Math Finan Econ (2015) 9:215–230
DOI 10.1007/s11579-015-0142-8

A note on utility-based pricing

Mark H. A. Davis · **Daisuke Yoshikawa**

Received: 12 August 2014 / Accepted: 12 February 2015 / Published online: 18 February 2015

Abstract In this short note, we discuss the features of utility-based pricing and indifference pricing. To do this, we introduce a utility-based curve that simply and simultaneously allows for a discussion of the graphical features of both prices. We also clarify features of these prices form an economics point of view; the introduction of this utility-based curve enables us to simply discuss the property of the partial equilibrium of the random endowment. We also discuss the availability of the analysis using income and substitution effects to clarify the quality of economic goods. This analysis is well-known in the context of economics. A utility-based curve shows us the impossibility of this analysis. In a sense, the implicit purpose of this paper is to show the limitations of a utility-based and indifference framework.

Keywords Utility-based price · Indifference pricing · Exponential utility · Utility-based curve · Partial equilibrium · Income and substitution effects

JEL Classification D52 · G13

1 Introduction

This paper discusses features of utility-based pricing and indifference pricing. In a complete market, the non-arbitrage principle is sufficient for asset pricing. When pricing a random endowment in an incomplete market, it is insufficient. Hence, another principle is required. Utility maximization is one of the most persuasive principles. Therefore, many authors have intensively developed the framework based on the utility maximization

M. H. A. Davis
Department of Mathematics, Imperial College London, London SW7 2AZ, UK
e-mail: mark.davis@imperial.ac.uk

D. Yoshikawa (✉)
Department of Business Administration, Hokkai-Gakuen University,
4-1-40 Asahimachi, Toyohira Ward, Sapporo, Hokkaido, Japan
e-mail: yoshikawa@ba.hokkai-s-u.ac.jp

principle [1,4–10,12,14,16–18]. Utility-based pricing and indifference pricing are two major frameworks that are based on the utility optimization problem; however, the settings differ from one another. Utility-based pricing is given by the problem where, for a fixed price of the random endowment, each investor maximizes his expected utility by optimizing the quantity of the random endowment. That is, this quantity is the solution to the optimization problem. The utility-based price is given as the price that is consistent with this optimized quantity. In contrast, a utility indifference framework is given by a problem in which the price of the random endowment is defined by the threshold price where the expected utility is constant for a given quantity of random endowment for selling (or buying).

This paper points out some features of these methods. For this purpose, we graphically demonstrate the relationship between these two frameworks using a utility-based curve which is a tool for simply and simultaneously depicting utility-based prices and utility indifference prices. The introduction of this utility-based curve enables us to simply discuss the property of the partial equilibrium of the random endowment. However, this partial equilibrium shows us that any trade will not be completed in this market. We also discuss the availability of the analysis using income and substitution effects to clarify the quality of economic goods. This analysis is well-known in the context of economics. A utility-based curve shows us the impossibility of this analysis. In a sense, the implicit purpose of this paper is to show the limitations of a utility-based and indifference framework and, hence, introduce our next paper [2].

The remainder of this paper is divided into four sections. Section 2 sets up the model. In Sect. 3, we define the utility-based curve and discuss the features of utility-based pricing and indifference pricing. Section 4 discusses the features of random endowment using utility-based pricing and indifference pricing to demonstrate three key points. One point is to show the positive effect of the introduction of random endowments in the financial market. The second is to deduce the partial equilibrium under the framework of a utility-based price. The third is to show the impossibility of analyzing the price change effect on random endowments. The final section provides concluding remarks.

2 The model

The mathematical framework is given by the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}$, $\mathcal{F} := \mathcal{F}_T$, and $\mathcal{F}_0$ is trivial. The stochastic process $X \in \mathbb{R}^d$ is defined as semimartingale, and the expected value of X is given by the probability measure P. We also assume that X is $\mathbb{F}$-locally bounded. Consider the $\mathcal{F}_T$-measurable random variable B which will generate some payoff at time T. Similarly to [5] and [1], we assume that the random variable B is bounded from below with $\mathbb{E}[e^{(\alpha+\epsilon)B}] < \infty$ and $\mathbb{E}[e^{-\epsilon B}] < \infty$ for some fixed $\alpha, \epsilon \in (0, \infty)$.

We consider the investor who manages his initial wealth to maximize his expected utility defined by the terminal wealth (implicitly indicating consumption at maturity). He can distribute his initial wealth to the risk-free asset as a numéraire, the discounted risky asset X, and the random endowment B. In this case, it is natural to consider the maximization problem of expected utility as follows:

$$\sup_{\theta \in \Theta, q \in \mathbb{R}} \mathbb{E}\left[U\left(x - pq + \int_0^T \theta_t^\top dX_t + Bq\right)\right], \tag{1}$$

where pq is the cost and revenue for random endowment B with amount $q \in \mathbb{R}$. Let $\theta := \{\theta_t;\ t \in [0, T]\} \in \Theta$ be an $\mathbb{R}^d$-valued admissible trading strategy[1] and Θ be the set of

[1] That is, $\theta \in L(X)$ and for some constant $c \in \mathbb{R}$, we have $\int_0^t \theta_s^\top dX_s \ge c,\ t \in [0, T]$.

X-integrable and predictable processes. We also assume that the wealth process $x - pq + \int_0^T \theta_t^\top dX_t + Bq$ is uniformly bounded from below for $\theta \in \Theta$. For problem (1), we call the price p as utility-based price[2]. Because the price p is given exogenously, problem (1) is not fundamentally a pricing framework. Rather, it shows the relationship between the optimal amount of q and the utility-based price. However, because the utility-based price is, indeed, a price, it has a consistent relationship with the pricing framework. The traditional pricing framework of mathematical finance is based on the non-arbitrage condition, indicating that the prices of financial assets are given by the martingale measure. Therefore, we define $\mathcal{M}$ as a set of Θ-martingale measures satisfying $H(Q|P) < \infty$, where $H(Q|P) := \int \frac{dQ}{dP} \ln \frac{dQ}{dP} dP$ is the relative entropy of $Q \in \mathcal{M}$ with respect to P, which is always non negative (c.f. Theorem 1.4.1 of [11]). Hereafter, we assume that

$$\mathcal{M} \neq \emptyset.$$

We write the solution of $\inf_{Q \in \mathcal{M}} H[Q|P]$ as $Q^0 \in \mathcal{M}$, which we call the minimal entropy martingale measure (MEMM). The existence of MEMM is shown by Proposition 2.2 (1) of [1]. We later show that the MEMM characterizes the utility-based price at $q = 0$.

Hereafter, we consider problem (1) under the exponential utility with risk-aversion $\gamma \in \mathbb{R}_+$ and,

$$U(x) = -e^{-\gamma x}.$$

The exponential utility is accepted as consistent with MEMM.

3 On the utility-based price

Section 3 is devoted to show the relationship between utility-based and indifference pricing. For this purpose, we introduce a tool, called utility-based curve, which gives graphical features of utility-based pricing and indifference pricing and makes available to discuss them simultaneously. In Sect. 3.1, we derive utility-based curve and give the utility-based pricing on the utility-based curve. Section 3.2 shows the relationship between utility-based and indifference pricing which is also given on the utility-based curve.

3.1 Graphical feature of utility-based pricing

First, we define $u(\theta, q; \gamma) := \ln \mathbb{E}\Big[e^{-\gamma\left(\int_0^T \theta_t^\top dX_t + Bq\right)}\Big]$. Theorem 4.1 of [12] shows that the consistency of the solution to $\inf_{\theta \in \Theta} \ln \mathbb{E}\Big[e^{-\gamma\left(\int_0^T \theta_t^\top dX_t + Bq\right)}\Big]$ coincides with the solution to $\sup_{\theta \in \Theta} \mathbb{E}\Big[-e^{-\gamma\left(\int_0^T \theta_t^\top dX_t + Bq\right)}\Big]$ for a given q (more precisely, Theorem 4.1 uses the conjugate function of the utility maximization problem). We then denote the solution to this problem by θ^q (the existence of the solution θ^q for the exponential utility have been established by the convexity of the expected utility). Therefore, we can define the function $u(q; \gamma) := \inf_{\theta \in \Theta}\{u(\theta, q; \gamma)\} = u(\theta^q, q; \gamma)$. The continuous differentiality of $u(q; \gamma)$ is also given by the Theorem 5.1 of [12]. We call this function u the utility-based curve, which has useful properties, as shown below.

[2] We refer to [9] and [13].

Lemma 1 *The utility-based curve is a convex function.*

Proof For $q, q' \in \mathbb{R}$ and $0 \le k \le 1$,

$$\begin{aligned} &ku(q;\gamma) + (1-k)u(q';\gamma) = ku(\theta^q, q;\gamma) + (1-k)u(\theta^{q'}, q';\gamma) \\ &= k \ln \mathbb{E}\left[e^{-\gamma\left(\int_0^T (\theta_t^q)^\top dX_t + Bq\right)}\right] + (1-k)\mathbb{E}\left[e^{-\gamma\left(\int_0^T (\theta_t^{q'})^\top dX_t + Bq'\right)}\right] \\ &= \ln \mathbb{E}\left[e^{-\gamma\left(\int_0^T (\theta_t^q)^\top dX_t + Bq\right)}\right]^k \mathbb{E}\left[e^{-\gamma\left(\int_0^T (\theta_t^{q'})^\top dX_t + Bq'\right)}\right]^{1-k} \\ &\ge \ln \mathbb{E}\left[e^{-k\gamma\left(\int_0^T (\theta_t^q)^\top dX_t + Bq\right)} e^{-(1-k)\gamma\left(\int_0^T (\theta_t^{q'})^\top dX_t + Bq'\right)}\right] \\ &= \ln \mathbb{E}\left[e^{-\gamma\left(\int_0^T (k\theta_t^q + (1-k)\theta_t^{q'})^\top dX_t + B(kq+(1-k)q')\right)}\right] \\ &= u(k\theta^q + (1-k)\theta^{q'}, kq + (1-k)q';\gamma) \\ &\ge u(\theta^{kq+(1-k)q'}, kq + (1-k)q';\gamma) = u(kq + (1-k)q';\gamma). \end{aligned}$$

We use Hölder's inequality on line 4. □

Remark 1 Even if the random endowment B is dependent on the amount q, the above lemma holds by introducing $b(q) := qB(q)$ which is assumed to be convex function of q. By this, it holds that

$$\begin{aligned} &ku(q;\gamma) + (1-k)u(q';\gamma) \\ &\ge \ln \mathbb{E}\left[e^{-\gamma\left(\int_0^T (k\theta_t^q + (1-k)\theta_t^{q'})^\top dX_t + (kB(q)q + (1-k)q'B(q'))\right)}\right] \\ &\ge \ln \mathbb{E}\left[e^{-\gamma\left(\int_0^T (k\theta_t^q + (1-k)\theta_t^{q'})^\top dX_t + B(kq+(1-k)q')(kq+(1-k)q')\right)}\right] \\ &= u(k\theta^q + (1-k)\theta^{q'}, kq + (1-k)q';\gamma). \end{aligned}$$

This leads to the statement of Lemma 1.

Lemma 2 *For all $q \in \mathbb{R}$, $u(\theta^{\tilde{q}}, q;\gamma) \ge u(q;\gamma)$, where $\tilde{q} \in \mathbb{R}$, and this utility-based curve $u(q;\gamma)$ makes contact with $u(\theta^{\tilde{q}}, q;\gamma)$ at $\tilde{q} = q$.*

Proof From the definition of θ^q, we have $u(\theta^q, q;\gamma) \le u(\theta, q;\gamma)$ for all $\theta \in \Theta$. Equality holds if and only if $\theta = \theta^q$, because of the convexity of $u(q;\gamma) = u(\theta^q, q;\gamma)$, where $u(\theta^q, q;\gamma)$ is uniquely given (see Theorem 5.1 of [12]). By the smoothness of $u(q;\gamma)$ and $u(\theta^{\tilde{q}}, q;\gamma)$, the utility-based curve $u(q;\gamma)$ makes contact with $u(\theta^{\tilde{q}}, q;\gamma)$ at $\tilde{q} = q$. □

These lemmas allow us to develop a graphical image of the utility-based curve and function $u(\theta^q, \tilde{q};\gamma)$. Figure 1 depicts two curves $u(\theta^q, \tilde{q};\gamma)$, denoted by $u(\theta^{q^1}, q;\gamma)$ and $u(\theta^{q^2}, q;\gamma)$, and the utility-based curve $u(q;\gamma)$, where θ^{q^1} is the optimal solution of the problem $\inf_{\theta\in\Theta} \ln \mathbb{E}\left[e^{-\gamma\left(\int_0^T \theta_t^\top dX_t + Bq^1\right)}\right]$. Similarly, θ^{q^2} is optimal for $\inf_{\theta\in\Theta} \ln \mathbb{E}\left[e^{-\gamma\left(\int_0^T \theta_t^\top dX_t + Bq^2\right)}\right]$. The utility-based curve shows the relationship between the amount q and optimal strategy θ^q. Therefore, $u(\theta^{q^1}, q;\gamma)$ and $u(\theta^{q^2}, q;\gamma)$ touch the utility-based curve only at q^1 and q^2, and $u(\theta^{q^1}, q;\gamma)$ and $u(\theta^{q^2}, q;\gamma)$ are placed above the utility-based

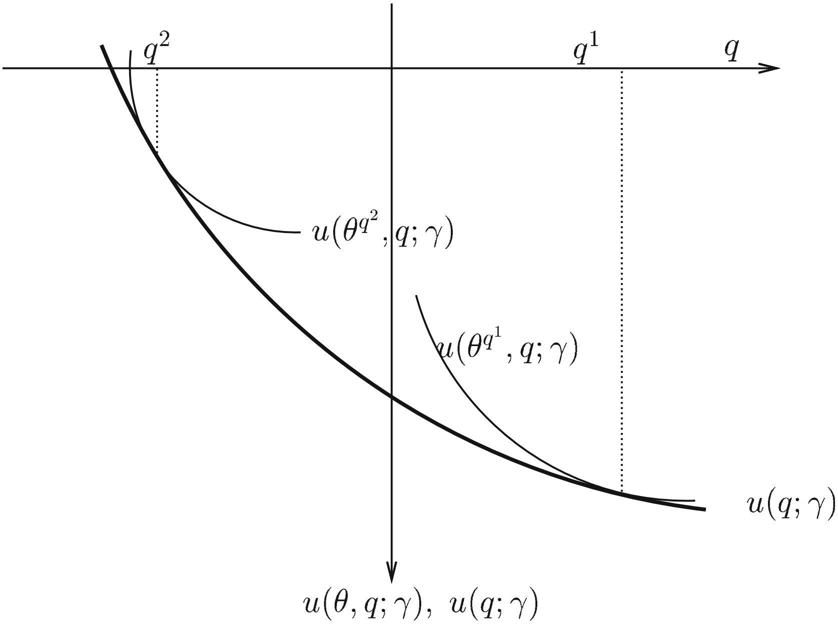

Fig. 1 Curves $u(\theta^{q^1}, q; \gamma)$ and $u(\theta^{q^2}, q; \gamma)$ imply the relationship between the amount q and the value of the expected utility. Each of their expected utilities is optimized only at $q = q^1$ and $q = q^2$, where they contact utility-based curve, $u(q; \gamma)$, respectively

curve $u(q; \gamma)$. This is because, from the definition of $u(\cdot; \gamma)$, the lower value of $u(\cdot; \gamma)$ is preferred for the investor with risk-aversion γ.

Using the utility-based curve, we also rewrite (1) to develop a graphical image of the utility-based price. First, the expected utility of (1) is transformed as follows,

$$u(\theta, q; \gamma) - \gamma(x - pq) = \ln\left(-\mathbb{E}\left[U(x - pq + \int_0^T \theta_t^\top dX_t + Bq)\right]\right).$$

This shows that the expected utility maximization problem is independent of initial capital $x - pq$. Using this transformation, (1) is rewritten as,

$$\begin{aligned}\inf_{\theta\in\Theta, q\in\mathbb{R}} \{u(\theta, q; \gamma) - \gamma(x - pq)\} &= \inf_{q\in\mathbb{R}} \left\{\inf_{\theta\in\Theta} u(\theta, q; \gamma) - \gamma(x - pq)\right\} \\ &= \inf_{q\in\mathbb{R}} \{u(q; \gamma) - \gamma(x - pq)\}\end{aligned}$$

For the amount q to be optimal, it has to satisfy

$$\frac{\partial u(q; \gamma)}{\partial q} = -\gamma p. \tag{2}$$

Equation (2) describes a graph of the utility-based price and the amount q, and we write the utility-based price satisfying (2) as $p^{HK}(B; q)$. Figure 2 shows that the utility-based price $p^{HK}(B; q^1)$ and $p^{HK}(B; q^2)$ correspond to the slope of the utility-based curve. We confirm that the smaller the optimal amount q, the larger the utility-based price, which is consistent with our intuition on price and demand. Figure 2 also shows that there exist utility-based prices corresponding to any amount $q \in \mathbb{R}$.

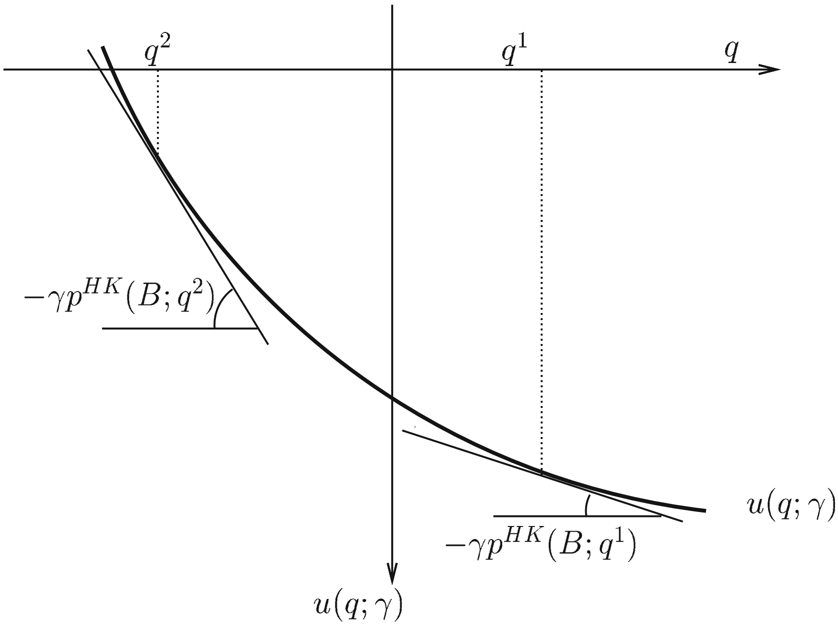

Fig. 2 Utility-based prices are given by points making contact with the utility-based curve. For the amount q^1, the utility-based price is $p^{HK}(B;q^1)$, and for q^2, it is $p^{HK}(B;q^2)$

We also display the utility-based price at $q = 0$ using MEMM. We show the Lemma below, which is well-known, but we use the utility-based curve to easily and clearly visualize its effect.

Lemma 3 *The slope of a tangent line to a utility-based curve* $u(q;\gamma)$ *at* $q = 0$ *is given by* $-\gamma\mathbb{E}^{Q^0}[B]$, *where* Q^0 *is MEMM.*

Proof From the definition of $u(\theta, q;\gamma)$, it holds that $\frac{\partial u(\theta^0,q;\gamma)}{\partial q} = -\gamma\,\mathbb{E}\left[B\frac{e^{-\gamma\left(\int_0^T(\theta_t^0)^\top dX_t+qB\right)}}{\mathbb{E}\left[e^{-\gamma\left(\int_0^T(\theta_t^0)^\top dX_t+qB\right)}\right]}\right]$ for fixed θ^0. Therefore,

$$\frac{\partial u(\theta^0,q;\gamma)}{\partial q}\Big|_{q=0} = -\gamma\mathbb{E}\left[B\frac{e^{-\gamma\left(\int_0^T(\theta_t^0)^\top dX_t\right)}}{\mathbb{E}\left[e^{-\gamma\left(\int_0^T(\theta_t^0)^\top dX_t\right)}\right]}\right] = -\gamma\mathbb{E}^{Q^0}[B],$$

where dQ^0/dP is given by $\frac{e^{-\gamma\left(\int_0^T(\theta_t^0)^\top dX_t\right)}}{\mathbb{E}\left[e^{-\gamma\left(\int_0^T(\theta_t^0)^\top dX_t\right)}\right]}$ by Proposition 2.2 of [1] and Lemma 3.1 of [3]. □

This Lemma shows that the utility-based price $p^{HK}(B;0) = \mathbb{E}^{Q^0}[B]$. In this sense, the utility-based price is connected with the pricing theory based on non-arbitrage.

3.2 Utility-based and indifference pricing

We also consider another pricing framework called indifference pricing and briefly discuss its relationship with utility-based pricing. The utility indifference price is defined as the price of the random endowment B that equals the maximized expected utility of terminal wealth without the random endowment B and the maximized expected utility of a terminal wealth with the random endowment B. That is, if the price p is the utility indifference price, then it satisfies,

$$\sup_{\theta\in\Theta}\mathbb{E}\left[U\left(x+\int_0^T\theta_t^\top dX_t\right)\right]=\sup_{\theta\in\Theta}\mathbb{E}\left[U\left(x-pq+\int_0^T\theta_t^\top dX_t+Bq\right)\right], \tag{3}$$

where $q\in\mathbb{R}$. The left-hand side of this equation is a maximized expected utility not including the random endowment B and the right hand side is a maximized expected utility including B. When $q>0$, we call the price p the utility indifference sell price; otherwise, it is the utility indifference buy price.

Using the function $u(\theta,q;\gamma)$, the utility indifference framework for $q\in\mathbb{R}$ is rewritten as,

$$\begin{aligned}\inf_{\theta\in\Theta}u(\theta,0;\gamma)-\gamma x&=\inf_{\theta\in\Theta}\{u(\theta,q;\gamma)-\gamma(x-pq)\}\\&=\inf_{\theta\in\Theta}\{u(\theta,q;\gamma)\}-\gamma(x-pq).\end{aligned} \tag{4}$$

Therefore, the utility indifference price $p^{UI}(B;q)$ is given by

$$p^{UI}(B;q)=\frac{1}{\gamma q}\left(u(\theta^0,0;\gamma)-u(\theta^q,q;\gamma)\right).$$

That is,

$$u(\theta^q,q;\gamma)=u(\theta^0,0;\gamma)-\gamma p^{UI}(B;q)q.$$

Using the utility-based curve, the above equation is rewritten as,

$$u(q;\gamma)=u(0,\gamma)-\gamma p^{UI}(B;q)q. \tag{5}$$

The right-hand side of the previous equation is a linear function of q. We have already specified the left hand side of this equation as a utility-based curve. Once the amount q is given, the utility indifference price is automatically determined by the utility-based curve. We can see this relationship in Fig. 3.

Since we use the utility-based curve to depict the utility-based price, the same logic for the utility indifference price is also applicable. In Fig. 4, when an amount q for random endowment B is set close to zero, the utility indifference price $p^{UI}(B;q)$ approaches $p^{UI}(B;0)$, which is given by $\mathbb{E}^{Q^0}[B]$, because of the smoothness of the utility-based curve. Therefore, the utility indifference price coincides with the utility-based price at $q=0$. Furthermore, for all domains, we can discuss the relationship between utility-based pricing and indifference pricing using the utility-based curve.

Proposition 1 *For some $q<0$, the utility-based price $p^{HK}(B;q)$ is larger than the utility indifference price $p^{UI}(B;q)$, and for $q>0$, the utility-based price $p^{HK}(B;q)$ is less than the utility indifference price $p^{UI}(B;q)$.*

Proof The proof is clear from the convexity of the indifference curve. □

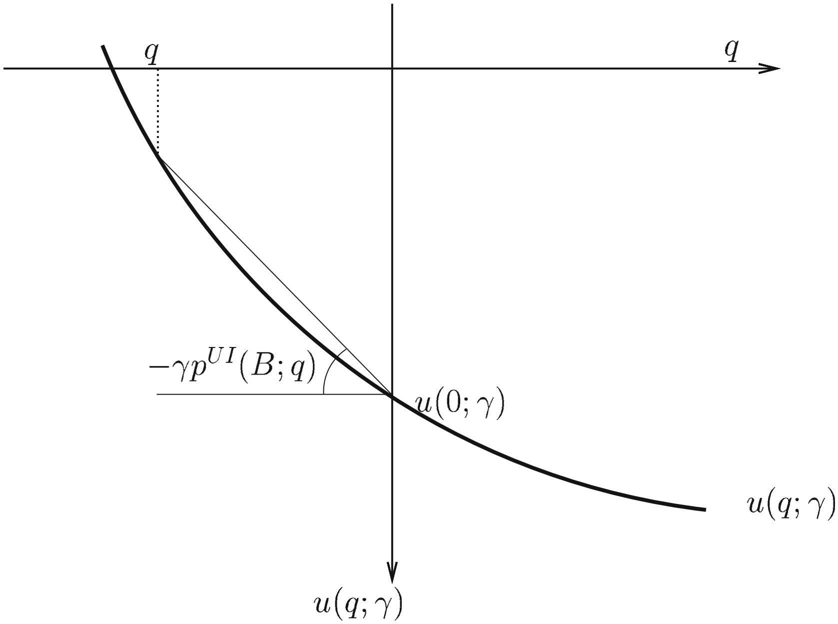

Fig. 3 Utility indifference price depicted on a utility-based curve. For given q, the utility indifference price is $p^{UI}(B;q)$

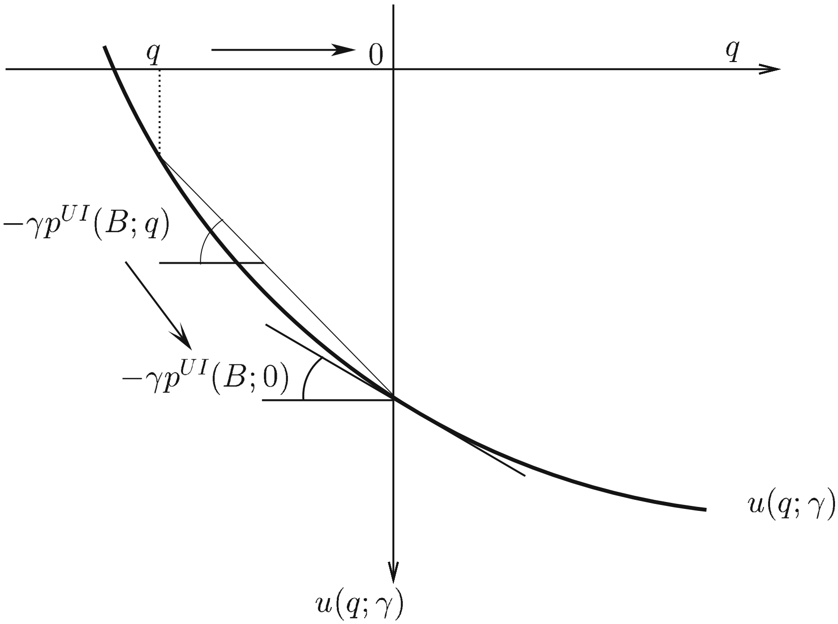

Fig. 4 The utility indifference price $p^{UI}(B;0)$ coincides with the utility-based price $p^{HK}(B;0)$ at $q=0$

From Proposition 1, $p^{HK}(B;q) = p^{UI}(B;q)$ for all $q \in \mathbb{R}$ if and only if the utility-based curve is linear. However, for any utility-based curve $u(q;\gamma)$, the slope of the tangent line at $q=0$ is $-\gamma\mathbb{E}^{Q_0}[B]$. Therefore, when a utility-based curve is linear, the slope of this curve has to be $-\gamma\mathbb{E}^{Q_0}[B]$, indicating that the utility indifference price is given by $\mathbb{E}^{Q_0}[B]$ for all q. As Proposition 3.2 of [1] shows that $\lim_{\gamma\downarrow 0} p^{UI}(B;q) = \mathbb{E}^{Q^0}[B]$, the risk-aversion of the investor is zero, when the utility-based curve is linear.

4 Economic interpretation of the utility-based price and the indifference price

In this section, we give three economic interpretations on the utility-based price and the indifference price. First, we discuss the significance of the introduction of random endowment in the market using the concept of utility-based pricing and indifference pricing. Second, we discuss the partial equilibrium in the market of random endowments and show that random endowments are not traded in this equilibrium. Third, we discuss the availability of the analysis of the income effect and the substitution effect, which are typically used to analyze the quality of goods in the context of economics.

4.1 The significance of the introduction of random endowment B in the market

Derivatives are very useful tools for investors to completely reduce market risks in complete markets. However, in the real market, many derivatives are not necessarily used as pure hedging tools, but are also used as assets for leveraged investments. Financial crises might be considered a typical result of derivative trading in a risk productive environment and have made a negative impression on derivative trading. However, derivatives are essentially not assets for widespread risks and they enable market participants to invest in various ways. By definition, derivatives are assets used to hedge risks in a complete market. Even in the incomplete market in which completely diminishing risks is impossible, the existence of derivatives is meaningful. We discuss this concept using the utility-based and indifference framework.

Proposition 2 *For a given price p, the optimized expected utility under the framework of a utility-based price is larger than the optimized expected utility under the framework of the utility indifference price.*

Proof For a given price p, let q^p be the optimal strategy satisfying (1). Then,

$$\begin{aligned}\inf_{\theta\in\Theta,q\in\mathbb{R}}\{u(\theta,q;\gamma)-\gamma(x-pq)\} &= u(q^p;\gamma)+\gamma p q^p-\gamma x\\ &= u(0;\gamma)-\gamma p^{UI}(B;q^p)q^p+\gamma p q^p-\gamma x\\ &\le u(0;\gamma)-\gamma x=\inf_{\theta\in\Theta}u(\theta,0;\gamma)-\gamma x, \end{aligned}\tag{6}$$

where we used (5) on line 2 and the convexity of a utility-based curve on line 3 [that is, if $q^p<0$, $0>-p^{UI}(B;q^p)\ge -p$; in contrast, if $q^p>0$, then $-p\ge -p^{UI}(B;q^p)$]. The right-hand side of (6) corresponds to the expected utility in the utility indifference framework [(see (4)]. □

This proposition is intuitively clear and well known. However, we note the implication of this proposition; introducing the random endowment in the market has a positive effect on every investor and clarifies the significance of the existence of random endowment in the financial market. This is due to the expected utility based on indifference pricing is indifferent to expected utility in the market without random endowment. Therefore, an investor's expected utility is improved by optimally holding the random endowment. This fact is intuitively natural. However, we could make the discussion very simple by using the utility-based curve as a tool to analyze utility-based price and indifference pricing simultaneously.

4.2 Partial equilibrium in the market of random endowments B

Davis and Yoshikawa [3] deduced the equilibrium under the utility indifference framework. Using the setting of the utility-based curve, we easily deduce the equilibrium even under the framework of the utility-based price. First, we give the definition of the equilibrium by referring to [15], the standard textbook of microeconomics.

Definition 1 Let an economy specify investors' preferences, which are described by the utility function $U := \{U_i(\cdot); U_i(x) := -e^{-\gamma_i x},\ i = 1, \dots, I + J\}$. An allocation $q^s := \{q_i^s, i = 1, \dots, I\}$, $q^b := \{q_j^b, j = 1, \dots, J\}$ and a price p of the random endowment B constitutes a price equilibrium if an assignment exists such that

(1) *Offer price condition* For any investor with utility function $\{U_i,\ i = 1, \dots, I\}$, when the investor sells q_i^s-units of the random endowment, (p, q_i^s) is preferred to all other allocations $(p, (q_i^s)')$; that is, an expected utility corresponding to the allocation (p, q_i^s) is larger than another expected utility corresponding to the allocation $(p, (q_i^s)')$.
(2) *Bid price condition* For any investor with utility function $\{U_{I+j},\ j = 1, \dots, J\}$, when the investor buys q_j^b-units of the random endowment, (p, q_j^b) is preferred to all other allocations $(p, (q_j^b)')$; that is, an expected utility corresponding to the allocation (p, q_j^b) is larger than another expected utility corresponding to the allocation $(p, (q_j^b)')$.
(3) *Market cleared condition* $\sum_{i=1}^{I} q_i^s = \sum_{j=1}^{J} q_j^b$.

From this definition, a proposition on equilibrium is deduced.

Proposition 3 *If investors in the market of the random endowment B act according to the utility maximization, then an equilibrium price is given by,*

$$p^* = \mathbb{E}^{Q^0}[B].$$

Furthermore, in the equilibrium, there is no trade on the random endowment. We call such an equilibrium a zero trade equilibrium.

Proof From Lemma 3, the optimal strategy of q for the price $p^* = \mathbb{E}^{Q^0}[B]$ is 0 for every investor. Because the utility-based curve is convex, the amount q that is optimal for $p > p^*$ is negative, and vice versa. As p^* is common for all investors, if $p > p^*$, selling (that is, $q < 0$) is optimal for all investors, and vice versa. Therefore, no trades are executed, giving equilibrium. Although trades are not executed, the price is given by MEMM Q^0 because if any other price is given, a trade will occur that will not be optimal for some investors. □

We note that the equilibrium discussed in the previous proposition is a partial equilibrium because we discuss only trading of the random endowment. In our model, in addition to the random endowment B, there are risky assets X and a numéraire, which might be a risk-free asset. We implicitly assume that the markets for these assets are in equilibrium. In future research, we plan to explore equilibrium in a more general setting, considering the equilibrium of the risky asset X and the numéraire.

4.3 The effect of price change on the random endowment

We consider the effect of price change on random endowment. The utility based curve displays the relationship between a price and the amount of random endowment. If the price of the random endowment changes, the corresponding amount of random endowment also changes.

Classical economics considered the effect that a price change might be divided into two parts. First is the effect on an investor's wealth; for example, if an asset price increases, the amount that an investor can purchase will decline, indicating that an investor's wealth has essentially decreased. Second is the effect on the demand for other assets; if an asset price increases, then the prices of other assets are observed to decline compared with the price-increasing asset. In the context of economics, the former effect is called the income effect and the latter is called the substitution effect. In the context of economics, the analysis using these two effects clarifies the quality of the asset and the relationship with other assets. Intuitively, if the price of an asset declines, demand for the asset increases. However, taking into account the income effect, this statement is not necessarily true. Compared with other assets, a declining price will give the effect of potentially increasing an investor's wealth, and this effect might help increase demand for other assets. Therefore, if this asset is less attractive than other assets, an investor might decrease demand for this price-declining asset or at least not increase it as much. Such assets are usually called inferior goods and their income effect is negative. Conversely, the assets with a positive income effect are called superior goods.

The question is whether this analysis is possible in the context of utility-based pricing. In this short section, we show that it is not.

First, we define substitution and income effects.

Definition 2 Let the initial purchasing amount q^0 be an optimal amount of the random endowment B for a given price p^0; we assume that the investor adopts a strategy based on the utility-based price.

(1) The substitution effect is the effect from making a change in demand q^0 to $\hat{q}$ via a price change from p^0 to p^1, where the strategy $(p^1, \hat{q})$ gives the same expected utility as the expected utility from the strategy (p^0, q^0).
(2) The income effect is the effect from making a change in demand from $\hat{q}$ to q^1, where q^1 is defined as the optimal strategy for a given price p^1; this optimal strategy is deduced from the utility-based price.

The definition of the substitution effect is referred to as the Hicks' substitution effect. This effects shows us the minimum change in demand to retain the expected utility when the price changes. With the income effect, the price is fixed as p^1; therefore, we can infer that the effect of a price change is extracted from the total change from q^0 to q^1.

We incorporate the analysis using these concepts in the framework of utility-based pricing. Let p^0 be the initial price of the random endowment B and let (θ^0, q^0) be the solution to the problem $\sup_{\theta\in\Theta, q\in\mathbb{R}} \mathbb{E}\left[U\left(x - p^0 q + \int_0^T \theta_t^\top dX_t + Bq\right)\right]$. For this price p^0 and the amount (θ^0, q^0), we define

$$U^*(p^0, q^0) := \mathbb{E}\left[U\left(x - p^0 q^0 + \int_0^T (\theta_t^0)^\top dX_t + Bq^0\right)\right].$$

Let $U^*(p^0, q^0)$ be the referenced expected utility. For this referenced expected utility $U^*(p^0, q^0)$, we consider the price change from p^0 to p^1 and consider the utility-based price $\sup_{\theta\in\Theta, q\in\mathbb{R}} \mathbb{E}\left[U\left(x - p^1 q + \int_0^T \theta_t^\top dX_t + Bq\right)\right]$. Let the solution to this problem be denoted by (θ^1, q^1) and define,

$$U^*(p^1, q^1) := \mathbb{E}\left[U\left(x - p^1 q^1 + \int_0^T (\theta_t^1)^\top dX_t + Bq^1\right)\right].$$

Next, we consider indifference pricing. Note that this indifference pricing is different from the usual type of indifference pricing at the point at which the referenced expected utility is defined by $U^*(p^0, q^0)$, for which the maximized expected utility includes the random endowment B. We call this a quasi-utility indifference pricing. Therefore, we consider the following problem.

$$\sup_{\theta\in\Theta} \mathbb{E}\left[U\left(x - \hat{p}^{UI}(B;q)q + \int_0^T \theta_t^\top dX_t + Bq\right)\right] = U^*(p^0, q^0),$$

where $\hat{p}^{UI}(B;q)$ is the quasi-utility indifference price. If we find an amount $\hat{q}$ satisfying, i.e.,

$$p^1 = \hat{p}^{UI}(B;\hat{q}),$$

then this amount $\hat{q}$ defines the substitution effect and the income effect. In fact, for this amount $\hat{q}$, the expected utility is retained as the expected utility with the strategy (p^0, q^0), even if the price is changed from p^0 to p^1. The rest of the demand effect change from $\hat{q}$ to q^1 represents the income effect. Next, we rewrite the previous procedure in the context of a utility-based curve.

Consider the utility-based price for p^0, that is, $\inf_{\theta\in\Theta,q\in\mathbb{R}}\left\{u(\theta,q;\gamma) - \gamma(x - p^0 q)\right\}$, where $\ln(-U^*(p^0,q^0)) = \inf_{\theta\in\Theta,q\in\mathbb{R}}\left\{u(\theta,q;\gamma) - \gamma(x - p^0 q)\right\}$. As shown in the proof of Proposition 2,

$$\inf_{\theta\in\Theta,q\in\mathbb{R}}\left\{u(\theta,q;\gamma) - \gamma(x - p^0 q)\right\} = u(0;\gamma) - \gamma p^{UI}(B;q^0)q^0 + \gamma p^0 q^0 - \gamma x$$

We show the right-hand side of this equation in Fig. 5. In the figure for the utility-based curve, initial wealth x is not depicted (initial wealth has no effect on the optimization of exponential utility). This figure clarifies that $\inf_{\theta\in\Theta,q\in\mathbb{R}}\left\{u(\theta,q;\gamma) + \gamma p^0 q\right\}$ is given by the points at which the line making contact with the utility-based curve at q^0 intersects with the axis line. That is, this point shows the magnitude of $\ln\left(-\frac{U^*(p^0,q^0)}{e^{-\gamma x}}\right)$.

Similarly $U^*(p^1,q^1)$ corresponds to $\inf_{\theta\in\Theta,q\in\mathbb{R}}\left\{u(\theta,q;\gamma) - \gamma(x - p^1 q)\right\}$ and we can specify the magnitude of $\ln\left(-\frac{U^*(p^1,q^1)}{e^{-\gamma x}}\right)$ as $\inf_{\theta\in\Theta,q\in\mathbb{R}}\left\{u(\theta,q;\gamma) + \gamma p^1 q\right\}$ which is shown in Fig. 6.

Next, we consider the quasi utility indifference pricing.

$$\ln\left(-\frac{U^*(p^0,q^0)}{e^{-\gamma x}}\right) = \ln\left(-\frac{\sup_{\theta\in\Theta}\mathbb{E}\left[U\left(x - \hat{p}^{UI}(B;q)q + \int_0^T \theta_t^\top dX_t + Bq\right)\right]}{e^{-\gamma x}}\right)$$
$$= u(q;\gamma) + \gamma\hat{p}^{UI}(B;q)q.$$

As the left-hand side of the above equation is $u(0;\gamma) - \gamma p^{UI}(B;q^0)q^0 + \gamma p^0 q^0$, we obtain

$$u(q;\gamma) = (u(0;\gamma) - \gamma p^{UI}(B;q^0)q^0 + \gamma p^0 q^0) - \gamma\hat{p}^{UI}(B;q)q. \tag{7}$$

As shown above, $u(0;\gamma) - \gamma p^{UI}(B;q^0)q^0 + \gamma p^0 q^0$ is given as the intersection with the axis line for the line with tangent $-\gamma p^0$ in Fig. 5. Therefore, the right-hand side of the Eq. (7) is the line with tangent $-\gamma\hat{p}^{UI}(B;q)$ and intercept $u(0;\gamma) - \gamma p^{UI}(B;q^0)q^0 + \gamma p^0 q^0$. Figure 7 shows that, although the quasi-utility indifference price is unique, two strategies exists for retaining the expected utility as $U^*(p^0,q^0)$. Actually, if the investor chooses the policy of $\hat{q}_a$ and attempts to retain the expected utility as $U^*(p^0,q^0)$, he or she has to trade the random endowment B at the price $\hat{p}^{UI}(B,\hat{q}_a)$. Simultaneously, the same effect is available for the

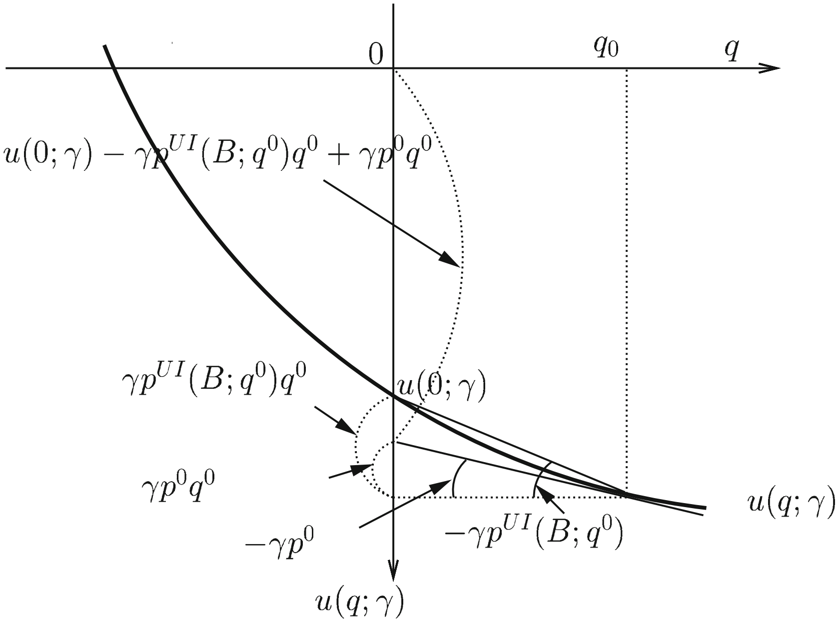

Fig. 5 The value of $u(0;\gamma) - \gamma p^{UI}(B;q^0)q^0 + \gamma p^0 q^0$ is given by the height of the intersection with the axis line and the tangential line at $q = q^0$

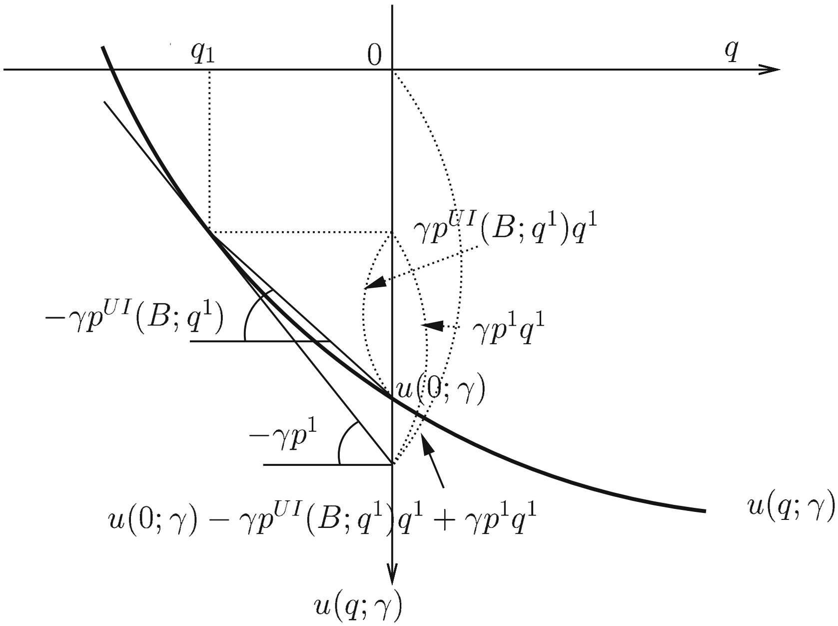

Fig. 6 For p^1, this figure shows the value of $u(0;\gamma) - \gamma p^{UI}(B;q^1)q^1 + \gamma p^1 q^1$, which implies the value of $\inf_{\theta\in\Theta,q\in\mathbb{R}}\left\{u(\theta,q;\gamma) + \gamma p^1 q\right\}$

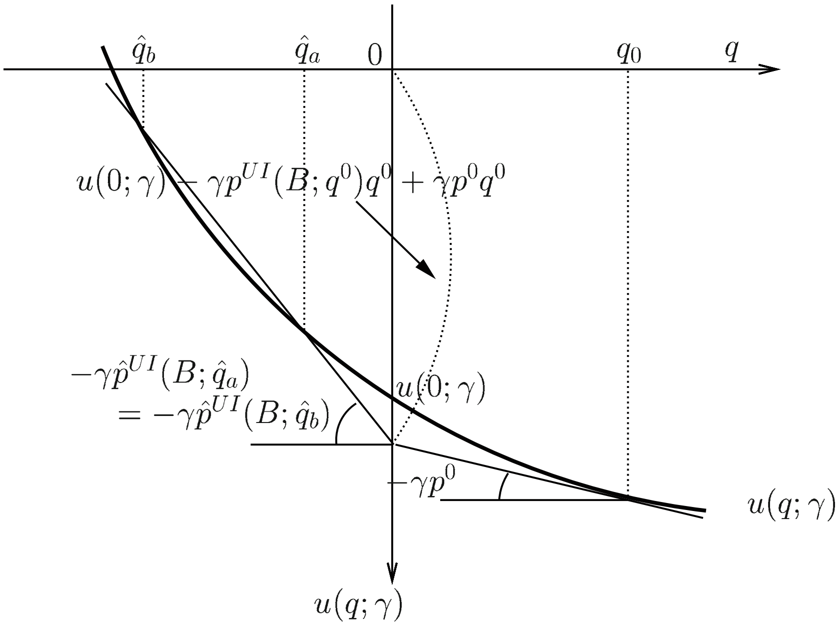

Fig. 7 Quasi utility indifference price $\hat{p}^{UI}(B;\hat{q}_a)$ is the same as $\hat{p}^{UI}(B;\hat{q}_b)$

policy $\hat{q}_b$, because the quasi-utility indifference price $\hat{p}^{UI}(B,\hat{q}_b)$ is the same as $\hat{p}^{UI}(B,\hat{q}_a)$, as shown in Fig. 7. To specify the income effect, we set $p^1 = \hat{p}^{UI}(B,\hat{q}_a) = \hat{p}^{UI}(B,\hat{q}_b)$. In Fig. 8, the optimal strategy q^1 for price p^1 is between $\hat{q}_a$ and $\hat{q}_b$. Therefore, if the investor chooses $\hat{q}_a$, then the income effect is positive. In contrast, if the investor chooses $\hat{q}_b$, the income effect is negative. That is, analyzing the quality of asset B from the viewpoint of the income and substitution effect is not feasible.

In Figs. 5, 6, 7 and 8, we consider the case of the price increasing from p^0 to p^1. The same logic is available for the case of declining prices. We also cannot specify the income effect uniquely.

5 Concluding remarks

Utility-based pricing and indifference pricing are well-developed methods of mathematical finance. However, as previously shown, difficulties exist in discussing them in the context of economics. The fact is that no trades are executed in equilibrium and even simple analysis using the income and substitution effect is not feasible, implicitly showing the limitations of utility-based pricing and indifference pricing. One reason for this issue is the setting of exponential utility. Although exponential utility has useful properties, such as mathematical tractability and independence of initial capital, in this case, such properties make hinder its incorporation into the equilibrium analysis and quality analysis using the income and substitution effect. However, the exponential utility is widely used in mathematical finance and economics. Then, under the framework of exponential utility, we considered the condition for making available utility-based and indifference pricing to connect in the context of economics. In our next paper, we will present the possibility of the appearance of non-zero trade

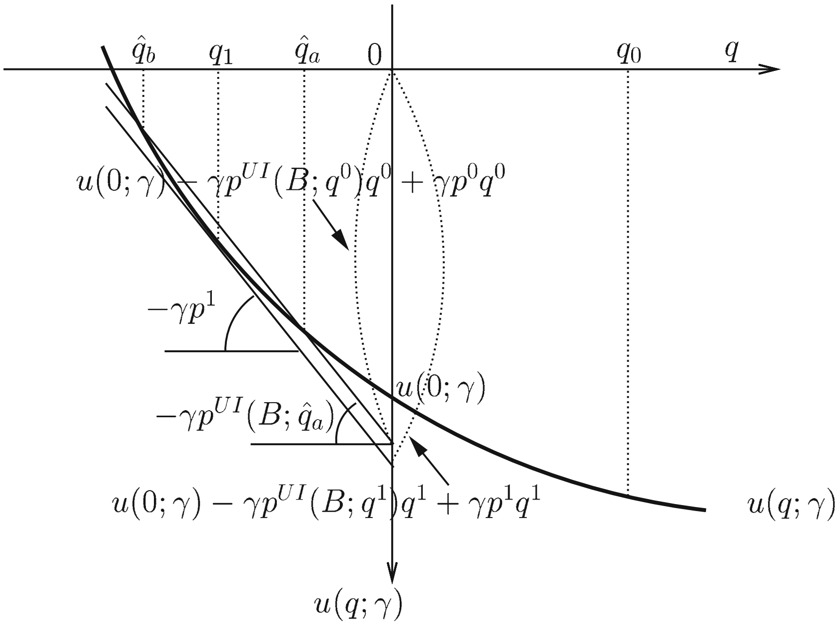

Fig. 8 This figure shows the situation for which $\hat{p}^{UI}(B;\hat{q}_a)$ is equal to p^1

equilibrium by introducing transaction costs into the model even in the case of exponential utility functions.

References

1. Becherer, D.: Rational hedging and valuation of integrated risks under constant absolute risk aversion. Insuarance **33**, 1–28 (2003)
2. Davis, M.A.H., Yoshikawa, D.: A note on utility-based pricing in models with transaction costs. Math. Financ. Econ. (forthcoming)
3. Davis, M.A.H., Yoshikawa, D.: An equilibrium approach to indifference pricing with model uncertainty. Preprint, (2015)
4. Davis, M.H.A.: Optimal hedging with basis risk. In: Kabanov, Y., Lipster, R., Stoyanov, J. (eds.) 2nd Bachelier Colloquium on Stochastic Calculus and Probability, 9–15 January 2005, pp. 169–187. Springer, Metabief (2006)
5. Delbaen, F., Grandits, P., Rheinländer, Th, Samperi, D., Schweizer, M., Stricker, Ch.: Exponential hedging and entropic penalties. Math. Financ. **12**(2), 99–123 (2002)
6. Kramkov, D., Schachermayer, W.: The asymptotic elasticity of utility functions and optimal investment in incomplete markets. Ann. Appl. Probab. **9**, 904–950 (1999)
7. Frittelli, M.: The minimal entropy martingale measure and the valuation problem in incomplete markets. Math. Financ. **10**(1), 39–52 (2000)
8. Henderson, V.: Valuation of claims on non-traded assets using utility maximization. Math. Financ. **12**, 351–373 (2002)
9. Hugonnier, J., Kramkov, D.: Optimal investment with random endowments in incomplete markets. Ann. Appl. Probab. **14**(2), 845–864 (2004)
10. Hugonnier, J., Kramkov, D., Shachermayer, W.: On utility-based pricing of contingent claims in incomplete markets. Math. Financ. **15**(2), 203–212 (2005)
11. Ihara, S.: Information Theory for Continuous System. World Scientific, Singapore (1993)
12. İlhan, A., Jonsson, M., Sircar, R.: Optimal investment with derivative securities. Financ. Stochast. **9**, 585–595 (2005)

13. Kramkov, D., Sîrbu, M.: Sensitivity analysis of utility-based prices and risk-tolerance wealth processes. Ann. Appl. Probab. **16**, 2140–2194 (2006)
14. Monoyios, M.: The minimal entropy measure and an esscher transform in an incomplete market model. Stat. Probab. Lett. **77**, 1070–1076 (2007)
15. Mss-Colell, A., Whinston, M.D., Green, J.R.: Microeconomic Theory. Oxford University Press, Oxford (1995)
16. Musiela, M., Zariphopoulou, T.: An example of indifference prices under exponential preferences. Financ. Stoch. **8**, 229–239 (2004)
17. Rouge, R., El Karoui, N.: Pricing via utility maximization and entropy. Math. Financ. **10**, 259–276 (2000)
18. Sircar, R., Zariphopoulou, T.: Bounds and asymptotic approximations for utility prices when volatility is random. SIAM J. Control Optim. **43**, 1328–1353 (2004)