

# Towards a fourth spatial dimension of brain activity

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**Abstract** Current advances in neurosciences deal with the functional architecture of the central nervous system, paving the way for general theories that improve our understanding of brain activity. From topology, a strong concept comes into play in understanding brain functions, namely, the 4D space of a “hypersphere’s torus”, undetectable by observers living in a 3D world. The torus may be compared with a video game with biplanes in aerial combat: when a biplane flies off one edge of gaming display, it does not crash but rather it comes back from the opposite edge of the screen. Our thoughts exhibit similar behaviour, i.e. the unique ability to connect past, present and future events in a single, coherent picture as if we were allowed to watch the three screens of past-present-future “glued” together in a mental kaleidoscope. Here we hypothesize that brain functions are embedded in a imperceptible fourth spatial dimension and propose a method to empirically assess its presence. Neuroimaging fMRI series can be evaluated, looking for the topological hallmark of the presence of a fourth dimension. Indeed, there is a typical feature which reveal the existence of a functional hypersphere: the simultaneous activation of areas opposite each other on the 3D cortical surface. Our suggestion—substantiated by recent findings—that brain

activity takes place on a closed, donut-like trajectory helps to solve long-standing mysteries concerning our psychological activities, such as mind-wandering, memory retrieval, consciousness and dreaming state.

**Keywords** Hypersphere · Brain · Central nervous system · Borsuk-Ulam theorem · Fourth dimension · Manifold

How do thoughts flow in the brain? Current advances in neuroscience emphasize the role of energetic landscapes (Watanabe et al. 2014; Sengupta et al. 2013), a sort of functional linens equipped with peaks, valleys and basins made of free-energy, where thoughts move, following erratic and/or constrained trajectories (Afraimovich et al. 2013; Zare and Grigolini 2013; Fraiman and Chialvo 2012; Deco and Jirsa 2012). Such “dynamical systems” theories lack however an explanation of the possible physiological mechanisms underlying brain activity. General theories are thus needed to further improving our knowledge of central nervous system activity. This review, based on recent findings, introduces the concept of a spatial fourth dimension, where brain functions might take place, as a general device underlying our thoughts’ dynamics. In particular, we hypothesize that brain activity is shaped in guise of a “3-sphere” which performs 4D movements on the cortical layers, giving rise to a functional “Clifford torus” where mental operations might take place. We will discuss how thoughts follow constrained, donut-like trajectories along preferential functional railways embedded in the very structure of the human brain connectome. This essay comprises five sections and an appendix organized as follows. The very first section will be devoted to give as simple as possible explanations of concepts from the far-

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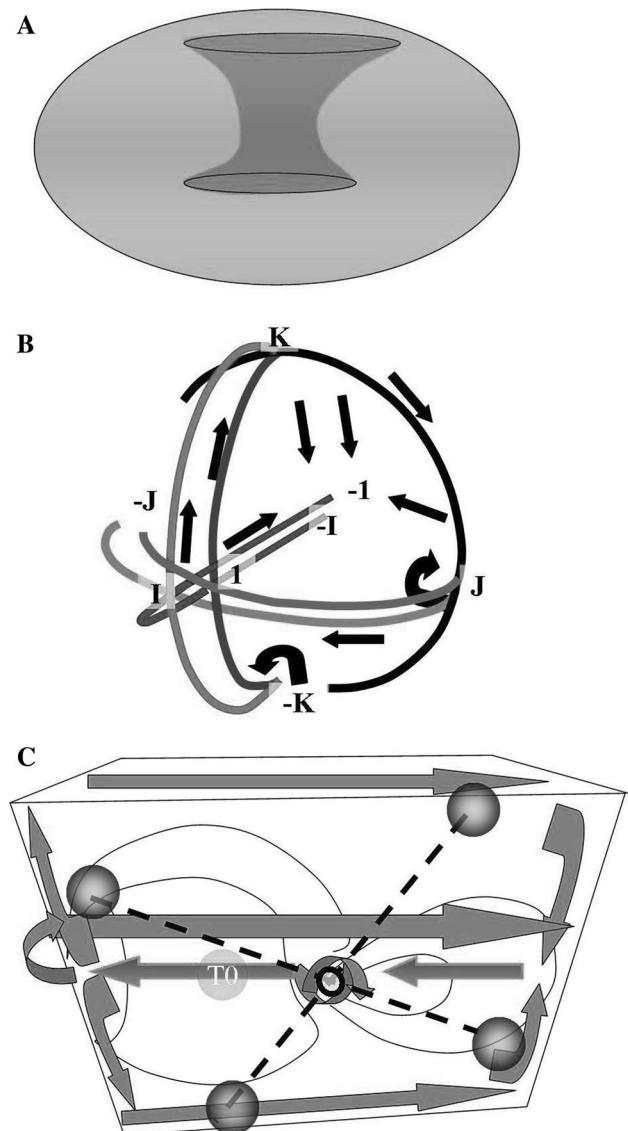
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fung branch of topology. Because this method of investigation of the laws of biology diverges from what was previously familiar, we prefer to use a language addressed to a broad interdisciplinary audience which encompasses experts from different fields (neuroscientists, medical doctors, biologists, epistemologists). Sections two and three will aim to enclose the concept of the 3-sphere in the framework of brain activity, in order to demonstrate the feasibility of our theory and to sketch a four-dimensional brain model in a way that is empirically testable. Section four, taking into account suggestions from recent papers, will provide quantitative methods for an experimental assessment of our hypothesis, which could be of use to researchers that investigate network connectivity. The final section discusses how a brain hypersphere could lead to novel approaches to organized brain models, improving our understanding of cortical functions and paving the way to innovative research strategies. In order to show how our theory can be operationalized, we also provide an appendix as supporting information for technical readers. This latter material unpacks 3-spheres and fourth dimensions afforded by a mathematical point of view.

### What is a hypersphere?

A  $n$ -sphere, also called  $S^n$ , is a  $n$ -dimensional structure embedded in a  $n + 1$  space (Henderson 1996). For example, a 1-sphere ( $S^1$ ) is the one-dimensional circumference surrounding a 2-dimensional disk, while a 2-sphere ( $S^2$ ) is the 2-dimensional surface of a 3-dimensional space (a beach ball's surface is a good illustration) (Henderson and Taimina 2001). In mathematical terms, a 3-sphere ( $S^3$ ), also called *glome* (from the Latin “glomus”, meaning ball of string), or generically *hypersphere*, is an artificial convex structure enclosed in a Euclidean 4-dimensional space called a *4-ball* (Marsaglia 1972). The prefix “hyper” refers to 3- (and higher-) dimensional analogues of  $n$ -spheres. A 3-sphere is thus the surface of a 4-dimensional ball, while a 4-dimensional ball is the interior (the content) of a 3-sphere, in the same way as a bottle of water is made of a glass surface and a liquid content. How to build a glome? In topology, a glome can be attained by superimposing two 2-spheres (two beach balls) whose opposite edges are abstractly glued together: we obtain a structure called the *Clifford torus* (Manetti 2015). A Clifford torus reminds a lifebelt, with its donut-like shape (Fig. 1a), nevertheless with an important difference: the Clifford torus displays the same local geometry as an “ordinary” three-dimensional space, but its global topology is different (Willard 1970; Krantz 2009). Our torus is constantly in movement, because it is equipped with intricate rotations called *quaternionic movements* (Ozdemir and Özkes 2013;



**Fig. 1** Different ways to depict a hypersphere. **a** How two 2-spheres glued together along their spherical boundary give rise to a donut-shaped Clifford torus. **b** Another way to depict a hypersphere: the superimposition of two 2-spheres (which circumferences are glued together) gives rise to a glome. Some of the quaternion rotations are depicted by the straight and curved arrows. **c** 3D projection of a hypersphere. The *lines* on the left enlarge in diameter, forming a circle of increasing circumference on the left surface of the 3D space. Conversely, on the opposite right side, the *lines* shrink and give rise to a circle of decreasing circumference on the right surface of the 3D space. See text for further details. The *dotted lines* and the *black spheres* depict some of the possible antipodal points predicted by the Borsuk Ulam Theorem (to give another example, J and  $-J$  are antipodal points in **a**)

Lemaître 1948). Imagine a skein made of rope circles, where each rope takes a different rotation: the general shape of the skein is always the same, but an ever changing motion occurs inside it. The internal shape of the glome is in ceaseless activity, depending on the number of ropes

taken into account and their trajectories. Complicated motions occur indeed on a hypersphere: four pairs of points opposite each other (called antipodal points  $l, -l, k, -k, j, -j$  and  $i, -i$ ) give rise to the “quaternion group” (Hart and Segerman 2014), which displays two possible types of reciprocal 4D rotations (Fig. 1b). For a mathematical treatment of hyperspheres, quaternions and the correlated Lie groups, see Appendix, Section 1. The glome, requiring four dimensions for its definition just as an ordinary sphere requires three, is not detectable in the usual spatial 3-dimensions and is thus very challenging to assess. Figures 1a–c show the possible ways to cope with a 3D visualization of a glome: observe how opposite sections of the rim fit together, rather than trying to visualize the whole thing at once the way you would visualize a common sphere. Each apparent line segment is really two, one arching upward into the third dimension and the other arching downward. To better understand the concept of a 3-sphere, the images should be watched during their complicated movements: a list of very useful videos is provided in Appendix, Section 1.

### Why an hypersphere to explain brain activity?

Experimental and theoretical clues allow us to conjecture that the brain activities (at least some of them) are embedded in a torus lying on the surface of a hypersphere. The theoretical claims of brain multidimensionality are widespread (Tononi 2008; Papo 2014; Ponce-Alvarez et al. 2015; Schoenberg and Speckens 2015). Models characterized by dimensionality reduction have been used in the study of human central nervous system, particularly through full-brain auto-regressive modelling (Garg et al. 2011). It has been demonstrated that high dimensionality spontaneous activity structures—termed “lag threads”—can be found in the brain, consisting of multiple, reproducible temporal sequences (Mitra et al. 2015). Moreover, it has been shown that the exceedingly diverse nonlinear selectivity in single-neuron activity in the prefrontal cortex, which encodes distributed information about all task-relevant aspects, is a signature of high-dimensional neural representations: crucially, this dimensionality is predictive of animal behaviour as it collapses in error trials (Rigotti et al. 2013). In addition, recent findings suggest that nervous structures process information through topological as well as spatial mechanisms. For example, it has been hypothesized that hippocampal place cells create topological templates to represent spatial information (Dabaghian et al. 2014). We stated that a glome displays a donut-like shape: it means that the trajectory followed by a particle inside the torus is closed. To make an example, if you walk along one of the curves of a 4-ball, you think are crossing a

straight trajectory, and do not recognize that your environment is embedded in an higher dimension. The torus may be compared with a video game with biplanes in aerial combat: when a biplane flies off one edge of gaming display, it does not crash but rather it comes back from the opposite edge of the screen. Mathematically speaking, the display edges have been “glued” together. Our brain exhibits similar behaviour, i.e., the unique ability to connect past, present and future events in a single, coherent picture (Fingelkurts and Fingelkurts 2014; Nyberg et al. 2010), as if we were allowed to watch the three screens of past-present-future glued together in a mental kaleidoscope. The same occurs during other brain functions, e.g., memory retrieval, recursive imagination and mind wandering (Andrews-Hanna et al. 2014), in which concepts flow from a state to another and appear to be “glued” together. As stated above, the torus is naturally visualized intrinsically, by ignoring any extrinsic properties a surface may have: it is thought that all the movements onto a torus surface are performed just by trajectories *internal* to its structure. For example, take a sheet of paper and bend it into a half-cylinder: the extrinsic geometry of the paper has obviously changed, but the paper itself has not been deformed and its intrinsic geometry has not varied. What would you see if you lived in a closed 4-dimensional manifold? You should be able to see yourself, via the intrinsic structure provided by the glued surfaces of a hypersphere, in an otherwise unperceivable 4D space (Weeks 2002). In the same way, we humans perceive our thoughts intrinsically and naturally adopt “private”, subjective standpoints.

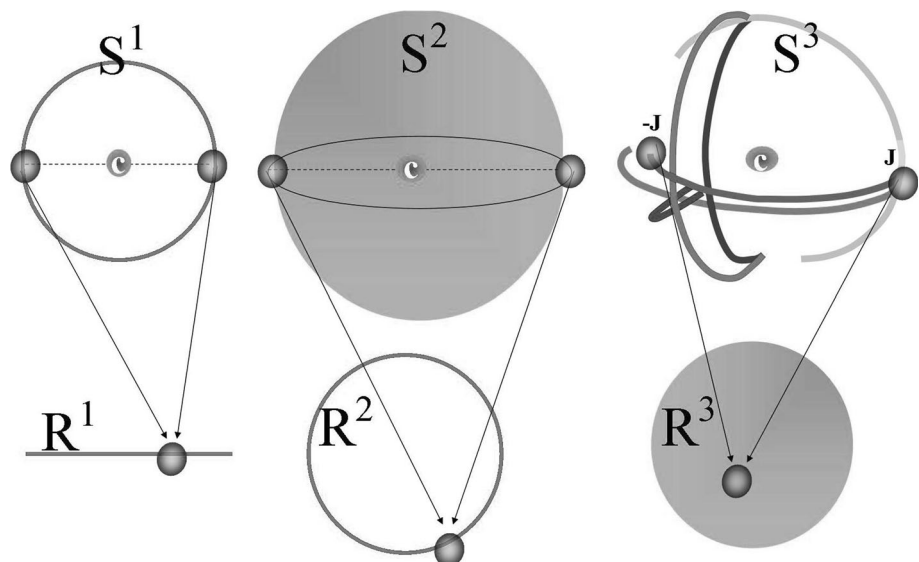
### How to demonstrate the presence of a brain hypersphere?

Brains equipped with a hypersphere is a counter-intuitive hypothesis, since we live in a 3D world with no immediate perception that 4D space exists at all. We need thus to evaluate indirect clues of the undetectable fourth dimension, such as signs of the glome rotations on a familiar 3D surface. In other words, rotations of a 4D torus embedded in a 4-ball can be identified through their “cross section” movements on a more accessible 3D surface (Fig. 1c), as if you recognized an object just from its shadow projected on a screen. In Fig. 1c, the arrows illustrate the trajectories followed by the 4D quaternionic movements of a Clifford torus, when projected onto the surface of the 3D space in which it has been artificially embedded. Note that the arrows (representing the quaternions’ flows) follow the external and medial surfaces of the 3D space in a way that is predictable. It must be emphasized that just one of the possible directions of the quaternion movements is

displayed in Fig. 1c: the flow on a Clifford torus may indeed occur in each of the four planes. The small circle labelled T0 depicts one of the possible conventional starting points of the flow. In Fig. 1c we showed just the trajectory from right to left, starting from T0; however, also the opposite trajectory, from left to right, and countless others, can be exploited by the torus during its 4D movements.

The presence of a glome can be detected invoking the Borsuk-Ulam Theorem (BUT) from topology, which is characterized by very helpful features (Borsuk 1933; Matoušek 2003). BUT states that we can identify on a  $n$ -sphere two “antipodal points” (i.e., points directly opposite each other) (Fig. 2). To make an example, there always exist a pair of opposite points on the earth’s equator (a 1-sphere) with the same temperature. It is worth of mention that the two antipodal points display similar characteristics and have matching descriptions (Cohen 1973; Giblin 2010). Such two opposite points can be used not just for the description of simple topological structures, but can also be generalized to the description of two antipodal shapes or signals. Thus, if you take into account physical and biological phenomena instead of “points”, BUT leads naturally to the possibility of a region-based, not simply point-based, geometry, in which we view collections of signals as surface shapes, where one shape maps to another antipodal one (Peters 2014; Collins 2004; Borsuk 1958–1959; Borsuk 1969; Borsuk and Gmurczyk 1980). A point on  $S^2$  (which is embedded in  $R^3$ ) projects to two antipodal points on  $R^4$ . It means that the projection from a lower dimension  $R^n$  (equipped with just one point) to an higher one  $R^{n+1}$  gives rise to two opposite, symmetrical points. It also means that different phenomena (including brain activity) could be studied in terms of antipodal points, if we embed them in just one dimension higher than the usual one.

**Fig. 2** The Borsuk-Ulam theorem for different values of  $S^n$ .  $S^1$  depicts a circumference,  $S^2$  a common sphere,  $S^3$  a hypersphere, while  $R^1$  portrays a line,  $R^2$  a circumference and  $R^3$  a common sphere. Note that the two antipodal points in every sphere  $S^n$  project to a single point in the corresponding space  $R^n$ , and vice versa



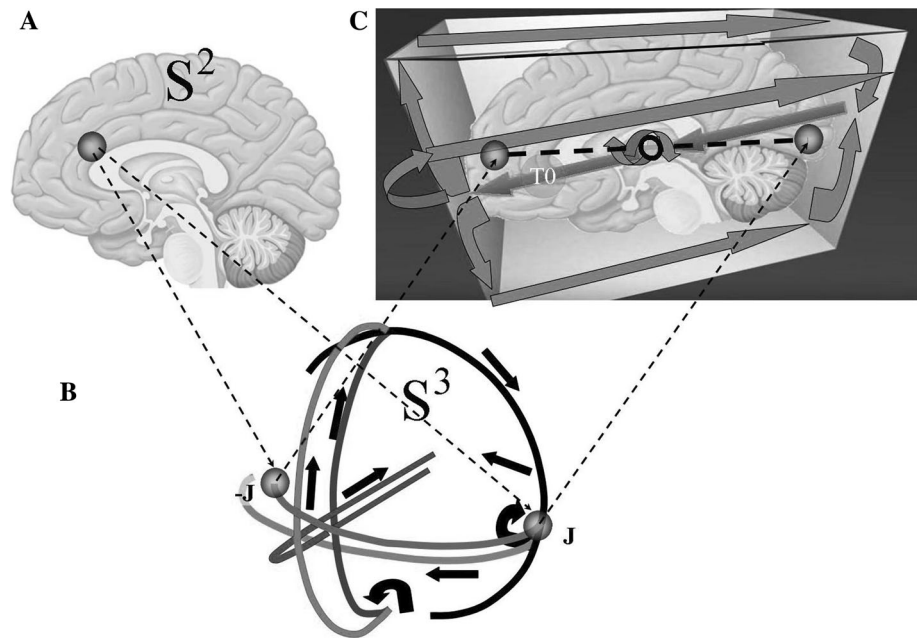
If we embed the brain in the 4D space of a Clifford torus, the presence of antipodal (opposite) points or shapes on 3D cortical surfaces could be evaluated (Fig. 3). We need to stress again that the hypersphere’s location displayed in Fig. 3 is just one of the countless possible: being the glome a *functional* structure equipped with many rotations and trajectories, it can be placed in different points of the brain surface. The antipodal points in the fourth dimension evoked by BUT can be viewed as brain signals opposite each other on the three-dimensional brain. For example, when a surface “fires”, the simultaneous firing of the opposite one is a proof of a perceivable “passing through” of the fourth dimension onto the brain 3D surface. The mechanism, elucidated in Fig. 3, is the following: the activation of a single point on the 3D brain  $S^2$  surface (Fig. 3a) leads to the activation of two antipodal points on the 4D brain  $S^3$  surface (Fig. 3b). In turn, the activation of two antipodal points on  $S^3$  leaves on the 3D brain  $S^2$  surface “hallmarks” (Fig. 3c) which can be detected by currently available neuroimaging techniques. For a mathematical treatment of the Borsuk-Ulam theorem, shapes, homotopies and their applications, see the Appendix, Section 2. In conclusion, the mathematical model of antipodal points can be casted in a biologically informed fashion, resulting in a framework that has the potential to be operationalized and assessed empirically.

### Are there proofs or clues of brain hypersphere?

Our brain hypersphere hypothesis could be corroborated either by prospective investigations, or by retrospective evaluation of published resting-state fMRI data, looking for the hallmarks of the hypothesized BUT. Why resting-state studies are more feasible for an evaluation of the possible



**Fig. 3** The concept of hypersphere in the framework of brain functional activity. According to the Borsuk-Ulam theorem, the activation of a single point on the  $S^2$  brain surface (Fig. 1a) leads to the activation of two antipodal points (corresponding in this case to the quaternionic points  $J$  and  $-J$ ) on the  $S^3$  brain surface (b). The simultaneous activation of the  $S^3$  antipodal points displayed in **b** can also be evaluated on the 3D cortical surface (c), provided the brain is embedded in a 3D space containing the 4D Clifford torus (the same 3D space described in Fig. 1c). The nomenclature is borrowed from c



presence of a glome in brain activity? Spontaneous oscillations are intrinsic, low-frequency fluctuations of cerebral activity which cannot be attributed to the experimental design or other explicit input or output (Fox and Raichle 2007). Among the networks exhibiting coherent fluctuations in spontaneous activity, the “default-mode network” (DMN) is worth of mentioning, because it includes functionally and structurally connected regions that show high metabolic activity at rest, but deactivate when specific goal-directed behavior is needed (Raichle et al. 2001). Spontaneous oscillations recapitulate the topographies of fMRI responses to a wide variety of sensory, motor and cognitive task paradigms, providing a powerful means of delineating brain functional organization without the need for subjects to perform tasks (Mitra et al. 2014). To evaluate our BUT hypothesis, we favour studies focused on intrinsic, instead of task-evoked activity, because the former is associated with mental operations that could be attributed to the activity of a glome—mental “screens” are glued together and the trajectories of particles (or thoughts!) follow the internal surface of a Clifford torus. For example, spontaneous brain activity has been associated with mind-wandering or day dreaming propensities (Kucyi et al. 2014), construction of coherent mental scenes, autobiographical memories (Conway and Pleydell-Pearce 2000), experiences focused on the future and dreaming state (Domhoff and Fox 2015). Recent evidence also suggests overlap between the DMN and regions involved in self- and other-related mental operations—such as affective and introspective processes (Amft et al. 2015; Philippi et al. 2015; Morewedge et al. 2014)—for a description of the terminology, see Andrews-Hanna et al. (2014). It has

also been hypothesized that spontaneous functional connectivity patterns at rest might constitute a “signature of consciousness”, reflecting a stream of ongoing cognitive processes (Bartfeld et al. 2015). Spontaneous activity is highly variable among individuals, depending on local brain differences, somatosensory awareness, age span, race, culture and so on (Gorgolewski and al 2014; Sadtler et al. 2014). We speculate that such variability might be correlated with those differences in Clifford torus’ structure and movements illustrated above. A brain 3-sphere has the potential to represent a conceptual bridge, because it exhibits both anatomical/functional (spontaneous brain activity and DMN) and psychological correlates (spontaneous, deliberate, self-generated thoughts).

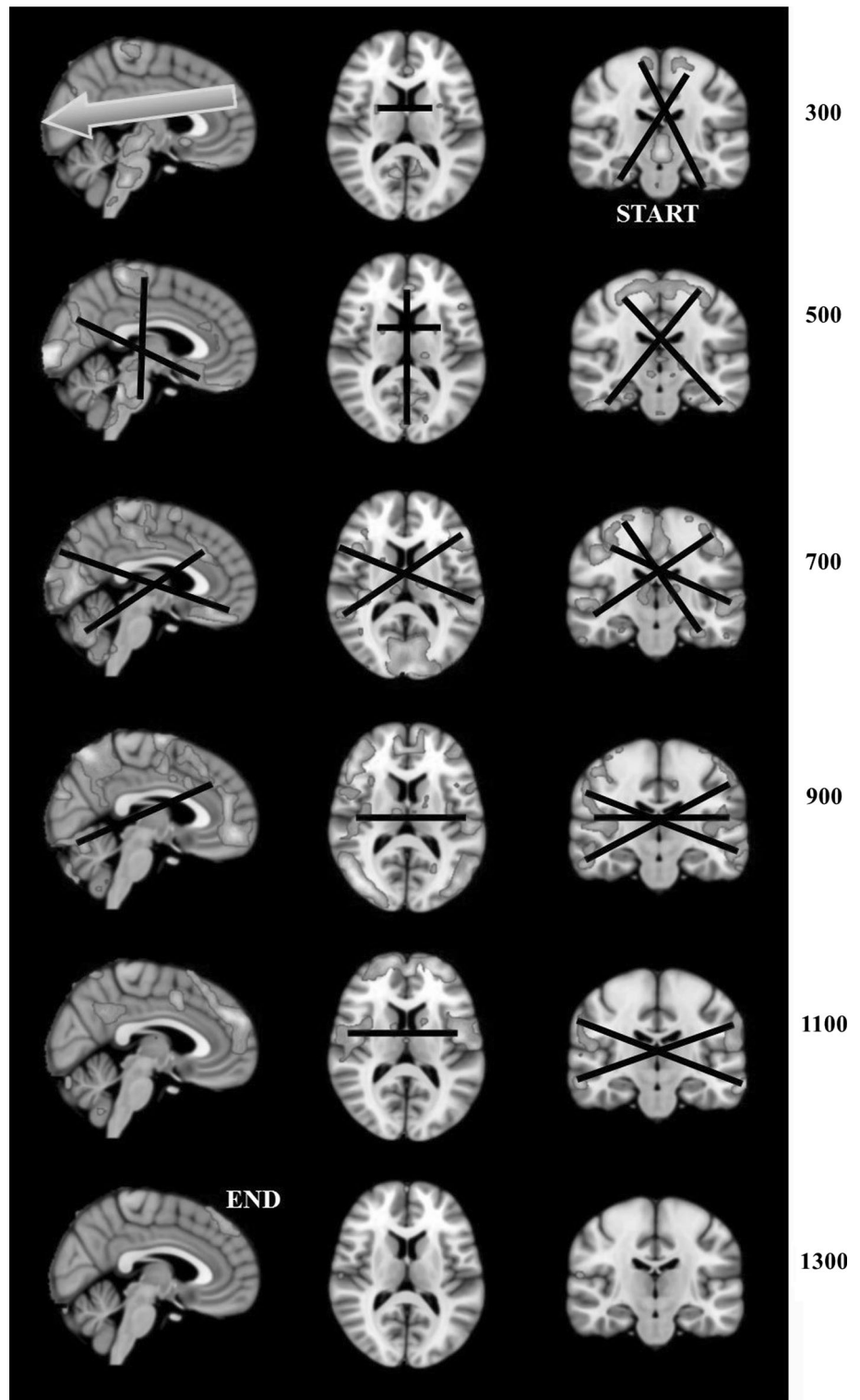
Several data (movies, figures and temporal series) extrapolated from different available experimental studies and/or meta-analyses describing brain spontaneous activity seem to corroborate our hypothesis. To make some examples:

- High dimensionality spontaneous activity structures—“lag threads” -, consisting of multiple highly reproducible temporal sequences, showed the frequent occurrence of the antipodal points’ activation at different times (Mitra et al. 2015) (Fig. 4).
- Decomposition of spontaneous brain activity into distinct fMRI co-activation patterns displayed a simultaneous activation of perisylvian and/or the opposite medial occipito-parietal areas (Liu et al. 2013) (Fig. 5a).
- Karahanoglu and Van De Ville (2015) decomposed resting-state fMRI using iCAPs, i.e. innovation-

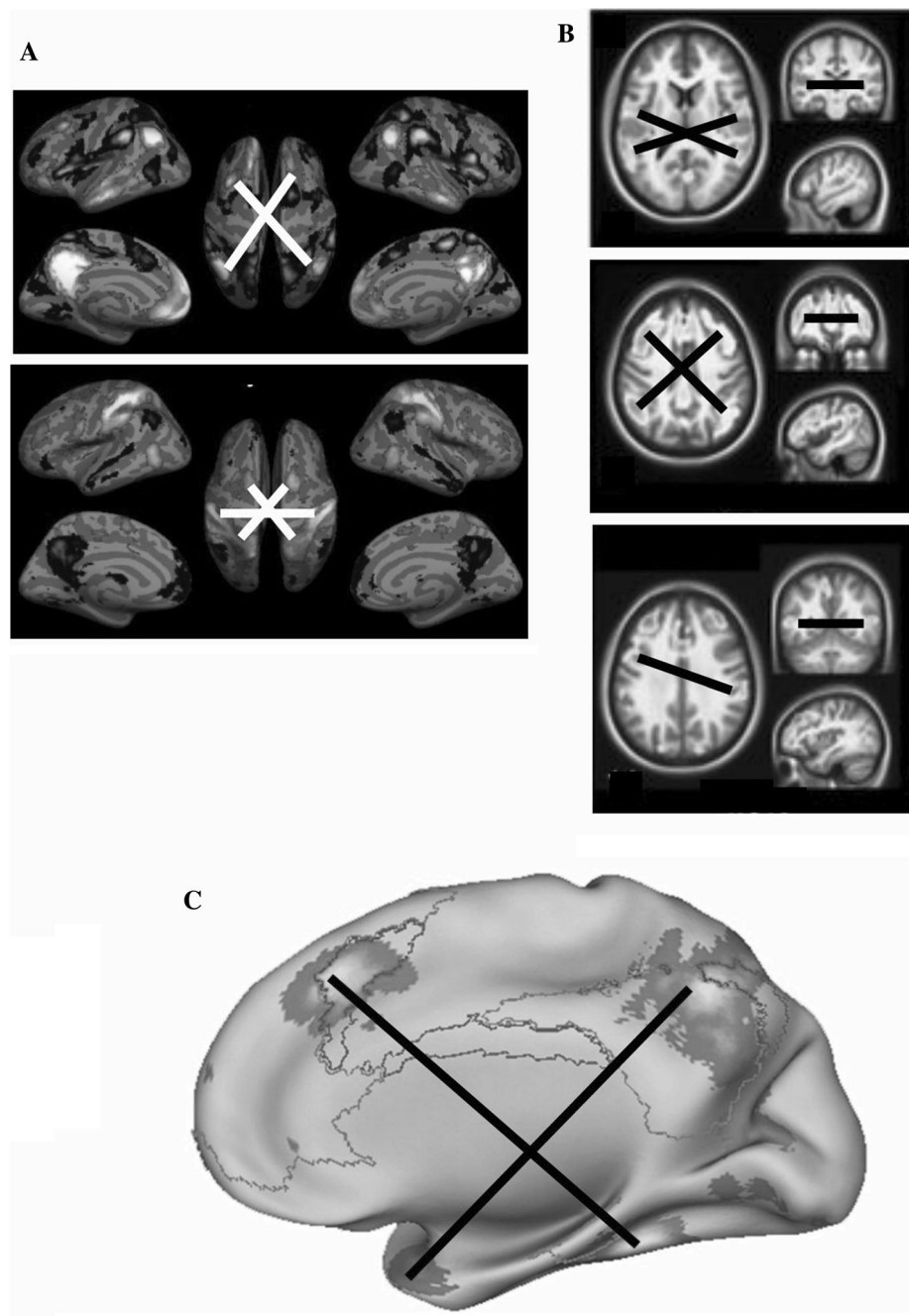
driven co-activation patterns. Their results show once again that opposite brain areas activate together: in particular, the pattern was clearly visible in bilateral posterior-frontal antipodal areas and in bilateral perisylvian antipodal areas (Fig. 5b).

(d) Meta-analytic clusters of fMRI temporal activation associated with mind-wandering and related spontaneous thought processes (Fox et al. 2015) clearly displayed the simultaneous activation of two antipodal zones (Fig. 5c).

**Fig. 4** Video frames, modified from Mitra et al. (2015), showing lag threads computed from real BOLD resting state rs-fMRI data in a group of 688 subjects, obtained from the Harvard-MGH Brain Genomics Superstruct Project (see Mitra et al. 2015 for further technical details). Note the widely diffused presence of BUT hallmarks (black lines) at different times and in different brain projections



**Fig. 5** Examples from functional neuroimaging real data showing how the predicted antipodal points can be correctly identified: given one point (a brain signal), there is a second point (another brain signal) at the opposite end of a straight line segment connecting them and passing for the center (*white or black lines*). **a** Decomposition of Spontaneous Brain Activity into Distinct fMRI Co-activation Patterns (Liu et al. 2013). **b** Different clusters during resting-state fMRI scanning, evaluated through innovation-driven co-activation patterns (called iCAPs) (Karahanoglu and Van De Ville 2015). **c** Significant meta-analytic clusters of fMRI temporal activation associated with mind-wandering and related spontaneous thought processes (Fox et al. 2015)



(e) Unnoticed features of antipodal points' activation can be found also in other data sets. For example, see Andrews-Hanna et al. (2014), Fox and Raichle (2007), Raichle (2009), Power et al. (2014), Mao et al. (2015), Harrison et al. (2015), Gusnard et al. (2001), Gravel et al. (2014), Barttfeld et al. (2015), Ajilore et al. (2013).

In sum, it is possible to find in literature several data confirming the occurrence of opposite brain zones' activation, which is the sign predicted by BUT. The fMRI

sequences of brain region activations, apart from differences depending on slight methodological distinctions among the papers, exhibited a stereotyped topographical firing pattern, such that brain loci are activated together with their opposite zones. As a result, highly reproducible topography and propagation through subsets of regions that are shared across multiple trajectories can be found in different data sets: it corroborates the prediction of BUT linked with the presence of a brain hypersphere.

## What now?

Our review uncovered evidence of hypersphere in experimental fMRI series obtained during spontaneous activity, raising the possibility that (at least some) brain functions lie on a glome enclosed in 4D space. Brain activity is thus temporally driven by a functional 3-sphere, intrinsic to the brain and (probably) embedded in the very anatomical structure of the connectome. Indeed, to evaluate glomes in terms of framework for brain functions, we first need to identify potential brain signal loci where quaternion rotations might take place. The natural candidate is the spatially embedded network of the human connectome (Sporns 2011; Jarman et al. 2014), a non-stationary, highly dynamical structure (de Pasquale et al. 2015; Fox et al. 2015) characterized by complex topological features and an ever-changing geometry (Roberts 2015). We may thus state that a 4D cap surrounds the brain, equipped with trajectories which follow quaternion movements along the nodes of the connectome. The reproducibility of the BUT hallmarks in different Authors suggests that this organizational feature is essential to normal brain physiology and function.

Further studies are needed to evaluate what happens when other techniques are used, *e.g.*, EEG and diffusion tensor imaging. Does evoked, task-related activity exhibit the same features displayed by resting-state? Additional investigations will elucidate whether, following the stimulus onset (either external from the environment or internal from the body milieu and the brain), the multidimensional space outlined by cortical activity is invariant or reduced (Papo 2014). Human fMRI studies have recently revealed that a co-activation of sensory and fronto-parietal areas (in particular, simultaneous firing of activated antipodal points in somatosensory, lateral and medial frontal, posterior parietal and in the cingulate cortex) is crucial for conscious sensory perception in the several second time-scale of BOLD signal fluctuations (Hirvonen and Palva 2015).

If you are embedded in a 4D torus containing a doll, you see in front of you side-to-side reversal (and/or three-quarters turn) doll's images. To be into a glome looks like to stand in front of a weird mirror: if you raise your left hand, you see in infinitely many screens countless copies of yourself raising the left hand, while other copies raise the right one. Because such neighboring images of the same object are related by glide reflections translations (Weeks 2002), it remains to be seen what the implications of the glome would have for consciousness, perception of time and nature of reality. Going a step further, our “deterministic” account of linear transformations needs to be contextualized, taking into account the suggestions of the brain as an energy-sparing, complex, nonlinear system equipped with attractors and/or random walks (Beggs and

Timme 2012; Friston 2010; Zhang et al. 2015). Also the role of electromagnetic currents needs to be re-evaluated, *i.e.*, do spikes contain the message, or, as recently suggested (Tozzi 2015), they serve other kinds of functions? For example, it has been proposed that features of a brain signal with spectral peaks in preferred bands (gamma, beta and so on) provide a basis for feature vectors in a 4D Euclidean space (Padhy et al. 2011). Further, the hypersphere, due to different movements of the quaternionic system, relentlessly varies its intrinsic structure: in this context, it is reasonable to speculate that each mental state corresponds to a different glome's topological space. In conclusion, our original approach to brain activity, starting from such a completely new point of view, contains the promise of great fruitfulness in the mere fact that it directs us into new ways, opens out new perspectives and shows how it is possible to get quite a new light upon the old problems which are constantly being turned over and over in the same way. Indeed, a deeper knowledge of a fourth brain dimension via experimental confirm will lead to novel approaches to organized biological models, will improve our understanding of cortical functions and will also pave the way to innovative therapeutic strategies.

## Appendix

### Section 1

#### *Movements of particles on 3-spheres*

At first, we need to mathematically define a hypersphere. It is an  $n$ -sphere formed by points which are constant distance from the origin in  $(n + 1)$ -dimensions (Henderson and Taimina 2001; Giblin 2010). A 3-sphere of radius  $r$  (where  $r$  may be any positive real number) is defined as the set of points in 4D Euclidean space at distance  $r$  from some fixed center point  $\mathbf{c}$  (which may be any point in the 4D space).

The notation  $S^n$  refers to an  $n$ -sphere, which is a generalization of the circle. A 1-sphere is a set points on the perimeter of a circle in a 2D space, while a 2-sphere is a sets of surface points in a 3D space and a 3-sphere is set of points on the surface of what is known as a “hypersphere”. From a geometer's perspective, we have the following  $n$ -spheres, starting with the perimeter of a circle ( $S^1$ ) and advancing to  $S^3$ , which is the smallest hypersphere, embedded in a 4-ball:

1-sphere  $S^1 : x_1^2 + x_2^2$ , embedded in  $R^2$  (circle perimeter, the common circumference),

2-sphere  $S^2 : x_1^2 + x_2^2 + x_3^2$ , embedded in  $R^3$  (surface of the common sphere, *i.e.* a beach ball),



3-sphere  $S^3 : x_1^2 + x_2^2 + x_3^2 + x_4^2$ , embedded in  $R^4$  (the smallest hypersphere surface),...,  
 n-sphere  $S^n : x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2$ , embedded in  $R^n$ .

In technical terms, a map of a glome equipped with Sp(1) or SU(2) Lie groups can be projected onto a 3-D surface. The 3-sphere is parallelizable as a differentiable manifold, with a principal U(1) bundle over the 2-sphere. Apart  $S^3$ , the only other spheres that admit the structure of a Lie group are the 0-sphere  $S^0$  (real numbers with absolute value 1), the circle  $S^1$  (complex numbers with absolute value 1) and  $S^7$ .

The 3-sphere's Lie group structure is Sp(1), which is a compact, simply connected symplectic group, equipped with quaternionic 1X1 unitary matrices. The glome  $S^3$  forms a Lie group by identification with the set of quaternions of unit norm, called versors (Ozdemir and Özekes 2013). The quaternionic manifold is a cube with each face glued to the opposite face with a one quarter clockwise turn. The name arises from the fact that its symmetries can be modeled in the quaternions, a number system similar to the complex numbers, but with three imaginary quantities, instead of just one (Lemaître 1948). For an affordable, less technical treatment of quaternions, see (Hart and Segerman 2014).

In addition: Sp(1)  $\approx$  SO(4)/SO(3)  $\approx$  Spin(3)  $\approx$  SU(2).

Thus, Sp(1) is equivalent to—and can be identified with—the special unitary group SU(2).

#### List of useful videos

They are very helpful in order to understand the hypersphere's movements in four dimensions.

- The superimposition of two 2-spheres (with circumferences glued together) gives rise to a 3-sphere equipped with quaternionic movements: <https://www.youtube.com/watch?v=XFW769hqa1U>
- a stereographic projection of a Clifford torus, performing a simple rotation through the xz plane: [https://en.wikipedia.org/wiki/Clifford\\_torus#/media/File:Clifford-torus.gif](https://en.wikipedia.org/wiki/Clifford_torus#/media/File:Clifford-torus.gif)
- 3-D Stereographic projection of the “toroidal parallels” of a 3-D sphere: <https://www.youtube.com/watch?v=QlcSITmc0Ts>
- The shape of the 3-sphere is ever-changing, depending on the number of circles taken into account and their trajectories: <http://nilesjohnson.net/hopf.html>
- A video correlated with the above mentioned Hart and Segerman's paper (2014) illustrates the quaternions' movements of... a group of monkeys: <http://blogs.scientificamerican.com/roots-of-unity/nothing-is-more-fun-than-a-hypercube-of-monkeys/>

## Section 2

### The Borsuk-Ulam Theorem

Continuous mappings from object spaces to feature spaces lead to various *incarnations* of the Borsuk-Ulam Theorem, a remarkable finding about Euclidean n-spheres and antipodal points by K. Borsuk (Borsuk 1958–1959). Briefly, antipodal points are points opposite each other on a  $S^n$  sphere. There are natural ties between Borsuk's result for antipodes and mappings called homotopies. In fact, the early work on n-spheres and antipodal points eventually led Borsuk to the study of retraction mappings and homotopic mappings (Borsuk and Gmurczyk 1980).

The Borsuk-Ulam Theorem states that:

Every continuous map  $f: S^n \rightarrow R^n$  must identify a pair of antipodal points.

Points on  $S^n$  are *antipodal*, provided they are diametrically opposite. Examples are opposite points along the circumference of a circle  $S^1$ , or poles of a sphere  $S^2$ . An  $n$ -dimensional Euclidean vector space is denoted by  $R^n$ . In terms of brain activity, a feature vector  $x \in R^n$  models the description of a brain signal.

### Borsuk-Ulam in brain signal analysis

In order to evaluate the possible applications of the Borsuk-Ulam Theorem in brain signal analysis, we view the surface of the brain as a n-sphere and the feature space for brain signals as finite Euclidean topological spaces. The Borsuk-Ulam Theorem tells us that for description  $f(x)$  for a brain signal  $x$ , we can expect to find an antipodal feature vector  $f(-x)$  that describes a brain signal on the opposite (antipodal) side of the brain. Moreover, the pair of antipodal brain signals have matching descriptions. Let  $X$  denote a nonempty set of points on the brain surface. A topological structure on  $X$  (called a brain topological space) is a structure given by a set of subsets  $\tau$  of  $X$ , having the following properties:

(Str.1) Every union of sets in  $\tau$  is a set in  $\tau$

(Str.2) Every finite intersection of sets in  $\tau$  is a set in  $\tau$

The pair  $(X, \tau)$  is called a topological space. Usually,  $X$  by itself is called a topological space, provided  $X$  has a topology  $\tau$  on it. Let  $X, Y$  be topological spaces. Recall that a function or map  $f: X \rightarrow Y$  on a set  $X$  to a set  $Y$  is a subset  $X \times Y$  so that for each  $x \in X$  there is a unique  $y \in Y$  such that  $(x, y) \in f$  (usually written  $y = f(x)$ ). The mapping  $f$  is defined by a rule that tells us how to find  $f(x)$ . For a good introduction to mappings, see (Willard 1970).

A mapping  $f: X \rightarrow Y$  is continuous, provided, when  $A \subset Y$  is open, then the inverse  $f^{-1}(A) \subset X$  is also open. For more about this, see Krantz (2009). In this view of

continuous mappings from the brain signal topological space  $X$  on the surface of the brain to the brain signal feature space  $R^n$ , we can consider not just one brain signal feature vector  $x \in R^n$ , but also mappings from  $X$  to a set of brain signal feature vectors  $f(X)$ . This expanded view of brain signals has interest, since every connected set of feature vectors  $f(X)$  has a shape. The significance of this is that brain signal shapes can be compared.

A consideration of  $f(X)$  (set of brain signal descriptions for a region  $X$ ) instead of  $f(x)$  (description of a single brain signal  $x$ ) leads to a region-based view of brain signals. This region-based view of the brain arises naturally in terms of a comparison of shapes produced by different mappings from  $X$  (brain object space) to the brain feature space  $R^n$ . An interest in continuous mappings from object spaces to feature spaces leads into homotopy theory and the study of shapes.

Let  $f, g: X \rightarrow Y$  be continuous mappings from  $X$  to  $Y$ . The continuous map  $H: X \times [0, 1] \rightarrow Y$  is defined by  $H(x, 0) = f(x)$ ,  $H(x, 1) = g(x)$ , for every  $x \in X$ .

The mapping  $H$  is a *homotopy*, provided there is a continuous transformation (called a deformation) from  $f$  to  $g$ . The continuous maps  $f, g$  are called homotopic maps, provided  $f(X)$  continuously deforms into  $g(X)$  (denoted by  $f(X) \rightarrow g(X)$ ). The sets of points  $f(X), g(X)$  are called shapes. For more about this, see Manetti (2015) and Cohen (1973).

For the mapping  $H: X \times [0, 1] \rightarrow R^n$ , where  $H(X, 0)$  and  $H(X, 1)$  are *homotopic*, provided  $f(X)$  and  $g(X)$  have the same shape. That is,  $f(X)$  and  $g(X)$  are homotopic, provided:  $\|f(X) - g(X)\| < \|f(X)\|$ , for all  $x \in X$ .

It was Borsuk who first associated the geometric notion of shape and homotopies. This leads into the geometry of shapes and shapes of space (Collins 2004). To make an example, a pair of connected planar subsets in Euclidean space  $R^2$  have equivalent shapes, if the planer sets have the same number of holes (Krantz 2009). The letters e, O, P and numerals 6, 9 belong to the same equivalence class of single-hole shapes. In terms of brain signals, this means that the connected graph for  $f(X)$  with, for example, an  $e$  shape, can be deformed into the 9 shape. This suggests yet another useful application of Borsuk's view of the transformation of shapes, one into the other, in terms of brain signal analysis. Sets of brain signals not only will have similar descriptions, but also dynamic character. Moreover, the deformation of one brain signal shape into another occurs when they are descriptively near (Peters 2014).

*A remark concerning brain activity on hyperspheres*

One of the formulations of BUT (the third one, in Borsuk 1933) states that there is no antipodal mapping  $f: S^n \rightarrow$

$S^{n-1}$ . It is not completely true: in case of antipodal points not represented by shapes or feature spaces, but by Lie groups, we can detect the hints of the movements of  $S^4$  on  $S^3$ , because the Lie group has at least one point  $S^n \cap S^{n-1}$ . The main benefit here is that, according to the BUT dictates, for each given brain signal we are allowed to find a counterpart in the cortical surface's antipodal position.

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