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RESEARCH ARTICLE

Power-rate synchronization of coupled genetic oscillators with unbounded time-varying delay

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Abstract In this paper, a new synchronization problem for the collective dynamics among genetic oscillators with unbounded time-varying delay is investigated. The dynamical system under consideration consists of an array of linearly coupled identical genetic oscillators with each oscillators having unbounded time-delays. A new concept called power-rate synchronization, which is different from both the asymptotical synchronization and the exponential synchronization, is put forward to facilitate handling the unbounded time-varying delays. By using a combination of the Lyapunov functional method, matrix inequality techniques and properties of Kronecker product, we derive several sufficient conditions that ensure the coupled genetic oscillators to be power-rate synchronized. The criteria obtained in this paper are in the form of matrix inequalities. Illustrative example is presented to show the effectiveness of the obtained results.

Keywords Power-rate synchronization · Genetic oscillators · Unbounded time-varying delay · Matrix inequality

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Introduction

It is well known that gene expression is a complex process regulated at several stages in the synthesis of proteins (Lewin 1999). DNA microarray technology has made it possible to measure gene expression levels on a genomic scale, and has therefore been widely applied to gene transcription analysis (Fraser et al. 2004). With the available huge amount of microarray data, researchers have been trying to understand how biological activities are governed by the connectivity of genes and proteins, which is described in terms of genetic regulatory networks (GRNs). A gene network consists of a group of genes that interact among themselves in order to synthesize certain products, i.e., proteins. The types and amount of proteins produced by a gene network have a fundamental effect on the development of the gene network itself, and on the biological systems with which the network interacts. Since genetic networks are biochemical dynamical systems, mathematical modelling of genetic networks as dynamical system models provides a powerful tool for studying gene regulation processes in living organisms (Wang et al. 2008). The study of genetic regulatory networks have become an attractive area and received great attention over past decade (Li et al. 2006a, b; Ren and Cao 2008; Wang et al. 2010; Xiao et al. 2014; Cao and Ren 2008; Ren et al. 2015).

Synchronization is an important dynamical behavior that has been intensively investigated in the last decade (Wu and

Chua 1995; Li and Chen 2004; Lu and Chen 2004; Wang and Cao 2006; Li et al. 2014; Huang et al. 2008; Qiu and Cao 2009). There are many benefits of having synchronization in some applications, such as secure communication, human heartbeat regulation, chemical reaction, power systems protection, ecological systems and so on. Recently, it has been discovered that the phase synchronization can also exist in many biologically plausible systems and such synchronization is a main mechanism to fulfil cellular communication. Therefore, it is not surprising that the synchronization problem of coupled genetic oscillators has begun to attract a great deal of research attention due to its potential applications in GRNs. For example, in McMillen et al. (2002), Kuznetsov et al. (2004), and Wang and Chen (2005), the synchronization was studied in biological networks of identical genetic oscillators and, in Garcia-Ojalvo et al. (2004), Gonze et al. (2005) and Li et al. (2006b), the synchronization was investigated for coupled nonidentical genetic oscillators. In Li et al. (2007), based on a system biology approach, the authors provided a general theoretical result on the synchronization of genetic oscillators with stochastic perturbations.

It is now well known that the slow processes of transcription, translation, and translocation or the finite switching speed of amplifiers will inevitably cause time delays, which should be taken into account in the biological systems or artificial genetic networks in order to have more accurate models (Li et al. 2014, 2008). The effects of transcriptional delays should be assessed in the dynamics of genetic networks whose time scales are short and transcription is regulated by feedback (Wang et al. 2008). The time-delay can be of a discrete nature, which assumes each macromolecule takes the same length of time to translocate from its place of synthesis to the location where it exerts an effect. The time-delay can also be time-varying where the delay changes as time changes. In the context of GRNs, both discrete time-delays and time-varying delays have recently been considered in Cao and Ren (2008), Monk (2003), Grammaticos et al. (2006) and Wang et al. (2008).

It should be pointed out that, in most existing literature considering delays in GRNs, only the stability has been considered, and the important synchronization behavior has not received much attention yet. Furthermore, almost all relevant research works have assumed that the time delays involved in GRNs are bounded. This is, unfortunately, not always the case in practice. If movement of mRNA from a transcription site to translation sites is an active process with a significant range of transport times for individual molecules, *unbounded* delay would be the proper modeling framework. For example, it has been shown in Smolen et al. (2000) that the time-delay can occur in a distributed hence unbounded way. Therefore, it is of great interest in both theory and applications to investigate the *synchronization* problem of the genetic regulatory networks with *unbounded delays*. To the best of the authors' knowledge, up to now, little effort has been made towards such a challenging problem, which motivates the present study.

In this paper, we investigate a new synchronization problem for the collective dynamics among genetic oscillators with unbounded time-varying delay. The dynamical system under consideration consists of an array of linearly coupled identical genetic oscillators with each oscillators having unbounded time-delays. It is worth noting that, in Chen and Wang (2007), pioneering research has been carried out for the power-rate stability of dynamical systems with unbounded time-varying delays and neat results have been reported. Motivated by the excellent work of Chen and Wang (2007), we put forward a new concept called power-rate synchronization in order to cater the unbounded time-delays in GRNs. Notice that the power-rate synchronization is significantly different from the traditional asymptotical synchronization and the exponential synchronization. By resorting to the Lyapunov functional method and properties of Kronecker product, we develop a new matrix inequality approach to derive several sufficient conditions under which the coupled genetic oscillators are synchronized in the power-rate sense. Illustrative example is presented to show the effectiveness of the obtained results.

The rest of this paper is organized as follows. In "Problem formulation and preliminaries" section, problem formulation and preliminaries are given. In "Power-rate synchronization" section, sufficient criteria are derived for the power-rate synchronization of coupled genetic oscillators with unbounded time-varying delay. In "Illustrative example" section, an example is given to show the effectiveness of the proposed results. Finally, conclusions are drawn in the last section.

Problem formulation and preliminaries

Consider the genetic oscillator with unbounded time-varying delay described by the following equation Li et al. (2006b):

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 f(x(t-\tau(t))) \\ &\quad + B_2 \widetilde{g}(x(t-\tau(t))),\end{aligned} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ represents the concentrations of proteins, RNAs and chemical complexes; A and B_i are matrices in $\mathbb{R}^{n\times n}$ while $Ax(t)$ includes the degradation terms and all the other linear terms in the genetic oscillator; $f(x(t-\tau(t))) = [f_1(x_1(t-\tau(t))), \ldots, f_n(x_n(t-\tau(t)))]^T$ and $\widetilde{g}(x(t-\tau(t))) = [\widetilde{g}_1(x_1(t-\tau(t))), \ldots, \widetilde{g}_n(x_n(t-\tau(t)))]^T$ with $f_i(x_i)$ and $\widetilde{g}_i(x_i)$ being monotonic increasing or decreasing regulatory functions that are usually of the Michaelis-Menten or Hill form; the unbounded time-varying delay function satisfies $\tau(t) \le \eta t$ with $0 < \eta < 1$. In this paper, we also assume that $f_i(x)$ is a monotonic increasing function of the Hill form

$$f_i(x_i(t)) = \frac{(x_i(t)/\beta_i)^{H_i}}{1 + (x_i(t)/\beta_i)^{H_i}},$$

and $\widetilde{g}_i(x)$ is a monotonic decreasing function of the following form

$$\widetilde{g}_i(x_i(t)) = \frac{1}{1 + (x_i(t)/\beta_i^*)^{H_i^*}}$$

where H_i and H_i^* are the Hill coefficients. We further assume that the ith column of B_1 and B_2 are zeros if $f_i, \widetilde{g}_i \equiv 0$.

Since

$$\begin{aligned}\widetilde{g}_i(x_i(t)) &= \frac{1}{1 + (x_i(t)/\beta_i^*)^{H_i^*}}\\ &= 1 - \frac{(x_i(t)/\beta_i^*)^{H_i^*}}{1 + (x_i(t)/\beta_i^*)^{H_i^*}}\\ &= 1 - g_i(x_i(t)),\end{aligned}$$

system (1) can be rewritten as follows:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 f(x(t-\tau(t))) - B_2 g(x(t-\tau(t))\\ &\quad + B_2 \cdot e_n,\end{aligned} \tag{2}$$

where $e_n = [1, 1, \ldots, 1]^T \in \mathbb{R}^n$. Noticing that f_i and g_i are monotonically increasing functions with saturation, we have

$$0 \le \frac{f_i(x) - f_i(y)}{x - y} \le F_i, \quad 0 \le \frac{g_i(x) - g_i(y)}{x - y} \le G_i,$$

for all x, $y \in \mathbb{R}$ with $x \neq y$, where F_i and G_i are real constants.

Remark 1 As indicated in Li et al. (2006a), the genetic oscillator (2) can be regarded as a kind of Lur'e system that can then be investigated by using the fruitful Lur'e system method in control theory.

In this paper, we consider the dynamical system consisting of N linearly coupled identical genetic oscillators with each oscillator being an n-dimensional dynamical system:

$$\begin{aligned}\dot{x}_i(t) &= Ax_i(t) + B_1 f(x_i(t-\tau(t)))\\ &\quad - B_2 g(x_i(t-\tau(t))) + B_2 \cdot e_n\\ &\quad + \sum_{j=1}^{N} c_{ij} D x_j(t), \quad i = 1, 2, \ldots, N,\end{aligned} \tag{3}$$

where $x_i(t) \in \mathbb{R}^n$ is the state vector of the ith genetic oscillator, $D = (d_{ij}) \in \mathbb{R}^{n \times n}$ is inner coupling matrix. $C = (c_{ij})_{N \times N}$ is the coupling configuration matrix representing the coupling strength and the topological structures of the network, in which $c_{ij} = c_{ji}$ is defined as follows:

$$c_{ij} = \begin{cases} \text{a positive constant,} & \text{if there is a link from oscillator } j \text{ to oscillator } i \ (j \neq i);\\ 0, & \text{otherwise.}\end{cases} \tag{4}$$

$$c_{ii} = -\sum_{j=1, j\neq i}^{N} c_{ij} = -\sum_{j=1, j\neq i}^{N} c_{ji}.$$

Remark 2 Similar to Li et al. (2006a), the synchronization problem considered in this paper is actually for coupled genetic oscillators with *SUM regulatory logic*, that is, each transcription factor acts additively to regulate a gene or, in other words, the regulatory function sums over all the inputs. As explained in Li et al. (2006a), such a SUM logic does exist in many natural genetic networks where an additive input function is to provide a gene with multiple promoters, each responding to one of the inputs.

In the following, we give the definition for power-rate synchronization.

Definition 1 Suppose that there exist constants K, $\epsilon > 0$ and $T > 0$ such that $\|x_i(t) - x_j(t)\| \le Kt^{-\epsilon}$ for all $t > T$ and $i, j = 1, 2, \ldots, N$. Then, the system (3) is said to be power-rate synchronized.

To obtain our main results, some definitions and lemmas in Wu and Chua (1995) and Wang and Cao (2006) are introduced here, which will be used throughout this paper.

Definition 2 Let $\hat{\mathbb{R}}$ denote a ring and $\mathbf{M}_{n\times m}(\hat{\mathbb{R}})$ be the set of $n \times m$ matrix with entries in $\hat{\mathbb{R}}$. $T(\hat{\mathbb{R}}, B)$ is defined as the set of matrices with entries in $\hat{\mathbb{R}}$ such that the sum of the entries in each row is equal to B for some $B \in \hat{\mathbb{R}}$.

Definition 3 $\mathbf{M}_1^N(1)$ is composed of matrices with N columns, and each row of $\tilde{M} \in \mathbf{M}_1^N(1)$ contains zeros and exactly one entry α_i and one entry $-\alpha_i$, $\alpha_i \neq 0$.

Definition 4 $\mathbf{M}_1^N(n)$ is defined as $\mathbf{M}_1^N(n) = \{\mathbf{M} = \tilde{M} \otimes I_n | \tilde{M} \in \mathbf{M}_1^N(1)\}$, i.e., $\mathbf{M} \in \mathbf{M}_1^N(n)$ is obtained by replacing entry m_{ij} in $\tilde{M} \in \mathbf{M}_1^N(1)$ with $m_{ij} I_n$, where I_n is the n-dimension identity matrix; the notation $\otimes$ indicates the Kronecker product of two matrices; $\mathbf{M}_2^N(n)$ are matrices $\mathbf{M}$ in $\mathbf{M}_1^N(n)$ such that for any pair of indexes i and j, there exist indexes $i_1, i_2, \ldots, i_l$ with $i_1 = i$ and $i_l = j$, and $p_1, p_2, \ldots, p_{l-1}$ such that $\mathbf{M}_{(p_q, i_q)} \neq 0$ and $\mathbf{M}_{(p_q, i_{q+1})} \neq 0$ for all $1 \le q < l$.

A nonnegative real-valued function is defined as follows to measures the distance between the various nodes:

$$d(x) = ||\mathbf{M}x||^2 = x^T \mathbf{M}^T \mathbf{M} x, \quad \mathbf{M} \in \mathbf{M}_2^N(n), \tag{5}$$

From the property of $\mathbf{M} \in \mathbf{M}_2^N(n)$, $d(x) \to 0$ if and only if $||x_i - x_j|| \to 0$ for all i and j, $1 \le i, j \le N$, which implies that the dynamical system (3) achieves synchronization when $d(x) \to 0$. Especially, if we take

$$\mathbf{M}_1 = \begin{pmatrix} I_n & -I_n & & & \\ & I_n & -I_n & & \\ & & & \ddots & \\ & & & I_n & -I_n \end{pmatrix}_{(N-1)\times N},$$

then $\mathbf{M}_1 \in \mathbf{M}_2^N(n)$ is a $(N-1)n \times Nn$ real-valued matrix, and $d(x) = \sum_{i=1}^{N-1} ||x_i - x_{i+1}||^2$.

Lemma 1 *Let W be a $N \times N$ matrix in $T(\hat{\mathbb{R}}, K)$, then the $(N-1)\times(N-1)$ matrix H defined by $H = MWV$ satisfies $MW = HM$ where M is the $(N-1)\times N$ matrix and V is the $N \times (N-1)$ matrix*

$$M = \begin{pmatrix} \mathbf{1} & -\mathbf{1} & & & \\ & \mathbf{1} & -\mathbf{1} & & \\ & & & \cdots & \\ & & & \mathbf{1} & -\mathbf{1} \end{pmatrix},$$

$$V = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \\ & & \ddots & & \mathbf{1} \\ & & & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

and $\mathbf{1}$ is the multiplicative identity of $\hat{\mathbb{R}}$. The matrix H can be written explicitly as $H_{(i,j)} = \sum_{k=1}^{j}[W_{(i,k)} - W_{(i+1,k)}]$ for $i,j \in \{1, 2, \ldots, N-1\}$.

Lemma 2 *A symmetric irreducible matrix B satisfies the condition (4) if and only if there exists a $p \times N$ matrix $M \in \mathbf{M}_2^N(1)$ such that $B = -M^T M$.*

Lemma 3 (Wang et al. 2006) *Let Σ, R, S be real matrices of appropriate dimensions and Σ is positive definite. Then, for any vectors x and y of appropriate dimensions, the following inequality holds:*

$$2x^T RSy \le x^T RS\Sigma^{-1}(RS)^T x + y^T \Sigma y.$$

Lemma 4 [Schur complement (Boyd et al. 1994)]. *The following linear matrix inequality (LMI)*

$$\begin{pmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{pmatrix} > 0,$$

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, and $S(x)$ depends affinely on x, is equivalent to the following condition:

$$Q(x) > 0,\ R(x) - S^T(x)Q^{-1}(x)S(x) > 0.$$

Power-rate synchronization

Theorem 1 *The network (3) is power-rate synchronized if there exist positive definite matrix $P = (p_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, positive definite diagonal matrices $Q = \mathrm{diag}\{q_1, q_2, \ldots, q_n\} \in \mathbb{R}^{n\times n}$, $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \in \mathbb{R}^{n\times n}$, a symmetric matrix $\Delta = (\Delta_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, such that*

$$\Omega = \begin{pmatrix} PA + A^T P - \Delta + 2P & PB_1 & -PB_2 \\ B_1^T P & -Q & 0 \\ -B_2^T P & 0 & -\Sigma \end{pmatrix} < 0, \quad P \ge FQF, \quad P \ge G\Sigma G, \tag{6}$$

and

$$-Nc_{ij}(PD + D^T P) + \Delta < 0, \quad 1 \le i < j \le N, \tag{7}$$

where $F = \mathrm{diag}\{F_1, F_2, \ldots, F_n\}$ and $G = \mathrm{diag}\{G_1, G_2, \ldots, G_n\}$.

Proof Let $e_N = [1, 1, \ldots, 1]^T \in \mathbb{R}^N$, $J = e_N e_N^T$, $U = J - NI_N$; Then $u_{ij} = 1(i \ne j)$, $u_{ii} = -(N-1)$, $i,j = 1, 2, \ldots, N$. Obviously, U is an irreducible symmetric matrix and satisfies condition (4). It follows from Lemma 2 that there exists a $p \times N$ matrix $\tilde{M} \in \mathbf{M}_2^N(1)$ such that $U = -\tilde{M}^T\tilde{M}$. Let $\mathbf{M} = \tilde{M} \otimes I_n$, then $\mathbf{M} \in \mathbf{M}_2^N(n)$ and denote $\mathbf{f}(x(t-\tau(t))) = [f^T(x_1(t-\tau(t))), f^T > (x_2(t-\tau(t))), \ldots, f^T(x_N(t-\tau(t)))]^T$, $\mathbf{g}(x(t-\tau(t))) = [g^T(x_1(t-\tau(t))), g^T(x_2(t-\tau(t))), \ldots, g^T(x_N(t-\tau(t)))]^T$, $x_i(t) = [x_{i,1}(t), x_{i,2}(t), \ldots, x_{i,n}(t)]^T$, $x(t) = [x_1^T(t), x_2^T(t), \ldots, x_N^T(t)]^T$, $\mathbf{A} = I_N \otimes A$, $\bar{\mathbf{A}} = I_p \otimes A$, $\mathbf{B}_1 = I_N \otimes B_1$, $\bar{\mathbf{B}}_1 = I_p \otimes B_1$, $\mathbf{B}_2 = I_N \otimes B_2$, $\bar{\mathbf{B}}_2 = I_p \otimes B_2$, $\mathbf{\Delta} = I_N \otimes \Delta$, $\bar{\mathbf{\Delta}} = I_p \otimes \Delta$, $\mathbf{P} = I_p \otimes P$, $\mathbf{Q} = I_p \otimes Q$, $\mathbf{C} = C \otimes D$, $\mathbf{u} = [(B_2 \cdot e_n)^T, (B_2 \cdot e_n)^T, \ldots, (B_2 \cdot e_n)^T]^T$. We can rewrite the network (3) as follows:

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}_1\mathbf{f}(x(t-\tau(t))) - \mathbf{B}_2\mathbf{g}(x(t-\tau(t))) + \mathbf{u} + \mathbf{C}x(t). \tag{8}$$

Let $y(t) = \mathbf{M}x(t) = [y_1^T(t), y_2^T(t), \ldots, y_p^T(t)]^T$, $y_i(t) = [y_{i,1}(t), y_{i,2}(t), \ldots, y_{i,n}(t)]^T$, where $y_i(t)$ is assumed to be $\alpha_i(x_{i_1}(t) - x_{i_2}(t))$ with $\alpha_i \ne 0$, $i = 1, 2, \ldots, p$. We further let $\bar{x}_j(t) = [x_{1,j}(t), x_{2,j}(t), \ldots, x_{N,j}(t)]^T$, then $\bar{y}_j(t) = \tilde{M}\bar{x}_j(t)$, $j = 1, 2, \ldots, n$.

By Lemma 4, condition (6) is equivalent to

$$PA + A^T P - \Delta + PB_1 Q^{-1} B_1^T P + PB_2\Sigma^{-1}B_2^T P + 2P < 0, \tag{9}$$

then we can find two sufficient small constants $\beta > 0$ and $\epsilon > 0$ such that

$$\widetilde{\Omega} = \beta P + PA + A^T P - \Delta + PB_1 Q^{-1} B_1^T P + PB_2\Sigma^{-1}B_2^T P + 2(1-\eta)^{-\epsilon} P \le 0. \tag{10}$$

In addition, a sufficiently large constant $T > 0$ can be found such that

$$\frac{\epsilon}{t} < \beta, \quad \text{for all } t > T. \tag{11}$$

We always assume that $t > T$ in the following proof. Consider the following Lyapunov function

$$V(x(t)) = t^{\epsilon} x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}x(t).$$

Defining a function $S(t)$ as $S(t) = \sup_{s \le t} V(x(s))$, we can see that $S(t)$ is non-decreasing, $V(x(t)) \le S(t)$ and $S(t)$ is bounded. Furthermore, for all $t \ge T$, one can obtain $S(t) = S(T)$. Clearly, for any $t_0 > T$, if $V(x(t_0)) < S(t_0)$, then $S(t)$ is non-increasing at t_0. Now, suppose that $V(x(t_0)) = S(t_0)$, we will need to prove that $S(t)$ is also non-increasing at t_0.

Calculating the time derivative of $V(x(t))$ along the trajectories of (8), one obtains

$$\begin{aligned}\dot{V}(x(t))\mid_{t=t_0} = {} & \epsilon t_0^{\epsilon-1} x^T(t_0)\mathbf{M}^T\mathbf{P}\mathbf{M}x(t_0) \\ & + t_0^{\epsilon} x^T(t_0)\mathbf{A}^T\mathbf{M}^T\mathbf{P}\mathbf{M}x(t_0) \\ & + t_0^{\epsilon} x^T(t_0)\mathbf{M}^T\mathbf{P}\mathbf{M}\mathbf{A}x(t_0) \\ & + 2t_0^{\epsilon} x^T(t_0)\mathbf{M}^T\mathbf{P}\mathbf{M}[\mathbf{B}_1\mathbf{f}(x(t_0-\tau(t_0))) \\ & - \mathbf{B}_2\mathbf{g}(x(t_0-\tau(t_0))) + \mathbf{u}] \\ & + 2t_0^{\epsilon} x^T(t_0)\mathbf{M}^T\mathbf{P}\mathbf{M}\mathbf{C}x(t_0) \\ & + t_0^{\epsilon} x^T(t_0)\mathbf{M}^T\mathbf{M}(\boldsymbol{\Delta} - \boldsymbol{\Delta})x(t_0).\end{aligned}$$

It follows from the properties of $\mathbf{M}$ that

$$\begin{aligned}&\mathbf{M}\mathbf{A} = \bar{\mathbf{A}}\mathbf{M},\ \mathbf{M}\mathbf{B}_1 = \bar{\mathbf{B}}_1\mathbf{M},\ \mathbf{M}\mathbf{B}_2 = \bar{\mathbf{B}}_2\mathbf{M},\\ &\mathbf{M}\mathbf{C} = \bar{\mathbf{C}}\mathbf{M},\ \mathbf{M}\mathbf{u} = 0,\ \mathbf{M}\boldsymbol{\Delta} = \bar{\boldsymbol{\Delta}}\mathbf{M},\end{aligned}$$

which leads to

$$\begin{aligned}&\dot{V}(x(t))\mid_{t=t_0} \\ &\quad = t_0^{\epsilon} x^T(t_0)\mathbf{M}^T[\mathbf{P}\bar{\mathbf{A}} + \bar{\mathbf{A}}^T\mathbf{P} - \bar{\boldsymbol{\Delta}} + \frac{\epsilon}{t_0}\mathbf{P}]\mathbf{M}x(t_0) \\ &\qquad + 2t_0^{\epsilon}\big[x^T(t_0)\mathbf{M}^T\mathbf{P}\bar{\mathbf{B}}_1\mathbf{M}\mathbf{f}(x(t_0-\tau(t_0))) \\ &\qquad - x^T(t_0)\mathbf{M}^T\mathbf{P}\bar{\mathbf{B}}_2\mathbf{M}\mathbf{g}(x(t_0-\tau(t_0)))\big] \\ &\qquad + 2t_0^{\epsilon} x^T(t_0)\mathbf{M}^T\mathbf{P}\mathbf{M}\mathbf{C}x(t_0) \\ &\qquad + t_0^{\epsilon} x^T(t_0)\mathbf{M}^T\bar{\boldsymbol{\Delta}}\mathbf{M}x(t_0).\end{aligned}$$

Noting that $\frac{\epsilon}{t_0} < \beta$, we obtain

$$\begin{aligned}&\dot{V}(x(t))\mid_{t=t_0} \\ &\quad \le t_0^{\epsilon} x^T(t_0)\mathbf{M}^T\big[\mathbf{P}\bar{\mathbf{A}} + \bar{\mathbf{A}}^T\mathbf{P} - \bar{\boldsymbol{\Delta}} + \beta\mathbf{P}\big]\mathbf{M}x(t_0) \\ &\qquad + 2t_0^{\epsilon}\big[x^T(t_0)\mathbf{M}^T\mathbf{P}\bar{\mathbf{B}}_1\mathbf{M}\mathbf{f}(x(t_0-\tau(t_0))) \\ &\qquad - x^T(t_0)\mathbf{M}^T\mathbf{P}\bar{\mathbf{B}}_2\mathbf{M}\mathbf{g}(x(t_0-\tau(t_0)))\big] \\ &\qquad + 2t_0^{\epsilon} x^T(t_0)\mathbf{M}^T\mathbf{P}\mathbf{M}\mathbf{C}x(t_0) \\ &\qquad + t_0^{\epsilon} x^T(t_0)\mathbf{M}^T\bar{\boldsymbol{\Delta}}\mathbf{M}x(t_0).\end{aligned}$$

An application of Lemma 3 yields

$$\begin{aligned}&2x^T(t_0)\mathbf{M}^T\mathbf{P}\bar{\mathbf{B}}_1\mathbf{M}\mathbf{f}(x(t_0-\tau(t_0))) = 2\sum_{j=1}^{p}\alpha_j(x_{j_1}(t_0)-x_{j_2}(t_0))^T PB_1\alpha_j \\ &\quad \times\big[f(x_{j_1}(t_0-\tau(t_0))) - f(x_{j_2}(t_0-\tau(t_0)))\big] \\ &\le \sum_{j=1}^{p}\alpha_j\big[f(x_{j_1}(t_0-\tau(t_0))) - f(x_{j_2}(t_0-\tau(t_0)))\big]^T \\ &\quad \times Q\alpha_j\big[f(x_{j_1}(t_0-\tau(t_0))) - f(x_{j_2}(t_0-\tau(t_0)))\big] \\ &\quad + \sum_{j=1}^{p} y_j^T(t_0)PB_1Q^{-1}B_1^TPy_j(t_0) \\ &\le \sum_{j=1}^{p}\alpha_j^2(x_{j_1}(t_0-\tau(t_0)) - x_{j_2}(t_0-\tau(t_0)))^T \\ &\quad \times FQF(x_{j_1}(t_0-\tau(t_0)) - x_{j_2}(t_0-\tau(t_0))) \\ &\quad + \sum_{j=1}^{p} y_j^T(t_0)PB_1Q^{-1}B_1^TPy_j(t_0),\end{aligned} \tag{12}$$

and

$$\begin{aligned}&-2x^T(t_0)\mathbf{M}^T\mathbf{P}\bar{\mathbf{B}}_2\mathbf{M}\mathbf{g}(x(t_0-\tau(t_0))) \\ &= -2\sum_{j=1}^{p}\alpha_j(x_{j_1}(t_0)-x_{j_2}(t_0))^T \\ &\quad \times PB_2\alpha_j\big[g(x_{j_1}(t_0-\tau(t_0))) - g(x_{j_2}(t_0-\tau(t_0)))\big] \\ &\le \sum_{j=1}^{p}\alpha_j\big[g(x_{j_1}(t_0-\tau(t_0))) - g(x_{j_2}(t_0-\tau(t_0)))\big]^T \\ &\quad \times \Sigma\alpha_j\big[g(x_{j_1}(t_0-\tau(t_0))) - g(x_{j_2}(t_0-\tau(t_0)))\big] \\ &\quad + \sum_{j=1}^{p} y_j^T(t_0)PB_2\Sigma^{-1}B_2^TPy_j(t_0) \\ &\le \sum_{j=1}^{p}\alpha_j^2(x_{j_1}(t_0-\tau(t_0))) - x_{j_2}(t_0-\tau(t_0)))^T \\ &\quad \times G\Sigma G(x_{j_1}(t_0-\tau(t_0))) - x_{j_2}(t_0-\tau(t_0))) \\ &\quad + \sum_{j=1}^{p} y_j^T(t_0)PB_2\Sigma^{-1}B_2^TPy_j(t_0).\end{aligned} \tag{13}$$

According to the assumptions that $P \ge FQF$ and $P \ge G\Sigma G$, one can obtain

$$\begin{aligned}&\sum_{j=1}^{p} t_0^{\epsilon}\alpha_j^2(x_{j_1}(t_0-\tau(t_0))) - x_{j_2}(t_0-\tau(t_0)))^T \\ &\quad \times FQF(x_{j_1}(t_0-\tau(t_0))) - x_{j_2}(t_0-\tau(t_0))) \\ &\le \sum_{j=1}^{p} t_0^{\epsilon}\alpha_j^2(x_{j_1}(t_0-\tau(t_0))) - x_{j_2}(t_0-\tau(t_0)))^T \\ &\quad \times P(x_{j_1}(t_0-\tau(t_0))) - x_{j_2}(t_0-\tau(t_0))) \\ &= t_0^{\epsilon} x^T(t_0-\tau(t_0))\mathbf{M}^T\mathbf{P}\mathbf{M}x(t_0-\tau(t_0)) \\ &= \left(\frac{t_0}{t_0-\tau(t_0)}\right)^{\epsilon} V(x(t_0-\tau(t_0))) \le (1-\eta)^{-\epsilon}V(x(t_0-\tau(t_0))) \\ &\le (1-\eta)^{-\epsilon}S(t_0) = (1-\eta)^{-\epsilon}V(x(t_0)) \\ &= (1-\eta)^{-\epsilon}t_0^{\epsilon}x^T(t_0)\mathbf{M}^T\mathbf{P}\mathbf{M}x(t_0),\end{aligned} \tag{14}$$

and

$$
\begin{aligned}
&\sum_{j=1}^{p} t_0^{\epsilon}\alpha_j^2(x_{j_1}(t_0-\tau(t_0)))-x_{j_2}(t_0-\tau(t_0)))^T \\
&\qquad \times G\Sigma G(x_{j_1}(t_0-\tau(t_0)))-x_{j_2}(t_0-\tau(t_0))) \\
&\quad \le \sum_{j=1}^{p} t_0^{\epsilon}\alpha_j^2(x_{j_1}(t_0-\tau(t_0)))-x_{j_2}(t_0-\tau(t_0)))^T \\
&\qquad \times P(x_{j_1}(t_0-\tau(t_0)))-x_{j_2}(t_0-\tau(t_0))) \\
&\quad = t_0^{\epsilon}x^T(t_0-\tau(t_0))\mathbf{M}^T\mathbf{PM}x(t_0-\tau(t_0)) \\
&\quad = \left(\frac{t_0}{t_0-\tau(t_0)}\right)^{\epsilon} V(x(t_0-\tau(t_0))) \\
&\quad \le (1-\eta)^{-\epsilon}V(x(t_0-\tau(t_0))) \\
&\quad \le (1-\eta)^{-\epsilon}S(t_0) = (1-\eta)^{-\epsilon}V(x(t_0)) \\
&\quad = (1-\eta)^{-\epsilon}t_0^{\epsilon}x^T(t_0)\mathbf{M}^T\mathbf{PM}x(t_0).
\end{aligned} \tag{15}
$$

It follows from (12), (13), (14) and (15) that

$$
\begin{aligned}
\dot V(x(t))\,|_{t=t_0} \le\; & t_0^{\epsilon}\sum_{j=1}^{p} y_j^T(t_0)[PA + A^TP - \Delta + \beta P \\
& + PB_1Q^{-1}B_1^TP + PB_2\Sigma^{-1}B_2^TP \\
& + 2(1-\eta)^{-\epsilon}P]y_j(t_0) \\
& + 2t_0^{\epsilon}x^T(t_0)\mathbf{M}^T\mathbf{PMC}x(t_0) \\
& + t_0^{\epsilon}x^T(t_0)\mathbf{M}^T\bar{\mathbf{\Delta}}\mathbf{M}x(t_0).
\end{aligned} \tag{16}
$$

Furthermore, one has

$$
\begin{aligned}
\sum_{k=1}^{N} u_{ik}c_{kj} &= (u_{ii}-1)c_{ij} + \sum_{k=1,k\ne i}^{N} u_{ik}c_{kj} + c_{ij} \\
&= -Nc_{ij} + \sum_{k=1}^{N} c_{kj} = -Nc_{ij},
\end{aligned}
$$

and

$$
\begin{aligned}
&2t_0^{\epsilon}x^T(t_0)\mathbf{M}^T\mathbf{PMC}x(t_0) + t_0^{\epsilon}x^T(t_0)\mathbf{M}^T\bar{\mathbf{\Delta}}\mathbf{M}x(t_0) \\
&\quad = 2t_0^{\epsilon}x^T(t_0)(\tilde M^T\otimes I_n)(I_p\otimes P)(\tilde M\otimes I_n) \\
&\qquad \times (C\otimes D)x(t_0) \\
&\qquad + t_0^{\epsilon}x^T(t_0)(\tilde M^T\otimes I_n)(I_p\otimes \Delta)(\tilde M\otimes I_n)x(t_0) \\
&\quad = 2t_0^{\epsilon}x^T(t_0)(\tilde M^T\tilde M C\otimes PD)x(t_0) + t_0^{\epsilon}x^T(t_0) \\
&\qquad \times (\tilde M^T\tilde M\otimes \Delta)x(t_0) \\
&\quad = -2t_0^{\epsilon}x^T(t_0)(UC\otimes PD)x(t_0) - t_0^{\epsilon}x^T(t_0) \\
&\qquad \times (U\otimes\Delta)x(t_0) \\
&\quad = -2t_0^{\epsilon}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i^T(t_0)(-Nc_{ij}PD)x_j(t_0) \\
&\qquad - t_0^{\epsilon}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i^T(t_0)(u_{ij}\Delta)x_j(t_0).
\end{aligned} \tag{17}
$$

Hence, we gets

$$
\begin{aligned}
\dot V(x(t))|_{t=t_0} \le\; & t_0^{\epsilon}\sum_{j=1}^{p} y_j^T(t_0)\widetilde{\Omega}y_j(t_0) \\
& - 2t_0^{\epsilon}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i^T(t_0)(-Nc_{ij}PD)x_j(t_0) \\
& - t_0^{\epsilon}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i^T(t_0)(u_{ij}\Delta)x_j(t_0) \\
\le\; & t_0^{\epsilon}\sum_{j=1}^{p} y_j^T(t_0)\widetilde{\Omega}y_j(t_0) \\
& - 2t_0^{\epsilon}\sum_{i=1}^{N}\Bigg(\sum_{j=1,j\ne i}^{N} x_i^T(t_0)(-Nc_{ij}PD)x_j(t_0) \\
& + x_i^T(t_0)(-Nc_{ii}PD)x_i(t_0)\Bigg) \\
& - t_0^{\epsilon}\sum_{i=1}^{N}\Bigg(\sum_{j=1,j\ne i}^{N} x_i^T(t_0)(u_{ij}\Delta)x_j(t_0) \\
& + x_i^T(t_0)(u_{ii}\Delta)x_i(t_0)\Bigg) \\
\le\; & t_0^{\epsilon}\sum_{j=1}^{p} y_j^T(t_0)\widetilde{\Omega}y_j(t_0) \\
& + 2t_0^{\epsilon}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(x_i(t_0)-x_j(t_0))^T(-Nc_{ij}PD) \\
& \times (x_i(t_0)-x_j(t_0)) \\
& + t_0^{\epsilon}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(x_i(t_0)-x_j(t_0))^T\Delta(x_i(t_0)-x_j(t_0)) \\
\le\; & t_0^{\epsilon}\sum_{j=1}^{p} y_j^T(t_0)\widetilde{\Omega}y_j(t_0) \\
& + t_0^{\epsilon}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}(x_i(t_0)-x_j(t_0))^T \\
& \times (-Nc_{ij}(PD+D^TP)+\Delta)(x_i(t_0)-x_j(t_0)) \le 0,
\end{aligned} \tag{18}
$$

which implies that $S(t)$ is also non-increasing at t_0. It follows from above analysis that $S(t) = S(T)$ for all $t \ge T$, which implies that $\|y(t)\| = O(t^{-\frac{\epsilon}{2}})$. This completes the proof. □

Remark 3 If the time delay is bounded and differentiable and $\dot\tau(t) \le \tau < 1$, following the similar method, we can obtain a sufficient condition for the delayed coupled genetic oscillators to be globally exponentially synchronized:

$$
\begin{aligned}
(i)\quad & PA + A^TP - \Delta + (1-\tau)^{-1}PB_1Q^{-1}B_1^TP \\
& + (1-\tau)^{-1}PB_2\Sigma^{-1}B_2^TP + FQF + G\Sigma G < 0, \\
(ii)\quad & -Nc_{ij}(PD + D^TP) + \Delta < 0,\ \ 1 \le i < j \le N,
\end{aligned}
$$

where the Lyapunov functional can be constructed as

$$V(x(t)) = e^{2\varepsilon t}x^T(t)\mathbf{M}^T\mathbf{P}\mathbf{M}x(t) + \int_{t-\tau(t)}^{t} \mathbf{f}^T(x(s))\mathbf{M}^T\mathbf{Q}\mathbf{M}\mathbf{f}(x(s))e^{2\varepsilon s}ds + \int_{t-\tau(t)}^{t} \mathbf{g}^T(x(s))\mathbf{M}^T\mathbf{\Sigma}\mathbf{M}\mathbf{g}(x(s))e^{2\varepsilon s}ds.$$

The proof is similar and omitted here.

If the coupling matrix D is a diagonal matrix, then the following result can be obtained.

Corollary 1 *The network* (3) *is power-rate synchronized if there exist diagonal matrices* $P = \mathrm{diag}\{p_1, p_2, \ldots, p_n\} > 0$, $Q = \mathrm{diag}\{q_1, q_2, \ldots, q_n\} > 0$, $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} > 0$, $\Delta = \mathrm{diag}\{\delta_1, \delta_2, \ldots, \delta_n\}$ *and an irreducible symmetric matrix* $U = (u_{ij}) \in \mathbb{R}^{N\times N}$ *satisfying condition* (4), such that

$$\begin{pmatrix} PA + A^TP + 2P - \Delta & PB_1 & -PB_2 \\ B_1^TP & -Q & 0 \\ -B_2^TP & 0 & -\Sigma \end{pmatrix} < 0,$$
$$P \geq FQF, \quad P \geq G\Sigma G, \tag{19}$$

and

$$p_jd_j(UC + CU) + \delta_jI_N \geq 0, \tag{20}$$

where $F = \mathrm{diag}\{F_1, F_2, \ldots, F_n\}$ *and* $G = \mathrm{diag}\{G_1, G_2, \ldots, G_n\}$.

Proof Since

$$\begin{aligned} &2x^T(t_0)\mathbf{M}^T\mathbf{P}\mathbf{M}\mathbf{C}x(t_0) + x^T(t_0)\mathbf{M}^T\mathbf{M}\Delta x(t_0) \\ &= \sum_{j=1}^{n} p_jd_j\bar{x}_j^T(t)\tilde{M}^T\tilde{M}C\bar{x}_j + \sum_{j=1}^{n}\bar{x}_j^T\tilde{M}^T\tilde{M}\delta_j\bar{x}_j(t) \\ &= -\sum_{j=1}^{n}\bar{x}_j^T(t)(p_jd_j(UC + CU) + \delta_jI_N)\bar{x}_j(t) \leq 0, \end{aligned}$$

and from (16), one has

$$\begin{aligned} \dot{V}(x(t))\mid_{t=t_0} \leq t_0^{\epsilon}\sum_{j=1}^{p} y_j^T(t_0)\big[&PA + A^TP - \Delta + \beta P \\ &+ PB_1Q^{-1}B_1^TP + PB_2\Sigma^{-1}B_2^TP \\ &+ 2(1-\eta)^{-\epsilon}P\big]y_j(t_0) \\ &- \sum_{j=1}^{n}\bar{x}_j^T(t)(p_jd_j(UC + CU) \\ &+ \delta_jI_N)\bar{x}_j(t) \leq 0. \end{aligned} \tag{21}$$

This completes the proof. □

Theorem 2 *The network* (3) *is power-rate synchronized if there exist positive definite matrix* $P = (p_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, *positive definite diagonal matrices* $Q = \mathrm{diag}\{q_1, q_2, \ldots, q_n\} \in \mathbb{R}^{n\times n}$, $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \in \mathbb{R}^{n\times n}$, *a symmetric matrix* $\Delta = (\Delta_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, *such that*

$$\begin{pmatrix} \tilde{\mathbf{P}}\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + 2\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\mathbf{H} + \mathbf{H}^T\tilde{\mathbf{P}} & \tilde{\mathbf{P}}\tilde{\mathbf{B}}_1 & -\tilde{\mathbf{P}}\tilde{\mathbf{B}}_2 \\ \tilde{\mathbf{B}}_1^T\tilde{\mathbf{P}} & -\tilde{\mathbf{Q}} & 0 \\ -\tilde{\mathbf{B}}_2^T\tilde{\mathbf{P}} & 0 & -\tilde{\mathbf{\Sigma}} \end{pmatrix} < 0, \tag{22}$$

and

$$\tilde{\mathbf{P}} \geq \mathbf{F}\tilde{\mathbf{Q}}\mathbf{F}, \quad \tilde{\mathbf{P}} \geq \mathbf{G}\tilde{\mathbf{\Sigma}}\mathbf{G}, \tag{23}$$

where $\mathbf{F} = I_{N-1}\otimes F$, $\mathbf{G} = I_{N-1}\otimes G$, $H = \tilde{M}CV$, $\mathbf{H} = H\otimes D$ *and* V *is defined in Lemma*1 *and* $\tilde{M}$ *is defined in the proof.*

Proof Suppose that

$$\tilde{M} = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & & \ddots & \\ & & & 1 & -1 \end{pmatrix}_{(N-1)\times N},$$

and $\tilde{M}$ is a $(N-1)\times N$ real-valued matrix. Let $\mathbf{M}_1 = \tilde{M}\otimes I_n$, $y(t) = \mathbf{M}_1x(t) = [y_1^T(t), y_2^T(t), \ldots, y_{N-1}^T(t)]^T$, $y_i(t) = [y_{i,1}(t), y_{i,2}(t), \ldots, y_{i,n}(t)]^T$. According to the structure of $\mathbf{M}_1$, it is easy to show that $y_i(t) = x_i(t) - x_{i+1}(t), i = 1, 2, \ldots, N-1$. We denote

$$\begin{aligned} &\tilde{\mathbf{A}} = I_{N-1}\otimes A, \ \tilde{\mathbf{B}}_1 = I_{N-1}\otimes B_1, \\ &\tilde{\mathbf{B}}_2 = I_{N-1}\otimes B_2, \ \tilde{\mathbf{P}} = I_{N-1}\otimes P, \\ &\tilde{\mathbf{Q}} = I_{N-1}\otimes Q, \ \tilde{\mathbf{\Sigma}} = I_{N-1}\otimes\Sigma, \\ &\mathbf{C} = C\otimes D, \ \mathbf{G} = I_{N-1}\otimes G, \\ &\mathbf{F} = I_{N-1}\otimes F, \ \tilde{\mathbf{\Omega}} = I_{N-1}\otimes\tilde{\Omega}, \\ &\mathbf{u} = \left[(B_2\cdot e_n)^T, (B_2\cdot e_n)^T, \ldots, (B_2\cdot e_n)^T\right]^T. \end{aligned}$$

By Lemma 4, condition (22) is equivalent to

$$\begin{aligned} &\tilde{\mathbf{P}}\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\mathbf{H} + \mathbf{H}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{B}}_1\tilde{\mathbf{Q}}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{P}} \\ &\quad + \tilde{\mathbf{P}}\tilde{\mathbf{B}}_2\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{B}}_2^T\tilde{\mathbf{P}} + 2\tilde{\mathbf{P}} < 0, \end{aligned} \tag{24}$$

then we can find two sufficient small constants $\beta > 0$ and $\epsilon > 0$ such that

$$\begin{aligned} \mathbf{\Omega} = &\beta\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\mathbf{H} + \mathbf{H}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{B}}_1\tilde{\mathbf{Q}}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{P}} \\ &+ \tilde{\mathbf{P}}\tilde{\mathbf{B}}_2\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{B}}_2^T\tilde{\mathbf{P}} + 2(1-\eta)^{-\epsilon}\tilde{\mathbf{P}} < 0, \end{aligned} \tag{25}$$

In addition, a sufficiently large constant $T > 0$ can be found such that

$$\frac{\epsilon}{t} < \beta, \quad \text{for all } t > T. \tag{26}$$

Consider the following Lyapunov function

$V(x(t)) = t^{\epsilon}x^T(t)\mathbf{M}_1^T\tilde{\mathbf{P}}\mathbf{M}_1x(t).$

Following the same method in Theorem 1, one obtains

$$\begin{aligned}\dot{V}(x(t))\mid_{t=t_0} &= \epsilon t_0^{\epsilon-1}x^T(t_0)\mathbf{M}_1^T\tilde{\mathbf{P}}\mathbf{M}_1x(t_0)\\ &+ t_0^{\epsilon}x^T(t_0)\left[\mathbf{M}_1^T\tilde{\mathbf{P}}\tilde{\mathbf{A}}\mathbf{M}_1 + \mathbf{M}_1^T\tilde{\mathbf{A}}^T\tilde{\mathbf{P}}\mathbf{M}_1\right]\\ &\times x(t_0) + 2t_0^{\epsilon}x^T(t_0)\mathbf{M}_1^T\tilde{\mathbf{P}}\\ &\times\left[\tilde{\mathbf{B}}_1\mathbf{M}_1\mathbf{f}(x(t_0-\tau(t_0)))\right.\\ &\left.-\tilde{\mathbf{B}}_2\mathbf{M}_1\mathbf{g}(x(t_0-\tau(t_0))) + \mathbf{M}_1\mathbf{u}\right]\\ &+ 2t_0^{\epsilon}x^T(t_0)\mathbf{M}_1^T\tilde{\mathbf{P}}\mathbf{M}_1\mathbf{C}x(t_0).\end{aligned}$$

Let $\phi(y(t-\tau(t))) = \mathbf{M}_1\mathbf{f}(x(t-\tau(t)))$ and $\varphi(y(t-\tau(t))) = \mathbf{M}_1\mathbf{g}(x(t-\tau(t)))$, it follows from Lemma 3 that

$$\begin{aligned}2y^T(t_0)\tilde{\mathbf{P}}\tilde{\mathbf{B}}_1\phi(y(t_0-\tau(t_0))) &\le y^T(t_0)\tilde{\mathbf{P}}\tilde{\mathbf{B}}_1\tilde{\mathbf{Q}}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{P}}y(t_0)\\ &+ \phi^T(y(t_0-\tau(t_0)))\tilde{\mathbf{Q}}\phi(y(t_0-\tau(t_0)))\\ &\le y^T(t_0)\tilde{\mathbf{P}}\tilde{\mathbf{B}}_1\tilde{\mathbf{Q}}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{P}}y(t_0)\\ &+ y^T(t_0-\tau(t_0))\mathbf{F}\tilde{\mathbf{Q}}\mathbf{F}y(t_0-\tau(t_0))\end{aligned}$$

and

$$\begin{aligned}&-2y^T(t_0)\tilde{\mathbf{P}}\tilde{\mathbf{B}}_2\varphi(y(t_0-\tau(t_0)))\\ &\le y^T(t_0)\tilde{\mathbf{P}}\tilde{\mathbf{B}}_2\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{B}}_2^T\tilde{\mathbf{P}}y(t_0)\\ &\quad+ \varphi^T(y(t_0-\tau(t_0)))\tilde{\mathbf{\Sigma}}\varphi(y(t_0-\tau(t_0)))\\ &\le y^T(t_0)\tilde{\mathbf{P}}\tilde{\mathbf{B}}_2\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{B}}_2^T\tilde{\mathbf{P}}y(t_0)\\ &\quad+ y^T(t_0-\tau(t_0))\mathbf{G}\tilde{\mathbf{\Sigma}}\mathbf{G}y(t_0-\tau(t_0)).\end{aligned}$$

Furthermore, since $\tilde{\mathbf{P}} \ge \mathbf{F}\tilde{\mathbf{Q}}\mathbf{F}$ and $\tilde{\mathbf{P}} \ge \mathbf{G}\tilde{\mathbf{\Sigma}}\mathbf{G}$, thus

$$\begin{aligned}&t_0^{\epsilon}y^T(t_0-\tau(t_0))\mathbf{F}\tilde{\mathbf{Q}}\mathbf{F}y(t_0-\tau(t_0))\\ &\le t_0^{\epsilon}y^T(t_0-\tau(t_0))\tilde{\mathbf{P}}y(t_0-\tau(t_0))\\ &= \left(\frac{t_0}{t_0-\tau(t_0)}\right)^{\epsilon}V(x(t_0-\tau(t_0)))\\ &\le (1-\eta)^{-\epsilon}V(x(t_0-\tau(t_0)))\\ &\le (1-\eta)^{-\epsilon}S(t_0) = (1-\eta)^{-\epsilon}V(x(t_0))\\ &= (1-\eta)^{-\epsilon}t_0^{\epsilon}x^T(t_0)\mathbf{M}_1^T\tilde{\mathbf{P}}\mathbf{M}_1x(t_0)\\ &= (1-\eta)^{-\epsilon}t_0^{\epsilon}y(t_0)^T\tilde{\mathbf{P}}y(t_0),\\ &t_0^{\epsilon}y^T(t_0-\tau(t_0))\mathbf{G}\tilde{\mathbf{\Sigma}}\mathbf{G}y(t_0-\tau(t_0))\\ &\le t_0^{\epsilon}y^T(t_0-\tau(t_0))\tilde{\mathbf{P}}y(t_0-\tau(t_0))\\ &= \left(\frac{t_0}{t_0-\tau(t_0)}\right)^{\epsilon}V(x(t_0-\tau(t_0)))\\ &\le (1-\eta)^{-\epsilon}V(x(t_0-\tau(t_0)))\\ &\le (1-\eta)^{-\epsilon}S(t_0) = (1-\eta)^{-\epsilon}V(x(t_0))\\ &= (1-\eta)^{-\epsilon}t_0^{\epsilon}x^T(t_0)\mathbf{M}_1^T\tilde{\mathbf{P}}\mathbf{M}_1x(t_0)\\ &= (1-\eta)^{-\epsilon}t_0^{\epsilon}y(t_0)^T\tilde{\mathbf{P}}y(t_0).\end{aligned}$$

Denote $\mathbf{\Omega}_1 = \beta\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{B}}_1\tilde{\mathbf{Q}}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{B}}_2\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{B}}_2^T\tilde{\mathbf{P}} + 2(1-\eta)^{-\epsilon}\tilde{\mathbf{P}}$, one has

$$\begin{aligned}\dot{V}(x(t))|_{t=t_0} &\le t_0^{\epsilon}y^T(t_0)\left[\frac{\epsilon}{t0}\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{B}}_1\tilde{\mathbf{Q}}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{P}}\right.\\ &\left.+\tilde{\mathbf{P}}\tilde{\mathbf{B}}_2\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{B}}_2^T\tilde{\mathbf{P}} + 2(1-\eta)^{-\epsilon}\tilde{\mathbf{P}}\right]y(t_0)\\ &+ 2t_0^{\epsilon}x^T(t_0)\mathbf{M}_1^T\tilde{\mathbf{P}}\mathbf{M}_1\mathbf{C}x(t_0)\\ &\le t_0^{\epsilon}y^T(t_0)\left[\beta\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{B}}_1\tilde{\mathbf{Q}}^{-1}\tilde{\mathbf{B}}_1^T\tilde{\mathbf{P}}\right.\\ &\left.+\tilde{\mathbf{P}}\tilde{\mathbf{B}}_2\tilde{\mathbf{\Sigma}}^{-1}\tilde{\mathbf{B}}_2^T\tilde{\mathbf{P}} + 2(1-\eta)^{-\epsilon}\tilde{\mathbf{P}}\right]y(t_0)\\ &+ 2t_0^{\epsilon}x^T(t_0)\mathbf{M}_1^T\tilde{\mathbf{P}}(\tilde{M}\otimes I_n)(C\otimes D)x(t_0)\\ &= t_0^{\epsilon}y^T(t_0)\mathbf{\Omega}_1y(t_0)\\ &+ 2t_0^{\epsilon}x^T(t_0)\mathbf{M}_1^T\tilde{\mathbf{P}}(\tilde{M}C\otimes D)x(t_0)\\ &= t_0^{\epsilon}y^T(t_0)\mathbf{\Omega}_1y(t_0)\\ &+ 2t_0^{\epsilon}x^T(t_0)\mathbf{M}_1^T\tilde{\mathbf{P}}(H\tilde{M}\otimes D)x(t_0)\\ &= t_0^{\epsilon}y^T(t_0)\mathbf{\Omega}_1y(t_0)\\ &+ 2t_0^{\epsilon}x^T(t_0)\mathbf{M}_1^T\tilde{\mathbf{P}}[(H\otimes D)(\tilde{M}\otimes I_n)]x(t_0)\\ &= t_0^{\epsilon}y^T(t_0)\mathbf{\Omega}_1y(t_0) + 2t_0^{\epsilon}x^T(t_0)\mathbf{M}_1^T\tilde{\mathbf{P}}\mathbf{H}\mathbf{M}_1x(t_0)\\ &= t_0^{\epsilon}y^T(t_0)\mathbf{\Omega}_1y(t_0)\\ &+ t_0^{\epsilon}x^T(t_0)\mathbf{M}_1^T(\tilde{\mathbf{P}}\mathbf{H} + \mathbf{H}^T\tilde{\mathbf{P}})\mathbf{M}_1x(t_0)\\ &= t_0^{\epsilon}y^T(t_0)\mathbf{\Omega}y(t_0) \le 0,\end{aligned}$$

which implies that $S(t)$ is also non-increasing at t_0. It follows from above analysis that $S(t) = S(T)$ for all $t \ge T$, which implies that $\|y(t)\| = O(t^{-\frac{\epsilon}{2}})$. This completes the proof. □

According to the proof of Theorem 2, one has the following corollary.

Corollary 2 *The network (3) is power-rate synchronized if there exist positive definite matrix $P = (p_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, positive definite diagonal matrices $Q = \mathrm{diag}\{q_1, q_2, \ldots, q_n\} \in \mathbb{R}^{n\times n}$, $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\} \in \mathbb{R}^{n\times n}$, a symmetric matrix $\Delta = (\Delta_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, such that*

$$\begin{pmatrix} PA + A^TP - \Delta + 2P & PB_1 & -PB_2\\ B_1^TP & -Q & 0\\ -B_2^TP & 0 & -\Sigma\end{pmatrix} < 0, \tag{27}$$

and

$$P \ge FQF, \quad P \ge G\Sigma G, \quad \tilde{\mathbf{P}}\mathbf{H} + \mathbf{H}^T\tilde{\mathbf{P}} + \tilde{\mathbf{\Delta}} < 0, \tag{28}$$

where $\mathbf{F} = I_{N-1}\otimes F$, $\mathbf{G} = I_{N-1}\otimes G$, $H = \tilde{M}CV$, $\mathbf{H} = H\otimes D$ and V is defined in Lemma 1 and $\tilde{M}$ is defined in Theorem 2, and $\tilde{\mathbf{\Delta}} = I_{N-1}\otimes\Delta$.

Remark 4 Likewise, if the time delay is bounded and differentiable and $\dot{\tau}(t) \le \tau < 1$, following the similar method, we can obtain a sufficient condition which is a minor revision of Theorem 2 for the delayed coupled genetic oscillators to be globally exponentially synchronized:

$$(i)\quad PA + A^TP - \Delta + (1-\tau)^{-1}PB_1Q^{-1}B_1^TP + (1-\tau)^{-1}PB_2\Sigma^{-1}B_2^TP + FQF + G\Sigma G < 0,$$

$$(ii)\quad \tilde{\mathbf{P}}\mathbf{H} + \mathbf{H}^T\tilde{\mathbf{P}} + \tilde{\Delta} < 0,$$

where the Lyapunov functional is constructed as

$$V(x(t)) = e^{2\varepsilon t}x^T(t)\mathbf{M}_1^T\tilde{\mathbf{P}}\mathbf{M}_1x(t) + \int_{t-\tau(t)}^{t}\mathbf{f}^T(x(s))\mathbf{M}_1^T\tilde{\mathbf{Q}}\mathbf{M}_1\mathbf{f}(x(s))e^{2\varepsilon s}ds + \int_{t-\tau(t)}^{t}\mathbf{g}^T(x(s))\mathbf{M}_1^T\tilde{\Sigma}\mathbf{M}_1\mathbf{g}(x(s))e^{2\varepsilon s}ds.$$

The proof is omitted.

Remark 5 A new synchronization problem for the collective dynamics among genetic oscillators with unbounded time-varying delay is investigated. A new concept called power-rate synchronization, which is different from both the asymptotical synchronization and the exponential synchronization, is put forward to facilitate handling the unbounded time-varying delays. An LMI-based sufficient condition is derived for the existence of the desired synchronization, which ensures the power-rate stability in the presence of time-delays. The synchronization of the addressed genetic oscillators can be readily checked by the solvability of a set of LMIs, which can be done by resorting to the Matlab LMI toolbox. It should be mentioned that, in the past decade, LMIs have gained much attention for their computational tractability and usefulness in many areas because the so-called interior point method has been proven to be numerically very efficient for solving the LMIs. In next section, an illustrative example will be provided to show the application potential of the proposed techniques in a population of N coupled SCN neuron model oscillators.

Illustrative example

Example. In the following, we consider a population of N coupled SCN neuron model oscillators. The single cell or genetic oscillator is described by the classical Goodwin model (Goodwin 1965). In this model, a clock gene mRNA (X) produces a clock protein (Y), which activates a transcriptional inhibitor (Z). Z inhibits the transcription of the clock gene, completing the cycle. The oscillators coupled through the release and receiving of neurotransmitter among neurons. The release of the neurotransmitter is supposed to be fast with respect to the 24 h timescale of the oscillators. In this paper, we assume that the light $L = 0$. V describes the evolution of the neurotransmitter in the neuron. The equation for each oscillator is given as follows:

$$\begin{aligned}\dot{X}(t) &= v_1\frac{1}{1.3 + Z(t-\tau(t))^m} - v_2X(t),\\ \dot{Y}(t) &= v_3X(t) - v_4Y(t),\\ \dot{Z}(t) &= v_5Y(t) - v_6Z(t),\\ \dot{V}(t) &= v_7X(t) - v_8V(t)\end{aligned}\qquad (29)$$

where $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are positive constants. The concentrations are express in nM, and the parameters are set as $m = 12$, $v_1 = 1\,\mathrm{nM\,h^{-1}}$; $v_2 = v_5 = v_6 = 1.2\,\mathrm{h^{-1}}$; $v_3 = 2\,\mathrm{h^{-1}}$; $v_4 = 1.5\,\mathrm{h^{-1}}$; $v_7 = 0.2\,\mathrm{h^{-1}}$; $v_8 = 2\,\mathrm{h^{-1}}$; $\tau(t) = 0.1t$, $N = 10$; the coupling matrix C are given as: $c_{ii} = -2 + \frac{2}{10}$; $c_{ij} = \frac{2}{10}$, $i \neq j$; and the inner linking matrix $D = (d_{ij})$ with $d_{14} = 3$ and other elements are 0.

By applying Theorem 1 and using Matlab LMI Toolbox, we can find the feasible solutions for (6) and (7), which indicate that the coupled network can achieve power-rate synchronization. In Fig. 1a, we plot the time evolution of the mRNA concentrations (X) of all the uncoupled oscillators with random initial values. In Fig. 1b, we show the error of the uncoupled oscillators. From Fig. 1a, b, we can see that

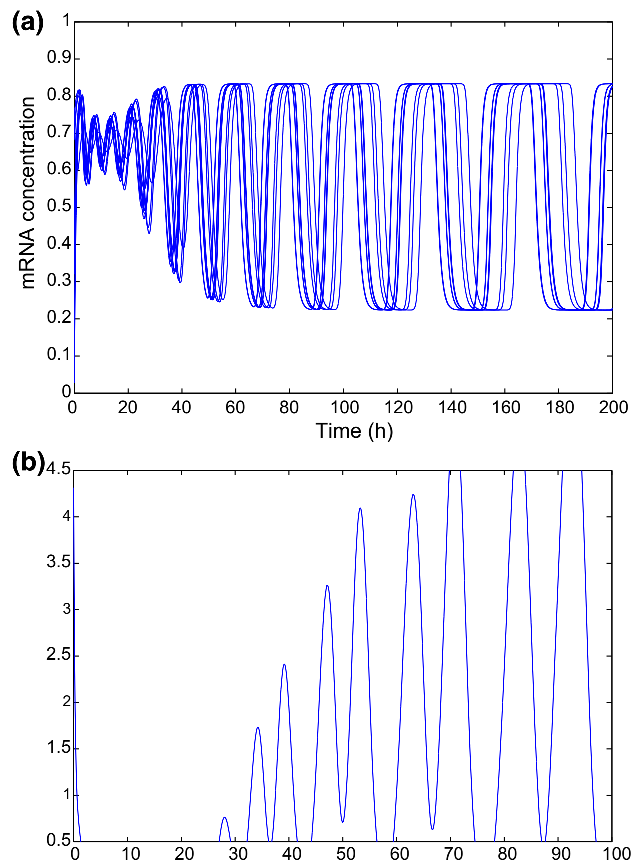

Fig. 1 Time response of all the uncoupled oscillators with random initial values

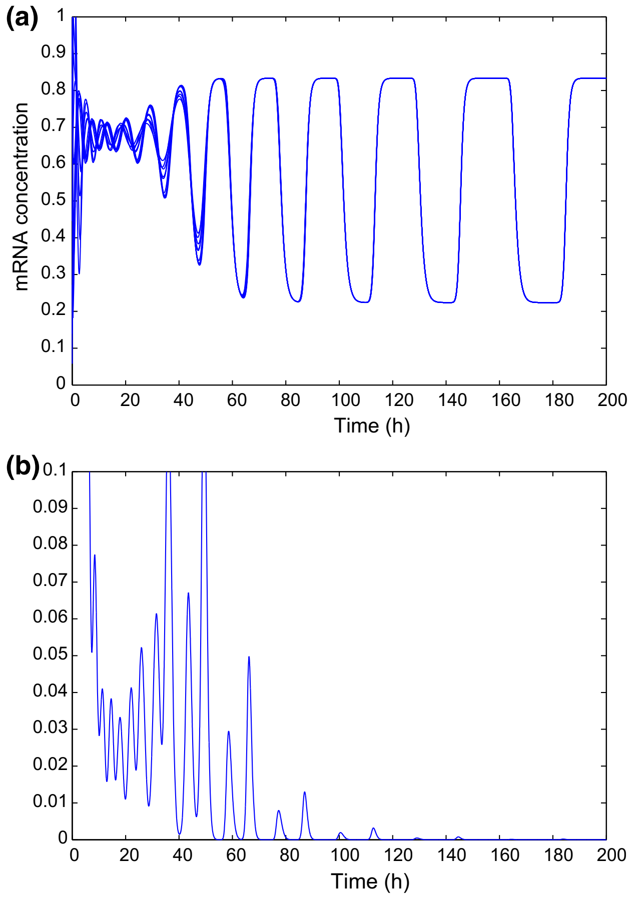

Fig. 2 Time response of all the coupled oscillators with random initial values

there is no synchronization between the oscillators without coupling. Figure 2a shows the time evolution of the mRNA concentration (X) of all the coupled oscillators with random initial values. Also, Fig. 2a shows that the synchronization is realized between the oscillators with linear coupling. So, linear coupling is an effective way to achieve synchronization. Figure 2b shows the time evolution of the synchronization error, which further confirms that the coupled network can achieve power-rate synchronization.

$$\text{error} = \sum_{i=1}^{N-1} [(X_i - X_{i+1})^2 + (Y_i - Y_{i+1})^2 + (Z_i - Z_{i+1})^2 + (V_i - V_{i+1})^2],$$

where X_i, Y_i, Z_i describe the dynamics of the oscillator in ith neuron, and V_i describes the evolution of the neurotransmitter in the ith neuron.

In our future research, it would be very interesting to see how promising it would be to apply our developed results to other well-known genetic oscillators, e.g. the ones described in Elowitz and Leibler (2000). Genetic regulatory network is a dynamical system that consists of a finite number of different subsystems rather than a single, isolated system. For example, there are many genomes in a cell which leads to the existence of different regulatory network. These regulatory networks are activated in the internal and external conditions which is variable. That is to say, in certain biological environment, gene regulatory network is a hybrid system. Therefore, in the near future, we will discuss the dynamics of switching GRNs.

Conclusions

In this paper, we have investigated the synchronization problem on the genetic oscillators with unbounded time-varying delays. Some criteria are given to ensure the power-rate synchronization of the coupled genetic oscillators. The obtained conditions are derived in terms of matrix inequalities. Synchronization with some power convergence rate is first proposed on genetic oscillators. From Chen and Wang (2007) and the concept of synchronization, we know that there indeed exist some dynamical systems which are power-rate synchronized but not exponentially synchronized. We have employed a simulation example to illustrate the effectiveness of the proposed results.

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