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Investment and consumption without commitment

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Abstract In this paper, we investigate the Merton portfolio management problem in the context of non-exponential discounting. This gives rise to time-inconsistency of the decision-maker. If the decision-maker at time $t = 0$ can commit her successors, she can choose the policy that is optimal from her point of view, and constrain the others to abide by it, although they do not see it as optimal for them. If there is no commitment mechanism, one must seek a subgame-perfect equilibrium policy between the successive decision-makers. In the line of the earlier work by Ekeland and Lazrak (Preprint, 2006) we give a precise definition of equilibrium policies in the context of the portfolio management problem, with finite horizon. We characterize them by a system of partial differential equations, and establish their existence in the case of CRRA utility. An explicit solution is provided for the case of logarithmic utility. We also investigate the infinite-horizon case and provide two different equilibrium policies for CRRA utility (in contrast with the case of exponential discounting, where there is only one optimal policy). Some of our results are proved under the assumption that the discount function $h(t)$ is a linear combination of two exponentials, or is the product of an exponential by a linear function.

Keywords Portfolio optimization · Merton problem · Equilibrium policies

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1 Introduction

The discounted utility model (DU) has been in use since the beginning of economic theory. Landmark papers are the ones by Ramsey in 1928 and Samuelson in 1937. There is by now a very rich literature, the common assumption being that the discount rate is constant over time so the discount function is exponential. The model makes it possible to compare outcomes occurring at different times by discounting future utility by some constant factor. A decision maker with high discount rates exhibits more impatience (care less about the future) than one with low discount rates. Most of financial-economics works have considered that the rate of time preference is constant (exponential discounting). However there is growing evidence to suggest that this may not be the case. Ainslie [1] performed empirical studies on human and animal behavior and found that discount functions are almost hyperbolic, that is they decrease like a negative power of time rather than an exponential. Loewenstein and Prelec [8] present four drawbacks of exponential discounting and propose a model which accounts for them. They analyze implications for savings behavior and estimation of discount rates.

As soon as discounting is non-exponential, the decision-maker becomes time-inconsistent: a policy, to be implemented after time $t > 0$, which is optimal when discounted at time 0, no longer is optimal if it is discounted at a later time, for instance t. If the decision-maker at time 0 can commit the later ones, that is, constrain them to follow the policy she has decided upon, then the policy which is optimal from her perspective can be implemented. But, apart for the dubious validity of doing that (why should her perspective be better than the one of decision-maker at time t, who, after all, will have to carry out a policy decided long before, and will be the one to bear its consequences?) it is not often the case that management decisions are irreversible; there will usually be many opportunities to reverse a decision which, as times goes by, seems ill-advised.

1.1 Existing research

Dynamic inconsistent behavior was first formalized analytically by Strotz [16]. Further work by Pollak [15], Peleg and Yaari [13], Goldmann [5] on this issue advocates that the policies to be followed should be the output of an intra-personal game among different temporal selves (today's self is a different player from tomorrow's self). Laibson [9] considers a discrete time consumption-investment economy without uncertainty. An agent (self) observes past consumption and financial wealth levels and chooses the consumption level for period t. They establish existence of a unique subgame perfect equilibrium in special cases, namely CRRA utilities. It is characterized by time-dependent consumption rules which are linear in wealth and should satisfy an Euler type equation. The work of Barro [2] is in a deterministic Ramsey model of economic growth with logarithmic utility; the discount function has a special form, whereby the rate of time preference is high in the near future but almost constant in the distant future. That is in [7] they consider instantaneous rates of time preference decreasing in time and vanishing at infinity. Krusell and Smith [7] also consider a discrete non-stochastic Ramsey paradigm with hyperbolic discounting (discount function is generalized hyperbola). They seek the equilibrium policy as the solution of a subgame-perfect equilibrium where the players are the agent and her future selves, and they show that there are several solutions to this problem. All these works are in the deterministic case, usually with discrete time.

Ekeland and Lazrak [4] consider a deterministic problem with continuous time, namely the Ramsey problem of economic growth with non-exponential discounting. They define a subgame perfect equilibrium strategies by letting the decision maker at time t build a coalition with her immediate successors s, with $s \in [t, t+\epsilon]$ and by letting $\epsilon \to 0$. They show that these strategies are characterized by two equivalent equations: a partial differential equation with a non-local term and an integral equation. The PDE coincide with the Hamilton–Jacobi–Bellmann equation of optimal control if the discounting is exponential.

1.2 Our contributions

The goal of this paper is to understand how non-exponential discounting affects an agent's investment-consumption policies in a Merton model.

Dynamic asset allocation has received a lot of attention lately. The ground-breaking paper in this literature is Merton [11]. He considered a model consisting of a risk-free asset with constant rate of return and one or more stocks, each with constant mean rate of return and volatility. An agent invests in this market and consumes to maximize her expected utility of intertemporal consumption and final wealth. Merton was able to derive a closed form solution for constant relative risk aversion (CRRA) and constant absolute risk aversion (CARA) preferences. It turns out that for CRRA preferences, the optimal consumption and investment in the risky asset are a constant proportion of wealth. This is not the case for CARA preferences although the optimal policy is still linear in wealth. Karatzas et al. [6], Cox and Huang [3] also solved the optimal investment and consumption problem by the static martingale method. In another paper Merton [10] also investigates the infinite horizon case. Although it yields the same quantitative results it is easier to handle -provided the psychological discount rate is related to the interest rate by a condition known in the literature as the transversality condition. It does not seem to have much empirical support, and what happens if it is not satisfied is a matter of debate.

All the papers mentioned above are in the exponential discounting paradigm which as we shown earlier has been challenged by some economic literature.

In a framework with non-exponential discounting we show that doing naive optimization does not work in the absence of commitment technologies. Then we follow the approach of [4] and introduce the concept of equilibrium policies in a stochastic context. In the finite horizon case we give a first description of the equilibrium policies for a general discount function through the solutions of a flow of BSDEs (one for every time instant). If the discount function is exponential, these conditions reduce to the classical HJB equation, so the equilibrium policy coincides with the optimal one given by dynamic programming. We then introduce special classes of discount functions, type I which is a linear combination of two exponentials, and type II which is the product of an exponential by a polynomial of degree one, and we show that for discount functions in that class, equilibrium solutions can be characterized by a system of HJB equations. We go one step further and characterize them through a parabolic PDE system which does not seem to have been considered before. Existence of a solution can be established in the case of CRRA utility. A closed form solution is presented for logarithmic utility. The equilibrium policies for this risk preferences and type I, II discounting are the same as the optimal policy in the case of exponential discounting namely to invest a constant proportion in the risky asset. However the equilibrium consumption rates are different. Furthermore we performed numerical experiments to emphasize this difference.

In an infinite horizon model the novel feature is stationarity. To borrow a sentence from [4], the decision maker at time t resets her watch so that time s becomes $s - t$, so she faces the same problem as the decision-maker at time 0. Keeping this in mind we define the equilibrium policies in this context. The infinite horizon causes some difficulties even for the traditional optimal control problems with exponential discounting (see eg transversality condition), hence we restrict our analysis to CRRA preferences. We follow the same approach as for finite horizon: first we describe the equilibrium policies through the adjoint processes and then for the special discounts through a system of ODEs and an integral equation. As one might probably have expected the equilibrium policies and optimal ones coincide if the discounting is exponential given some transversality condition. However we show that in the more general case, the equilibrium policy is not unique. More precisely for type I discounts we find two equilibrium policies. In this case the solution yielding a higher expected utility is preferred.

The equilibrium policies for CRRA preferences resemble the optimal one. It turns out that for all three discounts considered the agent invests the same constant proportion of her wealth in the risky asset. The equilibrium consumption rates, although constants, are different.

1.3 Organization of the paper

The reminder of this paper is organized as follow. In Sect. 2 we describe the model and formulate the objective. Section 3 treats the finite horizon and Sect. 4 treats the infinite horizon problem. The conclusions are summarized in Sect. 5. The paper ends with Appendix containing the proofs.

2 The model and problem formulation

2.1 Financial market

We adopt a model for the financial market consisting of a saving account and one stock (risky asset). The inclusion of more risky assets can be achieved by notational changes. The saving account accrues interest at the riskless rate $r > 0$. The stock price per share follows an exponential Brownian motion

$$dS(t) = S(t)\,[\alpha\,dt + \sigma\,dW(t)]\,, \quad 0 \le t \le \infty,$$

where $\{W(t)\}_{t\in[0,\infty)}$ is a $1-$dimensional Brownian motion on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{0\le t\le T}, \mathcal{F}, \mathbb{P})$. The filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ is the completed filtration generated by $\{W(t)\}_{t\in[0,\infty)}$. As usual α is *the mean rate of return* and $\sigma > 0$ is *the volatility*. Let us denote by $\mu \triangleq \alpha - r > 0$ *the excess return*.

2.2 Investment-consumption policies and wealth processes

A decision-maker in this market is continuously investing her wealth in the stock/bond and is consuming. An investment-consumption policy is determined by the proportion of current wealth she invests in the bond/stock and by the consumption rate. Formally we have:

Definition 2.1 An $\mathbb{R}^2$-valued stochastic process $\{\zeta(t), c(t)\}_{t\in[0,\infty)}$ is called an admissible policy process if it is progressively measurable, $c(t) \ge 0,$ for all $t \in [0,\infty)$, it satisfies

$$\sup_{0\le t\le T} \mathbb{E}[|\zeta(t)|^m + c^m(t)] < \infty \quad \forall m = 1, 2, \ldots, \quad \text{and } T > 0, \tag{2.1}$$

and the corresponding wealth process (see below) is positive $\mathbb{P}$ a.s.

This condition originates from [14] and is needed since we are using his result.

Given a policy process $\{\zeta(t), c(t)\}_{t\in[0,\infty)}$, $\zeta(t)$ is the proportion of wealth, denoted by $X^{\zeta,c}(t)$, invested in the stock at time t and $c(t)$ is the proportion of wealth consumed. The equation describing the dynamics of wealth $X^{\zeta,c}(t)$ is given by

$$\begin{aligned} dX^{\zeta,c}(t) &= X^{\zeta,c}(t)\left((\alpha\zeta(t)-c(t))\,dt+\sigma\zeta(t)\,dW(t)\right)+(1-\zeta(t))X^{\zeta,c}(t)r\,dt \\ &= X^{\zeta,c}(t)((r+\mu\zeta(t)-c(t))\,dt+\sigma\zeta(t)\,dW(t)). \end{aligned} \tag{2.2}$$

It simply says that the changes in wealth over time are due solely to gains/loses from investing in stock, from consumption and there is no cashflow coming in or out. This is usually referred to as the self-financing condition.

Under the regularity condition (2.1) imposed on $\{\zeta(t), c(t)\}_{t\in[0,\infty)}$ above, equation (2.2) admits a unique strong solution given by the explicit expression

$$X^{\zeta,c}(t) = X(0)\exp\left(\int_0^t \left(r+\mu\zeta(u)-c(u)-\frac{1}{2}|\sigma\zeta(u)|^2\right)du+\int_0^t \sigma\zeta(u)\,dW(u)\right),$$

The initial wealth $X^{\zeta,c}(0) = X(0) \in (0,\infty)$, is exogenously specified.

2.3 Utility Function

All decision-makers have the same von Neuman-Morgenstern utility. This is crucial for understanding the model: time-inconsistency arise, not from a change in preferences, but from the way the future is discounted. All decision-makers try to maximize the discounted expectation of a function $U : (0,\infty) \to \mathbb{R}$ strictly increasing and strictly concave, which is their (common) utility. We restrict ourselves to utility functions which are continuously differentiable and satisfy the Inada conditions

$$U'(0+) \triangleq \lim_{x\downarrow 0} U'(x) = \infty, \quad U'(\infty) \triangleq \lim_{x\uparrow\infty} U'(x) = 0. \tag{2.3}$$

We shall denote by $I(\cdot)$ the (continuous, strictly decreasing) inverse of the marginal utility function $U'(\cdot)$, and by (2.3) it follows that

$$I(0+) \triangleq \lim_{x\downarrow 0} I(x) = \infty, \quad I(\infty) \triangleq \lim_{x\uparrow\infty} I(x) = 0. \tag{2.4}$$

The agent is deriving utility from intertemporal consumption and final wealth. Let U be the utility of intertemporal consumption and $\hat{U}$ the utility of the terminal wealth at some non-random horizon T (which is a primitive of the model).

2.4 Discount function

Unlike other works in this area we do not restrict ourselves to the framework of exponential discounting. Following [4], a discount function $h : [0,\infty] \to \mathbb{R}$ is assumed to be continuously differentiable, with:

$$h(0) = 1, \quad h(s) \geq 0, \int_0^\infty h(t)\,dt < \infty. \tag{2.5}$$

The definition and characterization of equilibrium strategies (Theorems 3.1 and 4.4) hold for general discount functions. We will then particularize them to two special cases besides the exponential discounting, which we will call *pseudo-exponential* discount functions. They are of two types, type I:

$$h_1(t) = \lambda\exp(-\rho_1 t) + (1-\lambda)\exp(-\rho_2 t) \tag{2.6}$$

and type II:

$$h_2(t) = (1 + \lambda t) \exp(-\rho t) \tag{2.7}$$

Pseudo-exponential discount rates were first considered in the context of time-inconsistency in [4].

Why quasi-exponential discounts? Consider an investor, whose pure rate of time preference is ρ, and whose utility is $U(C)$. This investor is not eternal—the probability of his being alive at time $t > 0$ is $e^{-\lambda t}$. How does she value today an amount of consumption $C(t)$ at some future time $t > 0$? There are two possibilities:

- either she is alive at that time, in which case the present value would be $U(C(t))\, e^{-\rho t}$. The expected present value from that case is

$$e^{-\lambda t} e^{-\rho t} U(C(t)) = e^{-(\lambda+\rho)t} U(C(t))$$

- or she will die at some time $s \in (0, t)$, in which case the consumption will be someone else's presumably her heirs. We assume that this person will have the same risk preference and the same probability of dying, so that she will derive, from consuming $C(t)$, an expected utility $e^{-(\lambda+\rho)(t-s)} U(C(t))$. There is no reason why the original investor should value someone else's utility as much (or as little) as her own. She will therefore discount that utility at a rate $\delta > \rho$, leading us to a present value of:

$$e^{-\delta s} e^{-(\lambda+\rho)(t-s)} U(C(t))$$

Taking expectations, we find that the total present value for the investor will be:

$$U(C(t)) \left[e^{-(\lambda+\rho)t} + \int_0^t e^{-\delta s} e^{-(\lambda+\rho)(t-s)} \lambda e^{-\lambda s} ds \right] = h_\delta(t)\, U(C(t))$$

with

$$\begin{aligned} h_\delta(t) &= e^{-(\lambda+\rho)t} + e^{-(\lambda+\rho)t} \int_0^t \lambda e^{-(\delta-\rho)s} ds \\ &= e^{-(\lambda+\rho)t} + \frac{\lambda}{\delta-\rho} e^{-(\lambda+\rho)t} \left[1 - e^{-(\delta-\rho)t} \right] \\ &= \left(1 + \frac{\lambda}{\delta-\rho} \right) e^{-(\lambda+\rho)t} - \frac{\lambda}{\delta-\rho} e^{-(\lambda+\delta)t} \end{aligned}$$

We find a quasi-exponential discount of type I. Letting $\delta \longrightarrow \rho$, we find that:

$$\begin{aligned} e^{(\lambda+\rho)t} h_\delta(t) &= \left(1 + \frac{\lambda}{\delta-\rho} \right) - \frac{\lambda}{\delta-\rho} e^{-(\delta-\rho)t} \\ &= 1 + \frac{\lambda}{\delta-\rho} - \frac{\lambda}{\delta-\rho} \left(1 - (\delta-\rho)\, t + \frac{1}{2!} (\delta-\rho)^2 t^2 + \cdots \right) \\ &\longrightarrow 1 + \lambda t. \end{aligned}$$

so in the limit we get a quasi-exponential discount of type II.

2.5 Objective

Let us first note that optimal policies, although they exist, will not be time-consistent in general. Indeed, if the agent starts with a given positive wealth x, at some instant t, her optimal policy process $\{\tilde{\zeta}_t(s), \tilde{c}_t(s)\}_{s\in[t,T]}$ is chosen such that

$$\sup_{\zeta,c}\mathbb{E}\left[\int_t^T h(u-t)U(c(u)X^{\zeta,c}(u))\,du + h(T-t)\hat{U}(X^{\zeta,c}(T))\right]$$
$$= \mathbb{E}\left[\int_t^T h(u-t)U(\tilde{c}(u)X^{\tilde{\zeta},\tilde{c}}(u))\,du + h(T-t)\hat{U}(X^{\tilde{\zeta},\tilde{c}}(T))\right].$$

The value function associated with this stochastic control problem is

$$V(t,s,x) \triangleq \sup_{\zeta,c}\mathbb{E}\left[\int_s^T h(u-t)U(c(u)X^{\zeta,c}(u))\,du + h(T-s)\hat{U}(X^{\zeta,c}(T))\Big| X(s)=x\right],$$

$t \le s \le T$, and it solves the following Hamilton–Jacobi–Bellman equation

$$\frac{\partial V}{\partial s}(t,s,x) + \sup_{\zeta,c}\left[(r+\mu\zeta-c)x\frac{\partial V}{\partial x}(t,s,x) + \frac{1}{2}\sigma^2\zeta^2x^2\frac{\partial^2 V}{\partial x^2}(t,s,x)\right]$$
$$+ \frac{h'(s-t)}{h(s-t)}V(t,s,x) + U(xc) = 0,$$

with the boundary condition

$$V(t,T,x) = \hat{U}(x). \tag{2.8}$$

The first order necessary conditions yield the $t-$optimal policy $\{\tilde{\zeta}_t(s), \tilde{c}_t(s)\}_{s\in[t,T]}$

$$\tilde{\zeta}_t(s,x) = -\frac{\mu\frac{\partial V}{\partial x}(t,s,x)}{\sigma^2 x\frac{\partial^2 V}{\partial x^2}(t,s,x)}, \quad t \le s \le T, \tag{2.9}$$

$$\tilde{c}_t(s,x) = \frac{I(\frac{\partial V}{\partial x}(t,s,x))}{x}, \quad t \le s \le T. \tag{2.10}$$

The HJB equation contains the term $\frac{h'(s-t)}{h(s-t)}$, which depends not only on current time s but also on initial time t so the optimal policy will depend on t as well. Indeed let us consider the following example: $U(x) = \hat{U}(x) = \log x$, $T > 1$, and let $\frac{h'}{h}$ be a strictly monotone function. Then, from time 0 perspective, the optimal proportion of wealth consumed is

$$c_0(t) = \left[1 + \int_t^T \exp\left(k(s-t) + \log\left(\frac{h(s)}{h(t)}\right)\right) ds\right]^{-1},$$

while from time 1 perspective, the optimal proportion of wealth consumed is

$$c_1(t) = \left[1 + \int_t^T \exp\left(k(s-t) + \log\left(\frac{h(s-1)}{h(t-1)}\right)\right) ds\right]^{-1},$$

whence the time inconsistency. This failure to remain optimal across times can be regarded as time inconsistency.

Because of time-inconsistency, optimal solutions are irrelevant in practice (although they do exist mathematically) and one must look for an alternative way to solve the problem. This will be done by considering equilibrium policies, that is, policies such that, given that they will be implemented in the future, it is optimal to implement them right now. Let us consider:

$$\bar{\zeta}(s,x) = \frac{F_1(s,x)}{x}, \quad \bar{c}(s,x) = \frac{F_2(s,x)}{x}, \tag{2.11}$$

for some functions F_1, F_2. The corresponding wealth process $\{\bar{X}(s)\}_{s\in[0,T]}$ evolves according to

$$d\bar{X}(s) = \left[r\bar{X}(s) + \mu F_1\left(s, \bar{X}(s)\right) - F_2\left(s, \bar{X}(s)\right)\right] ds + \sigma F_1\left(s, \bar{X}(s)\right) dW(s). \tag{2.12}$$

Following [4] the functions F_1, F_2 will be chosen such that on $[t, t+\epsilon]$ it is optimal (this is made precise in our formal definition of equilibrium policies) to pick $\bar{\zeta}(t,x) = \frac{F_1(t,x)}{x}$, $\bar{c}(t,x) = \frac{F_2(t,x)}{x}$, given that the agent's wealth at time t is x, and that for every subsequent instance $s \geq t+\epsilon$ she follows (2.11). This intuition will be made precise in the following sections.

3 Finite horizon

3.1 General discount function

Let T be a finite, deterministic, time horizon exogenously specified. In general, for an admissible policy process $\{\zeta(s), c(s)\}_{s\in[0,T]}$ and its corresponding wealth process $\{X(s)\}_{s\in[0,T]}$ (see (2.2)) we denote the expected utility functional by

$$J(t,x,\zeta,c) \triangleq \mathbb{E}\left[\int_t^T h(s-t)U(c(s)X^{t,x}(s))\,ds + h(T-t)\hat{U}(X^{t,x}(T))\right]. \tag{3.1}$$

We give a rigorous mathematical formulation of the equilibrium policies in the formal definition below. Let $\{\bar{\zeta}(s), \bar{c}(s)\}_{s\in[t,T]}$ be an admissible policy whose corresponding wealth process is denoted $\{\bar{X}(s)\}_{s\in[t,T]}$. Let the process $\{\zeta_\epsilon(s), c_\epsilon(s)\}_{s\in[t,T]}$ be another investment-consumption policy defined by

$$\zeta_\epsilon(s) = \begin{cases} \bar{\zeta}(s), & s \in [t,T]\backslash E_{\epsilon,t} \\ \zeta(s), & s \in E_{\epsilon,t}, \end{cases} \tag{3.2}$$

$$c_\epsilon(s) = \begin{cases} \bar{c}(s), & s \in [t,T]\backslash E_{\epsilon,t} \\ c(s), & s \in E_{\epsilon,t}, \end{cases} \tag{3.3}$$

with $E_{\epsilon,t} = [t, t+\epsilon]$, and $\{\zeta(s), c(s)\}_{s\in E_{\epsilon,t}}$ is any policy for which $\{\zeta_\epsilon(s), c_\epsilon(s)\}_{s\in[t,T]}$ is an admissible policy.

Definition 3.2 A map $F = (F_1, F_2) : (0,\infty) \times [0,T] \to \mathbb{R} \times [0,\infty)$ is an equilibrium policy for the finite horizon investment-consumption problem, if for any $t, x > 0$

$$\lim_{\epsilon \downarrow 0} \frac{J(t,x,F_1,F_2) - J(t,x,\zeta_\epsilon,c_\epsilon)}{\epsilon} \geq 0, \tag{3.4}$$

where

$$J(t,x,F_1,F_2) \triangleq J(t,x,\bar{\zeta},\bar{c}),$$
$$\bar{\zeta}(s) = \frac{F_1\left(s,\bar{X}(s)\right)}{\bar{X}(s)} \quad \bar{c}(s) = \frac{F_2\left(s,\bar{X}(s)\right)}{\bar{X}(s)} \quad \mathbb{P}\text{ a.s.,} \tag{3.5}$$

and the equilibrium wealth process $\{\bar{X}(s)\}_{s\in[t,T]}$ is a solution of the stochastic differential equation (SDE)

$$dX(s) = [rX(s) + \mu F_1(s,X(s)) - F_2(s,X(s))]\,ds + \sigma F_1(s,X(s))\,dW(s). \tag{3.6}$$

The basic idea is the following: the agent (self) has a different rate of impatience $-\frac{h'(t)}{h(t)}$ as time goes by (unless the discounting is exponential) and can be regarded as a continuum of agents (selves); at every instant t she is building a coalition with her immediate selves s, with $s \in [t, t+\epsilon]$ and try to maximize expected utility of intertemporal consumption and terminal wealth given that the selves on $[t+\epsilon, T]$ agreed upon an equilibrium strategy.

Remark 3.1 We have chosen the proportions of wealth invested in the risky asset and of wealth consumed as our control variables. This is convenient since it leads naturally to non negative wealth processes and this is the framework we work on. However the notion of equilibrium policies can be defined in a context where the wealth may become negative. In that case we take the net amounts invested in the risky asset and consumed as control variables and define the equilibrium policies accordingally.

Our next item in the agenda is to compute explicitly the limit (3.4). This will be done in terms of some adjoint processes defined by a flow (one for every instant t) of backward stochastic differential equations (BSDE). More precisely for every $0 \le t \le T$ the process $\{M(t,s), N(t,s)\}_{s\in[t,T]}$ is a solution of the BSDE

$$\begin{cases} dM(t,s) = -\Big(M(t,s)\Big(r + \mu\frac{\partial F_1}{\partial x}(s,\bar{X}(s)) - \frac{\partial F_2}{\partial x}(s,\bar{X}(s))\Big) + \sigma\frac{\partial F_1}{\partial x}(s,\bar{X}(s))N(t,s) \\ \qquad\qquad + h(s-t)\frac{\partial F_2}{\partial x}(s,\bar{X}(s))U'(F_2(s,\bar{X}(s)))\Big)\,ds + N(t,s)dW(s) \\ M(t,T) = h(T-t)\hat{U}'(\bar{X}(T)), \end{cases} \tag{3.7}$$

where the equilibrium wealth process $\{\bar{X}(s)\}_{s\in[0,T]}$ follows (3.6).

The next central result does not depend on the choice of the discount function h satisfying (2.5), nor on the choice of preferences satisfying (2.3).

Theorem 3.1 *Assume there exists a map $F = (F_1, F_2) : (0,\infty) \times [0,T] \to \mathbb{R} \times (0,\infty)$, continuously differentiable with respect to x such that for every $t \in [0,T]$ there exists a solution $\{M(t,s), N(t,s)\}_{s\in[t,T]}$ of (3.7) which satisfies*

$$\mu M(t,t) + \sigma N(t,t) = 0, \tag{3.8}$$

and

$$F_2(t,x) = I(M(t,t)|X(t) = x). \tag{3.9}$$

Moreover assume F has bounded derivatives in x, i.e., for all $(t,x) \in (0,\infty)\times[0,T]$

$$\left|\frac{\partial F_i(t,x)}{\partial x}\right| \le K, \quad i = 1,2, \tag{3.10}$$

and $(\bar{\zeta},\bar{c})$ of (3.5) is admissible. Then F is an equilibrium policy.

Appendix A proves this Theorem. □

3.2 Special discount functions

Following [4] we would like to give a characterization of the equilibrium policies in terms of both a partial differential equation and an integral equation. We restrict ourselves to pseudo-exponential discounting, although our method extends beyond that case. Recall that type I is defined by (2.6), type II by (2.7), while exponential discounting is the special case where $\rho_1 = \rho_2$ (type I) or $\lambda = 0$ (type II). Let us introduce the Legendre transform of $-U(-x)$

$$\tilde{U}(y) \triangleq \sup_{x>0}[U(x) - xy] = U(I(y)) - yI(y), \quad 0 < y < \infty. \tag{3.11}$$

The function $\tilde{U}(\cdot)$ is strictly decreasing, strictly convex and satisfies the dual relationships

$$\tilde{U}'(y) = -x \quad \text{iff} \quad U'(x) = y. \tag{3.12}$$

The next Theorem is our main result. It describes the equilibrium policies through a coupled system of parabolic equations. Let the coefficients α_{ij}, β_{ij} correspond to different choices of discount functions. Thus for exponential discounting, $h_0(t) = \exp(-\delta t)$,

$$\alpha_{10} = \delta, \quad \alpha_{20} = 0, \ \beta_{10} = 0, \ \beta_{20} = 0, \tag{3.13}$$

for type I discounting, $h_1(t) = \lambda \exp(-\rho_1 t) + (1-\lambda)\exp(-\rho_2 t)$,

$$\begin{aligned} \alpha_{11} &= \lambda\rho_1 + (1-\lambda)\rho_2, \quad \alpha_{21} = \rho_1 - \rho_2, \\ \beta_{11} &= \lambda(1-\lambda)(\rho_1 - \rho_2), \quad \beta_{21} = \lambda\rho_2 + (1-\lambda)\rho_1, \end{aligned} \tag{3.14}$$

and for type II discounting, $h_2(t) = (1+\lambda t)\exp(-\rho t)$,

$$\alpha_{12} = \rho - \lambda, \quad \alpha_{22} = -\lambda, \quad \beta_{12} = \lambda, \quad \beta_{22} = \rho + \lambda. \tag{3.15}$$

Theorem 3.2 *Assume there exist two functions $v(t,x)$ and $w(t,x)$ three times continuously differentiable which satisfy*

$$\frac{\partial v}{\partial t}(t,x) + rx\frac{\partial v}{\partial x}(t,x) - \frac{\mu^2}{2\sigma^2}\frac{[\frac{\partial v}{\partial x}(t,x)]^2}{\frac{\partial^2 v}{\partial x^2}(t,x)} + \tilde{U}\left(\frac{\partial v}{\partial x}(t,x)\right) = \alpha_{1j}v(t,x) + \beta_{1j}w(t,x),$$

$$\frac{\partial w}{\partial t}(t,x) + \left(rx - I\left(\frac{\partial v}{\partial x}(t,x)\right)\right)\frac{\partial w}{\partial x}(t,x) - \frac{\mu^2}{\sigma^2}\frac{\frac{\partial v}{\partial x}(t,x)\frac{\partial w}{\partial x}(t,x)}{\frac{\partial^2 v}{\partial x^2}(t,x)}$$

$$+ \frac{\mu^2}{2\sigma^2}\frac{[\frac{\partial v}{\partial x}(t,x)]^2\frac{\partial^2 w}{\partial x^2}(t,x)}{[\frac{\partial^2 v}{\partial x^2}(t,x)]^2} = \alpha_{2j}v(t,x) + \beta_{2j}w(t,x),$$

for all $(t,x) \in [0,T] \times (0,\infty)$, and the boundary conditions

$$v(T,x) = \hat{U}(x), \quad w(T,x) = 0.$$

Let $F = (F_1, F_2)$ be defined by

$$F_1(t,x) = -\frac{\mu\frac{\partial v}{\partial x}(t,x)}{\sigma^2\frac{\partial^2 v}{\partial x^2}(t,x)}, \quad F_2(t,x) = I\left(\frac{\partial v}{\partial x}(t,x)\right), \ t \in [0,T]. \tag{3.16}$$

If F has bounded derivatives in x and $(\bar{\zeta}, \bar{c})$ of (3.5) is admissible, then $F = (F_1, F_2)$ is an equilibrium policy.

Appendix B proves this Theorem. □

Corollary 3.1 *Let $F=(F_1,F_2)$ be given as above and the discounting be exponential. If F has bounded derivatives in x and $(\bar{\zeta},\bar{c})$ of (3.5) is admissible, then $F=(F_1,F_2)$ is both an equilibrium and an optimal policy.*

The proof follows since the value function v

$$v(t,x)=\sup_{\zeta,c}\mathbb{E}\left[\int_t^T e^{-\delta(s-t)}U(c(s)X^{\zeta,c}(s))\,ds+e^{-\delta(T-t)}\hat{U}(X^{\zeta,c}(T))\Big|X(t)=x\right],\quad (3.17)$$

and $w=0$ satisfy the PDE system of Theorem 3.2. □

The next Proposition is in the spirit of Theorem 2 in [4] and gives another characterization of the equilibrium policy through an integral equation. It is stated only for the special discounts.

Proposition 3.1 *Assume there exist two functions $v(t,x)$ and $w(t,x)$ three times continuously differentiable which solve the PDE system of Theorem 3.2. Then $v(t,x)$ satisfies the integral equation*

$$v(t,x)=\mathbb{E}\left[\int_t^T h(s-t)U(F_2(s,\bar{X}^{t,x}(s)))\,ds+h(T-t)\hat{U}(\bar{X}^{t,x}(T))\right],\quad (3.18)$$

Recall that $\{\bar{X}(s)\}_{s\in[0,T]}$ is the equilibrium wealth process and it satisfies

$$d\bar{X}(s)=[r\bar{X}(s)+\mu F_1(s,\bar{X}(s))-F_2(s,\bar{X}(s))]ds+\sigma F_1(s,\bar{X}(s))dW(s).\quad (3.19)$$

Appendix C proves this Proposition. □

Remark 3.2 This Proposition suggests that for a general discount function $h(t)$ the equilibrium policies are of the form (3.16) for a function v as in (3.18). It is not hard to see that it holds for $h(t)=\sum_{i=1}^n P_i(t)\exp(-\rho_i t)$, where $P_i(t)$ are polynomials.

In the next Proposition stochastic representations for v and w are given.

Proposition 3.2 *Assume there exist two functions $v(t,x)$ and $w(t,x)$ three times continuously differentiable which solve the PDE system of Theorem 3.2. Then*

$$w(t,x)=\mathbb{E}\left[\alpha_{2j}\int_t^T \exp(-\beta_{2j}(s-t))v\left(s,\bar{X}^{t,x}(s)\right)\,ds\right],$$

$$v(t,x)=\mathbb{E}\left[\int_t^T \exp(-\alpha_{1j}(s-t))\left[U\left(I\left(\frac{\partial v}{\partial x}\left(s,\bar{X}^{t,x}(s)\right)\right)\right)\right.\right.$$
$$\left.\left.-\ \beta_{1j}w\left(s,\bar{X}^{t,x}(s)\right)\right]ds+h(T-t)\hat{U}\left(\bar{X}^{t,x}(T)\right)\right].$$

Therefore

$$v(t,x) = \mathbb{E}\left[\int_t^T \exp(-\alpha_{1j}(s-t))\left[U\left(I\left(\frac{\partial v}{\partial x}\left(s, \bar{X}^{t,x}(s)\right)\right)\right)\right.\right.$$
$$\left.\left. - \alpha_{2j}\beta_{1j}\int_s^T \exp(-\alpha_{2j}(z-s))v\left(z, \bar{X}^{t,x}(z)\right)\,dz\right]ds + h(T-t)\hat{U}(X^{t,x}(T))\right].$$

Proof It is a direct consequence of Feynman–Kac's formula. □

The main question (since all the results of this section are based on it) is: when does the system of Theorem 3.2 have solutions? Of course the answer is known for exponential discounting.

3.2.1 CRRA preferences

In the case of $U(x) = \hat{U}(x) = \frac{x^p}{p}$, one can look for equilibrium policies by decoupling time and wealth; take $v(t,x) = f(t)U(x)$ and $w(t,x) = g(t)U(x)$, where $f(t)$, $g(t)$ solve the ODE system

$$f'(s) + Kf(s) + (1-p)[f(s)]^{\frac{p}{p-1}} = \alpha_{1j}f(s) + \beta_{1j}g(s), \tag{3.20}$$

$$g'(s) + Kg(s) - pg(s)[f(s)]^{\frac{1}{p-1}} = \alpha_{2j}f(s) + \beta_{2j}g(s), \tag{3.21}$$

for all $t \in [0, T]$ with boundary condition

$$f(T) = 1, \quad g(T) = 0, \tag{3.22}$$

where $K = rp + \frac{\mu^2 p}{2\sigma^2(1-p)}$. The following Lemma yields existence and uniqueness for the above ODE system.

Lemma 3.1 *There exists a unique continuously differentiable solution* (f, g) *of Eqs.* (3.20), (3.21) *with boundary condition* (3.22).

Appendix D proves this Lemma. □

Theorem 3.3 *Let* $F = (F_1, F_2)$ *defined by*

$$F_1(t,x) = \frac{\mu x}{(1-p)\sigma^2}, \quad F_2(t,x) = [f(t)]^{\frac{1}{p-1}}x, \tag{3.23}$$

where (f, g) *is the unique solution of the above ODE system. Then* $F = (F_1, F_2)$ *is an equilibrium policy.*

The proof follows from Theorem 3.2. Notice that F has bounded derivatives in x and $(\bar{\zeta}, \bar{c})$ of (3.5) is admissible. Note also that since we do not prove uniqueness for the PDE system there may be other equilibrium policies, besides those we find by this ansatz. □

Remark 3.3 The case of logarithmic utility deserves a separate treatment since provides equilibrium policy in closed form. Indeed when $p = 0$ which correspond to logarithmic utility the above ODE system becomes linear

$$f'(s) + Kf(s) + 1 = \alpha_{1j}f(s) + \beta_{1j}g(s), \tag{3.24}$$

$$g'(s) + Kg(s) = \alpha_{2j}f(s) + \beta_{2j}g(s), \tag{3.25}$$

and has an explicit solution (the formula is lengthy so we omit it).

3.2.2 Numerical results

The equilibrium proportion of wealth consumed cannot be computed explicitly if $p \neq 0$, thus we perform some numerical experiments. We consider one stock following a geometric Brownian motion with drift $\alpha = 0.12$, volatility $\sigma = 0.2$. The interest rate $r = 0.05$, the discount factors $\rho_1 = 0.1$, $\rho_2 = 0.3$, the weighting parameter $\lambda = 0.25$, and the horizon $T = 3$. It follows from the integral equation (3.18) describing the equilibrium policies that the ODE system can be rewritten as an integral equation

$$\begin{cases} f(t) = \int_t^T h(s-t)e^{K(s-t)}[f(s)]^{\frac{p}{p-1}} e^{-\left(\int_t^s p[f(u)]^{\frac{1}{p-1}}\,du\right)}\,ds \\ \qquad\qquad + h(T-t)e^{K(T-t)} e^{-\left(\int_t^T p[f(u)]^{\frac{1}{p-1}}\,du\right)} \\ f(T) = 1, \end{cases} \tag{3.26}$$

where h is one of the three choices of discounting. We employ the following numerical scheme from [12]. Let us consider a regular partition of $[0, T]$ with step size $\frac{T}{N}$ and define $a \triangleq -\frac{T}{N}$, $q(t) \triangleq e^{-\left(\int_t^T p[f(u)]^{\frac{1}{p-1}}\,du\right)}$, and the sequences

$$q_{n+1}^N = q_n\left(1 + ph[f_n^N]^{\frac{1}{p-1}}\right), \quad q_0^N = 1, \; n = 0\cdots N,$$

$$f_{n+1}^N = f_n^N + h\left(-K + \frac{f_n^N h'(-na)}{h(-na)} + (p-1)[f_n^N]^{\frac{p}{p-1}} + s_n^N\right), \quad f_0^N = 1, \; n = 0\cdots N,$$

$$s_n^N = a\sum_{j=0}^{n} w_n^j \left[\frac{h'(-na)}{h(-na)} - \frac{h'((j-n)a)}{h((j-n)a)}\right] \exp\left(K(j-n)a\right)[f_n^N]^{\frac{p}{p-1}} \frac{q_n^N}{q_j^N} \quad s_0^N = 1,$$

where $n = 0, \ldots, N$, $w_0^n = w_n^n = \frac{1}{2}$, and $w_j^n = 1$, for $j = 1, \ldots, n-1$. The function $f^N(t)$ is obtained by linear interpolation of the values f_n^N at the points $T + na$. In [12] it is shown that $\sup_{t\in[0,T]} |f(t) - f^N(t)| \leq \frac{K}{N}$, for some positive constant K which depends solely on T and f. Next we graph $[f^N(t)]^{\frac{1}{p-1}}$ with $N = 100$, by using a C routine (Fig. 1).

Remark 3.4 As one of the referees points out all the equilibrium policies have the same form as the optimal policies from the standard case of exponential discounting. We suspect this is a special property of CRRA preferences which allows for special solutions $v(t, x) = f(t)U(x)$ and $w(t, x) = g(t)U(x)$. Note however, that even in this case, since we have no uniqueness for the PDE system, there may also be different types of equilibrium policies.

4 Infinite horizon

We next investigate stationary equilibrium policies in the infinite horizon framework. Before engaging into the formal definition let us point the following key fact. For an admissible time homogeneous policy process $\{\zeta(t), c(t)\}_{t\in[0,\infty)}$ and its corresponding wealth process $\{X(t)\}_{t\in[0,\infty)}$ (see (2.2)) the expected utility functional $J(x, \zeta, c)$ satisfies

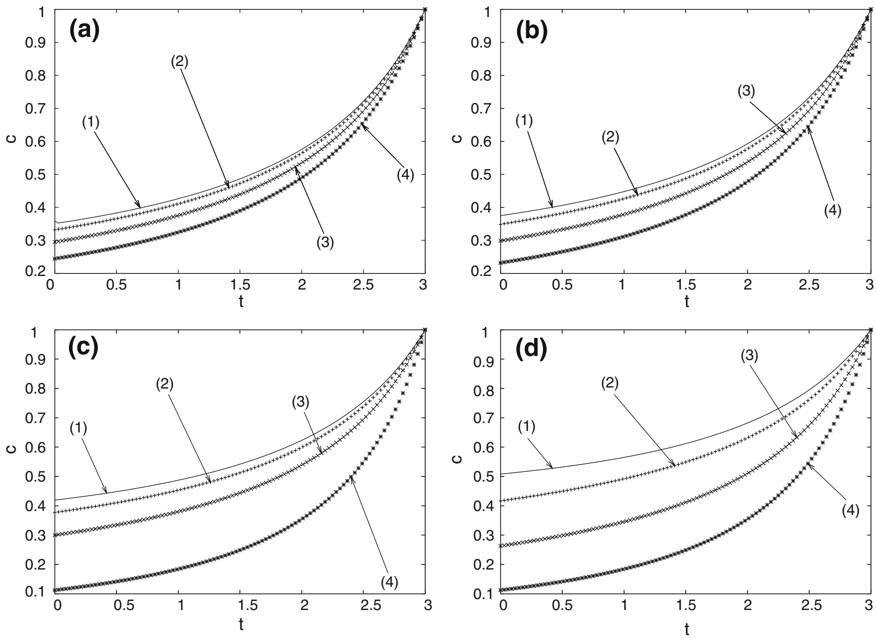

Fig. 1 Equilibrium proportion of wealth consumed for different discount functions *(1)* $\rightarrow$ ρ_2 exponential discounting; *(2)* $\rightarrow$ λ, ρ_1, ρ_2 type I discounting; *(3)* $\rightarrow$ ρ_1 exponential discounting; *(4)* $\rightarrow$ λ, ρ_1 type II discounting. The four pictures corresponds to different values of CRRA, p; for $p = -1 \rightarrow (a)$, $p = -0.5 \rightarrow (b)$, $p = 0 \rightarrow (c)$, and $p = 0.5 \rightarrow (d)$. The x axis represents the time t and the y axis the equilibrium proportion of wealth c consumed at time t. As $t \rightarrow T$ this proportion goes to 1 as it was to be expected. When p increases, i.e., when the agent becomes less risk averse the effect of using different discount functions becomes more significant

$$
\begin{aligned}
J(x, \zeta, c) &\triangleq \mathbb{E}\left[\int_t^{\infty} h(s-t)U\left(c(s)X^{t,x}(s)\right)\, ds\right] \\
&= \mathbb{E}\left[\int_0^{\infty} h(s)U\left(c(s)X^{t,x}(t+s)\right)\, ds\right] \\
&= \mathbb{E}\left[\int_0^{\infty} h(s)U\left(c(s)X^{0,x}(s)\right)\, ds\right]. \qquad (4.1)
\end{aligned}
$$

This is due to the fact that the processes $\{X^{t,x}(t+s)\}_{s\in[0,\infty)}$ and $\{X^{0,x}(s)\}_{s\in[0,\infty)}$ have the same $\mathbb{P}$ distribution. Let $\{\bar{\zeta}(t), \bar{c}(t)\}_{t\in[0,\infty)}$ be an admissible time homogeneous policy whose corresponding wealth process is denoted $\{\bar{X}(t)\}_{t\in[0,\infty)}$. Let $\{\zeta_\epsilon(t), c_\epsilon(t)\}_{t\in[0,\infty)}$ be another time homogeneous investment-consumption policy defined by

$$
\zeta_\epsilon(t) = \begin{cases} \bar{\zeta}(t), & t \in [0,\infty)\backslash E_\epsilon \\ \zeta(t), & t \in E_\epsilon, \end{cases} \qquad (4.2)
$$

$$
c_\epsilon(t) = \begin{cases} \bar{c}(t), & t \in [0,\infty)\backslash E_\epsilon \\ c(t), & t \in E_\epsilon. \end{cases} \qquad (4.3)
$$

Here $E_\epsilon \subset [0,\infty)$ is a measurable set with Lebesgue measure $|E_\epsilon| = \epsilon$, and $\{\zeta(t), c(t)\}_{t\in E_\epsilon}$ is any time homogeneous policy for which $\{\zeta_\epsilon(t), c_\epsilon(t)\}_{t\in[0,\infty)}$ is an admissible policy.

Definition 4.3 A map $F = (F_1, F_2) : (0,\infty) \to \mathbb{R} \times [0,\infty)$ is an equilibrium policy for the infinite horizon investment-consumption problem, if for any $x > 0$

$$\lim_{\epsilon\downarrow 0} \frac{J(x, F_1, F_2) - J(x, \zeta_\epsilon, c_\epsilon)}{\epsilon} \geq 0, \tag{4.4}$$

where

$$\begin{aligned} J(x, F_1, F_2) &\triangleq J(x, \bar{\zeta}, \bar{c}), \\ \bar{\zeta}(t) = \frac{F_1(\bar{X}(t))}{\bar{X}(t)} \quad & \bar{c}(t) = \frac{F_2(\bar{X}(t))}{\bar{X}(t)} \quad \mathbb{P} \text{ a.s.}, \end{aligned} \tag{4.5}$$

and the equilibrium wealth process $\{\bar{X}(t)\}_{t\in[0,\infty)}$ satisfies

$$d\bar{X}(t) = [r\bar{X}(t) + \mu F_1(\bar{X}(t)) - F_2(\bar{X}(t))]dt + \sigma F_1(\bar{X}(t))dW(t). \tag{4.6}$$

Following the finite horizon case we would like first to describe the equilibrium policies in terms of the adjoint process $\{M(t), N(t)\}_{t\in[0,\infty)}$. Its dynamics evolves according to

$$\begin{aligned} dM(t) = &-(M(t)(\mu F_1'(\bar{X}(t)) - F_2'(\bar{X}(t)) + \sigma F_1'(\bar{X}(t))N(t) \\ &+ h(t)F_2'(\bar{X}(t))U'(F_2(\bar{X}(t)))dt + N(t)dW(t), \end{aligned} \tag{4.7}$$

and the equilibrium wealth process $\{\bar{X}(t)\}_{t\in[0,\infty)}$ it is given by (4.6). Furthermore we impose the transversality condition at infinity

$$\lim_{t\to\infty} \mathbb{E}M(t)(Y^\epsilon(t) + Z^\epsilon(t)) = 0, \tag{4.8}$$

where the processes $\{Y^\epsilon(s)\}_{s\in[0,\infty)}$ and $\{Z^\epsilon(s)\}_{s\in[0,\infty)}$, defined by the SDE

$$\begin{cases} dY^\epsilon(s) = Y^\epsilon(s)(\mu F_1'(\bar{X}(s)) - F_2'(\bar{X}(s)))\,ds + \sigma[Y^\epsilon(s)F_1'(\bar{X}(s) \\ \qquad\qquad + (\bar{X}(s)\zeta(s) - F_1(\bar{X}(s))\chi_{E_\epsilon}(s)]\,dW(s) \\ Y^\epsilon(0) = 0 \end{cases} \tag{4.9}$$

and

$$\begin{cases} dZ^\epsilon(s) = [Z^\epsilon(s)(\mu F_1'(\bar{X}(s)) - F_2'(\bar{X}(s)) \\ \qquad\qquad + (\mu\bar{X}(s)(\zeta(s) - F_1'(\bar{X}(s))) - (c(s)\bar{X}(s) - F_2(\bar{X}(s))\chi_{E_\epsilon}(s)]\,ds \\ \qquad\qquad + [\sigma Z^\epsilon(s)F_2'(\bar{X}(s)) + (\sigma Y^\epsilon(s)(\zeta(s) - F_1'(\bar{X}(s))\chi_{E_\epsilon}(s)]\,dW(s) \\ Z^\epsilon(0) = 0 \end{cases}$$

are the first order and second order variation of $\{\bar{X}(s)\}_{s\in[0,\infty)}$. The next result is the infinite horizon counterpart of Theorem 3.1.

Theorem 4.4 *Assume there exists a map $F = (F_1, F_2) : (0,\infty) \to \mathbb{R} \times (0,\infty)$, continuously differentiable with respect to x such that there exists $\{M(t), N(t)\}_{t\in[0,\infty)}$ which satisfies* (4.7)*, the transversality condition* (4.8)*,*

$$\mu M(0) + \sigma N(0) = 0, \tag{4.10}$$

and

$$F_2(x) = I(M(0)|X(t) = x). \tag{4.11}$$

Moreover assume F has bounded derivatives in x, i.e., for all $x \in (0, \infty)$

$$|F_i'(x)| \le K, \quad i = 1, 2, \tag{4.12}$$

and $(\bar{\zeta}, \bar{c})$ of (4.5) is admissible. If

$$\begin{aligned} J(x, \zeta_\epsilon, c_\epsilon) \le\ & J(x, F_1, F_2) \\ & + \mathbb{E}\int_0^\infty \left(h(s)(U\left(c(s)\bar{X}^{0,x}(s)\right) - U\left(F_2\left(\bar{X}^{0,x}(s)\right)\right)\chi_{E_\epsilon}(s)\right) ds \\ & + \mathbb{E}\int_0^\infty \left(h(s)(Y^\epsilon(s) + Z^\epsilon(s))F_2'\left(\bar{X}^{0,x}(s)\right)U'\left(F_2\left(\bar{X}^{0,x}(s)\right)\right)\right) ds + o(\epsilon), \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} & \lim_{t\to\infty} \mathbb{E}\int_0^t \left(h(s)(Y^\epsilon(s) + Z^\epsilon(s))F_2'\left(\bar{X}^{0,x}(s)\right)U'\left(F_2\left(\bar{X}^{0,x}(s)\right)\right)\right) ds \\ & = \mathbb{E}\int_0^\infty \left(h(s)(Y^\epsilon(s) + Z^\epsilon(s))F_2'\left(\bar{X}^{0,x}(s)\right)U'\left(F_2\left(\bar{X}^{0,x}(s)\right)\right)\right) ds \end{aligned} \tag{4.14}$$

then F is an equilibrium policy.

The proof follows as in Theorem 3.1 (see Appendix A). □

Remark 4.5 Let us notice that some difficulties arise from the infinite horizon. Indeed a version of the asymptotic expansion (4.13) was established in the finite horizon case by means of estimates from Lemma 6.5 (see Appendix A). These estimates however do not hold for the infinite horizon thus we need to assume (4.13). In addition, in the finite horizon case the adjoint processes were defined by BSDEs with terminal condition which corresponds to the transversality condition (4.8). For a given policy $F = (F_1, F_2)$ one can easily construct $\{M(t), N(t)\}_{t\in[0,\infty)}$ to satisfy (4.7). Indeed let ζ be a square integrable random variable with respect to the probability measure which makes

$$d\tilde{W}(s) = d\,W(s) + \sigma F_1'(\bar{X}(s))\,ds,$$

a Brownian motion. From the integral representation of ζ under this probability measure it is straightforward how to construct $\{M(t), N(t)\}_{t\in[0,\infty)}$. However the equation (4.7) should be understood together with (4.5), (4.6), (4.8), (4.10) and (4.11) which together implicitly describe the equilibrium policies.

Given this implicit characterization we specify the choice of the discount function. As in the preceding Section we analyze the equilibrium policies for exponential, type I and type II discounting. With the coefficients α_{ij} and β_{ij} defined in (3.13), (3.14) and (3.15), we have the following formal result.

Theorem 4.5 *Assume there exist two functions $v(x)$ and $w(x)$ three times continuously differentiable which solve the following ODE system*

$$rxv'(x) - \frac{\mu^2}{2\sigma^2}\frac{v'^2(x)}{v''(x)} + \tilde{U}\left(v'(x)\right) = \alpha_{1j}v(x) + \beta_{1j}w(x),$$

$$(rx - I(v'(x)))w'(x) - \frac{\mu^2}{\sigma^2}\frac{v'(x)w'(x)}{w''(x)} + \frac{\mu^2}{2\sigma^2}\frac{[v'(x)]^2 w''(x)}{[v''(x)]^2} = \alpha_{2j}v(x) + \beta_{2j}w(x),$$

for all $x \in (0, \infty)$. Then $F = (F_1, F_2)$ given by

$$F_1(x) = -\frac{\mu v'(x)}{\sigma^2 v''(x)}, \quad F_2(x) = I(v'(x)), \tag{4.15}$$

is an equilibrium strategy if F has bounded derivatives, $(\bar{\zeta}, \bar{c})$ of (4.5) is admissible, and conditions (4.8), (4.13) and (4.14) hold true.

The proof follows as in Theorem 3.2. □

The next Proposition gives the description of equilibrium policies through an integral equation (IE) as in Proposition 3.1.

Proposition 4.3 *Assume there exist two functions $v(x)$ and $w(x)$ three times continuously differentiable which solve the ODE system of Theorem 4.5. Then $v(x)$ satisfies the integral equation*

$$v(x) = \mathbb{E}\left[\int_0^\infty h(t)U\left(F_2\left(\bar{X}^{0,x}(t)\right)\right)dt\right]. \tag{4.16}$$

Conversely a three times continuously differentiable solution of (4.16) yields a solution to the ODE system.

The proof follows as in Proposition 3.1. □

4.1 CRRA Preferences

In this subsection we are still in the paradigm of exponential, type I, type II discounts and further investigate the case of $U(x) = \frac{x^p}{p}$. Let us look for the function v of the form $v(x) = kU(x)$, for a constant k which is to be found. Of course there may be equilibrium policies of other forms, but we do not have a result towards this direction. The equilibrium policies of this form are linear in wealth

$$F_1(x) = \frac{\mu x}{(1-p)\sigma^2}, \quad F_2(x) = k^{\frac{1}{p-1}}x, \tag{4.17}$$

and the corresponding wealth process is

$$\bar{X}(t) = X(0)\exp\left(\left(r + \frac{(1-2p)\mu^2}{2(1-p)^2\sigma^2} - k^{\frac{1}{p-1}}\right)t + \frac{\mu}{(1-p)\sigma}W(t)\right). \tag{4.18}$$

According to Proposition 4.3 the value function v should satisfy the integral equation (4.16), which in this context becomes

$$k^{\frac{1}{1-p}} = \int_0^\infty h(u) e^{\tilde{k}u} \, du, \tag{4.19}$$

where

$$\tilde{k} = p \left(r + \frac{\mu^2}{2(1-p)\sigma^2} - k^{\frac{1}{p-1}} \right). \tag{4.20}$$

In the present framework the transversality equation (4.8) becomes

$$\lim_{t \to \infty} \mathbb{E} M(t) \bar{X}(t) = 0, \tag{4.21}$$

The next theorem shows that $F = (F_1, F_2)$ of (4.17) is an equilibrium policy for CRRA preferences.

Theorem 4.6 *Assume there exists a positive root k of* (4.19) *such that the transversality equation* (4.21) *holds true. Then $F = (F_1, F_2)$ is an equilibrium policy.*

Appendix E proves this Theorem. □

Let us elaborate more on the three cases independently.

4.1.1 Exponential discounting

With $h(t) = \exp(-\delta t)$, equation (4.19) reads

$$k^{\frac{1}{1-p}} (\delta - \tilde{k}) = 1, \tag{4.22}$$

if

$$\delta > \tilde{k}. \tag{4.23}$$

Lemma 4.2 *If*

$$\delta > (p \vee 0) \left[\frac{\mu^2}{2(1-p)\sigma^2} + r \right], \tag{4.24}$$

then (4.22) *has the positive solution*

$$k = \left[\frac{1}{1-p} \left(\delta - rp - \frac{p\mu^2}{2(1-p)\sigma^2} \right) \right]^{p-1}, \tag{4.25}$$

for which (4.23) *is satisfied. Moreover the transversality equation* (4.21) *holds true.*

Appendix F proves this Lemma. □

Remark 4.6 Condition (4.24) is weaker than the transversality condition

$$\delta > (p \vee 0) \left[\frac{(2-p)\mu^2}{2(1-p)\sigma^2} + r \right], \tag{4.26}$$

of [10].

4.1.2 Type I discounting

If $h(t) = \lambda \exp(-\rho_1 t) + (1-\lambda)\exp(-\rho_2 t)$, equation (4.19) leads to

$$k^{\frac{1}{1-p}} = \frac{\lambda}{\rho_1 - \tilde{k}} + \frac{1-\lambda}{\rho_2 - \tilde{k}}, \tag{4.27}$$

if

$$\rho_i - \tilde{k} > 0, \quad i = 1, 2. \tag{4.28}$$

Thus k can be expressed as the positive root of some quadratic equation (see Appendix G).

Let us search for the transversality condition sufficient to grant transversality equation (4.21). The process $\{M(t)\}_{t\in[0,\infty)}$ is given by

$$M(t) \triangleq \lambda \exp(-\rho_1 t) v_1'(\bar{X}(t)) + (1-\lambda)\exp(-\rho_2 t) v_2'(\bar{X}(t)), \tag{4.29}$$

where

$$v_i(x) = \mathbb{E}\left[\int_t^\infty e^{-\rho_i(s-t)} U(F_2(\bar{X}_s^{t,x}))\, ds\right], \quad i = 1, 2. \tag{4.30}$$

Consequently

$$v_i(x) = \frac{k^{\frac{p}{1-p}} x^p}{p(\rho_i - \tilde{k})}, \quad i = 1, 2. \tag{4.31}$$

It is interesting to point out that the function v_i is monotone increasing in ρ_i if p is negative and monotone decreasing if p is positive. Furthermore, for $i = 1, 2$

$$\begin{aligned}&\exp(-\rho_i s) v_i'(\bar{X}(t))\\ &= \frac{X(0)k^{\frac{p}{1-p}}}{(\rho_i - \tilde{k})}\left[\exp\left(\left(r(p-1) + \frac{(2p-1)\mu^2}{2(1-p)\sigma^2} - (p-1)k^{\frac{1}{p-1}} - \rho_i\right)t - \frac{\mu}{\sigma}W(t)\right)\right].\end{aligned}$$

The transversality equation reads (4.21)

$$pk^{\frac{1}{p-1}} + \rho_i > pr + \frac{p^2\mu^2}{2(1-p)^2\sigma^2}, \quad i = 1, 2. \tag{4.32}$$

The analysis is summarized in the following Lemma.

Lemma 4.3 *The equation* (4.27) *has a positive solution for which* (4.28) *and* (4.32) *are satisfied if p and ρ_i, $i = 1, 2$ are in one of the following three cases*:

- *case* 1. $p < 0$, $\rho_1 \simeq \rho$, *and* $\rho_2 \simeq \alpha\rho$ *for* ρ *large enough and* $0 < \alpha < \frac{p-1}{p}$.
- *case* 2. $p \in (0, \frac{1}{2}]$, $\rho_2 > \rho_1 > rp + \frac{p\mu^2}{2(1-p)\sigma^2}$, *and* ρ_2 *large enough*
- *case* 3. $p \in (\frac{1}{2}, 1)$, $\rho_2 > \rho_1 > rp + \frac{p^2\mu^2}{2(1-p)^2\sigma^2}$, *and* ρ_2 *large enough.*

The proof follows from direct computations. □

We are able to provide an example where there are two equilibrium policies as the next Lemma shows.

Lemma 4.4 *The equation* (4.27) *has two positive solutions for which* (4.28) *and* (4.32) *are satisfied if* p, λ *and* ρ_i, $i = 1, 2$ *are such that*

$$p > \frac{1}{2},\ \ p \simeq \frac{1}{2},\ \ \lambda > \frac{1+\sqrt{4p(1-p)}}{2},\ \ \lambda \simeq \frac{1+\sqrt{4p(1-p)}}{2},$$
$$\rho_1 = py + \epsilon,\ \ \rho_2 = py - \epsilon,\ \ y \triangleq r + \frac{\mu^2}{2(1-p)\sigma^2},\ \ \epsilon > \frac{2p\sqrt{p}}{p - \sqrt{p(1-p)}} \frac{\mu^2}{\sigma^2}.$$

The proof follows from direct computations. □

In this situation we choose the policy which yield a higher expected utility.

4.1.3 Type II discounting

If $h(t) = (1 + \lambda t)\exp(-\rho t)$, equation (4.19) leads to

$$k^{\frac{1}{1-p}} = \frac{1}{\rho - \tilde{k}} + \frac{\lambda}{(\rho - \tilde{k})^2}, \tag{4.33}$$

if

$$\rho - \tilde{k} > 0. \tag{4.34}$$

Thus k can be expressed as the positive root of some quadratic equation (see Appendix H).

Similar to the type I discounting it is possible to establish a positive solution of (4.33) which satisfies (4.34) and for which the transversality equation (4.21) holds true.

5 Concluding remarks

This paper introduced a novel concept in stochastic optimization namely the notion of equilibrium policies. We analyze the Merton portfolio management problem in the context of exponential and non-exponential discounting. Equilibrium policies are characterized in this context by means of dual processes. For the special cases of discounting they are further characterized by either an integral equation or a system of partial differential equations. We look for equilibrium policies of a certain form (namely linear) when risk preferences are CRRA, but this does not preclude the existence of other equilibrium policies. Thus, the linear equilibrium policies yield the same constant proportion of wealth to be invested in the risky asset for all three discounts. As for the equilibrium consumption rates they are deterministic functions of time but differ with the choice of discounting. Numerical work illustrates this difference. Closed form solutions are reported for the special case of logarithmic utility. The infinite horizon case is covered as well and it is shown that in some situations there are more equilibrium policies (in fact we conjecture there may be infinitely many). Moreover for CRRA preferences equilibrium policies corresponding to the three discounts are to consume, and invest in the risky asset, constant proportions of the wealth. These proportions are the same for investment but they are different for consumption.

6 Appendix

A Proof of Theorem 3.1 We assume $r = 0$ in order to ease the exposition. Let $\{X^\epsilon(s)\}_{s\in[0,T]}$ be the wealth corresponding to $\{\zeta_\epsilon(s), c_\epsilon(s)\}_{s\in[0,T]}$, i.e.,

$$dX^\epsilon(s) = X^\epsilon(s)((r + \mu\zeta_\epsilon(s) - c_\epsilon(s))\,ds + \sigma\zeta_\epsilon(s)\,dW(s)). \tag{6.1}$$

The processes $\{Y^\epsilon(s)\}_{s\in[0,T]}$ and $\{Z^\epsilon(s)\}_{s\in[0,T]}$, defined by the SDE

$$\begin{cases} dY^\epsilon(s) = Y^\epsilon(s)\left(\mu\frac{\partial F_1}{\partial x}\left(s,\bar{X}(s)\right) - \frac{\partial F_2}{\partial x}\left(s,\bar{X}(s)\right)\right)ds + \sigma[Y^\epsilon(t)\frac{\partial F_1}{\partial x}(s,\bar{X}(s) \\ \qquad\qquad + (\bar{X}(s)\zeta(s) - F_1(s,\bar{X}(s))\chi_{E_\epsilon}(s)]\,dW(s) \\ Y^\epsilon(0) = 0 \end{cases} \tag{6.2}$$

and

$$\begin{cases} dZ^\epsilon(s) = [Z^\epsilon(s)(\mu\frac{\partial F_1}{\partial x}(s,\bar{X}(s)) - \frac{\partial F_2}{\partial x}(s,\bar{X}(s))) \\ \qquad\qquad + (\mu\bar{X}(s)(\zeta(s) - \frac{\partial F_1}{\partial x}(s,\bar{X}(s))) - (c(s)\bar{X}(s) - F_2(s,\bar{X}(s))\chi_{E_\epsilon}(s)]\,ds \\ \qquad\qquad + [\sigma Z^\epsilon(s)\frac{\partial F_2}{\partial x}(s,\bar{X}(s)) + (\sigma Y^\epsilon(s)(\zeta(s) - \frac{\partial F_1}{\partial x}(s,\bar{X}(s))\chi_{E_\epsilon}(s)]\,dW(s) \\ Z^\epsilon(0) = 0 \end{cases}$$

can be regarded as first order and second order variation of the equilibrium wealth process $\{\bar{X}(s)\}_{s\in[0,T]}$ (its existence is granted by F_1 and F_2 being Lipschitz functions). At this point we need the Theorem 4.4 from [17] which we present below.

Lemma 6.5 *For any $k > 0$*

$$\sup_{s\in[0,T]} \mathbb{E}|X^\epsilon(s) - \bar{X}(s)|^{2k} = O(\epsilon^k), \tag{6.3}$$

$$\sup_{s\in[0,T]} \mathbb{E}|Y^\epsilon(s)|^{2k} = O(\epsilon^k), \tag{6.4}$$

$$\sup_{s\in[0,T]} \mathbb{E}|Z^\epsilon(s)|^{2k} = O(\epsilon^{2k}), \tag{6.5}$$

$$\sup_{s\in[0,T]} \mathbb{E}|X^\epsilon(s) - \bar{X}(s) - Y^\epsilon(s)|^{2k} = O(\epsilon^{2k}) \tag{6.6}$$

$$\sup_{s\in[0,T]} \mathbb{E}|X^\epsilon(s) - \bar{X}(s) - Y^\epsilon(s) - Z^\epsilon(s)|^{2k} = o(\epsilon^{2k}) \tag{6.7}$$

Moreover the following expansion hold:

$$\begin{aligned} J(t,x,\zeta_\epsilon,c_\epsilon) &= J(t,x,F_1,F_2) \\ &\quad + \mathbb{E}\int_t^T \left(h(s-t)(U(c(s)\bar{X}^{t,x}(s)) - U(F_2(s,\bar{X}^{t,x}(s)))\chi_{E_\epsilon}(s)\right)ds \\ &\quad + \mathbb{E}\int_t^T \left(h(s-t)\left(Y^\epsilon(s) + Z^\epsilon(s)\right)\frac{\partial F_2}{\partial x}\left(s,\bar{X}^{t,x}(s)\right)U'\left(F_2\left(s,\bar{X}^{t,x}(s)\right)\right)\right)ds \\ &\quad \times\mathbb{E}\int_t^T \left(h(s-t)\left(Y^\epsilon(s)\right)^2\frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x}\left(s,\bar{X}^{t,x}(s)\right)U'\left(F_2\left(s,\bar{X}^{t,x}(s)\right)\right)\right)\right)ds + o(\epsilon) \end{aligned}$$

In the above equation we would like to get rid of $\{Y^\epsilon(s)\}_{s\in[t,T]}$, $\{Z^\epsilon(s)\}_{s\in[t,T]}$ and the way to accomplish this is by using the adjoint processes $\{M(t,s), N(t,s)\}_{s\in[t,T]}$ and integration by parts. Let us recall the Hamiltonian function H is defined by

$$H(t,s,x,u,m,n) = mx(\zeta\mu - c) + nx\zeta\sigma + h(s-t)U(cx), \quad u = (\zeta, c). \tag{6.8}$$

Since the control variable enters into the volatility of the wealth process we have to introduce a second pair of adjoint processes $\{P(t,s), Q(t,s)\}_{s\in[t,T]}$ by

$$\begin{cases} dP(t,s) = -\Big(2P(t,s)\Big(\mu\frac{\partial F_1}{\partial x}\left(s,\bar{X}(s)\right) - \frac{\partial F_2}{\partial x}\left(s,\bar{X}(s)\right) + \sigma\left|\frac{\partial F_1}{\partial x}\left(s,\bar{X}(s)\right)\right|^2\Big) \\ \qquad\qquad + 2\sigma Q(t,s)\frac{\partial F_1}{\partial x}\left(s,\bar{X}(s)\right) + h(s-t)H_{xx}\left(t,s,\bar{X}^{t,x}(s),\bar{\zeta}(s),\bar{c}(s),M(t,s),N(t,s)\right)\Big)ds \\ \qquad\qquad + Q(t,s)dW(s) \\ P(t,T) = h(T-t)\hat{U}''(\bar{X}(T)), \end{cases} \tag{6.9}$$

where the equilibrium wealth process $\{\bar{X}(s)\}_{s\in[0,T]}$ follows (3.6). By using Lemma 4.5 and Lemma 4.6 in [17] we conclude that

$$\begin{aligned} &\lim_{\epsilon\downarrow 0}\frac{J(t,x,F_1,F_2) - J(t,x,\zeta_\epsilon,c_\epsilon)}{\epsilon} \\ &\quad = \mathbb{E}\Bigg[H\left(t,t,x,\frac{F_1(t,x)}{x},\frac{F_2(t,x)}{x},M(t,t),N(t,t)\right) \\ &\qquad - H(t,t,x,\zeta(t),c(t),M(t,t),N(t,t))\Bigg] - \frac{\sigma^2}{2}\mathbb{E}[(F_1(t,x))^2P(t,t)]. \end{aligned}$$

Next we claim that the process $\{P(t,s)\}_{s\in[t,T]}$ is negative. Indeed in the light of concavity of Hamiltonian H as a function of x

$$\begin{cases} dP(t,s) \le -\Big(2P(t,s)\Big(r + \mu\frac{\partial F_1}{\partial x}\left(s,\bar{X}(s)\right) - \frac{\partial F_2}{\partial x}\left(s,\bar{X}(s)\right) + \sigma\left|\frac{\partial F_1}{\partial x}\left(s,\bar{X}(s)\right)\right|^2\Big)\Big)ds \\ \qquad\qquad + Q(t,s)d\tilde{W}(s) \\ P(t,T) = h(T-t)\hat{U}''(\bar{X}(T)), \end{cases} \tag{6.10}$$

where

$$d\tilde{W}(s) = d\,W(s) + 2\sigma\frac{\partial F_1}{\partial x}\left(s,\bar{X}(s)\right)\,ds.$$

The claim follow due to concavity of function $\hat{U}$. Therefore a sufficient condition for $F = (F_1, F_2)$ to be a equilibrium policy is

$$\left(\frac{F_1(t,x)}{x},\frac{F_2(t,x)}{x}\right) = \arg\max_{\zeta,c} H(t,t,x,\zeta,c,M(t,t),N(t,t)). \tag{6.11}$$

Finally, the linearity of H in ζ implies

$$\mu M(t,t) + \sigma N(t,t) = 0, \tag{6.12}$$

and the first order condition for c (which are also sufficient due to concavity of U) yields

$$F_2(t,x) = I(M(t,t)|X(t) = x). \tag{6.13}$$

□

B Proof of Theorem 3.2 The result is established for all three cases (exponential, type I, type II) separately.

Exponential discounting: $h(t) = e^{-\delta t}$

Let $v(t, x)$ be the solution of the classical HJB

$$\frac{\partial v}{\partial t}(t, x) + rx\frac{\partial v}{\partial x}(t, x) - \frac{\mu^2}{2\sigma^2}\frac{\frac{\partial^2 v}{\partial x}(t, x)}{\frac{\partial^2 v}{\partial x^2}(t, x)} + \tilde{U}\left(\frac{\partial v}{\partial x}(t, x)\right) = \delta v(t, x), \tag{6.14}$$

for all $(t, x) \in [0, T] \times (0, \infty)$, with boundary condition

$$v(T, x) = \hat{U}(x).$$

Then $v(t, x)$ and $w(t, x) = 0$ solve the parabolic system. We show that the equilibrium policies are

$$F_1(t, x) = -\frac{\mu\frac{\partial v}{\partial x}(t, x)}{\sigma^2\frac{\partial^2 v}{\partial x^2}(t, x)}, \quad F_2(t, x) = I\left(\frac{\partial v}{\partial x}(t, x)\right), \quad t \in [0, T]. \tag{6.15}$$

Indeed let us consider the processes

$$M(t, s) = e^{-\delta(s-t)}\frac{\partial v}{\partial x}\left(s, \bar{X}(s)\right), \quad N(t, s) = \sigma e^{-\delta(s-t)}F_1\left(s, \bar{X}(s)\right)\frac{\partial^2 v}{\partial x^2}\left(s, \bar{X}(s)\right), \tag{6.16}$$

with $s \in [t, T]$. Recall that the equilibrium wealth process $\{\bar{X}(s)\}_{s\in[0,T]}$, is defined by

$$d\bar{X}(s) = \left[r\bar{X}(s) + \mu F_1\left(s, \bar{X}(s)\right) - F_2\left(s, \bar{X}(s)\right)\right]ds + \sigma F_1\left(s, \bar{X}(s)\right)dW(s). \tag{6.17}$$

It is a matter of direct calculations to prove that $\{M(t, s), N(t, s)\}_{s\in[t,T]}$ solves the BSDE (3.7). Next we observe

$$\mu M(t, t) + \sigma N(t, t) = 0, \tag{6.18}$$

and

$$F_2(t, x) = I(M(t, t)|X(t) = x), \tag{6.19}$$

so by Theorem 3.1, $F = (F_1, F_2)$ is an equilibrium strategy. □

Type I discounting: $h(t) = \lambda \exp(-\rho_1 t) + (1 - \lambda)\exp(-\rho_2 t)$

Let v and w be a solution of the PDE system. We show that the equilibrium strategies are

$$F_1(t, x) = -\frac{\mu\frac{\partial v}{\partial x}(t, x)}{\sigma^2\frac{\partial^2 v}{\partial x^2}(t, x)}, \quad F_2(t, x) = I\left(\frac{\partial v}{\partial x}(t, x)\right), \quad t \in [0, T]. \tag{6.20}$$

Let us define the process $\{M(t, s), N(t, s)\}_{s\in[t,T]}$ by

$$M(t, s) \triangleq \lambda \exp(-\rho_1(s - t))\frac{\partial v_1}{\partial x}(s, \bar{X}(s)) + (1 - \lambda)\exp(-\rho_2(s - t))\frac{\partial v_2}{\partial x}(s, \bar{X}(s)),$$

and

$$\begin{aligned}N(t, s) \triangleq\ & \lambda\sigma \exp(-\rho_1(s - t))F_1(s, \bar{X}(s))\frac{\partial^2 v_1}{\partial x^2}(s, \bar{X}(s)) \\ & + (1 - \lambda)\sigma \exp(-\rho_2(s - t))F_1(s, \bar{X}(s))\frac{\partial^2 v_2}{\partial x^2}(s, \bar{X}(s)),\end{aligned}$$

for some functions v_1 and v_2 which will be specified later on and $\{\bar{X}(s)\}_{s\in[0,T]}$ is the equilibrium wealth process of (6.17). By requesting $\{M(t,s), N(t,s)\}_{s\in[t,T]}$ to solve the BSDE (3.7), one finds a PDE system for (v_1, v_2). Indeed, on one hand by Itô's formula

$$\begin{aligned}
dM(t,s) = \Bigg(&\lambda \exp(-\rho_1(s-t))\bigg[-\rho_1 \frac{\partial v_1}{\partial x}(s,\bar{X}(s)) + \frac{\partial^2 v_1}{\partial x \partial s}(s,\bar{X}(s)) \\
&+ (\mu F_1(s,\bar{X}(s)) - F_2(s,\bar{X}(s)))\frac{\partial^2 v_1}{\partial x^2}(s,\bar{X}(s)) + \frac{\sigma^2}{2}F_1^2(s,\bar{X}(s))\frac{\partial^3 v_1}{\partial x^3}(s,\bar{X}(s))\bigg] \\
&+ (1-\lambda)\exp(-\rho_2(s-t))\bigg[-\rho_2 \frac{\partial v_2}{\partial x}(s,\bar{X}(s)) + \frac{\partial^2 v_2}{\partial x \partial s}(s,\bar{X}(s)) \\
&+ (\mu F_1(s,\bar{X}(s)) - F_2(s,\bar{X}(s)))\frac{\partial^2 v_2}{\partial x^2}(s,\bar{X}(s)) + \frac{\sigma^2}{2}F_1^2(s,\bar{X}(s))\frac{\partial^3 v_2}{\partial x^3}(s,\bar{X}(s))\bigg]\Bigg)\, ds \\
&+ N(t,s)dW(s).
\end{aligned}$$

On the other hand from the BSDE (3.7)

$$\begin{aligned}
dM(t,s) = \Bigg(&\lambda \exp(-\rho_1(s-t))\bigg[\frac{\partial v_1}{\partial x}(s,\bar{X}(s))\left(\mu\frac{\partial F_1}{\partial x}(s,\bar{X}(s)) - \frac{\partial F_2}{\partial x}(s,\bar{X}(s)\right) \\
&+ \sigma^2 F_1(s,\bar{X}(s))\frac{\partial F_1}{\partial x}(s,\bar{X}(s))\frac{\partial^2 v_1}{\partial x^2}(s,\bar{X}(s)) + \frac{\partial F_2}{\partial x}(s,\bar{X}(s))U'(F_2(s,\bar{X}(s)))\bigg] \\
&+ (1-\lambda)\exp(-\rho_2(s-t))\bigg[\frac{\partial v_2}{\partial x}(s,\bar{X}(s))\left(\mu\frac{\partial F_1}{\partial x}(s,\bar{X}(s)) - \frac{\partial F_2}{\partial x}(s,\bar{X}(s)\right) \\
&+ \sigma^2 F_1(s,\bar{X}(s))\frac{\partial F_1}{\partial x}(s,\bar{X}(s))\frac{\partial^2 v_2}{\partial x^2}(s,\bar{X}(s)) + \frac{\partial F_2}{\partial x}(s,\bar{X}(s))U'(F_2(s,\bar{X}(s)))\bigg]\Bigg)\, ds \\
&+ N(t,s)dW(s).
\end{aligned}$$

The two representations of $dM(t,s)$ yield

$$\begin{aligned}
&-\rho_1 \frac{\partial v_1}{\partial x}(s,x) + \frac{\partial^2 v_1}{\partial x \partial s}(s,x) + (\mu F_1(s,x) - F_2(s,x))\frac{\partial^2 v_1}{\partial x^2}(s,x) \\
&\quad + \frac{\sigma^2}{2}F_1^2(s,x)\frac{\partial^3 v_1}{\partial x^3}(s,x) = -\left(\mu\frac{\partial F_1}{\partial x}(s,x) - \frac{\partial F_2}{\partial x}(s,x)\right)\frac{\partial v}{\partial x}(s,x) \\
&\qquad\qquad - \sigma^2 F_1(s,x)\frac{\partial F_1}{\partial x}(s,x)\frac{\partial^2 v_1}{\partial x^2}(s,x) - \frac{\partial F_2}{\partial x}(s,x)\frac{\partial v}{\partial x}(s,x),
\end{aligned}$$

and

$$\begin{aligned}
&-\rho_2 \frac{\partial v_2}{\partial x}(s,x) + \frac{\partial^2 v_2}{\partial x \partial s}(s,x) + (\mu F_1(s,x) - F_2(s,x))\frac{\partial^2 v_2}{\partial x^2}(s,x) \\
&\quad + \frac{\sigma^2}{2}F_1^2(s,x)\frac{\partial^3 v_2}{\partial x^3}(s,x) = -\left(\mu\frac{\partial F_1}{\partial x}(s,x) - \frac{\partial F_2}{\partial x}(s,x)\right)\frac{\partial v_2}{\partial x}(s,x) \\
&\qquad\qquad - \sigma^2 F_1(s,x)\frac{\partial F_1}{\partial x}(s,x)\frac{\partial^2 v_2}{\partial x^2}(s,x) - \frac{\partial F_2}{\partial x}(s,x)\frac{\partial v}{\partial x}(s,x).
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
\frac{\partial}{\partial x}\bigg[&\frac{\partial v_1}{\partial s}(s,x) - \rho_1 v_1(s,x) + (\mu F_1(s,x) - F_2(s,x))\frac{\partial v_1}{\partial x}(s,x) \\
&+ \frac{\sigma^2}{2}F_1^2(s,x)\frac{\partial^2 v_1}{\partial x^2}(s,x) + U(F_2(s,x))\bigg] = 0,
\end{aligned}$$

and

$$\frac{\partial}{\partial x}\left[\frac{\partial v_2}{\partial s}(s,x)-\rho_2 v_2(s,x)+(\mu F_1(s,x)-F_2(s,x))\frac{\partial v_2}{\partial x}(s,x)\right.$$
$$\left.+\frac{\sigma^2}{2}F_1^2(s,x)\frac{\partial^2 v_2}{\partial x^2}(s,x)+U(F_2(s,x))\right]=0.$$

Recall that v and w is a solution of the PDE system. Thus

$$v_1 \triangleq v+(1-\lambda)w, \qquad v_2 \triangleq v-\lambda w \tag{6.21}$$

satisfy the above PDEs. Moreover

$$\mu M(t,t)+\sigma N(t,t)=0, \tag{6.22}$$

and

$$F_2(t,x)=I(M(t,t)|X(t)=x), \tag{6.23}$$

so by Theorem 3.1, $F=(F_1,F_2)$ is an equilibrium strategy. □

Type II discounting: $h(t)=(1+\lambda t)\exp(-\rho t)$

Let v and w be a solution of the PDE system. We show that the equilibrium strategies are

$$F_1(t,x)=-\frac{\mu\frac{\partial v}{\partial x}(t,x)}{\sigma^2\frac{\partial^2 v}{\partial x^2}(t,x)},\quad F_2(t,x)=I\left(\frac{\partial v}{\partial x}(t,x)\right),\quad t\in[0,T]. \tag{6.24}$$

Let us define the process $\{M(t,s),N(t,s)\}_{s\in[t,T]}$ by

$$M(t,s)\triangleq \exp(-\rho(s-t))\frac{\partial v_1}{\partial x}\left(s,\bar X(s)\right)+\lambda t\exp(-\rho(s-t))\frac{\partial v_2}{\partial x}\left(s,\bar X(s)\right),$$

and

$$N(t,s)\triangleq \sigma\exp(-\rho_1(s-t))F_1\left(s,\bar X(s)\right)\frac{\partial^2 v_1}{\partial x^2}\left(s,\bar X(s)\right)$$
$$+\sigma\lambda t\exp(-\rho_2(s-t))F_1\left(s,\bar X(s)\right)\frac{\partial^2 v_2}{\partial x^2}\left(s,\bar X(s)\right),$$

for the functions

$$v_1\triangleq v,\qquad v_2\triangleq v-w. \tag{6.25}$$

As for the case of type I one can check that $\{M(t,s),N(t,s)\}_{s\in[t,T]}$ solves BSDE (3.7). Moreover

$$\mu M(t,t)+\sigma N(t,t)=0, \tag{6.26}$$

and

$$F_2(t,x)=I(M(t,t)|X(t)=x), \tag{6.27}$$

so by Theorem 3.1, $F=(F_1,F_2)$ is an equilibrium strategy. □

C Proof of Proposition 3.1 If the discounting is exponential the result is a direct consequence of Feynman-Kac's formula. For type I, the functions v_1 and v_2 of (6.21) admit the following stochastic representations

$$v_i(t,x)=\mathbb{E}\left[\int_t^T e^{-\rho_i(s-t)}U(F_2(s,\bar X_s^{t,x}))\,ds\right],\quad i=1,2, \tag{6.28}$$

by Feynman-Kac's formula. Consequently

$$v(t,x) = \lambda v_1(t,x) + (1-\lambda) v_2(t,x) = \mathbb{E}\left[\int_t^T h(s-t) U\left(F_2(\bar{X}_s^{t,x})\right) ds\right]. \tag{6.29}$$

Similarly for type II, (6.29) holds true. □

D Proof of Lemma 3.1 Existence is granted on small time intervals. It follows from the integral equation (3.18) which describes the equilibrium policies that the ODE system can be rewritten as the integral equation (3.26). This can be written in differential form as

$$\begin{aligned} f'(t) &= -\left[\frac{h'(T-t)}{h(T-t)} + K\right] f(t) + (p-1)[f(t)]^{\frac{p}{p-1}} \\ &\quad + \int_t^T h(T-t) \frac{\partial}{\partial t}\left[\frac{h(s-t)}{h(T-t)}\right] [f(s)]^{\frac{p}{p-1}} e^{-\left(\int_t^s p[f(u)]^{\frac{1}{p-1}} du\right)} ds. \end{aligned} \tag{6.30}$$

From (3.26) we infer that

$$\int_t^T h(s-t) e^{K(s-t)} [f(s)]^{\frac{p}{p-1}} e^{-\left(\int_t^s p[f(u)]^{\frac{1}{p-1}} du\right)} ds \le f(t).$$

This together with the boundedness (from below and above) of the function h, yields a positive constant A (that depends on T and $h(\cdot)$) such that

$$f(t) \le A f(t) + (p-1)[f(t)]^{\frac{p}{p-1}}, \tag{6.31}$$

and

$$f'(t) \ge -A f(t) + (p-1)[f(t)]^{\frac{p}{p-1}}. \tag{6.32}$$

It follows from (6.31) that

$$\vartheta'(t) - \left(\frac{A}{1-p}\right) \vartheta(t) \le -1,$$

where

$$\vartheta(t) \triangleq \left(\frac{1}{[f(t)]^{\frac{1}{p-1}}}\right).$$

Thus

$$\left(e^{\left(\frac{At}{p-1}\right)} \vartheta(t)\right)' \le -e^{\left(\frac{At}{p-1}\right)} < 0.$$

Integrating this from t to T we obtain the lower estimate on $f(t)$. Following (6.31) and arguing similarly we can get the upper estimate on $f(t)$. This estimates show that we have global existence. As for uniqueness we argue by contraposition. Let f_1 and f_2 two solutions of (3.26). Then in the light of the estimates obtained above it follows that

$$\left| h(T-t)e^{K(T-t)}e^{-\left(\int_t^T p[f_1(u)]^{\frac{1}{p-1}}\,du\right)} - h(T-t)e^{K(T-t)}e^{-\left(\int_t^T p[f_2(u)]^{\frac{1}{p-1}}\,du\right)} \right|$$
$$\leq K_1 \int_t^T |f_1(u) - f_2(u)|\,du$$

and

$$|[f_1(t)]^{\frac{p}{p-1}} - [f_2(t)]^{\frac{p}{p-1}}| \leq K_2 |f_1(t) - f_2(t)|,$$

for some positive constants K_1 and K_2. Therefore there exists a positive constant K_3 such that

$$|f_1(t) - f_2(t)| \leq K_3 \int_t^T |f_1(u) - f_2(u)|\,du.$$

At this point one can argue as in the Gronwal's Lemma proof to get that $|f_1(t) - f_2(t)| = 0$. □

E Proof of Theorem 4.6 In a first step one needs to establish that

$$\begin{aligned} J(x, \zeta_\epsilon, c_\epsilon) \leq J(x, F_1, F_2) &+ \mathbb{E}\int_0^\infty \left(h(s)(U(c(s)\bar{X}^{0,x}(s)) - U(F_2(\bar{X}^{0,x}(s)))\chi_{E_\epsilon}(s)\right) ds \\ &+ \mathbb{E}\int_0^\infty \left(h(s)\left(Y^\epsilon(s) + Z^\epsilon(s)\right) F_2'\left(\bar{X}^{0,x}(s)\right) U'\left(F_2\left(\bar{X}^{0,x}(s)\right)\right)\right) ds + o(\epsilon). \end{aligned} \tag{6.33}$$

Direct computations show that this holds true. Indeed let us take $E_\epsilon = [0, \epsilon]$. Using the fact that $F_2(x) = k^{\frac{1}{p-1}} x$ and the inequality $U(a) - U(b) \leq (a-b)U'(b)$ for the concave function $U(x) = \frac{x^p}{p}$ one gets

$$\begin{aligned} J(x, \zeta_\epsilon, c_\epsilon) \leq J(x, F_1, F_2) &+ \mathbb{E}\int_0^\infty \left(h(s)(U(c(s)\bar{X}^{0,x}(s)) - U(F_2(\bar{X}^{0,x}(s)))\chi_{E_\epsilon}(s)\right) ds \\ &+ \mathbb{E}\int_\epsilon^\infty \left(k^{\frac{p}{p-1}} h(s)(X^\epsilon(s) - \bar{X}(s))(\bar{X}^{0,x}(s))^{p-1}\right) ds. \end{aligned} \tag{6.34}$$

Therefore to prove (6.33) it suffices to show that

$$\mathbb{E}\int_\epsilon^\infty \left(h(s)(X^\epsilon(s) - \bar{X}(s) - Y^\epsilon(s) - Z^\epsilon(s))(\bar{X}^{0,x}(s))^{p-1}\right) ds = o(\epsilon) \tag{6.35}$$

It turns out that

$$\frac{X^\epsilon(s)}{X^\epsilon(\epsilon)} = \frac{Y^\epsilon(s)}{Y^\epsilon(\epsilon)} = \frac{Z^\epsilon(s)}{Z^\epsilon(\epsilon)} = \frac{\bar{X}(s)}{\bar{X}(\epsilon)}, \quad \text{on } [\epsilon, \infty). \tag{6.36}$$

Thus (6.35) becomes

$$\mathbb{E}\int_{\epsilon}^{\infty}\left(h(s)(X^{\epsilon}(\epsilon)-\bar{X}(\epsilon)-Y^{\epsilon}(\epsilon)-Z^{\epsilon}(\epsilon))(\bar{X}^{0,x}(\epsilon))^{p-1}\left(\frac{\bar{X}^{0,x}(s)}{\bar{X}^{0,x}(\epsilon)}\right)^{p}\right)ds=o(\epsilon) \tag{6.37}$$

Since we consider stationary policies, the processes $(X^{\epsilon}(\epsilon)-\bar{X}(\epsilon)-Y^{\epsilon}(\epsilon)-Z^{\epsilon}(\epsilon))(\bar{X}^{0,x}(\epsilon))^{p-1}$ and $\frac{\bar{X}^{0,x}(s)}{\bar{X}^{0,x}(\epsilon)}$ are independent hence one needs to show that

$$\mathbb{E}\left[(X^{\epsilon}(\epsilon)-\bar{X}(\epsilon)-Y^{\epsilon}(\epsilon)-Z^{\epsilon}(\epsilon))(\bar{X}^{0,x}(\epsilon))^{p-1}\right]\left[\mathbb{E}\int_{\epsilon}^{\infty}\left(h(s)\left(\frac{\bar{X}^{0,x}(s)}{\bar{X}^{0,x}(\epsilon)}\right)^{p}\right)ds\right]=o(\epsilon). \tag{6.38}$$

In the light of (6.7) it suffices to prove that

$$\mathbb{E}\int_{\epsilon}^{\infty}\left(h(s)\left(\frac{\bar{X}^{0,x}(s)}{\bar{X}^{0,x}(\epsilon)}\right)^{p}\right)ds<\infty. \tag{6.39}$$

However $\mathbb{E}[\bar{X}^{0,x}(\epsilon)]^{p}<\infty$ and so (6.33) yields if we prove that

$$\mathbb{E}\int_{\epsilon}^{\infty}h(s)\left(\bar{X}^{0,x}(s)\right)^{p}ds=\mathbb{E}\left[\bar{X}^{0,x}(\epsilon)\right]^{p}\left[\mathbb{E}\int_{\epsilon}^{\infty}\left(h(s)\left(\frac{\bar{X}^{0,x}(s)}{\bar{X}^{0,x}(\epsilon)}\right)^{p}\right)ds\right]<\infty.$$

This follows from (4.16) and $v(x)=kU(x)=\frac{kx^{p}}{p}<\infty$. Next let us establish (4.14). This follows from the Dominated Convergence Theorem since

$$\int_{0}^{\infty}\left|\left(h(s)(Y^{\epsilon}(s)+Z^{\epsilon}(s))F_{2}'\left(\bar{X}^{0,x}(s)\right)U'\left(F_{2}\left(\bar{X}^{0,x}(s)\right)\right)\right)\right|ds$$

is integrable in the light of equation (6.36). Moreover in this context, due to (6.36) transversality equations (4.21) and (4.8) are equivalent. Notice that the equilibrium wealth process exists and $(\bar{\zeta},\bar{c})$ of (4.5) is admissible. Then Proposition 4.3 and Theorem 4.5 concludes the proof. □

F Proof of Lemma 4.2 The condition (4.24) yield the positivity of k and (4.23). The adjoint process $\{M(t)\}_{t\in[0,\infty)}$ is given by

$$\begin{aligned}M(t)&=\exp(-\delta t)v'(\bar{X}(t))\\&=kX(0)\left[\exp\left(\left(r(p-1)+\frac{(2p-1)\mu^{2}}{2(1-p)\sigma^{2}}-(p-1)k^{\frac{1}{p-1}}-\delta\right)t-\frac{\mu}{\sigma}W(t)\right)\right]\\&=kX(0)\left[\exp\left(-\left(r+\frac{\mu^{2}}{2\sigma^{2}}\right)t-\frac{\mu}{\sigma}W(t)\right)\right],\end{aligned}$$

whence the transversality equation (4.21) is automatically satisfied in the light of (4.24). □

G The quadratic equations describing the equilibrium policies for type I discounting

For type I discounting the equation (4.27) becomes for $z = k^{\frac{1}{p-1}}$

$$Q(z) = Az^2 + Bz + C = 0, \tag{6.40}$$

where

$$A \triangleq 1 - p, \tag{6.41}$$

$$B \triangleq (2p-1)\left(\frac{\mu^2}{2(1-p)\sigma^2} + r\right) + \frac{\lambda\rho_2 + (1-\lambda)\rho_1}{p} - (\rho_1 + \rho_2), \tag{6.42}$$

$$C \triangleq -\frac{1}{p}\left(\rho_1 - rp - \frac{p\mu^2}{2(1-p)\sigma^2}\right)\left(\rho_2 - rp - \frac{p\mu^2}{2(1-p)\sigma^2}\right). \tag{6.43}$$

When $\rho_1 = \rho_2 = \delta$, the equation (6.40) becomes

$$\left[(1-p)z - \left(\delta - pr - \frac{p\mu^2}{2(1-p)\sigma^2}\right)\right]\left[z + \frac{1}{p}\left(\delta - pr - \frac{p\mu^2}{2(1-p)\sigma^2}\right)\right] = 0, \tag{6.44}$$

which compared to (4.25) brings in a new solution

$$z_\delta = \left[-\frac{1}{p}\left(\delta - rp - \frac{p\mu^2}{2(1-p)\sigma^2}\right)\right]. \tag{6.45}$$

Thus when $\rho_1, \ \rho_2$ are close to δ the equation (6.40) has at least a positive solution provided that δ satisfies (4.24). □

H The quadratic equations describing the equilibrium policies for type II discounting

For type II discounting the equation (4.33) becomes for $z = k^{\frac{1}{p-1}}$

$$Q(z) = Az^2 + Bz + C = 0, \tag{6.46}$$

with

$$A \triangleq 1 - p, \tag{6.47}$$

$$B \triangleq (2p-1)\left(\frac{\mu^2}{2(1-p)\sigma^2} + r\right) + \frac{\rho(1-2p)}{p} + \frac{\lambda}{p}, \tag{6.48}$$

$$C = -\frac{1}{p}\left(\rho - rp - \frac{p\mu^2}{2(1-p)\sigma^2}\right)^2. \tag{6.49}$$

The transversality equation in this case is

$$pk^{\frac{1}{p-1}} + \rho_i > pr + \frac{p^2\mu^2}{2(1-p)^2\sigma^2}. \tag{6.50}$$

□

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