$. Thus if ρ and/or $\rho_{\mathcal{H}}$ in Theorem 4.4 are normalized on constants (and hence $f_\rho \equiv -\text{id}$ or $f_{\rho_{\mathcal{H}}} \equiv -\text{id}$), then ρ and/or $\rho_{\mathcal{H}}$ equal (minus) certainty equivalents. But (4.1) and (4.2) also comprise other prominent classes of risk measures. For instance if $f_\rho = -f_u$ or $f_{\rho_{\mathcal{H}}} = -f_u$, then $\rho_{\mathcal{H}}(X) = -\mathbb{E}_{\mathbb{P}}[u(X)]$ is an multivariate expected utility whereas $\rho_{\mathcal{H}}(X) = -\mathbb{E}_{\mathbb{P}}[u(X) | \mathcal{H}]$ is a multivariate conditional expected utility.

Example 4.5 We have seen in Theorem 4.4 that under some mild technical assumptions on the probability space every pair $\{\rho, \rho_{\mathcal{H}}\}$ of law-invariant CRMs is strongly consistent if and only if it is of the form (4.1) and (4.2). This class comprises two important subclasses: Firstly, if $f_\rho = -\text{id}$ and $f_{\rho_{\mathcal{H}}} = -\text{id}$, then

$$\rho(X) = -f_u^{-1}(\mathbb{E}_{\mathbb{P}}[u(X)]) \quad \text{and} \quad \rho_{\mathcal{H}}(X) = -f_u^{-1}(\mathbb{E}_{\mathbb{P}}[u(X) | \mathcal{H}]),$$

which are multivariate conditional certainty equivalents with deterministic utility function u . Secondly, if $f_\rho = -f_u$ and $f_{\rho_{\mathcal{H}}} = -f_u$, then

$$\rho(X) = -\mathbb{E}_{\mathbb{P}}[u(X)] \quad \text{and} \quad \rho_{\mathcal{H}}(X) = -\mathbb{E}_{\mathbb{P}}[u(X) | \mathcal{H}],$$

is a multivariate conditional expected utility.

Recall Theorem 3.9 where we proved that if a multivariate CRM $\rho_{\mathcal{H}}$ is strongly consistent in a forward looking way with an aggregation $\rho_{\mathcal{F}}$ under full information \mathcal{F} (and $\rho_{\mathcal{F}}$ fulfills (3.3)), then the multivariate CRM can be decomposed as in (3.4). The following Theorem 4.6 shows that we also obtain such a decomposition (3.4) under law-invariance by requiring strong consistency of $\rho_{\mathcal{H}}$ in a backward looking way with ρ given trivial information $\{\emptyset, \Omega\}$.

When stating Theorem 4.6 we will need an extension of $f_{\rho_{\mathcal{H}}}$ to $L^\infty(\mathcal{F})$: Suppose that the process $\mathbb{R} \ni a \mapsto f_{\rho_{\mathcal{H}}}(a)$ allows for a continuous realization. Due to the fact that $\rho_{\mathcal{H}}$ is strictly antitone and \mathcal{H} -local, we can find a possibly different realization $f_{\rho_{\mathcal{H}}}(\cdot, \cdot)$ such that $\tilde{f}_{\rho_{\mathcal{H}}} : \mathbb{R} \times \Omega \rightarrow \mathbb{R} : x \mapsto f_{\rho_{\mathcal{H}}}(x, \omega)$ is continuous and strictly decreasing in the first argument for all $\omega \in \Omega$. Note that there exists a well-defined inverse $\tilde{f}_{\rho_{\mathcal{H}}}^{-1}(\cdot, \omega)$ of $\tilde{f}_{\rho_{\mathcal{H}}}(\cdot, \omega)$ for all $\omega \in \Omega$. Now define the functions

$$\bar{f}_{\rho_{\mathcal{H}}} : L^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F}); \quad F \mapsto \tilde{f}_{\rho_{\mathcal{H}}}(F(\omega), \omega) \tag{4.3}$$

and

$$\bar{f}_{\rho_{\mathcal{H}}}^{-1} : \text{Im } \bar{f}_{\rho_{\mathcal{H}}} \rightarrow L^\infty(\mathcal{F}); \quad F \mapsto \tilde{f}_{\rho_{\mathcal{H}}}^{-1}(F(\omega), \omega),$$

where we with the standard abuse of notation identify the random variable $\tilde{f}_{\rho_{\mathcal{H}}}(F(\omega), \omega)$ or $\tilde{f}_{\rho_{\mathcal{H}}}^{-1}(F(\omega), \omega)$ with the equivalence classes they generate in $L^\infty(\mathcal{F})$.

By construction of $\bar{f}_{\rho_{\mathcal{H}}}$ we have that

$$\bar{f}_{\rho_{\mathcal{H}}}(L^\infty(\mathcal{J})) \subseteq L^\infty(\mathcal{J})$$

for all σ -algebras \mathcal{J} such that $\sigma(f_{\rho_{\mathcal{H}}}(a, \cdot), a \in \mathbb{R}) \subseteq \mathcal{J} \subseteq \mathcal{F}$, c.f. Hoffmann et al. [17, Lemma 3.1]. By definition $\bar{f}_{\rho_{\mathcal{H}}}$ is also \mathcal{F} -local and has the Lebesgue property due to continuity of $\mathbb{R} \ni a \mapsto \bar{f}_{\rho_{\mathcal{H}}}(a, \omega)$. Moreover, \mathcal{H} -locality and continuity also imply that indeed $\bar{f}_{\rho_{\mathcal{H}}}(X) = f_{\rho_{\mathcal{H}}}(X)$ for all $X \in \mathcal{H}$ (approximation by simple random variables), so $\bar{f}_{\rho_{\mathcal{H}}}$ is indeed an extension of $f_{\rho_{\mathcal{H}}}$ to $L^\infty(\mathcal{F})$.

Theorem 4.6 *Under the same conditions as in Theorem 4.4 let $\{\rho, \rho_{\mathcal{H}}\}$ be strongly consistent. Then ρ can be decomposed as*

$$\rho = \eta \circ \Lambda,$$

where

$$\Lambda : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F}); \quad X \mapsto -f_\rho(f_u^{-1}(u(X)))$$

is a $\{\emptyset, \Omega\}$ -conditional aggregation function,

$$\eta : \text{Im } \Lambda \rightarrow \mathbb{R}; \quad F \mapsto -U^{-1}(\mathbb{E}_{\mathbb{P}}[U(F)])$$

is a law-invariant univariate certainty equivalent given by the (deterministic) utility

$$U : \text{Im } \rho \rightarrow \mathbb{R}; \quad a \mapsto f_u(f_\rho^{-1}(-a))$$

which is strictly increasing and continuous. Here $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is the multivariate utility function from Theorem 4.4.

If the function $\mathbb{R} \ni a \mapsto f_{\rho_{\mathcal{H}}}(a)$ has a continuous realization, then $\rho_{\mathcal{H}}$ can be decomposed as

$$\rho_{\mathcal{H}} = \eta_{\mathcal{H}} \circ \Lambda_{\mathcal{H}},$$

with

$$\eta_{\mathcal{H}}(\Lambda_{\mathcal{H}}(X)) = -\Lambda_{\mathcal{H}}(X), \quad \text{for all } X \in L_d^\infty(\mathcal{H}),$$

where

- $\Lambda_{\mathcal{H}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F}); \quad X \mapsto -\bar{f}_{\rho_{\mathcal{H}}}(f_u^{-1}(u(X)))$ is a $\sigma(f_{\rho_{\mathcal{H}}}(a, \cdot) : a \in \mathbb{R})$ -conditional aggregation function ($f_{\rho_{\mathcal{H}}}(a, \cdot)$ denotes a continuous realization with strictly increasing paths);
- $\eta_{\mathcal{H}} : \text{Im } \Lambda_{\mathcal{H}} \rightarrow L^\infty(\mathcal{H}); \quad F \mapsto -U_{\mathcal{H}}^{-1}(\mathbb{E}_{\mathbb{P}}[U_{\mathcal{H}}(F) | \mathcal{H}])$ is a univariate conditional certainty equivalent;
- the stochastic utility $U_{\mathcal{H}} : \text{Im } \Lambda_{\mathcal{H}} \rightarrow L^\infty(\mathcal{F}); \quad F \mapsto f_u(\bar{f}_{\rho_{\mathcal{H}}}^{-1}(-F))$ is strictly isotone, \mathcal{F} -local, fulfills the Lebesgue property and $U_{\mathcal{H}}^{-1}(\text{Im } U_{\mathcal{H}} \cap L^\infty(\mathcal{H})) \subseteq L^\infty(\mathcal{H})$;
- $\bar{f}_{\rho_{\mathcal{H}}}$ is given in (4.3).

Moreover, it holds that

$$U_{\mathcal{H}} \circ \Lambda_{\mathcal{H}} = u = U \circ \Lambda \tag{4.4}$$

are deterministic and independent of the chosen information \mathcal{H} or $\{\Omega, \emptyset\}$.

Finally we also have that $f_{\Lambda_{\mathcal{H}}}^{-1} \circ \Lambda_{\mathcal{H}} = f_u^{-1} \circ u = f_\Lambda^{-1} \circ \Lambda$, i.e. $\{\Lambda, \Lambda_{\mathcal{H}}\}$ is strongly consistent as defined in (3.6).

Proof By Theorem 4.4 we have that

$$\begin{aligned} \rho_{\mathcal{H}}(X) &= f_{\rho_{\mathcal{H}}}(f_u^{-1}(\mathbb{E}_{\mathbb{P}}[u(X) | \mathcal{H}])) \\ &= \bar{f}_{\rho_{\mathcal{H}}}\left(f_u^{-1}\left(\mathbb{E}_{\mathbb{P}}\left[f_u\left(\bar{f}_{\rho_{\mathcal{H}}}^{-1}\left(\bar{f}_{\rho_{\mathcal{H}}}(f_u^{-1}(u(X))))\right)\right] \middle| \mathcal{H}\right)\right)\right), \end{aligned}$$

where u and f_u are given in Theorem 4.4. Hence, recalling the definitions of $U_{\mathcal{H}}$, $\eta_{\mathcal{H}}$, and $\Lambda_{\mathcal{H}}$, we have $\rho_{\mathcal{H}} = \eta_{\mathcal{H}} \circ \Lambda_{\mathcal{H}}$. It can be readily seen that $U_{\mathcal{H}}$ as well as $U_{\mathcal{H}}^{-1}$, and thus also $\Lambda_{\mathcal{H}}$, are \mathcal{F} -local, strictly isotone, and fulfill the Lebesgue property. As $\bar{f}_{\rho_{\mathcal{H}}}(L^\infty(\mathcal{J})) \subseteq L^\infty(\mathcal{J})$ for all σ -algebras \mathcal{J} such that $\sigma(f_{\rho_{\mathcal{H}}}(a, \omega) : a \in \mathbb{R}) \subseteq \mathcal{J} \subseteq \mathcal{F}$, the same also applies to $\Lambda_{\mathcal{H}} = -\bar{f}_{\rho_{\mathcal{H}}} \circ f_u^{-1} \circ u$ and we conclude that $\Lambda_{\mathcal{H}}$ is a $\sigma(f_{\rho_{\mathcal{H}}}(a, \omega) : a \in \mathbb{R})$ -conditional aggregation function. Moreover, for $X \in L_d^\infty(\mathcal{H})$

$$\eta_{\mathcal{H}}(\Lambda_{\mathcal{H}}(X)) = \bar{f}_{\rho_{\mathcal{H}}}(f_u^{-1}(u(X))) = -U_{\mathcal{H}}^{-1}(u(X)) = -\Lambda_{\mathcal{H}}(X).$$

The result for ρ follows similarly to the proof above without requiring a continuous realization and by using the canonical extension of f_ρ from \mathbb{R} to $L_d^\infty(\mathcal{F})$, i.e. $\bar{f}_\rho(F)(\omega) = f_\rho(F(\omega))$ for all $\omega \in \Omega$ and $F \in L^\infty(\mathcal{F})$. □

We remark that (4.4) is the crucial fact which ensures that ρ and $\rho_{\mathcal{H}}$ are strongly consistent and (conditionally) law-invariant.

In Theorem 4.6 we have seen that basically every CRM which is strongly consistent with a law-invariant CRM under trivial information can be decomposed into a conditional aggregation function and a univariate conditional certainty equivalent. For the rest of this section we study the effect of additional properties of the CRMs on this decomposition. For instance, we want to identify conditions under which the univariate conditional certainty equivalent is generated by a deterministic (instead of a stochastic) utility function; see Corollary 4.7. Also we study what happens if the univariate CRMs η and $\eta_{\mathcal{H}}$ from Theorem 4.6 are required to be strongly consistent; see Corollary 4.9.

Corollary 4.7 *In the situation of Theorem 4.6, if ρ is normalized on constants, then*

$$\Lambda(X) = f_u^{-1}(u(X)), \quad X \in L_d^\infty(\mathcal{F}),$$

and

$$\eta(F) = \rho(F\mathbf{1}_d) = -f_u^{-1}(\mathbb{E}_{\mathbb{P}}[f_u(F)]), \quad F \in L^\infty(\mathcal{F}).$$

If $\rho_{\mathcal{H}}$ is normalized on constants, then similarly

$$\Lambda_{\mathcal{H}}(X) = f_u^{-1}(u(X)), \quad X \in L_d^\infty(\mathcal{F}),$$

and

$$\eta_{\mathcal{H}}(F) = \rho_{\mathcal{H}}(F\mathbf{1}_d) = -f_u^{-1}(\mathbb{E}_{\mathbb{P}}[f_u(F) | \mathcal{H}]), \quad F \in L^\infty(\mathcal{F}).$$

In particular the univariate conditional certainty equivalent $\eta_{\mathcal{H}}$ is now given by the deterministic univariate utility function f_u , and thus $\eta_{\mathcal{H}}$ is conditionally law-invariant.

If both ρ and $\rho_{\mathcal{H}}$ are normalized on constants, then $\Lambda = \Lambda_{\mathcal{H}}$.

Example 4.8 Suppose that ρ and $\rho_{\mathcal{H}}$ from Theorem 4.6 are normalized on constants and that for all $F, G \in L^\infty(\mathcal{F})$, $m, \lambda \in \mathbb{R}$ with $\lambda \in (0, 1)$

$$\rho(F\mathbf{1}_d + m\mathbf{1}_d) = \rho(F\mathbf{1}_d) - m \tag{4.5}$$

as well as

$$\rho(\lambda F\mathbf{1}_d + (1 - \lambda)G\mathbf{1}_d) \leq \lambda\rho(F\mathbf{1}_d) + (1 - \lambda)\rho(G\mathbf{1}_d). \tag{4.6}$$

Recalling Corollary 4.7 it follows that $\eta(F) = \rho(F\mathbf{1}_d)$ is cash-additive (4.5) and convex (4.6). Since f_u is a deterministic function it can be easily checked that η and $\eta_{\mathcal{H}}$ are strongly consistent (conditionally) law-invariant univariate CRMs. Therefore we are in the framework

of Föllmer [12]. There it is shown that the univariate CRMs must be either linear or of entropic type, i.e.

$$f_u(x) = ax + b \text{ or } f_u(x) = -ae^{-\beta x} + b, \quad x \in \mathbb{R},$$

for constants $a, b, \beta \in \mathbb{R}$ with $a, \beta > 0$, which implies that

$$\eta_{\mathcal{H}}(F) = \mathbb{E}_{\mathbb{P}}[-F \mid \mathcal{H}] \text{ or } \eta_{\mathcal{H}}(F) = \frac{1}{\beta} \log \left(\mathbb{E}_{\mathbb{P}} \left[e^{-\beta F} \mid \mathcal{H} \right] \right)$$

and similarly for η . Clearly, this also has consequences for the aggregation function $\Lambda = \Lambda_{\mathcal{H}} = f_u^{-1} \circ u$ since $x \mapsto u(x\mathbf{1}_d) = f_u(x)$ is either of linear or exponential form. For instance, a possible aggregation would be given by $u(x_1, \dots, x_d) = a \sum_{i=1}^d w_i x_i + b$, where $w_i \in (0, 1)$ for $i = 1, \dots, d$ such that $\sum_{i=1}^d w_i = 1$, because $f_u(x) = ax + b$. In this case the aggregation function is simply $\Lambda(x) = \sum_{i=1}^d w_i x_i$.

Corollary 4.9 *In the situation of Theorem 4.6, suppose that η and $\eta_{\mathcal{H}}$ are defined on all of $L^\infty(\mathcal{F})$. Then $\{\eta, \eta_{\mathcal{H}}\}$ are strongly consistent if and only if*

$$\eta = -\tilde{u}^{-1}(\mathbb{E}_{\mathbb{P}}[\tilde{u}(F)]) \text{ and } \eta_{\mathcal{H}} = -\tilde{u}^{-1}(\mathbb{E}_{\mathbb{P}}[\tilde{u}(F) \mid \mathcal{H}])$$

for a continuous and strictly increasing utility function $\tilde{u} : \mathbb{R} \rightarrow \mathbb{R}$. Moreover, the corresponding (conditional) aggregation functions are given by

$$\Lambda = -f_\rho \circ f_u^{-1} \circ u \text{ and } \Lambda_{\mathcal{H}} = -f_\rho \circ f_u^{-1} \circ a_{\mathcal{H}} \circ u,$$

where $a_{\mathcal{H}}(F) = \alpha F + \beta$, $F \in L^\infty(\mathcal{F})$, is a positive affine transformation given by $\alpha, \beta \in L^\infty(\mathcal{H})$ with $\mathbb{P}(\alpha > 0) = 1$.

Proof As η is law-invariant, it follows from Lemma 4.3 that $\eta_{\mathcal{H}}$ is conditionally law-invariant. Moreover, $f_\eta \equiv f_{\eta_{\mathcal{H}}} \equiv -\text{id}$, i.e. η and $\eta_{\mathcal{H}}$ are normalized on constants. Thus by Theorem 4.4 we obtain that

$$\eta = -\tilde{u}^{-1}(\mathbb{E}_{\mathbb{P}}[\tilde{u}(F)]) \text{ and } \eta_{\mathcal{H}} = -\tilde{u}^{-1}(\mathbb{E}_{\mathbb{P}}[\tilde{u}(F) \mid \mathcal{H}])$$

for a continuous and strictly increasing function $\tilde{u} : \mathbb{R} \rightarrow \mathbb{R}$. It follows from Proposition D.1 that U as well as $U_{\mathcal{H}}$ are affine transformations of \tilde{u} . This in turn implies that $U_{\mathcal{H}} = \tilde{a}_{\mathcal{H}} \circ U$, where $\tilde{a}_{\mathcal{H}}(F) = \tilde{\alpha} F + \tilde{\beta}$ for $\tilde{\alpha}, \tilde{\beta} \in L^\infty(\mathcal{H})$ with $\mathbb{P}(\tilde{\alpha} > 0) = 1$. Finally we obtain that the $\sigma(f_{\rho_{\mathcal{H}}}(a, \omega), a \in \mathbb{R})$ -conditional aggregation function $\Lambda_{\mathcal{H}}$ is given by

$$\Lambda_{\mathcal{H}} = U_{\mathcal{H}}^{-1} \circ u = U^{-1} \circ \tilde{a}_{\mathcal{H}}^{-1} \circ u = -f_\rho \circ f_u^{-1} \circ \tilde{a}_{\mathcal{H}}^{-1} \circ u.$$

Since the inverse $a_{\mathcal{H}} := \tilde{a}_{\mathcal{H}}^{-1}$ of an affine function is affine the result follows. □

Remark 4.10 Our notion of consistency is defined in terms of the multivariate CRMs. In contrast in Kromer et al. [19] it is a priori assumed that the multivariate CRMs are of the decomposable form $\rho = \eta \circ \Lambda$ as in (3.4) and they define "consistency" of $\{\rho_{\mathcal{G}}, \rho_{\mathcal{H}}\}$ by requiring strong consistency of both pairs $\{\eta_{\mathcal{G}}, \eta_{\mathcal{H}}\}$ and $\{\Lambda_{\mathcal{G}}, \Lambda_{\mathcal{H}}\}$. Note that these definitions of consistency are not equivalent, in particular strong consistency of both $\{\eta_{\mathcal{G}}, \eta_{\mathcal{H}}\}$ and $\{\Lambda_{\mathcal{G}}, \Lambda_{\mathcal{H}}\}$ does not imply strong consistency of $\{\rho_{\mathcal{G}}, \rho_{\mathcal{H}}\}$. Kromer et al. [19] also study the interplay of the strong consistency of $\{\rho_{\mathcal{G}}, \rho_{\mathcal{H}}\}$ and of strong consistency of both $\{\eta_{\mathcal{G}}, \eta_{\mathcal{H}}\}$ and $\{\Lambda_{\mathcal{G}}, \Lambda_{\mathcal{H}}\}$. As Corollary 4.9 shows in the law-invariant case this requirement is quite restrictive.

5 Consistency of a family of conditional risk measures

So far we only considered consistency for two multivariate CRMs. In this section we extend our results on strong consistency to families of multivariate CRMs. We begin with some motivating examples.

Example 5.1 (Dynamic risk measures) If one is interested in a dynamic risk measurement under growing information in time up to a terminal time $T > 0$, this can be modeled by a family of CRMs $(\rho_t)_{t \in [0, T]}$ and a filtration $(\mathcal{F}_t)_{t \in [0, T]}$ such that $\rho_t : L^\infty_d(\mathcal{F}_T) \rightarrow L^\infty(\mathcal{F}_t)$.

In systemic risk measurement conditioning on varying information in space rather than in time is of interest. In that situation, as opposed to Example 5.1, the family of multivariate CRMs is not necessarily indexed by a filtration. To exemplify this we recall a multivariate version of the spatial risk measures which have been introduced by Föllmer [12] in a univariate framework.

Example 5.2 (Multivariate spatial risk measures) Let $I = \{1, \dots, d\}$ denote a set of financial institutions and let (S, \mathcal{S}) be a measurable space. Each financial institution $i \in I$ can be in some state $s \in S$, and $\Omega = S^I = \{\omega = (\omega_i)_{i \in I} : \omega_i \in S\}$ denotes all possible states of the system. Then the σ -algebra \mathcal{F}_J on Ω which is generated by the canonical projections on the j -th coordinate for $j \in J$ describes the observable information within the subsystem of financial institutions $J \subseteq I$. Finally let \mathbb{P} be a probability measure on (Ω, \mathcal{F}) , where $\mathcal{F} := \mathcal{F}_I$. Then the risk evolution under varying spatial information can be modeled by the family of CRMs $(\rho_J)_{J \subseteq I}$, where each $\rho_J : L^\infty_d(\mathcal{F}) \rightarrow L^\infty(\mathcal{F}_J)$, i.e. ρ_J is the risk of the system given the information on the state of the financial institutions within the subsystem J .

From the viewpoint of a regulator, systemic risk measurement contingent on information in space is helpful in identifying systemic relevant structures, i.e. in analyzing questions like: "How much is the system affected given that a specific institution or subgroup of institutions is in distress?", or "How resilient is a specific institution or subgroup of institutions given that the system is in distress?". In Example 5.2 the spatial conditioning is based on a σ -algebra which is generated by all possible states of the institutions within a given subsystem. To treat questions of the type mentioned before one might alternatively consider conditioning with respect to more granular information in space. For instance, in the spirit of the systemic risk measures CoVaR in Adrian and Brunnermeier [3] or Systemic Expected Shortfall in Acharya et al. [2] one could condition on a single crisis event with respect to a given subsystem, e.g. that all financial institutions within the subsystem are below their individual value-at-risk levels.

In Example 5.1 as well as Example 5.2 the families of CRMs are indexed by one-dimensional information structure. However, in Frittelli and Maggis [16], they propose conditional certainty equivalents based on a one-dimensional information structure caused by the fact that utilities of agents may vary over time:

Example 5.3 (Conditional certainty equivalents) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathbb{P})$ be an atomless filtered probability space and let $u_t : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ be a function which is strictly increasing and continuous in the first argument and \mathcal{F}_t -measurable in the second argument for all $t \in \mathbb{R}^+$. Suppose that the range $\mathcal{R}_t := \{u_t(x, \omega) : x \in \mathbb{R}\}$ is independent of $\omega \in \Omega$, that $\mathcal{R}_t \subseteq \mathcal{R}_s$ for all $s \leq t$, and denote the pathwise inverse function of u by $u_t^{-1}(y) \in L^\infty(\mathcal{F}_t)$ for all $y \in \mathcal{R}_t$, where $u_t(x)$ and $u_t^{-1}(y)$ is the shorthand for $u_t(x, \cdot)$ and $u_t^{-1}(y, \cdot)$, resp. Then the backward conditional certainty equivalent is given by

$$C_{s,t} : L^\infty(\mathcal{F}_t) \rightarrow L^\infty(\mathcal{F}_s); F \mapsto C_{s,t}(F) = -u_s^{-1}(\mathbb{E}_{\mathbb{P}} [u_t(F) | \mathcal{F}_s]).$$

It has been shown in Frittelli and Maggis [16, Proposition 1.1] that for a fixed $T \in \mathbb{R}^+$, we have that the family $(C_{t,T})_{t \leq T}$ is consistent, i.e. for all $s \leq t \leq T$

$$C_{t,T}(F) \geq C_{t,T}(G) \implies C_{s,T}(F) \geq C_{s,T}(G) \quad (F, G \in L^\infty(\mathcal{F}_T)).$$

Also in the context of conditioning on spatial information a two-dimensional information structure could be of interest, for example to represent risk measurement policies that differ locally in the financial system.

Example 5.4 (Local regulatory policies) In the context of Example 5.2, let $I = \{1, \dots, d\}$ be a network of financial institutions that is of interest for supervisory authorities associated to different levels with possibly different regulatory policies. For example, think of I as the European financial system. Then regulatory policies of authorities on the European level might differ from policies on the national levels which again might differ from regional policies. To include these different regulatory viewpoints into the framework of spatial risk measures one could consider a family of CRMs $(\rho_{J,K})_{J \subseteq K \subseteq I}$, where each $\rho_{J,K} : L^\infty_d(\mathcal{F}_K) \rightarrow L^\infty(\mathcal{F}_J)$. Here the first index J has the same meaning as in Example 5.2, i.e. the risk measurement is performed conditioned on the state of the institutions in subsystem J . The second index K identifies the type of regulatory policy on the risk management prevailing in subsystem K , for example expected shortfall measures at different significance levels according to European ($K = I$), national, or regional standards. Even though regulatory policies may differ depending on the level of authority, it might still be desirable that these policies behave consistently in some way, i.e. the family $(\rho_{J,K})_{J \subseteq K \subseteq I}$ should be consistent not only with respect to the contingent information implied by the index J but also with respect to the different policies implied by the index K . In the following, this question will be considered.

Motivated by the examples above, we will consider the following types of families of CRMs in this section: Let \mathcal{I}_1 and \mathcal{I}_2 be sets of sub- σ -algebras of \mathcal{F} such that \mathcal{I}_1 contains the trivial σ -algebra and denote by $\mathcal{E} := \{(\mathcal{H}, T) \in \mathcal{I}_1 \times \mathcal{I}_2 : \mathcal{H} \subseteq T\}$. In the following we denote by $\rho_{\mathcal{H},T}$ a multivariate CRM which maps $L^\infty_d(T)$ to $L^\infty(\mathcal{H})$ and we consider families of CRMs of type $(\rho_{\mathcal{H},T})_{(\mathcal{H},T) \in \mathcal{E}}$. In order to allow for a comparison of the risks of two random risk factors under different information, we assume for the rest of this section that $\rho_{\mathcal{H},T_1}(L^\infty_d(T_1)) = \rho_{\mathcal{H},T_2}(L^\infty_d(T_2))$ for all $(\mathcal{H}, T_1), (\mathcal{H}, T_2) \in \mathcal{E}$. Sometimes it will also be convenient to consider only a subfamily of \mathcal{E} where the second σ -algebra is fixed. In that case we denote the corresponding index set by $\mathcal{E}(T) := \{\mathcal{H} \in \mathcal{I}_1 : \mathcal{H} \subseteq T\}$ for $T \in \mathcal{I}_2$. Note that the structure of the families of CRMs discussed in Example 5.1 and Example 5.2 is covered by this framework by letting $\mathcal{I}_2 := \{\mathcal{F}\}$.

Definition 5.5 A family of CRMs $(\rho_{\mathcal{H},T})_{(\mathcal{H},T) \in \mathcal{E}}$ is strongly consistent if for all $\mathcal{G} \subseteq \mathcal{H} \subseteq T_1 \cap T_2$

$$\rho_{\mathcal{H},T_1}(X) \geq \rho_{\mathcal{H},T_2}(Y) \implies \rho_{\mathcal{G},T_1}(X) \geq \rho_{\mathcal{G},T_2}(Y), \quad (X \in L^\infty(T_1), Y \in L^\infty(T_2)).$$

It can be easily checked that the conditional certainty equivalents of Frittelli and Maggis [16] (see Example 5.3) are strongly consistent. Analogously to Lemma 3.2 strong consistency is equivalent to the following recursive relation between the CRMs.

Lemma 5.6 Let $(\rho_{\mathcal{H},T})_{(\mathcal{H},T) \in \mathcal{E}}$ be family of CRMs, then the following statements are equivalent:

- (i) $(\rho_{\mathcal{H},T})_{(\mathcal{H},T) \in \mathcal{E}}$ is strongly consistent;

(ii) For all $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$ and $X \in L_d^\infty(\mathcal{T}_1)$

$$\rho_{\mathcal{G}, \mathcal{T}_1}(X) = \rho_{\mathcal{G}, \mathcal{T}_2} \left(f_{\rho_{\mathcal{H}, \mathcal{T}_2}}^{-1} \left(\rho_{\mathcal{H}, \mathcal{T}_1}(X) \right) \mathbf{1}_d \right).$$

Clearly, our results from the previous sections carry over to families of CRM. We illustrate this in the following by giving the straightforward extensions of Theorem 3.9 and Theorem 4.4 to a family of CRMs.

Theorem 5.7 *Let $(\rho_{\mathcal{H}, \mathcal{T}})_{(\mathcal{H}, \mathcal{T}) \in \mathcal{E}}$ be a family of strongly consistent CRMs. Moreover, if there exists a $\mathcal{T} \in \mathcal{I}_2$ such that*

$$\bar{\rho}_{\mathcal{H}, \mathcal{T}}(x) = \bar{\rho}_{\mathcal{T}, \mathcal{T}}(x) \text{ for all } x \in \mathbb{R}^d \text{ and all } \mathcal{H} \in \mathcal{E}(\mathcal{T}), \tag{5.1}$$

then each multivariate CRM $\rho_{\mathcal{H}, \mathcal{T}}$ of the subfamily $(\rho_{\mathcal{H}, \mathcal{T}})_{\mathcal{H} \in \mathcal{E}(\mathcal{T})}$ which has a continuous realization $\rho_{\mathcal{H}, \mathcal{T}}(\cdot, \cdot)$ can be decomposed into a \mathcal{H} -conditional aggregation function $\Lambda_{\mathcal{H}, \mathcal{T}} : L_d^\infty(\mathcal{T}) \rightarrow L^\infty(\mathcal{T})$ and a univariate CRM $\eta_{\mathcal{H}, \mathcal{T}} : \text{Im } \Lambda_{\mathcal{H}, \mathcal{T}} \rightarrow L^\infty(\mathcal{H})$ such that

$$\rho_{\mathcal{H}, \mathcal{T}} = \eta_{\mathcal{H}, \mathcal{T}} \circ \Lambda_{\mathcal{H}, \mathcal{T}}$$

and $\rho_{\mathcal{H}, \mathcal{T}}(X) = \eta_{\mathcal{H}, \mathcal{T}}(\Lambda_{\mathcal{H}, \mathcal{T}}(X)) = -\Lambda_{\mathcal{H}, \mathcal{T}}(X)$ for all $X \in L_d^\infty(\mathcal{T})$. Moreover, for those $\rho_{\mathcal{H}, \mathcal{T}}, \mathcal{H} \in \mathcal{E}(\mathcal{T})$, for which a decomposition exists the corresponding conditional aggregation functions are strongly consistent.

As in the discussion of Theorem 3.9, if there is a minimal element \mathcal{G} in $\mathcal{E}(\mathcal{T})$, then (5.1) can be replaced by $\bar{\rho}_{\mathcal{T}, \mathcal{T}}(x) \in L^\infty(\mathcal{G})$ for all $x \in \mathbb{R}^d$. In particular if $\{\emptyset, \Omega\} \in \mathcal{E}(\mathcal{T})$, then (5.1) requires $\bar{\rho}_{\mathcal{T}, \mathcal{T}}(x) \in \mathbb{R}^d$ for all $x \in \mathbb{R}^d$, so all aggregation functions $\Lambda_{\mathcal{H}, \mathcal{T}}$ are necessarily deterministic, $\mathcal{H} \in \mathcal{E}(\mathcal{T})$.

Theorem 5.8 *Let $(\rho_{\mathcal{H}, \mathcal{T}})_{(\mathcal{H}, \mathcal{T}) \in \mathcal{E}}$ be a family of CRMs. Furthermore, suppose that there exists an $(\mathcal{G}, \mathcal{T}) \in \mathcal{E}$ such that $(\Omega, \mathcal{T}, \mathbb{P})$ is a conditionally atomless probability space given \mathcal{G} , $(\Omega, \mathcal{G}, \mathbb{P})$ is atomless and $\rho_{\mathcal{T}} := \rho_{\{\emptyset, \Omega\}, \mathcal{T}}$ is law-invariant. Then the subfamily $(\rho_{\mathcal{H}, \mathcal{T}})_{\mathcal{H} \in \mathcal{E}(\mathcal{T})}$ is strongly consistent if and only if for each $\mathcal{H} \in \mathcal{E}(\mathcal{T})$ the CRM $\rho_{\mathcal{H}, \mathcal{T}}$ is of the form*

$$\rho_{\mathcal{H}, \mathcal{T}}(X) = g_{\mathcal{H}, \mathcal{T}} \left(f_{u_{\mathcal{T}}}^{-1} \left(\mathbb{E}_{\mathbb{P}} [u_{\mathcal{T}}(X) \mid \mathcal{H}] \right) \right), \text{ for all } X \in L_d^\infty(\mathcal{T}), \tag{5.2}$$

where $u_{\mathcal{T}} : \mathbb{R}^d \rightarrow \mathbb{R}$ is strictly increasing and continuous, $f_{u_{\mathcal{T}}}^{-1} : \text{Im } f_{u_{\mathcal{T}}} \rightarrow \mathbb{R}$ is the unique inverse function of $f_{u_{\mathcal{T}}} : \mathbb{R} \rightarrow \mathbb{R}; x \mapsto u_{\mathcal{T}}(x\mathbf{1}_d)$ and $g_{\mathcal{H}, \mathcal{T}} : L^\infty(\mathcal{H}) \rightarrow L^\infty(\mathcal{H})$ is strictly antitone, \mathcal{H} -local, fulfills the Lebesgue property and $0 \in \text{Im } g_{\mathcal{H}, \mathcal{T}}$.

In particular, for any CRM of type (5.2) we have that $g_{\mathcal{H}, \mathcal{T}} = f_{\rho_{\mathcal{H}, \mathcal{T}}}$, where $f_{\rho_{\mathcal{H}, \mathcal{T}}}$ is defined in Definition 2.2.

Note that the latter results, being extensions from the two-CRM-case of the previous sections, only used the strong consistency as a pairwise strong consistency of the elements in subfamilies $(\rho_{\mathcal{H}, \mathcal{T}})_{(\mathcal{H}, \mathcal{T}) \in \mathcal{E}(\mathcal{T})}$ of $(\rho_{\mathcal{H}, \mathcal{T}})_{(\mathcal{H}, \mathcal{T}) \in \mathcal{E}}$. But if \mathcal{I}_2 contains more than just one σ -algebra, then the definition of strong consistency given in Definition 5.5 also has implications on the relations between these subfamilies corresponding to different sets $\mathcal{E}(\mathcal{T})$ for $\mathcal{T} \in \mathcal{I}_2$.

Assumption 2 In order to have sufficiently many subfamilies we suppose for the remainder of this section that $\mathcal{I}_1 = \mathcal{I}_2 =: \mathcal{I}$.

Proposition 5.9 *Let $(\rho_{\mathcal{H},\mathcal{T}})_{(\mathcal{H},\mathcal{T}) \in \mathcal{E}}$ be a strongly consistent family such that (5.2) holds for all $(\mathcal{H}, \mathcal{T}) \in \mathcal{E}$. Then for all $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{I}$ and $\mathcal{H} \in \mathcal{T}_1 \cap \mathcal{T}_2, \mathcal{H} \in \mathcal{I}$,*

$$\rho_{\mathcal{H},\mathcal{T}_1}(X) = f_{\rho_{\mathcal{H},\mathcal{T}_2}}^{-1} \left(f_{u_{\mathcal{T}_2}}^{-1} \left(a_{\mathcal{T}_1,\mathcal{T}_2} \mathbb{E}_{\mathbb{P}} [u_{\mathcal{T}_1}(X) \mid \mathcal{H}] + b_{\mathcal{H},\mathcal{T}_1,\mathcal{T}_2} \right) \right),$$

where $a_{\mathcal{T}_1,\mathcal{T}_2} \in \mathbb{R}^+ \setminus \{0\}, b_{\mathcal{H},\mathcal{T}_1,\mathcal{T}_2} \in L^\infty(\mathcal{H})$ and $\mathbb{E}_{\mathbb{P}} [b_{\mathcal{H},\mathcal{T}_1,\mathcal{T}_2} \mid \mathcal{G}] = b_{\mathcal{G},\mathcal{T}_1,\mathcal{T}_2}$ for all $\mathcal{G} \in \mathcal{I}$ with $\mathcal{G} \subseteq \mathcal{H}$.

The proof of Proposition 5.9 is provided in ‘‘Appendix E’’. From Proposition 5.9 it follows that any strongly consistent family $(\rho_{\mathcal{H},\mathcal{T}})_{(\mathcal{H},\mathcal{T}) \in \mathcal{E}}$ (under Assumption 2) is basically a family of conditional certainty equivalents as in Frittelli and Maggis [16]:

Corollary 5.10 *In the situation of Proposition 5.9, if $a_{\mathcal{T}_1,\mathcal{T}_2} = 1, b_{\mathcal{H},\mathcal{T}_1,\mathcal{T}_2} = 0$ for all $\mathcal{H} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$ where $\mathcal{H} \in \mathcal{I}$ and $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{I}$, and if $\rho_{\mathcal{T},\mathcal{T}}$ are normalized on constants for all $\mathcal{T} \in \mathcal{I}$, then $(\rho_{\mathcal{H},\mathcal{T}})_{(\mathcal{H},\mathcal{T}) \in \mathcal{E}}$ satisfies*

$$\rho_{\mathcal{H},\mathcal{T}}(X) = -f_{u_{\mathcal{H}}}^{-1} \left(\mathbb{E}_{\mathbb{P}} [u_{\mathcal{T}}(X) \mid \mathcal{H}] \right), \quad X \in L_d^\infty(\mathcal{T}). \tag{5.3}$$

Proof If $a_{\mathcal{T}_1,\mathcal{T}_2} = 1$ and $b_{\mathcal{H},\mathcal{T}_1,\mathcal{T}_2} = 0$ for all $\mathcal{H} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$, then

$$\rho_{\mathcal{H},\mathcal{T}_1}(X) = f_{\rho_{\mathcal{H},\mathcal{T}_2}}^{-1} \left(f_{u_{\mathcal{T}_2}}^{-1} \left(\mathbb{E}_{\mathbb{P}} [u_{\mathcal{T}_1}(X) \mid \mathcal{H}] \right) \right),$$

and thus by choosing $\mathcal{T}_2 = \mathcal{H}$ and since $\rho_{\mathcal{H},\mathcal{H}}$ is normalized on constants we get (5.3). \square

A Auxiliary results

Note that the strict antitonicity of $\rho_{\mathcal{G}}$ implies that the inverse function $f_{\rho_{\mathcal{G}}}^{-1}$ in Definition 2.2 is well-defined. Indeed let $\beta \in \text{Im } f_{\rho_{\mathcal{G}}}$ and $\alpha_1, \alpha_2 \in L^\infty(\mathcal{G})$ such that $f_{\rho_{\mathcal{G}}}(\alpha_1) = \beta = f_{\rho_{\mathcal{G}}}(\alpha_2)$. Suppose that $\mathbb{P}(A) > 0$ where $A := \{\alpha_1 > \alpha_2\} \in \mathcal{G}$. Then by strict antitonicity and \mathcal{G} -locality we obtain that

$$\begin{aligned} \beta \mathbb{1}_A + \rho_{\mathcal{G}}(\mathbf{0}_d) \mathbb{1}_{A^c} &= \rho_{\mathcal{G}}(\alpha_1 \mathbf{1}_d) \mathbb{1}_A + \rho_{\mathcal{G}}(\mathbf{0}_d) \mathbb{1}_{A^c} = \rho_{\mathcal{G}}(\alpha_1 \mathbf{1}_d \mathbb{1}_A) \\ &\leq \rho_{\mathcal{G}}(\alpha_2 \mathbf{1}_d \mathbb{1}_A) = \rho_{\mathcal{G}}(\alpha_2 \mathbf{1}_d) \mathbb{1}_A + \rho_{\mathcal{G}}(\mathbf{0}_d) \mathbb{1}_{A^c} \\ &= \beta \mathbb{1}_A + \rho_{\mathcal{G}}(\mathbf{0}_d) \mathbb{1}_{A^c}, \end{aligned}$$

and the inequality is strict with positive probability which is a contradiction. Thus we have that $\mathbb{P}(\alpha_1 > \alpha_2) = 0$. The same argument for $\{\alpha_1 < \alpha_2\}$ yields $\alpha_1 = \alpha_2$ \mathbb{P} -a.s.

Next we will show that properties of $\rho_{\mathcal{G}}$ transfer to $f_{\rho_{\mathcal{G}}}$ and $f_{\rho_{\mathcal{G}}}^{-1}$. Since the domain of $f_{\rho_{\mathcal{G}}}^{-1}$ might be only a subset of $L^\infty(\mathcal{G})$, we need to adapt the definition of the Lebesgue property for $f_{\rho_{\mathcal{G}}}^{-1}$ in the following way: If $(\beta_n)_{n \in \mathbb{N}} \subset \text{Im } f_{\rho_{\mathcal{G}}}$ is a sequence which is lower- and upper-bounded by some $\underline{\beta}, \bar{\beta} \in \text{Im } f_{\rho_{\mathcal{G}}}$, i.e. $\underline{\beta} \leq \beta_n \leq \bar{\beta}$ for all $n \in \mathbb{N}$, and such that $\beta_n \rightarrow \beta$ \mathbb{P} -a.s., then $f_{\rho_{\mathcal{G}}}^{-1}(\beta_n) \rightarrow f_{\rho_{\mathcal{G}}}^{-1}(\beta)$ \mathbb{P} -a.s. Note that this alternative definition of the Lebesgue property is equivalent to Definition 2.1 (iv) if the domain is $L^\infty(\mathcal{G})$. The properties ‘strict antitonicity’ and ‘locality’ of $f_{\rho_{\mathcal{G}}}$ or $f_{\rho_{\mathcal{G}}}^{-1}$ are defined analogous to Definition 2.1 (ii) and (iii).

Lemma A.1 *Let $f_{\rho_{\mathcal{G}}}$ and $f_{\rho_{\mathcal{G}}}^{-1}$ be as in Definition 2.2. Then $f_{\rho_{\mathcal{G}}}$ and $f_{\rho_{\mathcal{G}}}^{-1}$ are strictly antitone, \mathcal{G} -local and fulfill the Lebesgue property.*

Proof For f_{ρ_G} the statement follows immediately from the definition and the corresponding properties of ρ_G . Concerning the properties of $f_{\rho_G}^{-1}$, we start by proving strict antitonicity. Let $\beta_1, \beta_2 \in \text{Im } f_{\rho_G}$ such that $\beta_1 \geq \beta_2$ and $\mathbb{P}(\beta_1 > \beta_2) > 0$. Suppose that $\mathbb{P}(A) > 0$ where $A := \left\{ f_{\rho_G}^{-1}(\beta_1) > f_{\rho_G}^{-1}(\beta_2) \right\} \in \mathcal{G}$. Then

$$\begin{aligned} \beta_1 \mathbb{1}_A + f_{\rho_G}(0) \mathbb{1}_{A^c} &= f_{\rho_G} \left(f_{\rho_G}^{-1}(\beta_1) \right) \mathbb{1}_A + f_{\rho_G}(0) \mathbb{1}_{A^c} = f_{\rho_G} \left(f_{\rho_G}^{-1}(\beta_1) \mathbb{1}_A \right) \\ &\leq f_{\rho_G} \left(f_{\rho_G}^{-1}(\beta_2) \mathbb{1}_A \right) = \beta_2 \mathbb{1}_A + f_{\rho_G}(0) \mathbb{1}_{A^c}, \end{aligned}$$

and the inequality is strict on a set with positive probability since f_{ρ_G} is strictly antitone. This of course contradicts $\beta_1 \geq \beta_2$. Hence $f_{\rho_G}^{-1}(\beta_1) \leq f_{\rho_G}^{-1}(\beta_2)$. Moreover, as

$$\mathbb{P}(\beta_1 > \beta_2) = \mathbb{P} \left(f_{\rho_G} \left(f_{\rho_G}^{-1}(\beta_1) \right) > f_{\rho_G} \left(f_{\rho_G}^{-1}(\beta_2) \right) \right) > 0$$

we must have $f_{\rho_G}^{-1}(\beta_1) \neq f_{\rho_G}^{-1}(\beta_2)$ with positive probability, i.e.

$$\mathbb{P} \left(f_{\rho_G}^{-1}(\beta_1) < f_{\rho_G}^{-1}(\beta_2) \right) > 0.$$

Now we show that $f_{\rho_G}^{-1}$ is \mathcal{G} -local. Let $\beta_1, \beta_2 \in \text{Im } f_{\rho_G}$ as well as $A \in \mathcal{G}$ be arbitrary. Further let $\alpha_i = f_{\rho_G}^{-1}(\beta_i)$, $i = 1, 2$, i.e. $f_{\rho_G}(\alpha_i) = \beta_i$. Then we have that

$$f_{\rho_G}(\alpha_1 \mathbb{1}_A + \alpha_2 \mathbb{1}_{A^c}) = f_{\rho_G}(\alpha_1) \mathbb{1}_A + f_{\rho_G}(\alpha_2) \mathbb{1}_{A^c} = \beta_1 \mathbb{1}_A + \beta_2 \mathbb{1}_{A^c}.$$

Thus $f_{\rho_G}^{-1}(\beta_1 \mathbb{1}_A + \beta_2 \mathbb{1}_{A^c}) = \alpha_1 \mathbb{1}_A + \alpha_2 \mathbb{1}_{A^c}$.

Finally for the Lebesgue property let $\underline{\beta}, \bar{\beta} \in \text{Im } f_{\rho_G}$ and let $(\beta_n)_{n \in \mathbb{N}} \subset \text{Im } f_{\rho_G}$ be a sequence with $\underline{\beta} \leq \beta_n \leq \bar{\beta}$ for all $n \in \mathbb{N}$ and $\beta_n \rightarrow \beta$ \mathbb{P} -a.s. Consider the bounded sequences $\beta_n^u := \sup_{k \geq n} \beta_k$ and $\beta_n^d := \inf_{k \geq n} \beta_k$, $n \in \mathbb{N}$ which converge monotonically almost surely to β , i.e. $\beta_n^u \downarrow \beta$ \mathbb{P} -a.s. and $\beta_n^d \uparrow \beta$ \mathbb{P} -a.s. Since $\underline{\beta} \leq \beta_n^u \leq \bar{\beta}$ for all $n \in \mathbb{N}$ which by antitonicity of $f_{\rho_G}^{-1}$ yields $f_{\rho_G}^{-1}(\bar{\beta}) \leq f_{\rho_G}^{-1}(\beta_n^u) \leq f_{\rho_G}^{-1}(\underline{\beta})$, we observe that the sequence $\left(f_{\rho_G}^{-1}(\beta_n^u) \right)_{n \in \mathbb{N}}$ is uniformly bounded in $L^\infty(\mathcal{G})$. Note that by the same argumentation also the sequences $\left(f_{\rho_G}^{-1}(\beta_n^d) \right)_{n \in \mathbb{N}}$ and $\left(f_{\rho_G}^{-1}(\beta_n) \right)_{n \in \mathbb{N}}$ are uniformly bounded in $L^\infty(\mathcal{G})$. Next we will show that $\beta_n^u \in \text{Im } f_{\rho_G}$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and set recursively

$$A_{n-1}^n := \{\beta_n^u = \beta\} \quad \text{and} \quad A_k^n := \{\beta_n^u = \beta_k\} \setminus \bigcup_{i=n-1}^{k-1} A_i^n, \quad k \geq n,$$

then it follows from induction that $A_k^n \in \mathcal{G}$, $k \geq n - 1$. Since $\sup\{\beta, \beta_k : k \geq n\} = \max\{\beta, \beta_k : k \geq n\}$, we have that $\left(\bigcup_{k \geq n-1} A_k^n \right)^C$ is a \mathbb{P} -nullset. It follows from \mathcal{G} -locality and the Lebesgue property of f_{ρ_G} that

$$\begin{aligned}
 & f_{\rho_G} \left(f_{\rho_G}^{-1}(\beta) \mathbb{1}_{A_{n-1}^n} + \sum_{k \geq n} f_{\rho_G}^{-1}(\beta_k) \mathbb{1}_{A_k^n} \right) \\
 &= \beta \mathbb{1}_{A_{n-1}^n} + f_{\rho_G} \left(\lim_{m \rightarrow \infty} \sum_{k=n}^m f_{\rho_G}^{-1}(\beta_k) \mathbb{1}_{A_k^n} \right) \mathbb{1}_{\cup_{k \geq n} A_k^n} \\
 &= \beta \mathbb{1}_{A_{n-1}^n} + \lim_{m \rightarrow \infty} \left(\sum_{k=n}^m \beta_k \mathbb{1}_{A_k^n} + f_{\rho_G}(0) \mathbb{1}_{\cup_{k \geq m} A_k^n} \right) \\
 &= \beta \mathbb{1}_{A_{n-1}^n} + \sum_{k \geq n} \beta_k \mathbb{1}_{A_k^n} = \beta_n^u,
 \end{aligned}$$

which implies $\beta_n^u \in \text{Im } f_{\rho_G}$. By a similar argumentation we obtain $\beta_n^d \in \text{Im } f_{\rho_G}$. Recall that $\beta_n^u \downarrow \beta$ \mathbb{P} -a.s. which by antitonicity of $f_{\rho_G}^{-1}$ implies that the sequence $(f_{\rho_G}^{-1}(\beta_n^u))_{n \in \mathbb{N}}$ is isotone and thus $\alpha = \lim_{n \rightarrow \infty} f_{\rho_G}^{-1}(\beta_n^u)$ exists in $L^\infty(\mathcal{G})$. It follows from antitonicity and the Lebesgue property of f_{ρ_G} that

$$\beta = \lim_{n \rightarrow \infty} \beta_n^u = \lim_{n \rightarrow \infty} f_{\rho_G} \left(f_{\rho_G}^{-1}(\beta_n^u) \right) = f_{\rho_G}(\alpha),$$

and hence that indeed $\alpha = f_{\rho_G}^{-1}(\beta)$. Analogously, we obtain that $f_{\rho_G}(\hat{\alpha}) = \beta$ for $\hat{\alpha} = \lim_{n \rightarrow \infty} f_{\rho_G}^{-1}(\beta_n^d)$, and thus $\hat{\alpha} = \alpha = f_{\rho_G}^{-1}(\beta)$. Hence, by antitonicity of $f_{\rho_G}^{-1}$

$$\begin{aligned}
 f_{\rho_G}^{-1}(\beta) &= \lim_{n \rightarrow \infty} f_{\rho_G}^{-1}(\beta_n^u) \leq \liminf_{n \rightarrow \infty} f_{\rho_G}^{-1}(\beta_n) \\
 &\leq \limsup_{n \rightarrow \infty} f_{\rho_G}^{-1}(\beta_n) \leq \lim_{n \rightarrow \infty} f_{\rho_G}^{-1}(\beta_n^d) = f_{\rho_G}^{-1}(\beta),
 \end{aligned}$$

so $\lim_{n \rightarrow \infty} f_{\rho_G}^{-1}(\beta_n) = f_{\rho_G}^{-1}(\beta)$, i.e. $f_{\rho_G}^{-1}$ has the Lebesgue property. □

An important observation is that the domain of $f_{\rho_G}^{-1}$ is equal to the image of ρ_G , i.e. $f_{\rho_G}^{-1}(\rho_G(X))$ is well-defined for all $X \in L_d^\infty(\mathcal{F})$.

Lemma A.2 For a CRM $\rho_G : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$ it holds that

$$\rho_G(L_d^\infty(\mathcal{F})) = f_{\rho_G}(L^\infty(\mathcal{G})).$$

Proof Clearly, $\rho_G(L_d^\infty(\mathcal{F})) \supseteq f_{\rho_G}(L^\infty(\mathcal{G}))$.

For the reverse inclusion let $X \in L_d^\infty(\mathcal{F})$. Our aim is to show that there exists an $\alpha^* \in L^\infty(\mathcal{G})$ such that

$$\rho_G(X) = f_{\rho_G}(\alpha^*). \tag{A.1}$$

Define

$$P := \{ \alpha \in L^\infty(\mathcal{G}) : f_{\rho_G}(\alpha) \geq \rho_G(X) \}.$$

As $-\|X\|_{d,\infty} \mathbf{1}_d \leq X \leq \|X\|_{d,\infty} \mathbf{1}_d$ we have that $-\|X\|_{d,\infty} \in P$, so $P \neq \emptyset$. Moreover, P is bounded from above by $\|X\|_{d,\infty}$ since if $A := \{ \alpha > \|X\|_{d,\infty} \}$ for $\alpha \in L^\infty(\mathcal{G})$ has positive probability, then by \mathcal{G} -locality and strict antitonicity

$$f_{\rho_G}(\alpha) \mathbb{1}_A = f_{\rho_G}(\alpha \mathbb{1}_A) \mathbb{1}_A \leq f_{\rho_G}(\|X\|_{d,\infty} \mathbb{1}_A) \mathbb{1}_A = f_{\rho_G}(\|X\|_{d,\infty}) \mathbb{1}_A \leq \rho_G(X) \mathbb{1}_A$$

where the first inequality is strict with positive probability, so $\alpha \notin P$. By \mathcal{G} -locality it also follows that P is upwards directed. Hence, for

$$\alpha^* := \text{esssup } P$$

there is a uniformly bounded sequence $(\alpha_n)_{n \in \mathbb{N}} \subset P$ such that $\alpha^* = \lim_{n \rightarrow \infty} \alpha_n$ \mathbb{P} -a.s.; see Föllmer and Schied [15, Theorem A.33]. Thus it follows that $\alpha^* \in L^\infty(\mathcal{G})$ and

$$f_{\rho_{\mathcal{G}}}(\alpha^*) = \lim_{n \rightarrow \infty} f_{\rho_{\mathcal{G}}}(\alpha_n) \geq \rho_{\mathcal{G}}(X),$$

i.e. $\alpha^* \in P$. Let

$$B := \{f_{\rho_{\mathcal{G}}}(\alpha^*) > \rho_{\mathcal{G}}(X)\}$$

and note that by the Lebesgue property

$$B = \bigcup_{n \in \mathbb{N}} \{f_{\rho_{\mathcal{G}}}(\alpha^* + 1/n) > \rho_{\mathcal{G}}(X)\} \quad \mathbb{P}\text{-a.s.}$$

Hence, if $\mathbb{P}(B) > 0$ it follows that $\mathbb{P}(B_n) > 0$ for some $B_n := \{f_{\rho_{\mathcal{G}}}(\alpha^* + 1/n) > \rho_{\mathcal{G}}(X)\}$. Note that $B_n \in \mathcal{G}$ and that

$$\alpha^* \mathbb{1}_{B_n^c} + (\alpha^* + 1/n) \mathbb{1}_{B_n} \in P$$

by \mathcal{G} -locality of $f_{\rho_{\mathcal{G}}}$. But this contradicts the definition of α^* . Hence, $\mathbb{P}(B) = 0$. □

Lemma A.3 *Let $\Lambda : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F})$ be a conditional aggregation function. Then f_Λ and f_Λ^{-1} are strictly isotone, \mathcal{F} -local, and fulfill the Lebesgue property. Moreover, $\Lambda(L_d^\infty(\mathcal{F})) = f_\Lambda(L^\infty(\mathcal{F}))$ and $\Lambda(X) = \Lambda(f_\Lambda^{-1}(\Lambda(X))\mathbf{1}_d)$ for all $X \in L_d^\infty(\mathcal{F})$.*

The well-definedness of f_Λ^{-1} follows similarly to the well-definedness of $f_{\rho_{\mathcal{G}}}^{-1}$. Further the proof of Lemma A.3 is analogous to the proofs of Lemma A.1 and Lemma A.2 and therefore omitted here.

B Proof of Theorem 3.9

The Proof of Theorem 3.9 is based on a result from Hoffmann et al. [17] which we in the following present in a version adapted to the framework of this paper.

Proposition B.1 *Let $\rho_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{G})$ be a CRM and suppose that there exists a continuous realization $\rho_{\mathcal{G}}(\cdot, \cdot)$ which satisfies risk-antitonicity:*

$$\rho_{\mathcal{G}}(X(\omega), \omega) \geq \rho_{\mathcal{G}}(Y(\omega), \omega) \quad \mathbb{P}\text{-a.s.}, \text{ implies } \rho_{\mathcal{G}}(X) \geq \rho_{\mathcal{G}}(Y).$$

Then there exists a \mathcal{G} -conditional aggregation function $\Lambda_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F})$ and a univariate CRM $\eta_{\mathcal{G}} : \text{Im } \Lambda_{\mathcal{G}} \rightarrow L^\infty(\mathcal{G})$ such that

$$\rho_{\mathcal{G}}(X) = \eta_{\mathcal{G}}(\Lambda_{\mathcal{G}}(X)) \quad \text{for all } X \in L_d^\infty(\mathcal{F})$$

and

$$\eta_{\mathcal{G}}(\Lambda_{\mathcal{G}}(X)) = -\Lambda_{\mathcal{G}}(X) \quad \text{for all } X \in L_d^\infty(\mathcal{G}). \tag{B.1}$$

This decomposition is unique.

Proof Since ρ_G is antitone, $\mathbb{R}^d \ni x \mapsto \rho_G(x)$ is antitone. It has been shown in Hoffmann et al. [17, Theorem 2.10] that this property in conjunction with the fact that ρ_G has a continuous realization which fulfills risk-antitonicity is sufficient for the existence and uniqueness of a function $\Lambda_G : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F})$ which is isotone, \mathcal{F} -local and fulfills the Lebesgue property and a function $\eta_G : \text{Im } \Lambda_G \rightarrow L^\infty(\mathcal{G})$ which is antitone such that

$$\rho_G = \eta_G \circ \Lambda_G \quad \text{and} \quad \eta_G(\Lambda_G(x)) = -\Lambda_G(x) \text{ for all } x \in \mathbb{R}^d. \tag{B.2}$$

Note that in the proof of Theorem 2.10 in Hoffmann et al. [17] Λ_G is basically constructed by setting $\Lambda_G(X)(\omega) = -\rho_G(X(\omega), \omega)$, which implies that Λ_G is necessarily \mathcal{F} -local even though this is not directly mentioned in the paper. Indeed in Hoffmann et al. [17] we do not require or mention locality at all.

It remains to be shown that Λ_G is a \mathcal{G} -conditional aggregation function, η_G is a univariate CRM on $\text{Im } \Lambda_G$, and that (B.1) holds. First of all, we show that \mathcal{F} -locality and (B.2) imply (B.1). To this end denote by \mathcal{S} the set of \mathcal{F} -measurable simple random vectors, i.e. $X \in \mathcal{S}$ if X is of the form $X = \sum_{i=1}^k x_i \mathbb{1}_{A_i}$, where $k \in \mathbb{N}$, $x_i \in \mathbb{R}^d$ and $A_i \in \mathcal{F}$, $i = 1, \dots, k$, are disjoint sets such that $\mathbb{P}(A_i) > 0$ and $\mathbb{P}(\bigcup_{i=1}^k A_i) = 1$. Now let $X \in L_d^\infty(\mathcal{G})$. Pick a uniformly bounded sequence $(X_n)_{n \in \mathbb{N}} = \left(\sum_{i=1}^{k_n} x_i^n \mathbb{1}_{A_i^n} \right)_{n \in \mathbb{N}} \subset \mathcal{S}$ such that $A_i^n \in \mathcal{G}$ for all $i = 1, \dots, k_n$, $n \in \mathbb{N}$, and $X_n \rightarrow X$ \mathbb{P} -a.s. Then by (B.2), \mathcal{F} -locality and the Lebesgue property of Λ_G and ρ_G we infer that

$$\begin{aligned} -\Lambda_G(X) &= -\lim_{n \rightarrow \infty} \Lambda_G(X_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} -\Lambda_G(x_i^n) \mathbb{1}_{A_i^n} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} \rho_G(x_i^n) \mathbb{1}_{A_i^n} = \lim_{n \rightarrow \infty} \rho_G(X_n) = \rho_G(X), \end{aligned}$$

which proves (B.1). Next we show that Λ_G is a \mathcal{G} -conditional aggregation function. The yet missing properties which need to be verified are strict antitonicity and that Λ_G is \mathcal{G} -conditional. The latter follows from Hoffmann et al. [17, Lemma 3.1]. As for strict antitonicity let $X, Y \in L_d^\infty(\mathcal{F})$ with $X \geq Y$ such that $\mathbb{P}(X > Y) > 0$. Then by isotonicity of Λ_G we have that $\Lambda_G(X) \geq \Lambda_G(Y)$. Suppose that $\Lambda_G(X) = \Lambda_G(Y)$ \mathbb{P} -a.s., then

$$\rho_G(X) = \eta_G(\Lambda_G(X)) = \eta_G(\Lambda_G(Y)) = \rho_G(Y)$$

which contradicts strict antitonicity of ρ_G . Thus Λ_G fulfills all properties of a \mathcal{G} -conditional aggregation function.

As for η_G , note that by Lemma A.3 for all $F \in \text{Im } \Lambda_G$ we have that

$$\eta_G(F) = \eta_G(\Lambda_G(f_{\Lambda_G}^{-1}(F)\mathbf{1}_d)) = \rho_G(f_{\Lambda_G}^{-1}(F)\mathbf{1}_d). \tag{B.3}$$

Since ρ_G and $f_{\Lambda_G}^{-1}$ are strictly monotone, \mathcal{G} -local, and fulfill the Lebesgue property, so does η_G , i.e. η_G is a univariate CRM on $\text{Im } \Lambda_G$. □

The proof of Theorem 3.9 is now based on the following observations: $\rho_{\mathcal{F}}$ is necessarily risk-antitone as defined in Proposition B.1. Strong consistency in turn implies that risk-antitonicity of $\rho_{\mathcal{F}}$ is passed on (backwards) to ρ_G , and hence Proposition B.1 applies.

Proof of Theorem 3.9 In case we already know that (3.4) holds, then by antitonicity of η_G it follows that $\{\rho_G, -\Lambda_G\}$ is strongly consistent, and clearly $-\Lambda_G : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F})$ is also a CRM. Thus the last assertion of Theorem 3.9 is proved.

In order to show the first part of Theorem 3.9, we recall that the only property which remains to be shown in order to apply Proposition B.1 is risk-antitonicity of $\rho_{\mathcal{G}}$: For this purpose we first consider simple random vectors $X, Y \in \mathcal{S}$ where \mathcal{S} was defined in the proof of Proposition B.1. Note that there is no loss of generality by assuming that $X = \sum_{i=1}^n x_i \mathbb{1}_{A_i} \in \mathcal{S}$ and $Y = \sum_{i=1}^n y_i \mathbb{1}_{A_i} \in \mathcal{S}$, i.e. the partition $(A_i)_{i=1, \dots, n}$ of Ω is the same for X and Y . Suppose that $\rho_{\mathcal{G}}(X(\omega), \omega) \geq \rho_{\mathcal{G}}(Y(\omega), \omega)$ \mathbb{P} -a.s. It follows that $\rho_{\mathcal{G}}(x_i, \omega) \geq \rho_{\mathcal{G}}(y_i, \omega)$ for all $\omega \in A_i \setminus N, i = 1, \dots, n$, where N is a \mathbb{P} -nullset. Let $B_i := \{\omega \in \Omega \mid \rho_{\mathcal{G}}(x_i, \omega) \geq \rho_{\mathcal{G}}(y_i, \omega)\} \in \mathcal{G}$. As $(A_i \setminus N) \subseteq B_i$, using antitonicity and \mathcal{G} -locality of $f_{\rho_{\mathcal{G}}}^{-1}$ we obtain

$$f_{\rho_{\mathcal{G}}}^{-1}(\rho_{\mathcal{G}}(x_i)) \mathbb{1}_{A_i} = f_{\rho_{\mathcal{G}}}^{-1}(\rho_{\mathcal{G}}(x_i) \mathbb{1}_{B_i}) \mathbb{1}_{A_i} \leq f_{\rho_{\mathcal{G}}}^{-1}(\rho_{\mathcal{G}}(y_i) \mathbb{1}_{B_i}) \mathbb{1}_{A_i} = f_{\rho_{\mathcal{G}}}^{-1}(\rho_{\mathcal{G}}(y_i)) \mathbb{1}_{A_i}.$$

Now by strong consistency of $\{\rho_{\mathcal{G}}, \rho_{\mathcal{F}}\}$, \mathcal{F} -locality of $\rho_{\mathcal{F}}$ and $f_{\rho_{\mathcal{F}}}^{-1}$, and by (3.3) as well as antitonicity of $\rho_{\mathcal{G}}$ we arrive at

$$\begin{aligned} \rho_{\mathcal{G}}(X) &= \rho_{\mathcal{G}}\left(f_{\rho_{\mathcal{F}}}^{-1}(\rho_{\mathcal{F}}(X)) \mathbf{1}_d\right) = \rho_{\mathcal{G}}\left(\sum_{i=1}^n f_{\rho_{\mathcal{F}}}^{-1}(\rho_{\mathcal{F}}(x_i)) \mathbb{1}_{A_i} \mathbf{1}_d\right) \\ &= \rho_{\mathcal{G}}\left(\sum_{i=1}^n f_{\rho_{\mathcal{G}}}^{-1}(\rho_{\mathcal{G}}(x_i)) \mathbb{1}_{A_i} \mathbf{1}_d\right) \geq \rho_{\mathcal{G}}\left(\sum_{i=1}^n f_{\rho_{\mathcal{G}}}^{-1}(\rho_{\mathcal{G}}(y_i)) \mathbb{1}_{A_i} \mathbf{1}_d\right) \\ &= \rho_{\mathcal{G}}(Y), \end{aligned}$$

which proves risk-antitonicity for simple random vectors $X, Y \in \mathcal{S}$. For general $X, Y \in L_d^\infty(\mathcal{F})$ with $\rho_{\mathcal{G}}(X(\omega), \omega) \geq \rho_{\mathcal{G}}(Y(\omega), \omega)$ for \mathbb{P} -a.e. $\omega \in \Omega$ we can find uniformly bounded sequences $(X_n)_{n \in \mathbb{N}}, (Y_n)_{n \in \mathbb{N}} \subset \mathcal{S}$ such that $X_n \nearrow X$ and $Y_n \searrow Y$ \mathbb{P} -a.s. for $n \rightarrow \infty$. Then by antitonicity

$$\rho_{\mathcal{G}}(X_n(\omega), \omega) \geq \rho_{\mathcal{G}}(X(\omega), \omega) \geq \rho_{\mathcal{G}}(Y(\omega), \omega) \geq \rho_{\mathcal{G}}(Y_n(\omega), \omega) \quad \mathbb{P}\text{-a.s.}$$

Therefore, $\rho_{\mathcal{G}}(X_n) \geq \rho_{\mathcal{G}}(Y_n)$ and the Lebegue property of $\rho_{\mathcal{G}}$ yields

$$\rho_{\mathcal{G}}(X) = \lim_{n \rightarrow \infty} \rho_{\mathcal{G}}(X_n) \geq \lim_{n \rightarrow \infty} \rho_{\mathcal{G}}(Y_n) = \rho_{\mathcal{G}}(Y).$$

Thus $\rho_{\mathcal{G}}$ is risk-antitone and we apply Proposition B.1. Hence, there is a \mathcal{G} -conditional aggregation function $\Lambda_{\mathcal{G}} : L_d^\infty(\mathcal{F}) \rightarrow L^\infty(\mathcal{F})$ and a univariate CRM $\eta_{\mathcal{G}} : \text{Im } \Lambda_{\mathcal{G}} \rightarrow L^\infty(\mathcal{G})$ such that $\rho_{\mathcal{G}} = \eta_{\mathcal{G}} \circ \Lambda_{\mathcal{G}}$ and $\eta_{\mathcal{G}}(\Lambda_{\mathcal{G}}(X)) = -\Lambda_{\mathcal{G}}(X)$ for all $X \in L_d^\infty(\mathcal{G})$.

Using locality it follows that (3.3) indeed holds for all $\alpha \in L_d^\infty(\mathcal{G}) \cap \mathcal{S}$ and thus by continuity $\bar{\rho}_{\mathcal{F}}(\alpha) = \bar{\rho}_{\mathcal{G}}(\alpha) \in L^\infty(\mathcal{G})$ for all $\alpha \in L_d^\infty(\mathcal{G})$. Thus also $\Lambda_{\mathcal{F}}(\alpha) = -\rho_{\mathcal{F}}(\alpha) \in L^\infty(\mathcal{G})$ for all $\alpha \in L_d^\infty(\mathcal{G})$. Finally by the same procedure as above, i.e. approximation via elements in \mathcal{S} , using locality, strong consistency, and continuity, we obtain (3.6). \square

C Proof of Theorem 4.4

Lemma C.1 *Let $\{\rho, \rho_{\mathcal{H}}\}$ be strongly consistent and suppose that ρ is law-invariant (and thus $\rho_{\mathcal{H}}$ is conditionally law-invariant by Lemma 4.3). If $(\Omega, \mathcal{H}, \mathbb{P})$ is an atomless probability space and $X \in L_d^\infty(\mathcal{F})$ is independent of \mathcal{H} , then*

$$f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X)) = f_{\rho}^{-1}(\rho(X)).$$

The proof of Lemma C.1 is adapted from Kupper and Schachermayer [20].

Proof We distinguish three cases:

- Suppose that $f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X)) \leq f_{\rho}^{-1}(\rho(X))$ and strictly smaller with positive probability. Then by strong consistency

$$\begin{aligned} f_{\rho}^{-1}(\rho(X)) &= f_{\rho}^{-1}\left(\rho\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X))\mathbf{1}_d\right)\right) \\ &< f_{\rho}^{-1}\left(\rho\left(f_{\rho}^{-1}(\rho(X))\mathbf{1}_d\right)\right) = f_{\rho}^{-1}(\rho(X)), \end{aligned}$$

by strict antitonicity of ρ which is a contradiction.

- Analogously it follows that it is not possible that $f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X)) \geq f_{\rho}^{-1}(\rho(X))$ and $\mathbb{P}(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X)) > f_{\rho}^{-1}(\rho(X))) > 0$.
- There exist $A, B \in \mathcal{H}$ such that $\mathbb{P}(A) = \mathbb{P}(B) > 0$ and

$$f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X)) > f_{\rho}^{-1}(\rho(X)) \text{ on } A \text{ and } f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X)) < f_{\rho}^{-1}(\rho(X)) \text{ on } B.$$

Then we have for an arbitrary $m = a\mathbf{1}_d$ where $a \in \mathbb{R}$ that

$$\begin{aligned} \rho(X\mathbf{1}_A + m\mathbf{1}_{A^c}) &= \rho\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X\mathbf{1}_A + m\mathbf{1}_{A^c}))\mathbf{1}_d\right) \\ &= \rho\left(f_{\rho_{\mathcal{H}}}^{-1}(\rho_{\mathcal{H}}(X))\mathbf{1}_A\mathbf{1}_d + m\mathbf{1}_{A^c}\right) \\ &< \rho\left(f_{\rho}^{-1}(\rho(X))\mathbf{1}_A\mathbf{1}_d + m\mathbf{1}_{A^c}\right) \end{aligned} \tag{C.1}$$

and similarly

$$\rho(X\mathbf{1}_B + m\mathbf{1}_{B^c}) > \rho\left(f_{\rho}^{-1}(\rho(X))\mathbf{1}_B\mathbf{1}_d + m\mathbf{1}_{B^c}\right). \tag{C.2}$$

However, as X is independent of \mathcal{H} the random vector $X\mathbf{1}_A + m\mathbf{1}_{A^c}$ has the same distribution under \mathbb{P} as $X\mathbf{1}_B + m\mathbf{1}_{B^c}$. Note that also $f_{\rho}^{-1}(\rho(X))\mathbf{1}_A + a\mathbf{1}_{A^c}$ and $f_{\rho}^{-1}(\rho(X))\mathbf{1}_B + a\mathbf{1}_{B^c}$ share the same distribution under \mathbb{P} . Hence, as ρ is law-invariant, (C.1) and (C.2) yield a contradiction. □

Proof of Theorem 4.4 For the last assertion of the theorem note that since u is a deterministic function, we have for $\alpha \in L^\infty(\mathcal{H})$ that

$$\begin{aligned} f_{\rho_{\mathcal{H}}}(\alpha) &= \rho_{\mathcal{H}}(\alpha\mathbf{1}_d) = g_{\mathcal{H}}\left(f_u^{-1}(\mathbb{E}_{\mathbb{P}}[u(\alpha\mathbf{1}_d) \mid \mathcal{H}])\right) \\ &= g_{\mathcal{H}}\left(f_u^{-1}(f_u(\alpha))\right) = g_{\mathcal{H}}(\alpha) \end{aligned}$$

and analogously we obtain $f_{\rho} \equiv g$.

Next we prove sufficiency in the first statement of the theorem: Let $\rho_{\mathcal{H}}$ and ρ be as in (4.2) and (4.1). It is easily verified that $\rho_{\mathcal{H}}$ and ρ are (conditionally) law-invariant CRMs. Furthermore, since f_u^{-1} is strictly increasing and $g_{\mathcal{H}}$ is strictly antitone and \mathcal{H} -local, we have for each $X, Y \in L_d^\infty(\mathcal{F})$ with $\rho_{\mathcal{H}}(X) \geq \rho_{\mathcal{H}}(Y)$ that

$$\mathbb{E}_{\mathbb{P}}[u(X) \mid \mathcal{H}] \leq \mathbb{E}_{\mathbb{P}}[u(Y) \mid \mathcal{H}].$$

But this implies that also $\mathbb{E}_{\mathbb{P}}[u(X)] \leq \mathbb{E}_{\mathbb{P}}[u(Y)]$ and thus that $\rho(X) \geq \rho(Y)$, i.e. $\{\rho, \rho_{\mathcal{H}}\}$ is strongly consistent.

Now we prove necessity in the first statement of the theorem: We assume in the following that ρ and $\rho_{\mathcal{H}}$ are normalized on constants and follow the approach of Föllmer [12, Theorem 3.4]. The idea is to introduce a preference order $<$ on multivariate distributions μ, ν on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with bounded support given by

$$\mu < \nu \iff \rho(X) > \rho(Y), \text{ with } X \sim \mu \text{ and } Y \sim \nu.$$

Here $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel- σ -algebra on \mathbb{R}^d and $X \sim \mu$ means that the distribution of $X \in L_d^\infty(\mathcal{F})$ under \mathbb{P} is μ . It is well-known that if this preference order fulfills a set of conditions, then there exists a von Neumann-Morgenstern representation, that is

$$\mu < \nu \iff \int u(x) \mu(dx) < \int u(x) \nu(dx), \tag{C.3}$$

where $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is a continuous function. Sufficient conditions to guarantee (C.3) are that $<$ is continuous and fulfills the independence axiom; cf. Föllmer and Schied [15, Corollary 2.28]. We refer to Föllmer and Schied [15] for a definition and comprehensive discussion of preference orders and the mentioned properties. Suppose for the moment that we have already proved (C.3). Note that strict antitonicity of ρ implies that $\delta_x > \delta_y$ whenever $x, y \in \mathbb{R}^d$ satisfy $x \geq y$ and $x \neq y$. Hence $u(x) = \int u(s) \delta_x(ds) > \int u(s) \delta_y(ds) = u(y)$, and we conclude that u is necessarily strictly increasing as claimed.

Now we prove (C.3): The proof of continuity of $<$ is completely analogous to the corresponding proof in Föllmer [12, Theorem 3.4], so we omit it here. The crucial property is the independence axiom, which states that for any three distributions μ, ν, ϑ such that $\mu \leq \nu$ and for all $\lambda \in (0, 1]$, we have

$$\lambda\mu + (1 - \lambda)\vartheta \leq \lambda\nu + (1 - \lambda)\vartheta.$$

Since $(\Omega, \mathcal{F}, \mathbb{P})$ is conditionally atomless given \mathcal{H} , we can find $X, Y, Z \in L_d^\infty(\mathcal{F})$ which are independent of \mathcal{H} such that $X \sim \mu, Y \sim \nu$ and $Z \sim \vartheta$. Furthermore, since $(\Omega, \mathcal{H}, \mathbb{P})$ is atomless, we can find an $A \in \mathcal{H}$ with $\mathbb{P}(A) = \lambda$. It can be easily seen that $X\mathbb{1}_A + Z\mathbb{1}_{A^c} \sim \lambda\mu + (1 - \lambda)\vartheta$ and $Y\mathbb{1}_A + Z\mathbb{1}_{A^c} \sim \lambda\nu + (1 - \lambda)\vartheta$. Moreover, since $\mu \leq \nu$, we have that $\rho(X) \geq \rho(Y)$. As $\{\rho, \rho_{\mathcal{H}}\}$ is strongly consistent and as ρ is law-invariant, we know from Lemma 4.3 that $\rho_{\mathcal{H}}$ is conditionally law-invariant. This ensures that we can apply Lemma C.1 to the random vectors X and Y which are independent of \mathcal{H} . Therefore, by \mathcal{H} -locality of $\rho_{\mathcal{H}}$ and recalling Remark 3.4

$$\begin{aligned} \rho(X\mathbb{1}_A + Z\mathbb{1}_{A^c}) &= \rho(-\rho_{\mathcal{H}}(X\mathbb{1}_A + Z\mathbb{1}_{A^c})\mathbf{1}_d) \\ &= \rho(-\rho_{\mathcal{H}}(X)\mathbb{1}_A\mathbf{1}_d - \rho_{\mathcal{H}}(Z)\mathbb{1}_{A^c}\mathbf{1}_d) \\ &= \rho(-\rho(X)\mathbb{1}_A\mathbf{1}_d - \rho_{\mathcal{H}}(Z)\mathbb{1}_{A^c}\mathbf{1}_d) \\ &\geq \rho(-\rho(Y)\mathbb{1}_A\mathbf{1}_d - \rho_{\mathcal{H}}(Z)\mathbb{1}_{A^c}\mathbf{1}_d) = \rho(Y\mathbb{1}_A + Z\mathbb{1}_{A^c}), \end{aligned}$$

which is equivalent to $\lambda\mu + (1 - \lambda)\vartheta \leq \lambda\nu + (1 - \lambda)\vartheta$. Thus there exists a von Neumann-Morgenstern representation (C.3) with a continuous and strictly increasing utility function $u : \mathbb{R}^d \rightarrow \mathbb{R}$.

In the next step we define $f_u : \mathbb{R} \rightarrow \mathbb{R}; x \mapsto u(x\mathbf{1}_d)$. Then f_u is strictly increasing and continuous and thus f_u^{-1} exists. Let μ be an arbitrary distribution on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with bounded support and $X \sim \mu$. Then

$$\rho(\|X\|_{d,\infty}\mathbf{1}_d) \leq \rho(X) \leq \rho(-\|X\|_{d,\infty}\mathbf{1}_d)$$

and hence

$$\begin{aligned} f_u(-\|X\|_{d,\infty}) &= \int u(x) \delta_{-\|X\|_{d,\infty}}\mathbf{1}_d(dx) \leq \int u(x) \mu(dx) \\ &\leq \int u(x) \delta_{\|X\|_{d,\infty}}\mathbf{1}_d(dx) = f_u(\|X\|_{d,\infty}). \end{aligned}$$

The intermediate value theorem now implies the existence of a constant $c(\mu) \in \mathbb{R}$ such that

$$f_u(c(\mu)) = \int u(x) \mu(dx) \iff c(\mu) = f_u^{-1} \left(\int u(x) \mu(dx) \right).$$

Finally, since $\delta_{c(\mu)} \mathbf{1}_d \approx \mu$, we have

$$\rho(X) = \rho(c(\mu) \mathbf{1}_d) = -c(\mu) = -f_u^{-1} \left(\int u(x) \mu(dx) \right) = -f_u^{-1} (\mathbb{E}_{\mathbb{P}} [u(X)]).$$

Hence, we have proved (4.1) (with $g \equiv -\text{id}$). Define

$$\psi_{\mathcal{H}}(X) := -f_u^{-1} (\mathbb{E}_{\mathbb{P}} [u(X) | \mathcal{H}]), \quad X \in L_d^{\infty}(\mathcal{F}),$$

then we have seen in the first part of the proof that $\psi_{\mathcal{H}}$ is a CRM which is strongly consistent with ρ . Moreover, $\psi_{\mathcal{H}}$ is normalized on constants. Thus it follows by Lemma 3.5 that $\rho_{\mathcal{H}} = \psi_{\mathcal{H}}$. If ρ and/or $\rho_{\mathcal{H}}$ are not normalized on constants, then considering the normalized CRMs $-f_{\rho}^{-1} \circ \rho$ and $-f_{\rho_{\mathcal{H}}}^{-1} \circ \rho_{\mathcal{H}}$, the result follows from $\rho = f_{\rho} \circ (-(-f_{\rho}^{-1} \circ \rho))$ and $\rho_{\mathcal{H}} = f_{\rho_{\mathcal{H}}} \circ (-(-f_{\rho_{\mathcal{H}}}^{-1} \circ \rho_{\mathcal{H}}))$, i.e. $g = f_{\rho}$ and $g_{\mathcal{H}} = f_{\rho_{\mathcal{H}}}$. \square

D Positive affine transformations of stochastic utilities

Proposition D.1 *Let $U_{\mathcal{H}}$ be the stochastic utility from Theorem 4.6 and let $\tilde{U}_{\mathcal{H}} : \text{Im } \Lambda_{\mathcal{H}} \rightarrow L^{\infty}(\mathcal{F})$ be another function which is strictly isotone, \mathcal{F} -local, fulfills the Lebesgue property and $\tilde{U}_{\mathcal{H}}(\text{Im } \Lambda_{\mathcal{H}} \cap L^{\infty}(\mathcal{H})) \subseteq L^{\infty}(\mathcal{H})$, such that*

$$\tilde{U}_{\mathcal{H}}^{-1} (\mathbb{E}_{\mathbb{P}} [\tilde{U}_{\mathcal{H}}(F) | \mathcal{H}]) = U_{\mathcal{H}}^{-1} (\mathbb{E}_{\mathbb{P}} [U_{\mathcal{H}}(F) | \mathcal{H}]), \quad \text{for all } F \in \text{Im } \Lambda_{\mathcal{H}}. \quad (\text{D.1})$$

Then $\tilde{U}_{\mathcal{H}}$ is an \mathcal{H} -measurable positive affine transformation of $U_{\mathcal{H}}$, i.e. there exist $\alpha, \beta \in L^{\infty}(\mathcal{H})$ with $\mathbb{P}(\alpha > 0) = 1$ such that $\tilde{U}_{\mathcal{H}}(F) = \alpha U_{\mathcal{H}}(F) + \beta$ for all $F \in \text{Im } \Lambda_{\mathcal{H}}$.

Proof We have seen in Theorem 4.6 that $U_{\mathcal{H}} \circ \Lambda_{\mathcal{H}} = u$, where u is strictly increasing and continuous. Thus

$$\mathcal{X} := \text{Im } U_{\mathcal{H}} = u(L_d^{\infty}(\mathcal{F})) \subseteq L^{\infty}(\mathcal{F})$$

and it follows that for all $F \in \mathcal{X}$ there exists a sequence of \mathcal{F} -simple random variables $(F_n)_{n \in \mathbb{N}} \subseteq \mathcal{X}$ such that $F_n \rightarrow F$ \mathbb{P} -a.s. Moreover, by the intermediate value theorem we can find for each $X, Y \in L_d^{\infty}(\mathcal{F})$ and $\lambda \in L^{\infty}(\mathcal{F})$ with $0 \leq \lambda \leq 1$ a random variable Z such that $\min\{-\|X\|_{d,\infty}, -\|Y\|_{d,\infty}\} \leq Z \leq \max\{\|X\|_{d,\infty}, \|Y\|_{d,\infty}\}$ and for all \mathbb{P} -almost all $\omega \in \Omega$

$$\lambda(\omega)u(X(\omega)) + (1 - \lambda)u(Y(\omega)) = u(Z(\omega) \mathbf{1}_d)$$

where $X(\cdot), Y(\cdot)$ and $\lambda(\cdot)$ are arbitrary representatives of X, Y and λ . Indeed, it can be shown by a measurable selection argument that Z can be chosen to be \mathcal{F} -measurable and hence \mathcal{X} is \mathcal{F} -conditionally convex in the sense that $\lambda F + (1 - \lambda)G \in \mathcal{X}$ for all $F, G \in \mathcal{X}$ and $\lambda \in L^{\infty}(\mathcal{F})$ with $0 \leq \lambda \leq 1$.

Next define the strictly isotone and \mathcal{F} -local function

$$V_{\mathcal{H}} : \mathcal{X} \rightarrow L^{\infty}(\mathcal{F}); \quad X \mapsto \tilde{U}_{\mathcal{H}} \left(U_{\mathcal{H}}^{-1}(F) \right),$$

that is $\tilde{U}_{\mathcal{H}} = V_{\mathcal{H}} \circ U_{\mathcal{H}}$. Moreover, it easily follows that $V_{\mathcal{H}}$ fulfills the Lebesgue property and $V_{\mathcal{H}}(\mathcal{X} \cap L^{\infty}(\mathcal{H})) \subseteq L^{\infty}(\mathcal{H})$. We show that $V_{\mathcal{H}}$ is an affine function, that is $V_{\mathcal{H}}(F) = \alpha F + \beta$

for all $F \in \mathcal{X}$, where $\alpha, \beta \in L^\infty(\mathcal{F})$. Note that affinity can be equivalently expressed via $V_{\mathcal{H}}(\lambda F + (1 - \lambda)G) = \lambda V_{\mathcal{H}}(F) + (1 - \lambda)V_{\mathcal{H}}(G)$ for all $F, G \in \mathcal{X}$ and $\lambda \in L^\infty(\mathcal{F})$ with $0 \leq \lambda \leq 1$.

We suppose that $V_{\mathcal{H}}$ is not affine, i.e. there are $F, G \in \mathcal{X}$ and $\lambda \in L^\infty(\mathcal{F})$ with $0 \leq \lambda \leq 1$ such that

$$\mathbb{P}(V_{\mathcal{H}}(\lambda F + (1 - \lambda)G) \neq \lambda V_{\mathcal{H}}(F) + (1 - \lambda)V_{\mathcal{H}}(G)) > 0. \tag{D.2}$$

First note that it suffices to assume that (D.2) holds for deterministic F, G and λ . To see this suppose that $V_{\mathcal{H}}$ is affine on deterministic values, but not on the whole of \mathcal{X} , i.e. (D.2) holds for some $F, G \in \mathcal{X}$ and $\lambda \in L^\infty(\mathcal{F})$ with $0 \leq \lambda \leq 1$. We know that there exist sequences of \mathcal{F} -simple functions $(F_n)_{n \in \mathbb{N}}, (G_n)_{n \in \mathbb{N}} \subset \mathcal{X} \cap \mathcal{S}$ and $(\lambda_n)_{n \in \mathbb{N}} \subset L^\infty(\mathcal{F}) \cap \mathcal{S}$ with $0 \leq \lambda_n \leq 1$ for all $n \in \mathbb{N}$ such that $F_n \rightarrow F, G_n \rightarrow G, \lambda_n \rightarrow \lambda$ \mathbb{P} -a.s., where \mathcal{S} was defined in the proof of Proposition B.1. Without loss of generality we might assume that $F_n = \sum_{i=1}^{k_n} F_i^n \mathbb{1}_{A_i^n}, G_n = \sum_{i=1}^{k_n} G_i^n \mathbb{1}_{A_i^n}$ and $\lambda_n = \sum_{i=1}^{k_n} \lambda_i^n \mathbb{1}_{A_i^n}$ have the same disjoint \mathcal{F} -partition $(A_i^n)_{i=1, \dots, k_n}$. By the \mathcal{F} -locality and Lebesgue property and since $F_i^n, G_i^n, \lambda_i^n \in \mathbb{R}$ for all $i = 1, \dots, k_n$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} V_{\mathcal{H}}(\lambda F + (1 - \lambda)G) &= \lim_{n \rightarrow \infty} V_{\mathcal{H}}(\lambda_n F_n + (1 - \lambda_n)G_n) \\ &= \lim_{n \rightarrow \infty} V_{\mathcal{H}}\left(\sum_{i=1}^{k_n} (\lambda_i^n F_i^n + (1 - \lambda_i^n)G_i^n) \mathbb{1}_{A_i^n}\right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} V_{\mathcal{H}}(\lambda_i^n F_i^n + (1 - \lambda_i^n)G_i^n) \mathbb{1}_{A_i^n} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} (\lambda_i^n V_{\mathcal{H}}(F_i^n) + (1 - \lambda_i^n)V_{\mathcal{H}}(G_i^n)) \mathbb{1}_{A_i^n} \\ &= \lim_{n \rightarrow \infty} \lambda_n V_{\mathcal{H}}(F_n) + (1 - \lambda_n)V_{\mathcal{H}}(G_n) \\ &= \lambda V_{\mathcal{H}}(F) + (1 - \lambda)V_{\mathcal{H}}(G), \end{aligned}$$

which contradicts (D.2). Moreover we assume that $0 < \lambda < 1$ since otherwise this would also contradict (D.2). Finally, we assume w.l.o.g. that

$$A := \{V_{\mathcal{H}}(\lambda F + (1 - \lambda)G) < \lambda V_{\mathcal{H}}(F) + (1 - \lambda)V_{\mathcal{H}}(G)\} \in \mathcal{H}$$

has positive probability. Next define $H_1 := F \mathbb{1}_A + G \mathbb{1}_{A^c}$ and $H_2 := G$, then $H_i \in \mathcal{X} \cap L^\infty(\mathcal{H}), i = 1, 2$ and by \mathcal{F} -locality of $V_{\mathcal{H}}$

$$V_{\mathcal{H}}(\lambda H_1 + (1 - \lambda)H_2) \leq \lambda V_{\mathcal{H}}(H_1) + (1 - \lambda)V_{\mathcal{H}}(H_2)$$

and the inequality is strict with positive probability.

Since $(\Omega, \mathbb{P}, \mathcal{F})$ is conditionally atomless given \mathcal{H} there exists a $B \in \mathcal{F}$ with $\mathbb{P}(B) = \lambda$ and which is independent of \mathcal{H} . Since $H_1, H_2 \in \mathcal{X}$ and \mathcal{X} is \mathcal{F} -conditionally convex

$$H := H_1 \mathbb{1}_B + H_2 \mathbb{1}_{B^c} \in \mathcal{X}.$$

Now by \mathcal{F} -locality of $V_{\mathcal{H}}$, $V_{\mathcal{H}}(\mathcal{X} \cap L^\infty(\mathcal{H})) \subseteq L^\infty(\mathcal{H})$ and $B \perp \mathcal{H}$ we get

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [V_{\mathcal{H}}(H) \mid \mathcal{H}] &= \mathbb{E}_{\mathbb{P}} [V_{\mathcal{H}}(H_1 \mathbb{1}_B + H_2 \mathbb{1}_{B^c}) \mid \mathcal{H}] \\ &= V_{\mathcal{H}}(H_1) \mathbb{E}_{\mathbb{P}} [\mathbb{1}_B \mid \mathcal{H}] + V_{\mathcal{H}}(H_2) \mathbb{E}_{\mathbb{P}} [\mathbb{1}_{B^c} \mid \mathcal{H}] \\ &= V_{\mathcal{H}}(H_1) \mathbb{E}_{\mathbb{P}} [\mathbb{1}_B] + V_{\mathcal{H}}(H_2) \mathbb{E}_{\mathbb{P}} [\mathbb{1}_{B^c}] \\ &= \lambda V_{\mathcal{H}}(H_1) + (1 - \lambda) V_{\mathcal{H}}(H_2) \\ &\geq V_{\mathcal{H}}(\lambda H_1 + (1 - \lambda) H_2) \\ &= V_{\mathcal{H}}(\mathbb{E}_{\mathbb{P}} [H_1 \mathbb{1}_B + H_2 \mathbb{1}_{B^c} \mid \mathcal{H}]) \\ &= V_{\mathcal{H}}(\mathbb{E}_{\mathbb{P}} [H \mid \mathcal{H}]), \end{aligned}$$

and the inequality is strict with positive probability. Moreover $\mathcal{X} = \text{Im } U_{\mathcal{H}}$ implies the existence of a $\tilde{H} \in \text{Im } \Lambda_{\mathcal{H}}$ such that $H = U_{\mathcal{H}}(\tilde{H})$. Finally we get

$$\begin{aligned} \tilde{U}_{\mathcal{H}}^{-1}(\mathbb{E}_{\mathbb{P}} [\tilde{U}_{\mathcal{H}}(\tilde{H}) \mid \mathcal{H}]) &= U_{\mathcal{H}}^{-1}(V_{\mathcal{H}}^{-1}(\mathbb{E}_{\mathbb{P}} [V_{\mathcal{H}}(U_{\mathcal{H}}(\tilde{H})) \mid \mathcal{H}])) \\ &= U_{\mathcal{H}}^{-1}(V_{\mathcal{H}}^{-1}(\mathbb{E}_{\mathbb{P}} [V_{\mathcal{H}}(H) \mid \mathcal{H}])) \\ &\geq U_{\mathcal{H}}^{-1}(V_{\mathcal{H}}^{-1}(V_{\mathcal{H}}(\mathbb{E}_{\mathbb{P}} [H \mid \mathcal{H}]))) \\ &= U_{\mathcal{H}}^{-1}(\mathbb{E}_{\mathbb{P}} [H \mid \mathcal{H}]) \\ &= U_{\mathcal{H}}^{-1}(\mathbb{E}_{\mathbb{P}} [U_{\mathcal{H}}(\tilde{H}) \mid \mathcal{H}]), \end{aligned}$$

and the inequality is strict with positive probability, since $\tilde{U}_{\mathcal{H}}^{-1}$ and $U_{\mathcal{H}}^{-1}$ are strictly isotone (c.f. Lemma A.1). Thus we have the desired contradiction of (D.1) and hence $V_{\mathcal{H}}$ is affine, i.e. $V_{\mathcal{H}}(F) = \alpha F + \beta$ for all $F \in \mathcal{X}$, where $\alpha, \beta \in L^\infty(\mathcal{F})$. Moreover, since we know that $V_{\mathcal{H}}(x) \in L^\infty(\mathcal{H})$ for all $x \in \mathbb{R} \cap \mathcal{X}$, we obtain that α, β are actually \mathcal{H} -measurable. That $\alpha > 0$ follows immediately from the fact that $\tilde{U}_{\mathcal{H}}, U_{\mathcal{H}}^{-1}$ are strictly isotone. \square

E Proof of Proposition 5.9

Lemma E.1 *Let $u : \mathbb{R}^d \rightarrow \mathbb{R}$ be a deterministic utility, i.e. u is strictly increasing and continuous, and let \mathcal{G} and \mathcal{H} be a sub- σ -algebras of \mathcal{F} such that $\mathcal{G} \subseteq \mathcal{H}$. Then*

$$\mathbb{E}_{\mathbb{P}} [u(L_d^\infty(\mathcal{H})) \mid \mathcal{G}] = u(L_d^\infty(\mathcal{G})).$$

Proof "≥": Obvious. "≤": Define the CRM $\rho_{\mathcal{G}} : L_d^\infty(\mathcal{H}) \rightarrow L^\infty(\mathcal{G}); X \mapsto -\mathbb{E}_{\mathbb{P}} [u(X) \mid \mathcal{G}]$. By Lemma A.2 it follows that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [u(L_d^\infty(\mathcal{H})) \mid \mathcal{G}] &= -\rho_{\mathcal{G}}(L_d^\infty(\mathcal{H})) = -f_{\rho_{\mathcal{G}}}(L^\infty(\mathcal{G})) = \mathbb{E}_{\mathbb{P}} [u(L^\infty(\mathcal{G}) \mathbf{1}_d) \mid \mathcal{G}] \\ &\subseteq \mathbb{E}_{\mathbb{P}} [u(L_d^\infty(\mathcal{G})) \mid \mathcal{G}] = u(L_d^\infty(\mathcal{G})). \end{aligned}$$

\square

Lemma E.2 *For an arbitrary $\mathcal{T} \in \mathcal{I}$ let $u_{\mathcal{T}} : \mathbb{R}^d \rightarrow \mathbb{R}$ be a deterministic utility and define $\mathcal{X}_{\mathcal{H}} := u_{\mathcal{T}}(L_d^\infty(\mathcal{H}))$ for all $\mathcal{H} \in \mathcal{E}(\mathcal{T})$. Moreover, let $p_{\mathcal{H}} : \mathcal{X}_{\mathcal{H}} \rightarrow L^\infty(\mathcal{H})$ be functions such that $p_{\mathcal{H}}$ is \mathcal{H} -local, strictly isotone and fulfills the Lebesgue-property. If for all $\mathcal{G}, \mathcal{H} \in \mathcal{E}(\mathcal{T})$ with $\mathcal{G} \subseteq \mathcal{H}$ and \mathcal{H} atomless it holds that*

$$p_{\mathcal{G}}(\mathbb{E}_{\mathbb{P}} [F \mid \mathcal{G}]) = \mathbb{E}_{\mathbb{P}} [p_{\mathcal{H}}(F) \mid \mathcal{G}] \text{ for all } F \in \mathcal{X}_{\mathcal{H}}, \tag{E.1}$$

then

$$p_{\mathcal{H}}(F) = aF + \beta_{\mathcal{H}},$$

where $a \in \mathbb{R}^+ \setminus \{0\}$ and $\beta_{\mathcal{H}} \in L^\infty(\mathcal{H})$ such that $\mathbb{E}_{\mathbb{P}}[\beta_{\mathcal{H}} | \mathcal{G}] = \beta_{\mathcal{G}}$.

Note that (E.1) is well-defined by Lemma E.1.

Proof Firstly, we consider the case where \mathcal{G} is the trivial σ -algebra. We write $p := p_{\{\Omega, \emptyset\}}$. Note that, since p is a deterministic function, $p(\mathbb{E}_{\mathbb{P}}[F])$ is law-invariant and thus by (E.1) also $\mathbb{E}_{\mathbb{P}}[p_{\mathcal{H}}(F)]$.

Now suppose that there exist $x, y \in \mathcal{X} := \mathcal{X}_{\{\Omega, \emptyset\}}$ with $p_{\mathcal{H}}(x) - p_{\mathcal{H}}(y) \notin \mathbb{R}$, i.e. there exists a $c \in \mathbb{R}$ such that $\mathbb{P}(p_{\mathcal{H}}(x) \leq p_{\mathcal{H}}(y) + c) \in (0, 1)$. Since \mathcal{H} is an atomless space we can choose $A_1, A_2, A_3 \in \mathcal{H}$ with

$$\mathbb{P}(A_1) = \mathbb{P}(A_2) := q > 0$$

such that

$$A_1 \subseteq \{p_{\mathcal{H}}(x) \leq p_{\mathcal{H}}(y) + c\}, A_2 \subseteq \{p_{\mathcal{H}}(x) > p_{\mathcal{H}}(y) + c\}, A_3 := (A_1 \cup A_2)^C.$$

Moreover, we define

$$F_1 := x \mathbb{1}_{A_1} + y \mathbb{1}_{A_2} + x \mathbb{1}_{A_3} \quad \text{and} \quad F_2 := y \mathbb{1}_{A_1} + x \mathbb{1}_{A_2} + x \mathbb{1}_{A_3}.$$

Obviously $F_1, F_2 \sim q\delta_y + (1 - q)\delta_x$, that is $F_1 \stackrel{d}{=} F_2$. However, since $p_{\mathcal{H}}$ is \mathcal{H} -local, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[p_{\mathcal{H}}(F_1)] + cq &= \mathbb{E}_{\mathbb{P}}[p_{\mathcal{H}}(x)\mathbb{1}_{A_1}] + \mathbb{E}_{\mathbb{P}}[(p_{\mathcal{H}}(y) + c)\mathbb{1}_{A_2}] + \mathbb{E}_{\mathbb{P}}[p_{\mathcal{H}}(x)\mathbb{1}_{A_3}] \\ &< \mathbb{E}_{\mathbb{P}}[(p_{\mathcal{H}}(y) + c)\mathbb{1}_{A_1}] + \mathbb{E}_{\mathbb{P}}[p_{\mathcal{H}}(x)\mathbb{1}_{A_2}] + \mathbb{E}_{\mathbb{P}}[p_{\mathcal{H}}(x)\mathbb{1}_{A_3}] \\ &= \mathbb{E}_{\mathbb{P}}[p_{\mathcal{H}}(F_2)] + cq, \end{aligned}$$

which contradicts the law-invariance of $F \mapsto \mathbb{E}_{\mathbb{P}}[p_{\mathcal{H}}(F)]$.

Hence we have that $p_{\mathcal{H}}(x) - p_{\mathcal{H}}(y) \in \mathbb{R}$ for all $x, y \in \mathcal{X}$. Choose an arbitrary $\tilde{x} \in \mathcal{X}$, and let

$$a(x) := p_{\mathcal{H}}(x) - p_{\mathcal{H}}(\tilde{x}), \quad x \in \mathcal{X},$$

so $a : \mathcal{X} \rightarrow \mathbb{R}$. Define $\tilde{\beta}_{\mathcal{H}} := p_{\mathcal{H}}(\tilde{x}) \in L^\infty(\mathcal{H})$, then $p_{\mathcal{H}}(x) = a(x) + \tilde{\beta}_{\mathcal{H}}$. The function a is continuous, since otherwise there would exist a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathcal{X}$ with $x_n \rightarrow x \in \mathcal{X}$, but $a(x_n) \not\rightarrow a(x)$ and the Lebesgue-property would imply the contradiction

$$p_{\mathcal{H}}(x) = \lim_{n \rightarrow \infty} p_{\mathcal{H}}(x_n) = \lim_{n \rightarrow \infty} a(x_n) + \tilde{\beta}_{\mathcal{H}} \neq a(x) + \tilde{\beta}_{\mathcal{H}} = p_{\mathcal{H}}(x).$$

Let $F \in \mathcal{X}_{\mathcal{H}}$. Since the \mathcal{H} -measurable simple random vectors are dense in $L_d^\infty(\mathcal{H})$ and by the definition of $\mathcal{X}_{\mathcal{H}}$ there exists a sequence of \mathcal{H} -measurable simple random variables $(F_n)_{n \in \mathbb{N}} \subset \mathcal{X}_{\mathcal{H}} \cap \mathcal{S}$ with $F_n = \sum_{i=1}^{k_n} x_i^n \mathbb{1}_{A_i^n} \rightarrow F$ \mathbb{P} -a.s. Thus

$$\begin{aligned} p_{\mathcal{H}}(F) &= \lim_{n \rightarrow \infty} p_{\mathcal{H}}(F_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} p_{\mathcal{H}}(x_i^n) \mathbb{1}_{A_i^n} = \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} a(x_i^n) \mathbb{1}_{A_i^n} + \tilde{\beta}_{\mathcal{H}} \\ &= \lim_{n \rightarrow \infty} a\left(\sum_{i=1}^{k_n} x_i^n \mathbb{1}_{A_i^n}\right) + \tilde{\beta}_{\mathcal{H}} = \lim_{n \rightarrow \infty} a(F_n) + \tilde{\beta}_{\mathcal{H}} = a(F) + \tilde{\beta}_{\mathcal{H}}. \end{aligned}$$

The function $\mathcal{X}_{\mathcal{H}} \ni F \mapsto \mathbb{E}_{\mathbb{P}} [F]$ induces a preference relation on $\mathcal{M} := \{\mu : \exists F \in \mathcal{X}_{\mathcal{H}} \text{ such that } F \sim \mu\}$ via

$$\mu \succsim \nu \iff \mathbb{E}_{\mathbb{P}} [F] \geq \mathbb{E}_{\mathbb{P}} [G], F \sim \mu, G \sim \nu.$$

Moreover the function $x \mapsto p^{-1}(x + \mathbb{E}[\tilde{\beta}_{\mathcal{H}}])$ is strictly increasing and by (E.1)

$$\mathbb{E}_{\mathbb{P}} [F] = p^{-1}(\mathbb{E}_{\mathbb{P}} [p_{\mathcal{H}}(F)]) = p^{-1}(\mathbb{E}_{\mathbb{P}} [a(F)] + \mathbb{E}[\tilde{\beta}_{\mathcal{H}}]).$$

Thus $\mathbb{E}_{\mathbb{P}} [a(F)]$ is another affine numerical representation of \succsim . It is well-known that the affine numerical representation of \succsim is unique up to a positive affine transformation (see e.g. Föllmer and Schied [15, Theorem 2.21]), i.e. there exist $\tilde{a}, b \in \mathbb{R}, \tilde{a} > 0$ such that $\mathbb{E}_{\mathbb{P}} [a(F)] = \tilde{a}\mathbb{E}_{\mathbb{P}} [F] + b$ for all $F \in \mathcal{X}_{\mathcal{H}}$. In particular this implies that for all $x \in \mathcal{X}$

$$a(x) = \mathbb{E}_{\mathbb{P}} [a(x)] = a\mathbb{E}_{\mathbb{P}} [x] + b = \tilde{a}x + b.$$

By setting $b + \tilde{\beta}_{\mathcal{H}} =: \beta_{\mathcal{H}} \in L^{\infty}(\mathcal{H})$ we get for all $F \in \mathcal{X}_{\mathcal{H}}$ that

$$p_{\mathcal{H}}(F) = a(F) + \tilde{\beta}_{\mathcal{H}} = \tilde{a}F + b + \tilde{\beta}_{\mathcal{H}} = \tilde{a}F + \beta_{\mathcal{H}}.$$

Finally we obtain by (E.1) that for every $\mathcal{G} \subseteq \mathcal{H}$ and for all $F \in \mathcal{X}_{\mathcal{G}}$

$$p_{\mathcal{G}}(F) = p_{\mathcal{G}}(\mathbb{E}_{\mathbb{P}} [F | \mathcal{G}]) = \mathbb{E}_{\mathbb{P}} [p_{\mathcal{H}}(F) | \mathcal{G}] = aF + \mathbb{E}_{\mathbb{P}} [\beta_{\mathcal{H}} | \mathcal{G}],$$

which proves the martingale property of $(\beta_{\mathcal{G}})_{\mathcal{G} \subseteq \mathcal{H}}$.

Proof of Proposition 5.9 Let $(\rho_{\mathcal{H}, \mathcal{T}})_{(\mathcal{H}, \mathcal{T}) \in \mathcal{E}}$ be a strongly consistent family such that (5.2) holds for all $(\mathcal{H}, \mathcal{T}) \in \mathcal{E}$, i.e.

$$\rho_{\mathcal{H}, \mathcal{T}}(X) = f_{\rho_{\mathcal{H}, \mathcal{T}}} \left(f_{u_{\mathcal{T}}}^{-1}(\mathbb{E}_{\mathbb{P}} [u_{\mathcal{T}}(X) | \mathcal{H}]) \right), \text{ for all } X \in L_d^{\infty}(\mathcal{T}),$$

We define the functions

$$h_{\mathcal{H}, \mathcal{T}} : u_{\mathcal{T}}(L_d^{\infty}(\mathcal{H})) \rightarrow L^{\infty}(\mathcal{H}); F \mapsto f_{\rho_{\mathcal{H}, \mathcal{T}}} \circ f_{u_{\mathcal{T}}}^{-1}(F)$$

and

$$p_{\mathcal{H}, \mathcal{T}_1, \mathcal{T}_2} : u_{\mathcal{T}_1}(L_d^{\infty}(\mathcal{H})) \rightarrow L^{\infty}(\mathcal{H}); F \mapsto h_{\mathcal{H}, \mathcal{T}_2}^{-1} \circ h_{\mathcal{H}, \mathcal{T}_1}(F).$$

By strong consistency, we obtain for $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{T}_1 \cap \mathcal{T}_2, X \in L_d^{\infty}(\mathcal{T}_1)$ and $F := \mathbb{E}_{\mathbb{P}} [u_{\mathcal{T}_1}(X) | \mathcal{H}]$ that

$$\begin{aligned} p_{\mathcal{G}, \mathcal{T}_1, \mathcal{T}_2}(\mathbb{E}_{\mathbb{P}} [F | \mathcal{G}]) &= h_{\mathcal{G}, \mathcal{T}_2}^{-1} (h_{\mathcal{G}, \mathcal{T}_1}(\mathbb{E}_{\mathbb{P}} [\mathbb{E}_{\mathbb{P}} [u_{\mathcal{T}_1}(X) | \mathcal{H}] | \mathcal{G}])) \\ &= h_{\mathcal{G}, \mathcal{T}_2}^{-1} (\rho_{\mathcal{G}, \mathcal{T}_1}(X)) \\ &= h_{\mathcal{G}, \mathcal{T}_2}^{-1} \left(\rho_{\mathcal{G}, \mathcal{T}_2} \left(f_{\rho_{\mathcal{H}, \mathcal{T}_2}}^{-1} (\rho_{\mathcal{H}, \mathcal{T}_1}(X)) \mathbf{1}_d \right) \right) \\ &= \mathbb{E}_{\mathbb{P}} \left[h_{\mathcal{H}, \mathcal{T}_2}^{-1} (h_{\mathcal{H}, \mathcal{T}_1}(\mathbb{E}_{\mathbb{P}} [u_{\mathcal{T}_1}(X) | \mathcal{H}])) \mid \mathcal{G} \right] \\ &= \mathbb{E}_{\mathbb{P}} [p_{\mathcal{H}, \mathcal{T}_1, \mathcal{T}_2}(F) | \mathcal{G}]. \end{aligned} \tag{E.2}$$

By Lemma E.2 (E.2) is fulfilled, if and only if

$$p_{\mathcal{H}, \mathcal{T}_1, \mathcal{T}_2}(F) = a_{\mathcal{T}_1, \mathcal{T}_2}F + b_{\mathcal{H}, \mathcal{T}_1, \mathcal{T}_2}, \text{ for all } F \in u_{\mathcal{T}_1}(L_d^{\infty}(\mathcal{H})),$$

where $a_{\mathcal{T}_1, \mathcal{T}_2} \in \mathbb{R}^+ \setminus \{0\}, b_{\mathcal{H}, \mathcal{T}_1, \mathcal{T}_2} \in L^{\infty}(\mathcal{H})$ and $\mathbb{E}_{\mathbb{P}} [b_{\mathcal{H}, \mathcal{T}_1, \mathcal{T}_2} | \mathcal{G}] = b_{\mathcal{G}, \mathcal{T}_1, \mathcal{T}_2}$ for all $\mathcal{G} \in \mathcal{I}$ with $\mathcal{G} \subseteq \mathcal{H}$. Thus

$$h_{\mathcal{H}, \mathcal{T}_1}(F) = h_{\mathcal{H}, \mathcal{T}_2}(a_{\mathcal{T}_1, \mathcal{T}_2}F + b_{\mathcal{H}, \mathcal{T}_1, \mathcal{T}_2}), \text{ } F \in u_{\mathcal{T}_1}(L_d^{\infty}(\mathcal{H})),$$

which implies that

$$\rho_{\mathcal{H}, \mathcal{T}_1}(X) = f_{\rho_{\mathcal{H}, \mathcal{T}_2}} \left(f_{u_{\mathcal{T}_2}}^{-1} \left(a_{\mathcal{T}_1, \mathcal{T}_2} \mathbb{E}_{\mathbb{P}} [u_{\mathcal{T}_1}(X) \mid \mathcal{H}] + b_{\mathcal{H}, \mathcal{T}_1, \mathcal{T}_2} \right) \right).$$

References

1. Acciaio, B., Penner, I.: Dynamic risk measures. In: Nunno, J.Di, Øksendal, B. (eds.) *Advanced Mathematical Methods for Finance*, Chapter 1, pp. 11–44. Springer, Berlin (2011)
2. Acharya, V., Pedersen, L., Philippon, T., Richardson, M.: Measuring systemic risk. Available at SSRN 1573171 (2010)
3. Adrian, T., Brunnermeier, M.K.: CoVaR. Technical report, National Bureau of Economic Research (2011)
4. Bion-Nadal, J.: Dynamic risk measures: time consistency and risk measures from BMO martingales. *Finance Stoch.* **12**(2), 219–244 (2008)
5. Brunnermeier, M.K., Cheridito, P.: Measuring and allocating systemic risk. Available at SSRN 2372472 (2014)
6. Chen, C., Iyengar, G., Moallemi, C.C.: An axiomatic approach to systemic risk. *Manag. Sci.* **59**(6), 1373–1388 (2013)
7. Cheridito, P., Delbaen, F., Kupper, M.: Dynamic monetary risk measures for bounded discrete-time processes. *Electron. J. Probab.* **11**(3), 57–106 (2006)
8. Cheridito, P., Kupper, M.: Composition of time-consistent dynamic monetary risk measures in discrete time. *Int. J. Theor. Appl. Finance* **14**(01), 137–162 (2011)
9. Cont, R., Moussa, A., Santos, E.B.: Network structure and systemic risk in banking systems. In: Fouque, J.-P., Langsam, J.A. (eds.) *Handbook on Systemic Risk*, Chapter 13, pp. 327–368. Cambridge University Press, Cambridge (2013)
10. Detlefsen, K., Scandolo, G.: Conditional and dynamic convex risk measures. *Finance Stoch.* **9**(4), 539–561 (2005)
11. Farkas, E.W., Koch Medina, P., Munari, C.-A.: Measuring risk with multiple eligible assets. *Math. Financ. Econ.* **9**(1), 3–27 (2015)
12. Föllmer, H.: Spatial risk measures and their local specification: the locally law-invariant case. *Stat. Risk Model.* **31**(1), 79–103 (2014)
13. Föllmer, H., Klüppelberg, C.: Spatial risk measures: local specification and boundary risk. In: Crisan, D., Hambly, B., Zariphopoulou, T. (eds.) *Stochastic Analysis and Applications 2014: In Honour of Terry Lyons*. Springer, Berlin (2014)
14. Föllmer, H., Penner, I.: Convex risk measures and the dynamics of their penalty functions. *Stat. Decis.* **24**(1), 61–96 (2006)
15. Föllmer, H., Schied, A.: *Stochastic Finance: An Introduction in Discrete Time*, 3rd edn. De Gruyter, Berlin (2011)
16. Frittelli, M., Maggis, M.: Conditional certainty equivalent. *Int. J. Theor. Appl. Finance* **14**(01), 41–59 (2011)
17. Hoffmann, H., Meyer-Brandis, T., Svindland, G.: Risk-consistent conditional systemic risk measures. *Stoch. Process. Appl.* **126**(7), 2014–2037 (2016)
18. Kromer, E., Overbeck, L., Zilch, K.: Systemic risk measures on general measurable spaces. *Math. Methods Oper. Res.* **84**(2), 323–357 (2016)
19. Kromer, E., Overbeck, L., Zilch, K.A.: Dynamic systemic risk measures for bounded discrete-time processes. Available at SSRN 2469475 (2014)
20. Kupper, M., Schachermayer, W.: Representation results for law invariant time consistent functions. *Math. Financ. Econ.* **2**(3), 189–210 (2009)
21. Penner, I.: Dynamic convex risk measures: time consistency, prudence, and sustainability. Ph. D. thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät II (2007)
22. Revuz, D., Yor, M.: *Continuous Martingales and Brownian Motion*, vol. 293. Springer, Berlin (1999)
23. Tutsch, S.: Konsistente und konsequente dynamische Risikomaße und das Problem der Aktualisierung. Ph.D. thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät II (2007)
24. Weber, S.: Distribution-invariant risk measures, information, and dynamic consistency. *Math. Finance* **16**(2), 419–442 (2006)