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Pareto optimal allocations and optimal risk sharing for quasiconvex risk measures

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Abstract The main goal of this paper is to generalize the characterization of Pareto optimal allocations known for convex risk measures (see, among others, Jouini et al., in Math Financ 18(2):269–292, 2008 and Filipovic and Kupper, in Int J Theor Appl Financ, 11:325–343, 2008) to the wider class of quasiconvex risk measures. Following the approach of Jouini et al., in Math Financ 18(2):269–292, 2008 for convex risk measures, in the quasiconvex case we provide sufficient conditions for allocations to be (weakly) Pareto optimal in terms of exactness of the so-called quasiconvex inf-convolution as well as an existence result for weakly Pareto optimal allocations. Moreover, we give a necessary condition for weakly optimal risk sharing that is also sufficient under cash-additivity of at least one between the risk measures.

Keywords Risk measures · Quasiconvex · Pareto optimal · Risk sharing · Inf-convolution

JEL classification D81 · G11 · G13 · G22

1 Introduction

Pareto optimal allocations and optimal risk sharing have been firstly studied in the insurance literature and dates back to Borch [4] (see also Bühlmann and Jewell [5] and Deprez and Gerber [9]). Roughly speaking, the main idea of the problems above is to find the best way to share the total risk (called aggregate risk) between two parties (for instance, between insurer and reinsurer). In the recent literature, many papers are devoted to the problems of finding Pareto optimal allocations and optimal risk sharing for coherent or convex risk measures.

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As we will see in a while, the main tool to solve these problems is inf-convolution. See, among many others, Delbaen [8], Föllmer and Schied [13], Frittelli and Rosazza Gianin [15] (for coherent and convex risk measures), and Acciaio [1], Barrieu and El Karoui [2], [3], Chateauneuf et al. [7], Filipovic and Kupper [12], Jouini et al. [16], Klöppel and Schweizer [17] and Ravanelli and Svindland [24] (for Pareto optimality and optimal risk sharing for Choquet-expected-utility, coherent/convex risk measures and robust utilities).

Quite recently, an increasing interest has been devoted to quasiconvex risk measures, that is risk measures where convexity is replaced by quasiconvexity and cash-additivity is dropped. The main motivation to the introduction of such risk measures is that the right formulation of diversification of risk is quasiconvexity (instead of convexity) even if, under cash-additivity of the risk measures, the two axioms are equivalent. For a deep discussion on quasiconvex risk measures we refer to Cerreia-Vioglio et al. [6], Drapeau and Kupper [10] and Frittelli and Maggis [14].

In this paper, we focus on Pareto optimal allocations and optimal risk sharing for quasiconvex risk measures, in the perspective of generalizing the results established in the literature for convex risk measures. The main difficulties can be summarized as follows. For quasiconvex risk measures, convexity is replaced by quasiconvexity and cash-invariance is dropped, hence Fenchel-Moreau biconjugate Theorem cannot be applied. Moreover, inf-convolution of quasiconvex functionals is not always quasiconvex. In the following, the notion of quasiconvex inf-convolution (or, shortly, qco-convolution) will be used instead of the classical inf-convolution, since more appropriate in our setting because of its stability with respect to convexity and quasiconvexity. See Rockafellar [25], Volle [27], Elqortobi [11] and Seeger and Volle [26], among others, for more details on quasiconvex inf-convolution (also known as level sum).

We provide some sufficient conditions for allocations to be *(weakly) Pareto optimal for quasiconvex risk measures*, in terms of exactness of the quasiconvex inf-convolution. On the one hand, our result can be seen as an extension to quasiconvex risk measures of Theorem 3.1 of Jouini et al. [16]; on the other hand, it is weaker than it since the equivalence between exactness and (weakly) Pareto optimality does not hold any more. We prove indeed that, given two quasiconvex risk functionals π_1 and π_2 satisfying some further assumptions, exactness of the qco-convolution $\pi_1 \nabla \pi_2$ at (ξ_1, ξ_2) implies that (ξ_1, ξ_2) is weakly Pareto. We provide also some counterexamples showing that exactness of the qco-convolution does not guarantee that any allocation attaining the infimum in the qco-convolution is Pareto optimal, but only weakly Pareto optimal; and that weakly Pareto optimality does not imply exactness in general. The existence of weakly Pareto optimal allocations can be also proved under suitable assumptions and similarly to Jouini et al. [16].

Finally, we focus on optimal risk sharing in the quasiconvex setting. Inspired by the characterization of Jouini et al. [16] and by the non-equivalence between Pareto optimality and exactness of quasiconvex inf-convolution in the present setting, we define *weakly optimal risk sharing* any admissible allocation satisfying individual rationality and at which the qco inf-convolution is exact. We provide a necessary condition for weakly optimal risk sharing that is also sufficient under cash-additivity of at least one between the risk measures. We emphasize that, because of the lack of cash-additivity, in the quasiconvex setting one cannot expect a good behavior and interpretation of optimal risk sharing in terms of prices as in the convex case. As soon as cash-additivity of at least one risk measure is imposed, something more can be obtained.

The results found in this paper let us draw the following conclusions: first, it seems to be more appropriate to use quasiconvex inf-convolution than classical inf-convolution when dealing with quasiconvex risk measures; second, quasiconvex inf-convolution allows us to provide necessary or sufficient conditions of weakly Pareto optimal allocations and weakly optimal risk sharing for quasiconvex risk measures, but not a complete characterization of them (except under stronger assumptions). The reason of this last point is due to the definition of quasiconvex inf-convolution.

The paper is organized as follows. In Sect 2 we introduce the assumptions used in the paper and recall the dual representation of quasiconvex risk measures as well as the notions of (quasiconvex) inf-convolution. Some basic results on the properties of quasiconvex inf-convolution are proved in Sect 3. The main results of the paper can be found in Sect 4, where we provide necessary and sufficient conditions for exactness of the quasiconvex inf-convolution and its relation with Pareto optimality for quasiconvex risk measures. Some counterexamples about Pareto optimality are provided in Sect 4.1, while Sect 4.3 deals with optimal risk sharing. Some well-known definitions and results about quasiconvex functionals are summarized in the Appendix.

2 Preliminaries and assumptions

In the following, we will consider a probability space $(\Omega, \mathcal{F}, P)$ and an ordered locally convex topological vector space $L = L(\Omega, \mathcal{F}, P)$ of random variables defined on $(\Omega, \mathcal{F}, P)$. For instance, the space $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ will be often used in the following.

Let $\pi : L \to \bar{\mathbb{R}}$ (where $\bar{\mathbb{R}} \triangleq \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$) represent an insurance premium or correspond to a risk measure $\rho(X) \triangleq \pi(-X)$. It is usually supposed to satisfy some among the following axioms:

- quasiconvexity: $\pi(\alpha X + (1-\alpha)Y) \leq \pi(X) \vee \pi(Y)$ for any $\alpha \in [0, 1]$ and $X, Y \in L$, where $a \vee b \triangleq \max\{a; b\}$.
- convexity: $\pi(\alpha X + (1-\alpha)Y) \leq \alpha\pi(X) + (1-\alpha)\pi(Y)$ for any $\alpha \in [0, 1]$ and $X, Y \in L$
- monotonicity: if $X, Y \in L$ and $X \leq Y$, then $\pi(X) \leq \pi(Y)$
- cash-subadditivity: $\pi(X + c) \leq \pi(X) + c$ for any $X \in L$ and $c \geq 0$
- cash-additivity: $\pi(X + c) = \pi(X) + c$ for any $X \in L$ and $c \in \mathbb{R}$
- continuity from below (resp. from above): if $(X_n)_{n\geq 0} \subseteq L$, $X_n \nearrow_n X \in L$ (resp. $X_n \searrow_n X \in L$), then $\lim_n \pi(X_n) = \pi(X)$

By Penot and Volle [22], it is well known that a functional π is quasiconvex iff any level set $\{X \in L : \pi(X) \leq c\}$ (with $c \in \mathbb{R}$) is convex (or, equivalently, iff any strict level set $\{X \in L : \pi(X) < c\}$ (with $c \in \mathbb{R}$) is convex). For simplicity of notation, the level sets above will be sometimes denoted, respectively, by $\{\pi \leq c\}$ and $\{\pi < c\}$.

We refer to [6,8,10,13,15], among many others, for a detailed discussion on the axioms above.

We remind that for convex risk functionals π (on L^∞) satisfying monotonicity, cash-additivity and $\pi(0) = 0$, continuity from above implies continuity from below (equivalent to lower semi-continuity with respect to the weak topology $\sigma(L^\infty, L^1)$)—see, for instance, Barrieu and El Karoui [3], Klöppel and Schweizer [17]. The same does not hold anymore for quasiconvex and monotone risk measures.

We recall (from Penot and Volle [22], Cerreia-Vioglio et al. [6] and Frittelli and Maggis [14]) that any quasiconvex, monotone, continuous from below risk measure $\pi : L^\infty \to \bar{\mathbb{R}}$ can be represented as:

$$\pi(X) = \sup_{Q \in \mathcal{M}_1} R\left(E_Q[X], Q\right), \quad \forall X \in L^\infty, \tag{1}$$

where

$$R(t, Q) = \inf\{\pi(Y) | E_Q[Y] = t\}, \quad \forall (t, Q) \in \mathbb{R} \times \mathcal{M}_{1,f}, \tag{2}$$

$\mathcal{M}_{1,f}$ denotes the set formed by all finitely additive probability measures that are absolutely continuous with respect to P, while $\mathcal{M}_1$ denotes the subset of $\mathcal{M}_{1,f}$ consisting of all countably additive elements. With an abuse of notation, $\mathcal{M}_1$ can be considered a subset of L^1 by identifying any $Q \in \mathcal{M}_1$ with its Radon-Nikodym density $\frac{dQ}{dP}$.

By [14], continuity from below of π is equivalent to $\sigma\left(L^\infty, L^1\right)$-lower semi-continuity of π. When continuity from below is replaced by continuity from above, the dual representation of π remains as in (1) but the supremum is attained and the functional R belongs to the set $\mathcal{R}_0$ of functionals $R : \mathbb{R} \times \mathcal{M}_{1,f} \to [-\infty, +\infty]$ that are upper semi-continuous, quasiconcave and increasing in t and satisfy $\inf_{t\in\mathbb{R}} R(t, Q) = \inf_{t\in\mathbb{R}} R\left(t, Q'\right)$ for any $Q, Q' \in \mathcal{M}_{1,f}$. See Penot and Volle [22], Cerreia-Vioglio et al. [6] and Frittelli and Maggis [14] for further details.

It is also worth to stress that any quasiconvex and monotone risk measure π satisfying cash-additivity is necessarily convex and that the functional R associated to a convex, monotone, cash-additive and continuous from above (or below) π is given by

$$R(t, Q) = t - F(Q) = t - \pi^*(Q), \tag{3}$$

where π^* stands, as usual, for the convex conjugate of π (see Cerreia-Vioglio et al. [6] and Frittelli and Maggis [14] for the proof).

Our aim is to focus on Pareto optimal allocations and optimal risk sharing and, in particular, to extend the results of Jouini et al. [16]—based on inf-convolution and on dual representations of convex risk measures—to quasiconvex risk measures.

We recall (see, for instance, Moreau [19] and Barrieu and El Karoui [2]) that, given two functionals $\pi_1, \pi_2 : L \to \bar{\mathbb{R}}$, the inf-convolution of π_1 and π_2 is defined as

$$(\pi_1 \Box \pi_2)(X) \triangleq \inf_{Y\in L} \{\pi_1(X - Y) + \pi_2(Y)\}, \quad \forall X \in L.$$

It is well known that inf-convolution is stable with respect to convexity, that is the inf-convolution of two convex functions is also convex. Nevertheless, the same does not hold any more for quasiconvexity (see Luc and Volle [18] for a counterexample on $\mathbb{R}^2$). For this reason, in the present paper we will use the notion of quasiconvex inf-convolution (see Rockafellar [25] and Volle [27]). Given two extended real-valued functionals $\pi_1, \pi_2 : L \to \bar{\mathbb{R}}$, the *quasiconvex inf-convolution* (qco-convolution, for short) of π_1 and π_2 is defined as

$$(\pi_1 \nabla \pi_2)(X) \triangleq \inf_{Y\in L} \{\pi_1(X - Y) \vee \pi_2(Y)\} = \inf_{Y\in L} \{\pi_1(Y) \vee \pi_2(X - Y)\} \tag{4}$$

for any $X \in L$. It is well known (see, for instance, Rockafellar [25], Elqortobi [11] and Seeger and Volle [26]) that quasiconvex inf-convolution is stable with respect to quasiconvexity (and convexity), that is the quasiconvex inf-convolution of two quasiconvex (resp. convex) functions is also quasiconvex (resp. convex).

Other notions on quasiconvex functionals and on subdifferentiability useful in the paper can be found in the Appendix.

3 Properties of quasiconvex inf-convolution

Let L be an ordered locally convex topological vector space.

Given two functionals $\pi_1, \pi_2 : L \to \bar{\mathbb{R}}$, we focus now on the quasiconvex inf-convolution of π_1 and π_2 defined as in (4) and on its properties. Some financial/economic motivations and interpretations of the qco-convolution can be found in Seeger and Volle [26]. Focusing on risk measures or, better, on insurance premiums, $\pi_1(X-Y) \vee \pi_2(Y)$ represents the maximal premium to be paid for insurance and reinsurance separately, when Y is the risk transferred by the insurance to the reinsurance. Hence, the qco-convolution of the insurance premium π_1 and the reinsurance premium π_2 corresponds to the minimization of the maximal premium to be paid for (each) insurance and reinsurance contract.

Proposition 1 *Let $\pi_1, \pi_2 : L \to \bar{\mathbb{R}}$ be quasiconvex and monotone.*
Then $\pi_1 \nabla \pi_2$ is quasiconvex and monotone. Moreover:

(i) if at least one between π_1 and π_2 is continuous from above (resp. cash-subadditive), then also $\pi_1 \nabla \pi_2$ is continuous from above (resp. cash-subadditive).
(ii) $(\pi_1 \nabla \pi_2)(X) \leq \min\{\pi_1(0) \vee \pi_2(X); \pi_1(X) \vee \pi_2(0)\}$ for any $X \in L$. In particular, if $\pi_1(0) = \pi_2(0) = 0$, then $(\pi_1 \nabla \pi_2)(0) \leq 0$.

Proof Quasi-convexity of $\pi_1 \nabla \pi_2$ is due to Rockafellar [25]. The proof of monotonicity is trivial.
(i) Since $(\pi_1 \nabla \pi_2)(X) = \inf_{Y \in L} \{\pi_1(X-Y) \vee \pi_2(Y)\} = \inf_{Y \in L} \{\pi_1(Y) \vee \pi_2(X-Y)\}$, we may assume without loss of generality that π_1 is continuous from above. By monotonicity of $\pi_1 \nabla \pi_2$ and by continuity from above of π_1, $\inf_{n \in \mathbb{N}} (\pi_1 \nabla \pi_2)(X_n) = (\pi_1 \nabla \pi_2)(X)$ can be verified easily for any sequence $(X_n)_{n \geq 0} \subseteq L$ such that $X_n \searrow_n X \in L$.

Assume now without loss of generality that π_1 is cash-subadditive. Then, for any $X \in L$ and $c \geq 0$

$$\begin{aligned}(\pi_1 \nabla \pi_2)(X+c) - c &= \inf_{Y \in L} \{[\pi_1(X+c-Y) - c] \vee [\pi_2(Y) - c]\} \\ &\leq \inf_{Y \in L} \{\pi_1(X-Y) \vee [\pi_2(Y) - c]\} \\ &\leq (\pi_1 \nabla \pi_2)(X),\end{aligned}$$

hence cash-subadditivity of $\pi_1 \nabla \pi_2$.
The proof of (ii) is straightforward. □

The following result establishes how qco-convolution works at the level of the functional R.

Theorem 2 *Let $\pi_1, \pi_2 : L^\infty \to \bar{\mathbb{R}}$ be two quasiconvex, monotone risk measures. Suppose that at least one of them is continuous from above and, eventually, the other one is continuous from below, and let R_1, R_2 be their corresponding functionals.*

Then $\pi^\nabla = \pi_1 \nabla \pi_2$ is quasiconvex, monotone and continuous from above and its corresponding functional $R^\nabla \in \mathcal{R}_0$ is given by

$$R^\nabla(t, Q) = (R_1 \nabla_t R_2)(t, Q) = \inf_{t_1 + t_2 = t} \{R_1(t_1, Q) \vee R_2(t_2, Q)\}, \tag{5}$$

where ∇_t means that the qco-convolution is only in t.

Proof Quasi-convexity, monotonicity and continuity from above of π^∇ are due to Proposition 1. Hence π^∇ can be represented as in (1) (with supremum attained) by means of R^∇. By definition of π^∇ and in the same spirit of the proof of Proposition 2.3 of Seeger and Volle [26],

$$
\begin{aligned}
R^{\nabla}(t,Q) &= \inf_{Y\in L^\infty : E_Q[Y]=t} \inf_{Z\in L^\infty} \{\pi_1(Z)\vee\pi_2(Y-Z)\} \\
&= \inf_{Z\in L^\infty} \inf_{Y : E_Q[Y]=t} \{\pi_1(Z)\vee\pi_2(Y-Z)\} \\
&= \inf_{Y_1,Y_2 : E_Q[Y_1+Y_2]=t} \{\pi_1(Y_1)\vee\pi_2(Y_2)\} \\
&= \inf_{\substack{t_1,t_2:\\ t_1+t_2=t}} \inf_{\substack{Y_1,Y_2:\\ E_Q[Y_i]=t_i,\text{ for } i=1,2}} \{\pi_1(Y_1)\vee\pi_2(Y_2)\} \\
&= \inf_{t_1,t_2:t_1+t_2=t} \left\{\left(\inf_{Y_1:E_Q[Y_1]=t_1}\pi_1(Y_1)\right)\vee\left(\inf_{Y_2:E_Q[Y_2]=t_2}\pi_2(Y_2)\right)\right\} \\
&= \inf_{t_1,t_2:t_1+t_2=t} \{R_1(t_1,Q)\vee R_2(t_2,Q)\}.
\end{aligned}
$$

By the arguments above and by Cerreia-Vioglio et al. [6], $R^{\nabla}\in\mathcal{R}_0$. □

We consider now the particular case of convex risk measures.

Proposition 3 (convex case) *Let $\pi_1,\pi_2 : L^\infty\to\mathbb{R}$ be two convex, monotone and cash-additive risk measures such that at least one of them is continuous from above and, eventually, the other one is continuous from below, and let R_1, R_2 [or, equivalently, F_1, F_2 - see (3)] be their corresponding functionals.*

Then either $\pi^{\nabla}=\pi_1\nabla\pi_2$ is identically equal to $-\infty$ on L^∞ or $\pi^{\nabla}: L^\infty\to\mathbb{R}$ is convex, monotone and continuous from above and its corresponding functional R^{∇} is given by

$$R^{\nabla}(t,Q)=\frac{1}{2}\left[t-F_1(Q)-F_2(Q)\right]. \tag{6}$$

Moreover, when π^{∇} is real-valued it can be represented as

$$\pi^{\nabla}(X)=\frac{1}{2}\max_{Q\in\mathcal{M}_1}\left\{E_Q[X]-F^{\square}(Q)\right\}, \tag{7}$$

with $F^{\square}(Q)=(\pi_1\square\pi_2)^*(Q)$.

Proof By Rockafellar [25] and by Proposition 1 we know that $\pi^{\nabla}: L^\infty\to\bar{\mathbb{R}}$ is convex, monotone and continuous from above. Since both π_1 and π_2 are real-valued, it holds $(\pi_1\nabla\pi_2)(X)\le\pi_1(X)\vee\pi_2(0)<+\infty$ for any $X\in L^\infty$. Moreover, since π_1 and π_2 are cash-additive (hence cash-subadditive), also $\pi_1\nabla\pi_2$ is cash-subadditive [see Proposition 1(i)].

Suppose now that there exists $\bar{X}\in L^\infty$ such that $(\pi_1\nabla\pi_2)(\bar{X})=-\infty$. We are going to show that $(\pi_1\nabla\pi_2)(Y)=-\infty$ for any $Y\in L^\infty$. Indeed, for any $Y\in L^\infty$ it holds that $s\triangleq ess.\sup(Y-\bar{X})\in\mathbb{R}$ and $Y\le\bar{X}+s$. If $s<0$, then $Y\le\bar{X}$ and, by monotonicity, $(\pi_1\nabla\pi_2)(Y)\le(\pi_1\nabla\pi_2)(\bar{X})=-\infty$. If $s\ge 0$, then by monotonicity and cash-subadditivity it follows that

$$(\pi_1\nabla\pi_2)(Y)\le(\pi_1\nabla\pi_2)(\bar{X}+s)\le(\pi_1\nabla\pi_2)(\bar{X})+s=-\infty.$$

By the arguments above, either $\pi^{\nabla}=\pi_1\nabla\pi_2$ is identically equal to $-\infty$ on L^∞ or $\pi^{\nabla}: L^\infty\to\mathbb{R}$. In the second case, from (5) and (3) it follows that

$$R^{\nabla}(t,Q)=\inf_{t_2\in\mathbb{R}}\{(t-t_2-F_1(Q))\vee(t_2-F_2(Q))\}=\frac{1}{2}\left[t-F_1(Q)-F_2(Q)\right].$$

Hence, $\pi^{\nabla}(X) = \frac{1}{2}\max_{Q\in\mathcal{M}_1}\left\{E_Q[X] - F^{\Box}(Q)\right\}$, where $F^{\Box}(Q) \triangleq F_1(Q) + F_2(Q)$. Since π_1 and π_2 are proper and convex, then $F^{\Box}(Q) = (\pi_1 \Box \pi_2)^*(Q)$ (see Rockafellar [25]). □

The following example shows that the qco-convolution of a convex risk measure and a quasiconvex risk measure may be not convex.

Example 4 Consider two risk measures π_1 and π_2 whose corresponding functionals are, respectively,

$$R_1(t, Q) = t - F(Q); \qquad R_2(t, Q) = \begin{cases} t - F(Q); & t \le 0 \\ t^2 - F(Q); & t > 0 \end{cases}$$

where $F(Q) \in \{0, +\infty\}$ with $F = 0$ on a weakly compact set.

It is easy to check that both R_1 and R_2 are increasing, continuous and quasiconcave in t and $\inf_t R_i(t, Q) = \inf_t R_i\left(t, Q'\right) = -\infty$ for any Q, Q' and $i = 1, 2$, so $R_1, R_2 \in \mathcal{R}_0$. Furthermore, both π_1 and π_2 are monotone, π_1 is convex and cash-additive, while π_2 is only quasiconvex. Since R_1 and R_2 are continuous in t, π_1 and π_2 are continuous from below. By the definition of R_1 and F, it follows that π_1 is also continuous from above and coherent.

By the arguments above, it follows that π^{∇} is monotone, quasiconvex and continuous from above. By (5), its corresponding functional R^{∇} is

$$R^{\nabla}(t, Q) = \begin{cases} t/2 - F(Q); & t \le 0 \\ t + \frac{1}{2} - \frac{\sqrt{1+4t}}{2} - F(Q); & t > 0 \end{cases}$$

It can be checked easily that π^{∇} is not convex $\left(\text{take e.g. } \alpha = \frac{1}{2},\ X = -\frac{1}{2} \text{ and } Y = \frac{1}{2}\right)$.

4 Pareto optimal allocations and optimal risk sharing for quasiconvex risk measures

In this section, we focus on Pareto optimal allocations and on optimal risk sharing for quasiconvex risk measures. Our main goal is twofold: first, to translate the problem in the quasiconvex case; second, to investigate if results similar to those true in the convex case still hold for quasiconvex risk measures.

Let X_1 be the initial risk for an insurer, let X_2 be the initial risk for a reinsurer and let $X = X_1 + X_2$ be the total risk (called aggregate risk). Let $\pi_1(Y)$ be the premium to be paid to the insurer because of the risk Y and $\pi_2(Y)$ be the premium to be paid to the reinsurer because of Y.

The basic idea of Pareto optimal allocations and optimal risk sharing is to find the best way (in some sense to be precised) for two agents (say, insurer and reinsurer) to share the aggregate risk.

Here below, we recall the following definitions, well known in the economical literature. As previously, we consider $\pi_1, \pi_2 : L^{\infty} \to \bar{\mathbb{R}}$ and risky positions belonging to $L = L^{\infty}$.

Definition 5 The set $A(X) \triangleq \{(\xi_1, \xi_2) \in L^{\infty} \times L^{\infty} : \xi_1 + \xi_2 = X\}$ is called set of attainable allocations.

An attainable allocation (ξ_1, ξ_2) is said to be:

(a) a Pareto optimal allocation if: whenever there exists $(\eta_1, \eta_2) \in A(X)$ such that $\pi_1(\eta_1) \le \pi_1(\xi_1)$ and $\pi_2(\eta_2) \le \pi_2(\xi_2)$, then $\pi_1(\eta_1) = \pi_1(\xi_1)$ and $\pi_2(\eta_2) = \pi_2(\xi_2)$;

(b) a weakly Pareto optimal allocation if there does not exist any $(\eta_1, \eta_2) \in A(X)$ satisfying $\pi_1(\eta_1) < \pi_1(\xi_1)$ and $\pi_2(\eta_2) < \pi_2(\xi_2)$;
(c) an optimal risk sharing if it is Pareto optimal and it satisfies the Individual Rationality constraint (IR for short), that is $\pi_1(\xi_1) \leq \pi_1(X_1)$ and $\pi_2(\xi_2) \leq \pi_2(X_2)$.

We recall that any Pareto optimal allocation is also weakly Pareto optimal, while the converse is not true.

It is already known that for convex risk functionals π_1, π_2 satisfying monotonicity, lower semi-continuity, cash-additivity and $\pi_1(0) = \pi_2(0) = 0$, Pareto optimal allocations can be characterized as below, by means of inf-convolution of π_1 and π_2.

Theorem 6 (see Jouini et al. [16]) *Let $\pi_1, \pi_2 : L^\infty \to \mathbb{R}$ be two convex risk measures satisfying monotonicity, $\sigma(L^\infty, L^1)$-lsc, cash-additivity and $\pi_1(0) = \pi_2(0) = 0$ and let F_1, F_2 be their convex conjugate.*

Let $X \in L^\infty$ be a given aggregate risk.

The following conditions are equivalent:

(i) an attainable allocation (ξ_1, ξ_2) is Pareto optimal;
(ii) the inf-convolution $(\pi_1 \Box \pi_2)(X)$ is exact, with $(\pi_1 \Box \pi_2)(X) = \pi_1(\xi_1) + \pi_2(\xi_2)$;
(iii) $\pi_i(\xi_i) = E_{\bar{Q}}[\xi_i] - F_i(\bar{Q})$ for some $\bar{Q} \in \mathcal{M}_1$;
(iv) $\partial\pi_1(\xi_1) \cap \partial\pi_2(\xi_2) \neq \emptyset$, where $\partial\pi$ stands for the Fenchel-Moreau subdifferential of π.

See also Filipovic and Kupper [12] in a more general framework.

In the following, we will investigate what happens for quasiconvex risk functionals and, in particular, whether a characterization similar to the one above still holds (once inf-convolution is replaced by qco-convolution and Fenchel-Moreau subdifferential by a suitable subdifferential). Some definitions and results on different notions of subdifferentials for quasiconvex functions (as well as some basic references) can be found in the Appendix.

From now on, we consider two risk measures satisfying the following:

Assumption (Aπ): $\pi_1, \pi_2 : L^\infty \to \bar{\mathbb{R}}$ are quasiconvex risk functionals satisfying monotonicity and continuity from above. Hence, π_1 and π_2 can be represented as in (1) where supremum is attained.

We remind that the qco-convolution

$$(\pi_1 \nabla \pi_2)(X) = \inf_{Y \in L^\infty} \{\pi_1(Y) \vee \pi_2(X - Y)\} = \inf_{(\xi_1, \xi_2) \in A(X)} \{\pi_1(\xi_1) \vee \pi_2(\xi_2)\} \tag{8}$$

is said to be exact at $(\xi_1^*, \xi_2^*) \in A(X)$ if the infimum in (8) is realized for (ξ_1^*, ξ_2^*), that is $(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1^*) \vee \pi_2(\xi_2^*)$.

The following result emphasizes the link between weakly Pareto optimal allocations, exactness of qco-convolutions and representations of quasiconvex risk measures. On the one hand, it is weaker than Theorem 6. Indeed, we are not able to prove the equivalence between exactness and Pareto optimality nor the representations similar to those given in Theorem 6(iii). Nevertheless, we prove that exactness at (ξ_1, ξ_2) implies that the allocation (ξ_1, ξ_2) is weakly Pareto and we characterize R^∇ (respectively, the Greenberg-Pierskalla subdifferential $\partial^{GP}(\pi_1 \nabla \pi_2)(X)$—see Definition 17) in terms of R_1, R_2 (respectively, of $\partial^{GP}\pi_i, i = 1, 2$). On the other hand, the next theorem can be seen as an extension of the result of Jouini et al. [16] to quasiconvex risk measures.

Theorem 7 *Let π_1, π_2 be two risk measures satisfying Assumption (Aπ) and let R_1, R_2 be the corresponding functionals defined on $\mathbb{R} \times \mathcal{M}_{1,f}$. Let $X \in L^\infty$ be a given aggregate risk.*

(i) If $(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2)$ *for some* $(\xi_1, \xi_2) \in A(X)$*, then* (ξ_1, ξ_2) *is a weakly Pareto optimal allocation.*

(ii) $(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2)$ *holds for* $(\xi_1, \xi_2) \in A(X)$ *if and only if the following conditions are both satisfied:*

(ii-r) $R^{\nabla}(E_{\bar{Q}}(X), \bar{Q}) = R_1(E_{Q_1}(\xi_1), Q_1) \vee R_2(E_{Q_2}(\xi_2), Q_2)$*;*

(ii-p) $\pi_i(\xi_i) = R_i(E_{\bar{Q}}(\xi_i), \bar{Q})$ *whenever:* $\pi_i(\xi_i) > \pi_j(\xi_j)$*, or* $\pi_i(\xi_i) = \pi_j(\xi_j)$ *and* $R_i(E_{\bar{Q}}(\xi_i), \bar{Q}) > R_j(E_{\bar{Q}}(\xi_j), \bar{Q})$ *(for* $i, j = 1, 2$*),*

where $\bar{Q} \in \mathcal{M}_1$ *(respectively,* $Q_i \in \mathcal{M}_1$ *for* $i = 1, 2$*) are such that* $(\pi_1 \nabla \pi_2)(X) = R^{\nabla}(E_{\bar{Q}}(X), \bar{Q})$ *(resp.* $\pi_i(X) = R_i(E_{Q_i}(X), Q_i)$*).*

(iii) Let π_1 *and* π_2 *be* $\sigma(L^\infty, L^1)$*-upper semi-continuous.*

If $\pi_1 \nabla \pi_2(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2)$ *for* $\xi_1, \xi_2 \in A(X)$ *and* ξ_1, ξ_2 *are not local minimizers of* π_1, π_2*, then* $\partial^{GP}\pi_1(\xi_1) \cap \partial^{GP}\pi_2(\xi_2) = \partial^{GP}(\pi_1 \nabla \pi_2)(X)$*.*

Remark 8 (a) Item (i) of Theorem 7 would hold even without the hypothesis of continuity from above of both π_1 and π_2. Notice also that the assumption of $\sigma(L^\infty, L^1)$-upper semi-continuity taken in item (iii) (together with monotonicity) is stronger than continuity from above.

(b) By Proposition 3.23 of Penot and Zalinescu [23], in Theorem 7(iii) the assumption that ξ_1, ξ_2 are not local minimizers of π_1, π_2 (together with $(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2)$ and $\sigma(L^\infty, L^1)$-usc of both π_1 and π_2) implies that $\pi_1(\xi_1) = \pi_2(\xi_2)$.

(c) A discussion on conditions (ii-r) and (ii-p) is needed. First of all, $\bar{Q}$ represents the probability measure which maximizes the function $R^{\nabla}(\mathbb{E}_{\cdot}[X], \cdot)$, while Q_1, Q_2 realize the maximum respectively for R_1, R_2. Condition (ii-r) requires then that the maximum of $R^{\nabla}(\mathbb{E}_{\cdot}[X], \cdot)$, considered as a function of $\bar{Q}$, coincides with the biggest between the maximum of $R_1(\mathbb{E}_{\cdot}[\xi_1], \cdot)$ and $R_2(\mathbb{E}_{\cdot}[\xi_2], \cdot)$.

Condition (ii-p) says something more. Assume, without loss of generality, that $R^{\nabla}(\mathbb{E}_{\bar{Q}}[X], \bar{Q}) = R_1(\mathbb{E}_{Q_1}[\xi_1], Q_1)$ (hence $R_1(\mathbb{E}_{Q_1}[\xi_1], Q_1) \geq R_2(\mathbb{E}_{Q_2}[\xi_1], Q_2)$). Then (ii-p) guarantees that $R_1(\mathbb{E}_{\cdot}[\xi_1], \cdot)$ realizes its maximum also in correspondence of $\bar{Q}$. Hence both $\bar{Q}$ and Q_1 belong to argmax $R_1(\mathbb{E}_{\cdot}[\xi_1], \cdot)$.

Proof of Theorem 7 (i) Suppose by contradiction that there exist $\tilde{X}_1, \tilde{X}_2 \in L^\infty$ such that $\tilde{X}_1 + \tilde{X}_2 = X$ and

$$\pi_1(\tilde{X}_1) < \pi_1(\xi_1) \quad \text{and} \quad \pi_2(\tilde{X}_2) < \pi_2(\xi_2). \tag{9}$$

Hence

$$\pi_1(\xi_1) \vee \pi_2(\xi_2) = (\pi_1 \nabla \pi_2)(X) \leq \pi_1(\tilde{X}_1) \vee \pi_2(\tilde{X}_2) < \pi_1(\xi_1) \vee \pi_2(\xi_2),$$

that is a contradiction. (ξ_1, ξ_2) is therefore a weakly Pareto optimal allocation.

(ii) By Proposition 1(i), continuity from above of π_1 and π_2 guarantees that also $\pi_1 \nabla \pi_2$ is continuous from above. Hence $\pi_i(\xi_i) = R_i(E_{Q_i}(\xi_i), Q_i)$ for some $Q_i = Q_i(\xi_i) \in \mathcal{M}_1$ (with $i = 1, 2$) and $(\pi_1 \nabla \pi_2)(X) = R^{\nabla}(E_{\bar{Q}}(X), \bar{Q})$ for some $\bar{Q} = \bar{Q}(X) \in \mathcal{M}_1$.

Only if part. Suppose that both (ii-r) and (ii-p) are satisfied for some $(\xi_1, \xi_2) \in A(X)$. Hence

$$\begin{aligned} \pi_1(\xi_1) \vee \pi_2(\xi_2) &= R_1(E_{Q_1}(\xi_1), Q_1) \vee R_2(E_{Q_2}(\xi_2), Q_2) \\ &= R^{\nabla}(E_{\bar{Q}}(X), \bar{Q}) \\ &= (\pi_1 \nabla \pi_2)(X), \end{aligned} \tag{10}$$

where equality (10) is due to condition (ii-r).

If part. Suppose that $\pi_1 \nabla \pi_2(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2)$ holds for some $(\xi_1, \xi_2) \in A(X)$. It follows that

$$\begin{aligned}(\pi_1 \nabla \pi_2)(X) &= \pi_1(\xi_1) \vee \pi_2(\xi_2)\\ &\geq R_1(E_{\bar{Q}}(\xi_1), \bar{Q}) \vee R_2(E_{\bar{Q}}(\xi_2), \bar{Q}) \qquad (11)\\ &\geq R^{\nabla}(E_{\bar{Q}}(\xi_1) + E_{\bar{Q}}(\xi_2), \bar{Q}) \qquad (12)\\ &= R^{\nabla}(E_{\bar{Q}}(X), \bar{Q}) = (\pi_1 \nabla \pi_2)(X),\end{aligned}$$

where inequality (12) is due to (5). Hence the inequalities (11) and (12) are indeed equalities and

$$\begin{aligned}R_1(E_{Q_1}(\xi_1), Q_1) \vee R_2(E_{Q_2}(\xi_2), Q_2) &= \pi_1(\xi_1) \vee \pi_2(\xi_2)\\ &= R_1(E_{\bar{Q}}(\xi_1), \bar{Q}) \vee R_2(E_{\bar{Q}}(\xi_2), \bar{Q}) \qquad (13)\\ &= R^{\nabla}(E_{\bar{Q}}(X), \bar{Q}) = (\pi_1 \nabla \pi_2)(X), \qquad (14)\end{aligned}$$

that implies condition (ii-r).

It remains to show that also condition (ii-p) follows by exactness. In order to verify this, we will distinguish the following cases: (a) $\pi_1(\xi_1) < \pi_2(\xi_2)$ [or $\pi_1(\xi_1) > \pi_2(\xi_2)$]; (b) $\pi_1(\xi_1) = \pi_2(\xi_2)$.

(a) Suppose that $\pi_1(\xi_1) < \pi_2(\xi_2)$ (for $\pi_1(\xi_1) > \pi_2(\xi_2)$ the proof can be driven similarly). Since $\pi_1(\xi_1) \vee \pi_2(\xi_2) = \pi_2(\xi_2)$ and $R_1(E_{\bar{Q}}(\xi_1), \bar{Q}) \leq \pi_1(\xi_1) < \pi_2(\xi_2)$, by (13) and $\pi_2(\xi_2) \geq R_2(E_{\bar{Q}}(\xi_2), \bar{Q})$ we deduce that $R_1(E_{\bar{Q}}(\xi_1), \bar{Q}) < R_2(E_{\bar{Q}}(\xi_2), \bar{Q}) = \pi_2(\xi_2)$. Condition (ii-p) is therefore established whenever $\pi_1(\xi_1) < \pi_2(\xi_2)$.

(b) Suppose now that $\pi_1(\xi_1) = \pi_2(\xi_2)$ and, without loss of generality, that $R_2(E_{\bar{Q}}(\xi_2), \bar{Q}) \geq R_1(E_{\bar{Q}}(\xi_1), \bar{Q})$. Since

$$\pi_1(\xi_1) \vee \pi_2(\xi_2) \geq R_1(E_{\bar{Q}}(\xi_1), \bar{Q}) \vee R_2(E_{\bar{Q}}(\xi_2), \bar{Q}) = \pi_1(\xi_1) \vee \pi_2(\xi_2),$$

it holds that

$$\pi_2(\xi_2) = \pi_1(\xi_1) \vee \pi_2(\xi_2) = R_1(E_{\bar{Q}}(\xi_1), \bar{Q}) \vee R_2(E_{\bar{Q}}(\xi_2), \bar{Q}) = R_2(E_{\bar{Q}}(\xi_2), \bar{Q}),$$

corresponding to condition (ii-p) under the assumptions above.

(iii) is a direct consequence of Proposition 18. Indeed, we recall (see [20], page 628) that if π_1, π_2 are $\sigma(L^\infty, L^1)$-upper semi-continuous, then it holds that $\partial^{(*)}\pi_i(\xi_i) = \partial^{GP}\pi_i(\xi_i) \cup \{0\}$, where $\partial^{(*)}$ stands for the star-subdifferential (see Definition 17). Furthermore, if ξ_1, ξ_2 are not local minimizers of π_1, π_2 respectively, then (by Proposition 18) equality (33) becomes $\partial^{GP}\pi_1(\xi_1) \cap \partial^{GP}\pi_2(\xi_2) = \partial^{GP}(\pi_1 \nabla \pi_2)(X)$. □

As we will show in Example 10, an allocation attaining the qco-convolution is not necessarily Pareto optimal but only weakly Pareto optimal. Nevertheless, in the following result we prove that when both π_1 and π_2 are supposed to satisfy the additional assumption of cash-additivity, then in Theorem 7(i) exactness of qco-convolution implies that $(\xi_1, \xi_2) \in A(X)$ is also Pareto optimal. Notice that Assumption (Aπ) and cash-additivity of π_1 and π_2 together imply convexity of π_1 and π_2 (see Prop. 2.1 of Cerreia-Vioglio et al. [6]).

Proposition 9 (Convex case) *Let π_1 and π_2 satisfy Assumption (Aπ) and cash-additivity, and let $X \in L^\infty$ be a given aggregate risk.*

If $(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2)$ *for some* $(\xi_1, \xi_2) \in A(X)$*, then* (ξ_1, ξ_2) *is a Pareto optimal allocation.*

Proof The proof uses extensively and is based on Remark 3.1 of Jouini et al. [16].

We may suppose without loss of generality that $\pi_1(\xi_1) \leq \pi_2(\xi_2)$. We will consider separately the following cases: (a) $\pi_1(\xi_1) < \pi_2(\xi_2)$; (b) $\pi_1(\xi_1) = \pi_2(\xi_2)$.

(a) Assume that $\pi_1(\xi_1) < \pi_2(\xi_2)$. Suppose by contradiction that (ξ_1, ξ_2) is not a Pareto optimal allocation, i.e. that there exists $(\tilde{\xi}_1, \tilde{\xi}_2) \in A(X)$ such that

$$\pi_1(\tilde{\xi}_1) \leq \pi_1(\xi_1) \text{ and } \pi_2(\tilde{\xi}_2) \leq \pi_2(\xi_2) \tag{15}$$

where at least one of the inequalities is strict.

(a1) Whenever $\pi_1(\tilde{\xi}_1) = \pi_1(\xi_1)$ and $\pi_1(\xi_1) \leq \pi_2(\tilde{\xi}_2) < \pi_2(\xi_2)$,

$$\pi_1(\tilde{\xi}_1) \vee \pi_2(\tilde{\xi}_2) = \pi_1(\xi_1) \vee \pi_2(\tilde{\xi}_2) < \pi_1(\xi_1) \vee \pi_2(\xi_2) = (\pi_1 \nabla \pi_2)(X)$$

that is a contradiction.

(a2) When $\pi_1(\tilde{\xi}_1) = \pi_1(\xi_1)$ and $\pi_2(\tilde{\xi}_2) < \pi_1(\xi_1) \leq \pi_2(\xi_2)$, we immediately get a contradiction since $\pi_1(\tilde{\xi}_1) \vee \pi_2(\tilde{\xi}_2) = \pi_1(\xi_1) \vee \pi_2(\tilde{\xi}_2) < \pi_1(\xi_1) \vee \pi_2(\xi_2)$.

(a3) Whenever $\pi_1(\tilde{\xi}_1) < \pi_1(\xi_1)$ and $\pi_2(\tilde{\xi}_2) = \pi_2(\xi_2)$,

$$\pi_1(\tilde{\xi}_1) \vee \pi_2(\tilde{\xi}_2) = \pi_2(\tilde{\xi}_2) = \pi_2(\xi_2) = \pi_1(\xi_1) \vee \pi_2(\xi_2) = (\pi_1 \nabla \pi_2)(X).$$

Take now $\eta_1 = \tilde{\xi}_1 + c$ and $\eta_2 = \tilde{\xi}_2 - c$ for $c = \min\{\frac{\pi_1(\xi_1) - \pi_1(\tilde{\xi}_1)}{2}; \frac{\pi_2(\xi_2) - \pi_1(\xi_1)}{2}\} > 0$. By cash-additivity of π_1 and π_2 and by definition of c, it is immediate to check that $(\eta_1, \eta_2) \in A(X)$ satisfies

$$\pi_1(\eta_1) = \pi_1(\tilde{\xi}_1 + c) = \pi_1(\tilde{\xi}_1) + c < \pi_1(\xi_1)$$
$$\pi_2(\eta_2) = \pi_2(\tilde{\xi}_2 - c) = \pi_2(\tilde{\xi}_2) - c < \pi_2(\xi_2).$$

It follows that $\pi_1(\eta_1) \vee \pi_2(\eta_2) < \pi_1(\xi_1) \vee \pi_2(\xi_2) = (\pi_1 \nabla \pi_2)(X)$, hence a contradiction.

(a4) Whenever $\pi_1(\tilde{\xi}_1) < \pi_1(\xi_1)$ and $\pi_2(\tilde{\xi}_2) < \pi_2(\xi_2)$, a contradiction can be found immediately by proceeding as above.

(b) Assume now that $\pi_1(\xi_1) = \pi_2(\xi_2) = p$.

(b1) Whenever $\pi_1(\tilde{\xi}_1) = \pi_1(\xi_1) = p$ and $\pi_2(\tilde{\xi}_2) < \pi_2(\xi_2) = p$, consider $\eta_1 = \tilde{\xi}_1 - c$ and $\eta_2 = \tilde{\xi}_2 + c$ for $c = \frac{\pi_2(\xi_2) - \pi_2(\tilde{\xi}_2)}{2} > 0$. Proceeding as in case (a3), it can be checked that

$$\pi_1(\eta_1) = \pi_1(\tilde{\xi}_1 - c) = \pi_1(\tilde{\xi}_1) - c < \pi_1(\xi_1) = p$$
$$\pi_2(\eta_2) = \pi_2(\tilde{\xi}_2 + c) = \pi_2(\tilde{\xi}_2) + c < \pi_2(\xi_2) = p.$$

As seen above, a contradiction follows immediately by $\pi_1(\eta_1) \vee \pi_2(\eta_2) < p = \pi_1(\xi_1) \vee \pi_2(\xi_2) = (\pi_1 \nabla \pi_2)(X)$.

Cases (b2) ($\pi_1(\tilde{\xi}_1) < \pi_1(\xi_1) = p$ and $\pi_2(\tilde{\xi}_2) = \pi_2(\xi_2) = p$) and (b3) ($\pi_1(\tilde{\xi}_1) < \pi_1(\xi_1) = p$ and $\pi_2(\tilde{\xi}_2) < \pi_2(\xi_2) = p$) can be driven similarly.

Pareto optimality of (ξ_1, ξ_2) follows then by the arguments above. □

4.1 Counterexamples

In this section, we provide some examples that emphasize the following facts: exactness of the qco-convolution does not guarantee that any allocation attaining the infimum in the qco-convolution is Pareto optimal, but only weakly Pareto optimal (Example 10); in Theorem 7(i) the converse implication does not hold in general (Example 11); and, differently from the classical case, exactness of qco-convolution at (ξ_1, ξ_2) does not guarantee that $\partial^{GP}\pi_1(\xi_1) \cap \partial^{GP}\pi_2(\xi_2) \neq \emptyset$ (Example 12).

Example 10 Consider the following functionals $\pi_1, \pi_2 : \mathbb{R} \to \mathbb{R}$:

$$\pi_1(x) = x \quad \text{and} \quad \pi_2(x) = k(x) \triangleq \begin{cases} -1, & x \leq -1 \\ x, & x \in (-1, 1) \\ 1, & x \geq 1 \end{cases} \tag{16}$$

Clearly, π_1 is increasing and linear (hence, quasiconvex), while π_2 is non-decreasing and quasiconvex.

Consider now $x \in [-2, 2]$. It is easy to check that

$$\pi_1(y) \vee \pi_2(x-y) = \begin{cases} 1; & y \leq x-1 \\ x-y; & x-1 < y \leq \frac{x}{2} \\ y; & y \geq \frac{x}{2} \end{cases} \tag{17}$$

Hence for any $x \in [-2, 2]$ the qco-convolution $(\pi_1 \nabla \pi_2)(x)$ is exact, with $(\pi_1 \nabla \pi_2)(x) = \frac{x}{2} = \pi_1\left(\frac{x}{2}\right) \vee \pi_2\left(x - \frac{x}{2}\right)$. It can be deduced by Theorem 7 (or it can be checked easily) that $(\xi, x-\xi) = (\frac{x}{2}, \frac{x}{2})$ is weakly Pareto optimal for any $x \in [-2, 2]$.

Take now $x = 2$. By the arguments above, $(\xi, x - \xi) = (1, 1)$ is a weakly Pareto optimal allocation. Nevertheless, we will show that it is not Pareto optimal. Consider, indeed, any admissible allocation $(\eta, x - \eta)$ with $\eta < 1$. It holds that $\pi_1(\eta) < \pi_1(\xi) = \pi_1(1) = 1$ and $\pi_2(x - \eta) = \pi_2(x - \xi) = \pi_2(1) = 1$. It follows that $(\xi, x - \xi) = (1, 1)$ is not a Pareto optimal allocation since there exist some pairs $(\eta, x - \eta)$ (e.g. with $\eta < 1$) satisfying $\pi_1(\eta) < \pi_1(1)$ and $\pi_2(x - \eta) \leq \pi_2(1)$.

The following example underlines that in Theorem 7(i) the converse implication does not hold, i.e.

$$(\xi_1, \xi_2) \in A(X) \text{ weakly Pareto} \quad \not\Rightarrow \quad (\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2).$$

Example 11 Take π_1 and π_2 as in (16) and $x \in [-2, 2]$.

For any $x \in [-2, 2]$ it holds that $(\pi_1 \nabla \pi_2)(x) = \frac{x}{2}$ (see Example 10). Furthermore, it can be checked easily that any pair $(\xi_1, \xi_2) \in A(x)$ with $\xi_1 \in (-\infty, x-1) \cup (x+1, +\infty)$ is a weakly Pareto optimal allocation. Nevertheless, for $\xi_1 > x + 1$ we get

$$\pi_1(\xi_1) \vee \pi_2(\xi_2) = \pi_1(\xi_1) \vee \pi_2(x - \xi_1) = \xi_1 \vee (-1) = \xi_1 > x + 1 \geq \frac{x}{2} = (\pi_1 \nabla \pi_2)(x),$$

where the third equality is due to $\xi_1 > x + 1 \geq -1$, while the last inequality holds because of $x \in [-2, 2]$.

Even if any pair $(\xi_1, \xi_2) \in A(x)$ with $\xi_1 \in (x + 1, +\infty)$ is a weakly Pareto optimal allocation (as well as $(\pi_1 \nabla \pi_2)(x)$ is exact at $(\xi_1, \xi_2) \in A(x)$), it follows that

$$(\pi_1 \nabla \pi_2)(x) \neq \pi_1(\xi_1) \vee \pi_2(\xi_2).$$

As recalled in Theorem 6, Jouini et al. [16] proved that, for convex risk measures, the exactness of inf-convolution at (ξ_1, ξ_2) is equivalent to the nonemptyness of $\partial\pi_1(\xi_1) \cap \partial\pi_2(\xi_2)$. In the following example, we show that a similar result does not hold in general for quasiconvex risk measures. Namely,

$$(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2) \quad \not\Rightarrow \quad \partial^{GP}\pi_1(\xi_1) \cap \partial^{GP}\pi_2(\xi_2) \neq \emptyset.$$

As we will see in the next section, the converse is true under suitable assumptions.

Example 12 Consider the functions $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ defined as $\pi_1(u) = e_1 \cdot u$ and $\pi_2(u) = k(e_2 \cdot u)$, where $k : \mathbb{R} \to \mathbb{R}$ is defined as in (16) and e_1, e_2 is the canonical basis in $\mathbb{R}^2$, i.e. $e_1 = (1, 0),\ e_2 = (0, 1)$. It is easy to check that π_1 and π_2 are quasiconvex (better, π_1 is also convex), monotone and continuous from above.

Let $x = (x_1, x_2) \in \mathbb{R}^2$ be fixed arbitrarily. To prove exactness of $(\pi_1 \nabla \pi_2)(x)$, we need to find a pair $(\bar{\xi}, \bar{\zeta}) \in A(x) \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ such that $(\pi_1 \nabla \pi_2)(x) = \pi_1(\bar{\xi}) \vee \pi_2(\bar{\zeta})$. For any pair $(\xi, \zeta) \in A(x)$ it is easy to check that

$$\pi_1(\xi) \vee \pi_2(\zeta) = \pi_1(x - \zeta) \vee \pi_2(\zeta)$$
$$= \begin{cases} -1, & \text{if } e_2 \cdot \zeta < -1 \text{ and } e_1 \cdot (x - \zeta) \le -1 \\ e_2 \cdot \zeta, & \text{if } e_2 \cdot \zeta \in [-1, 1] \text{ and } e_1 \cdot (x - \zeta) \le e_2 \cdot \zeta \\ 1, & \text{if } e_2 \cdot \zeta > 1 \text{ and } e_1 \cdot (x - \zeta) \le 1 \\ e_1 \cdot (x - \zeta), & \text{if } e_2 \cdot \zeta < -1 \text{ and } e_1 \cdot (x - \zeta) \ge -1 \\ e_1 \cdot (x - \zeta), & \text{if } e_2 \cdot \zeta \in [-1, 1] \text{ and } e_1 \cdot (x - \zeta) \ge e_2 \cdot \zeta \\ e_1 \cdot (x - \zeta), & \text{if } e_2 \cdot \zeta > 1 \text{ and } e_1 \cdot (x - \zeta) \ge 1 \end{cases}$$

Hence $\inf_{\zeta \in \mathbb{R}^2}\{\pi_1(x - \zeta) \vee \pi_2(\zeta)\} = -1$, where the infimum is attained at any $\zeta \in \mathbb{R}^2$ satisfying $e_2 \cdot \zeta \le -1$ *and* $e_1 \cdot (x - \zeta) \le -1$. This means that $(\pi_1 \nabla \pi_2)(x)$ is exact for any $x \in \mathbb{R}^2$, with $(\pi_1 \nabla \pi_2)(x) = \pi_1(x - \zeta) \vee \pi_2(\zeta)$ for $\zeta \in \mathbb{R}^2$ as above (in particular, for $\zeta = (\zeta_1, -1)$ and $x_1 - \zeta_1 \le -1$).

About the Greenberg-Pierskalla subdifferentials of π_1, π_2, it can be checked easily that $\partial^{GP}\pi_1(u) = (\mathbb{R}_+ \setminus \{0\})e_1$ for any $u \in \mathbb{R}^2$, while $\partial^{GP}\pi_2(u) = (\mathbb{R}_+ \setminus \{0\})e_2$ for any u with $e_2 \cdot u \in [-1, 1]$.

By the arguments above, it follows that $(\pi_1 \nabla \pi_2)(x)$ is exact for any $x \in \mathbb{R}^2$, with $(\pi_1 \nabla \pi_2)(x) = \pi_1(x - \zeta) \vee \pi_2(\zeta)$ e.g. for $\zeta = (\zeta_1, -1)$ and $x_1 - \zeta_1 \le -1$. Nevertheless, for ζ as above it holds that $\partial^{GP}\pi_1(x - \zeta) \cap \partial^{GP}\pi_2(\zeta) = \emptyset$.

4.2 Exactness and existence of weak Pareto optimal allocations

Example 12 shows that, for quasiconvex functions, exactness of $(\pi_1 \nabla \pi_2)(X)$ at (ξ_1, ξ_2) does not imply that $\partial^{GP}\pi_1(\xi_1) \cap \partial^{GP}\pi_2(\xi_2) \neq \emptyset$. Nevertheless, the converse is true under further assumptions on π_1 and π_2.

Theorem 13 *Let π_1, π_2 be two quasiconvex risk measures satisfying assumption $(A\pi)$ (with associated functionals R_1, R_2). Assume also that π_1 and π_2 are radially continuous on L^∞, i.e. for any $X \in L^\infty$ it holds that: $f_i(t) = \pi_i(X + tY)$ is continuous at 0 for any $Y \in L^\infty$ (for $i = 1, 2$).*

Let $X \in L^\infty$ be an aggregate risk and $\xi_1, \xi_2 \in A(X)$ be such that π_1, π_2 are finite at ξ_1, ξ_2 and Lipschitz[1] *on $\{\pi_1 < \pi_1(\xi_1)\}$ and $\{\pi_2 < \pi_2(\xi_2)\}$, respectively.*

If $\partial^{GP}\pi_1(\xi_1) \cap \partial^{GP}\pi_2(\xi_2) \neq \emptyset$ and $\pi_1(\xi_1) = \pi_2(\xi_2)$, then $(\pi_1 \nabla \pi_2)(X)$ is exact and $(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2)$.

Proof Notice that, under the assumptions above, $\partial^{GP}\pi_1(\xi_1) \cap \partial^{GP}\pi_2(\xi_2) \neq \emptyset$ implies that $\partial^{<}\pi_i(\xi_i) \neq \emptyset$, for $i = 1, 2$, where $\partial^{<}$ denotes the Plastria subdifferential (see Definition 17). By Proposition 21 of Penot [21], indeed, under our assumptions it holds that

$$\partial^{GP}\pi_i(\xi_i) = \partial^{(*)}\pi_i(\xi_i) = (0, 1]\partial^{<}\pi_i(\xi_i), \qquad i = 1, 2. \tag{18}$$

[1] π is said to be Lipschitz on the strict lower level set $\{\pi < \pi(\xi)\}$ if there exists $c > 0$ such that $|\pi(X) - \pi(Y)| \le c\|X - Y\|_\infty$ for any $X, Y \in L^\infty$ such that $\pi(X), \pi(Y) < \pi(\xi)$. Roughly speaking, this means that π is $\|\cdot\|_\infty$-continuous on the strict lower level set.

Now we want to show that the nonemptyness of both $\partial^< \pi_i(\xi_i)$ implies that $\partial^< \pi_1(\xi_1) \nabla \partial^< \pi_2(\xi_2)$ (defined in (34)) is nonempty too. By (34) and by the arguments above, it is sufficient to verify that

$$(\partial^{GP} \pi_1(\xi_1) \cap \partial^< \pi_2(\xi_2)) \cup (\partial^< \pi_1(\xi_1) \cap \partial^{GP} \pi_2(\xi_2)) \neq \emptyset. \tag{19}$$

Let $\bar{\xi}$ be an element of $\partial^{GP} \pi_1(\xi_1) \cap \partial^{GP} \pi_2(\xi_2)$. Then, by (18), there exist $\lambda_i \in (0, 1]$ and $\bar{\xi}_i \in \partial^< \pi_i(\xi_i)$ $(i = 1, 2)$ such that $\bar{\xi} = \lambda_1 \bar{\xi}_1$ and $\bar{\xi} = \lambda_2 \bar{\xi}_2$.

Suppose now that $\lambda_1 \leq \lambda_2$. We then have $\bar{\xi}_2 = \lambda^{1,2} \bar{\xi}_1$, with $\lambda^{1,2} \triangleq \frac{\lambda_1}{\lambda_2} \leq 1$. We can conclude that $\bar{\xi}_2 \in (0, 1]\partial^< \pi_1(\xi_1)$, hence $\bar{\xi}_2 \in \partial^{GP} \pi_1(\xi_1) \cap \partial^< \pi_2(\xi_2)$. Consequently,

$$\partial^{GP} \pi_1(\xi_1) \cap \partial^< \pi_2(\xi_2) \neq \emptyset. \tag{20}$$

If $\lambda_1 \geq \lambda_2$, we can proceed similarly so to obtain that

$$\partial^< \pi_1(\xi_1) \cap \partial^{GP} \pi_2(\xi_2) \neq \emptyset. \tag{21}$$

The arguments above imply that at least one between (20) and (21) is true. Hence (19) is verified and, because of (34), $\partial^< \pi_1(\xi_1) \nabla \partial^< \pi_2(\xi_2)$ is nonempty. Applying Proposition 19, we conclude that $(\pi_1 \nabla \pi_2)(X)$ is exact with $(\pi_1 \nabla \pi_2)(X) = \pi_1(\xi_1) \vee \pi_2(\xi_2)$. □

Under suitable assumptions, an existence result similar to the one established in Jouini et al. [16] for Pareto optimal allocations with convex risk measures can be proved even in the quasiconvex case for weakly Pareto optimal allocations.

Theorem 14 (Existence) *Let π_1, π_2 be two law invariant risk measures satisfying Assumption (Aπ), $\|\cdot\|_\infty$-continuous and consistent with respect to the Second Order Stochastic Dominance. Let $X \in L^\infty$ be a given aggregate risk.*

Then there exists a weakly Pareto optimal allocation.

Proof The proof can be driven as in Lemma 6.1 and Theorem 3.2 of Jouini et al. [16] (up to small changes) and by applying Theorem 7(i). □

Similarly to the convex case, the assumption of consistency with respect to the Second Order Stochastic Dominance may be dropped when the probability space is, for instance, non atomic (see Theorem 5.1 of Cerreia-Vioglio et al. [6]).

4.3 Optimal risk sharing

In this section, we will focus on (weakly) optimal risk sharing for quasiconvex risk measures. Note that, by Theorem 3.1 of Jouini et al. [16] (see also Theorem 6), the classical definition of an optimal risk sharing (ξ_1^*, ξ_2^*) is equivalent to requiring that $(\pi_1 \Box \pi_2)(X_1 + X_2)$ is exact at $(\xi_1^*, \xi_2^*) \in A(X)$ and that individual rationality (i.e. $\pi_i(\xi_i^*) \leq \pi_i(X_i)$ for $i = 1, 2$) is verified. Supported by this fact, in the present setting we will focus on *weakly optimal risk sharing*, where with weakly optimal risk sharing we mean any allocation $(\xi_1^*, \xi_2^*) \in A(X)$ at which $(\pi_1 \nabla \pi_2)(X)$ is exact and satisfying individual rationality.

For convex risk measures, Jouini et al. [16] proved that an optimal risk sharing may be obtained starting from a Pareto optimal allocation and taking into account a "suitable" price. We wonder whether a similar result holds true also for quasiconvex risk measures.

In order to characterize weakly optimal risk sharing for quasiconvex functions and following the approach of Jouini et al. [16], we define

$$p_1(\eta) \triangleq \pi_1(X_1) - \pi_1(X_1 - \eta); \qquad p_2(\eta) \triangleq \pi_2(X_2 + \eta) - \pi_2(X_2), \tag{22}$$

for any $\eta \in L^\infty$. As already pointed out, η can be seen as the risk transferred from insurer to reinsurer (say, from agent 1 to agent 2). In the framework of convex risk measures (where cash-additivity holds), $p_1(\eta)$ can be understood as the maximal price that agent 1 would pay because of the "risk exchange", while $p_2(\eta)$ represents the reinsurance premium or the minimal price that agent 2 would like to receive because of the additional risk η. Always in the convex setting, Jouini et al. [16] proved that, given a Pareto optimal allocation $(X_1 - \xi^*, X_2 + \xi^*)$, the allocation $(X_1 - \xi^* + p, X_2 + \xi^* - p)$ (with $p \in \mathbb{R}$) is an optimal risk sharing if and only if $p \in [p_1(\xi^*), p_2(\xi^*)]$ (see Theorem 3.3 of Jouini et al. [16]), that is iff p is a price at which both agents would agree to exchange the risk ξ^*.

In the quasiconvex case, the situation is a little bit different and difficult because of the lack of cash-additivity. In the next theorem we give a necessary condition for a weakly optimal risk sharing; further, a sufficient condition can be given by imposing on π_2 the additional assumption of cash-additivity.

Theorem 15 *Let $\pi_1, \pi_2 : L^\infty \to \mathbb{R}$ be two risk functionals (or insurance premiums) satisfying assumption $(A\pi)$ and cash-subadditivity, and let $X_1, X_2 \in L^\infty$ with aggregate risk $X = X_1 + X_2$.*

Assume that $\pi_1(X_1) \geq \pi_2(X_2)$ and that $(\pi_1 \nabla \pi_2)(X)$ is exact at a pair $(X_1 - \xi^*, X_2 + \xi^*)$.

(i) If $\pi_1(X_1) = \pi_2(X_2)$, then $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly optimal risk sharing.

(ii) If $\pi_1(X_1) > \pi_2(X_2)$, then either $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly optimal risk sharing or the following holds:
if $(X_1 - \xi^* + p, X_2 + \xi^* - p)$ is a weakly optimal risk sharing for some $p > 0$, then p satisfies:

$$\pi_1(X_1 - \xi^* + p) = \pi_1(X_1 - \xi^*) \vee \pi_2(X_2 + \xi^*) \tag{23}$$

and

$$p \geq \max\left\{\pi_2(X_2 + \xi^*) - \pi_2(X_2); \pi_2(X_2 + \xi^*) - \pi_1(X_1 - \xi^*)\right\}. \tag{24}$$

If, in addition, π_2 is cash-additive, then also the converse holds true. More precisely, if $p > 0$ satisfies

$$\pi_1(X_1 - \xi^* + p) = \pi_1(X_1 - \xi^*) \vee \pi_2(X_2 + \xi^*) \tag{25}$$

and

$$p \geq \max\left\{\pi_2(X_2 + \xi^*) - \pi_2(X_2); \pi_2(X_2 + \xi^*) - \pi_1(X_1 - \xi^*)\right\}, \tag{26}$$

then $(X_1 - \xi^* + p, X_2 + \xi^* - p)$ is a weakly optimal risk sharing allocation.

When $\pi_1(X_1) \leq \pi_2(X_2)$, the result above can be reformulated similarly.

Notice that inequality (24) can be rewritten as

$$p \geq \max\{p_2(\xi^*); p_2(\xi^*) + p_1(\xi^*) + \pi_2(X_2) - \pi_1(X_1)\}.$$

Differently from Jouini et al. [16], in the quasiconvex case the constraint on p depends not only on $p_1(\xi^*)$ and $p_2(\xi^*)$, but also on the difference between $\pi_1(X_1)$ and $\pi_2(X_2)$.

Proof of Theorem 15 (i) Suppose that $\pi_1(X_1) = \pi_2(X_2)$. Since, by assumption, $(\pi_1 \nabla \pi_2)(X)$ is exact at $(X_1 - \xi^*, X_2 + \xi^*)$, we have

$$\pi_1(X_1 - \xi^*) \vee \pi_2(X_2 + \xi^*) \leq \pi_1(X_1) \vee \pi_2(X_2) = \pi_1(X_1) = \pi_2(X_2).$$

Hence

$$\pi_1(X_1 - \xi^*) \leq \pi_1(X_1) \vee \pi_2(X_2) = \pi_1(X_1)$$

and

$$\pi_2(X_2 + \xi^*) \leq \pi_1(X_1) \vee \pi_2(X_2) = \pi_2(X_2).$$

We can conclude that $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly optimal risk sharing rule.
(ii) Suppose now that $\pi_1(X_1) > \pi_2(X_2)$. Since, by assumption, $(\pi_1 \nabla \pi_2)(X)$ is exact at $(X_1 - \xi^*, X_2 + \xi^*)$, we have necessarily that $\pi_1(X_1 - \xi^*), \pi_2(X_2 + \xi^*) \leq \pi_1(X_1)$. One of the following cases can then occur:

1. $\pi_1(X_1 - \xi^*) \leq \pi_1(X_1)$ and $\pi_2(X_2 + \xi^*) \leq \pi_2(X_2)$
2. $\pi_1(X_1 - \xi^*) \leq \pi_2(X_2) \leq \pi_2(X_2 + \xi^*) \leq \pi_1(X_1)$
3. $\pi_2(X_2) \leq \pi_1(X_1 - \xi^*) \leq \pi_2(X_2 + \xi^*) \leq \pi_1(X_1)$
4. $\pi_2(X_2) \leq \pi_2(X_2 + \xi^*) \leq \pi_1(X_1 - \xi^*) \leq \pi_1(X_1)$

Let us consider separately the four cases above. In the first one, we see immediately that $(X_1 - \xi^*, X_2 + \xi^*)$ is a weakly optimal risk sharing allocation. Notice that the same is no more true in the remaining cases.

Concerning the second and the third case, let us assume that $(X_1 - \xi^* + p, X_2 + \xi^* - p)$, for $p > 0$, is a weakly optimal risk sharing allocation. This implies that

$$\pi_2(X_2 + \xi^* - p) \leq \pi_2(X_2) \leq \pi_2(X_2 + \xi^*)$$

and

$$\pi_1(X_1 - \xi^* + p) = \pi_2(X_2 + \xi^*) \leq \pi_1(X_1), \tag{27}$$

where the equality in (27) follows from exactness of $(\pi_1 \nabla \pi_2)(X)$ at both $(X_1 - \xi^* + p, X_2 + \xi^* - p)$ and $(X_1 - \xi^*, X_2 + \xi^*)$, hence

$$\pi_1(X_1 - \xi^* + p) \vee \pi_2(X_2 + \xi^* - p) = \pi_1(X_1 - \xi^*) \vee \pi_2(X_2 + \xi^*) = \pi_2(X_2 + \xi^*).$$

Notice that condition (23) follows immediately from (27) and $\pi_1(X_1 - \xi^*) \leq \pi_2(X_2 + \xi^*)$ (true in the present cases).

Moreover, by cash-subadditivity of π_1 and π_2 we get

$$\pi_2(X_2 + \xi^* - p) \geq \pi_2(X_2 + \xi^*) - p \tag{28}$$

$$\pi_1(X_1 - \xi^* + p) \leq \pi_1(X_1 - \xi^*) + p \tag{29}$$

for any $p \geq 0$. Notice that (28) can be rewritten as

$$p \geq \pi_2(X_2 + \xi^*) - \pi_2(X_2 + \xi^* - p) \geq \pi_2(X_2 + \xi^*) - \pi_2(X_2) \tag{30}$$

because of monotonicity of π_2, while, combining (27) with (29), we get

$$p \geq \pi_2(X_2 + \xi^*) - \pi_1(X_1 - \xi^*).$$

Condition (24) follows therefore by the last two inequalities.

Now, let us consider the fourth case. Let us assume that $(X_1 - \xi^* + p, X_2 + \xi^* - p)$ is a weakly optimal risk sharing allocation for some $p > 0$. Since $\pi_2(X_2 + \xi^* - p) \leq \pi_2(X_2) \leq \pi_2(X_2 + \xi^*) \leq \pi_1(X_1 - \xi^*)$, it follows that

$$\pi_1(X_1 - \xi^* + p) = \pi_1(X_1 - \xi^*) = \pi_1(X_1 - \xi^*) \vee \pi_2(X_2 + \xi^*) \leq \pi_1(X_1),$$

where the first equality is due to exactness of $(\pi_1 \nabla \pi_2)(X)$ at $(X_1 - \xi^* + p, X_2 + \xi^* - p)$. Equality (23) has been therefore established. Moreover, by proceeding as above we deduce again (30). This implies immediately condition (24) since, in the present case, $\pi_2(X_2 + \xi^*) - \pi_1(X_1 - \xi^*) \leq 0$, while $\pi_2(X_2 + \xi^*) - \pi_2(X_2) \geq 0$.

Finally, let us assume that π_2 is cash-additive and that $\pi_1(X_1-\xi^*+p)=\pi_2(X_2+\xi^*)\vee\pi_1(X_1-\xi^*)$.

If both $\pi_1(X_1-\xi^*)$ and $\pi_2(X_2+\xi^*)$ are smaller than (or equal to) $\pi_2(X_2)$, then exactness and individual rationality of $(X_1-\xi^*+p, X_2+\xi^*-p)$ are trivial. By exactness of $(\pi_1\nabla\pi_2)(X)$ at $(X_1-\xi^*, X_2+\xi^*)$, it can never happen that $\pi_2(X_2)<\pi_1(X_1)<\pi_1(X_1-\xi^*)\vee\pi_2(X_2+\xi^*)$. If $\pi_2(X_2)\le\pi_2(X_2+\xi^*)\vee\pi_1(X_1-\xi^*)\le\pi_1(X_1)$, then we can consider the following cases:

1. $\pi_2(X_2+\xi^*)\le\pi_2(X_2)\le\pi_1(X_1-\xi^*)\le\pi_1(X_1)$
2. $\pi_1(X_1-\xi^*)\le\pi_2(X_2)\le\pi_2(X_2+\xi^*)\le\pi_1(X_1)$
3. $\pi_2(X_2)\le\pi_2(X_2+\xi^*)\le\pi_1(X_1-\xi^*)\le\pi_1(X_1)$
4. $\pi_2(X_2)\le\pi_1(X_1-\xi^*)\le\pi_2(X_2+\xi^*)\le\pi_1(X_1)$

It is easy to check that in all the four cases above, condition (25) implies exactness at $(X_1-\xi^*+p, X_2+\xi^*-p)$. Moreover, condition (26) and cash-additivity of π_2 imply that $\pi_2(X_2+\xi^*-p)=\pi_2(X_2+\xi^*)-p\le\pi_2(X_2)$. The condition $\pi_1(X_1-\xi^*+p)\le\pi_1(X_1)$ follows immediately from the arguments above. We can conclude that, for $p>0$ satisfying (25) and (26), $(X_1-\xi^*+p, X_2+\xi^*-p)$ is a weakly optimal risk sharing allocation. □

Differently from Jouini et al. [16], the existence of a weakly optimal risk sharing is not guaranteed a priori. This fact is related to the lack of cash-additivity of at least one between the risk measures taken into account.

In order to illustrate the meaning of Theorem 15 we provide the following example.

Example 16 Consider the risk measures π_1 and π_2 defined as

$$\pi_1(X)=g(\mathbb{E}[X])=\mathbb{E}[X]\wedge 0;\quad \pi_2(X)=\mathbb{E}[X].$$

We notice that π_1 is cash-subadditive while π_2 is cash-additive. Moreover, it is easy to check that both π_1 and π_2 are increasing, continuous and quasiconvex. By the arguments given in Section 3, it follows that π^∇ is monotone, continuous from above and quasiconvex.

Let now $X, X_1, X_2\in L^\infty$ such that $\mathbb{E}[X_1]=2$, $\mathbb{E}[X_2]=-1$ and $X=X_1+X_2$. Then $\pi_1(X_1)=0$ while $\pi_2(X_2)=-1$. Moreover,

$$\begin{aligned}(\pi_1\nabla\pi_2)(X)&=\inf_{\xi\in L^\infty}\{(\mathbb{E}[X_1-\xi]\wedge 0)\vee\mathbb{E}[X_2+\xi]\}\\&=\inf_{\xi\in L^\infty}\{((2-\mathbb{E}[\xi])\wedge 0)\vee(\mathbb{E}[\xi]-1)\}=0.\end{aligned}$$

The minimum in the expression above is realized for any $\xi^*\in L^\infty$ such that $x^*=\mathbb{E}[\xi^*]\le 1$. Hence any pair of the form $(X_1-\xi^*, X_2+\xi^*)$ such that $x^*=\mathbb{E}[\xi^*]\le 1$ is exact; nevertheless, for any $\xi^*\in L^\infty$ such that $x^*>0$ this pair is not an optimal risk sharing. In fact, $\pi_1(X_1-\xi^*)=(2-x^*)\wedge 0=0=\pi_1(X_1)$ while $\pi_2(X_2+\xi^*)=x^*-1>-1=\pi_2(X_2)$.

In the spirit of Theorem 15, we construct a weakly optimal risk sharing $(X_1-\xi^*+p; X_2+\xi^*-p)$ taking into account a suitable "price" $p>0$. More precisely, we look for $p>0$ satisfying

$$\pi_1(X_1-\xi^*+p)=\pi_1(X_1-\xi^*)\vee\pi_2(X_2+\xi^*)\tag{31}$$

and

$$p\ge\max\left\{\pi_2(X_2+\xi^*)-\pi_2(X_2), \pi_2(X_2+\xi^*)-\pi_1(X_1-\xi^*)\right\}.\tag{32}$$

Equality (31) requires that $(2-x^*+p)\wedge 0=0$, that is $p\ge x^*-2$, while inequality (32) implies that $p\ge x^*$. Hence any $p\ge x^*=\mathbb{E}[\xi^*]$ is the suitable price we were looking for.

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5 Appendix

In the following, we recall some basic definitions and results on different notions of subdifferentiability for quasiconvex functions. We refer to Penot [21] and to Penot and Zalinescu [23] for a deep and wide treatment.

Definition 17 *(*see Penot [21], Penot and Zalinescu [23]) Let $\mathcal{X}$ be a locally convex topological vector space and $\mathcal{X}^*$ be its topological dual space. Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a quasiconvex function and let $x_0 \in \mathcal{X}$ be such that $f(x_0)$ is finite.

The Greenberg-Pierskalla subdifferential $\partial^{GP} f(x_0)$, the star subdifferential $\partial^{(*)} f(x_0)$ and the lower subdifferential of Plastria $\partial^< f(x_0)$ of f at $x_0 \in \mathcal{X}$ are defined, respectively, as

$$\partial^{GP} f(x_0) \triangleq \left\{x^* \in \mathcal{X}^* : \langle x^*, x - x_0\rangle < 0 \ \text{ for any } \ x \in \{f < f(x_0)\}\right\}$$
$$\partial^{(*)} f(x_0) \triangleq \left\{x^* \in \mathcal{X}^* : \langle x^*, x - x_0\rangle \leq 0 \ \text{ for any } x \in \{f < f(x_0)\}\right\}$$
$$\partial^{<} f(x_0) \triangleq \{x^* \in \mathcal{X}^* : \langle x^*, x - x_0\rangle \leq f(x) - f(x_0) \ \text{ for any } x \in \{f < f(x_0)\}\}.$$

Notice that the last definition is similar but weaker than the one of Fenchel-Moreau subdifferential. Other relations among the different notions above can be found in Penot [21] and Penot and Zalinescu [23], among others.

Proposition 18 (see Prop. 2.8 of Penot and Zalinescu [23]) *Let $f : \mathcal{X} \to \mathbb{R}$ and $g : \mathcal{X} \to \mathbb{R}$ be two given functions and let $z_0 \in \mathcal{X}$. If the qco-convolution $(f \nabla g)(z_0)$ is exact with $(f\nabla g)(z_0) = f(x_0) \vee g(y_0)$ for $x_0, y_0 \in \mathcal{X}$ such that $x_0 + y_0 = z_0$ and $f(x_0) = g(y_0)$, then*

$$(\partial^{(*)} f(x_0) \cap \partial^{GP} g(y_0)) \cup (\partial^{GP} f(x_0) \cap \partial^{(*)} g(y_0)) \subseteq \partial^{GP} (f\nabla g)(z_0), \tag{33}$$

where equality holds if x_0, y_0 are not local minimizers of f, g, respectively.

Given two functionals $f, g : \mathcal{X} \to \bar{\mathbb{R}}$, given $x_0, y_0 \in \mathcal{X}$ and the subdifferentials $\partial^< f(x_0)$ and $\partial^< g(y_0)$, the set $(\partial^< f(x_0))\nabla(\partial^< g(y_0))$ is defined as

$$(\partial^< f(x_0)) \nabla (\partial^< g(y_0)) \triangleq \left(\bigcup_{\lambda\in(0,1)} [\lambda\partial^< f(x_0) \cap (1-\lambda)\partial^< g(y_0)]\right)$$
$$\cup (\partial^{(*)} f(x_0) \cap \partial^< g(y_0)) \cup (\partial^< f(x_0) \cap \partial^{(*)} g(y_0)). \tag{34}$$

See Penot and Zalinescu [23] for a general definition and for further explanations.

Proposition 19 (see Prop. 3.23 of Penot and Zalinescu [23]) *Let $f, g : \mathcal{X} \to \bar{\mathbb{R}}$ be two quasiconvex functions, $x_0 \in \operatorname{dom} f$ and $y_0 \in \operatorname{dom} g$.*

If $(\partial^< f(x_0)) \nabla (\partial^< g(y_0))$ is nonempty and $f(x_0) = g(y_0)$, then $(f\nabla g)(x_0 + y_0)$ is exact with $(f\nabla g)(x_0 + y_0) = f(x_0) \vee g(y_0)$, and $(\partial^< f(x_0)) \nabla (\partial^< g(y_0)) \subseteq \partial^<(f\nabla g)(x_0 + y_0)$.

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