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A note on Cline's formula for some generalized inverses in a ring

Soufiane Hadji[1]

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Abstract
In this note, we generalize Cline's formula for some new generalized inverses such as strong Drazin inverse, generalized strong Drazin inverse, Hirano inverse and generalized Hirano inverse. As a particular case, some recent results on this subject are recovered.

Keywords Cline's formula · Ring · Strong Drazin inverse · Generalized strong Drazin inverse · Hirano inverse · Generalized Hirano inverse

Mathematics Subject Classification 15A09 · 16U50

1 Introduction

In this paper, $\mathcal{R}$ denotes an associative ring with a unit element denoted by 1. We denote by $\mathcal{R}^{inv}$ the set of all invertible elements in $\mathcal{R}$, $\mathcal{R}^{nil}$ the set of all nilpotent elements in $\mathcal{R}$. Let $a \in \mathcal{R}$. We use the following notations

$$\text{comm}(a) = \{x \in \mathcal{R} : ax = xa\},$$

and

$$\text{comm}^2(a) = \{x \in \mathcal{R} : xy = yx \text{ for all } y \in \text{comm}(a)\}.$$

Following Harte[14], an element $a \in \mathcal{R}$ is *quasinilpotent* if, for all $x \in \text{comm}(a)$, $1 + ax \in \mathcal{R}^{inv}$. The set of all quasinilpotent elements in $\mathcal{R}$ is denoted by $\mathcal{R}^{qnil}$. The *Jacobson radical* of $\mathcal{R}$, denoted as $J(\mathcal{R})$, is defined as the set of elements a for which $1 + ax$ is invertible for all x in $\mathcal{R}$.

✉ Soufiane Hadji
soufiane.hadji@usmba.ac.ma

[1] Department of Mathematics, Dhar El Mahraz Faculty of Science, Sidi Mohamed Ben Abdellah University, 30003 Fez, Morocco

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In [27], Wang introduced a subclass of the Drazin inverse. An element $a \in \mathcal{R}$ is *strongly Drazin invertible* (or, *s-Drazin invertible*) if there exists $x \in \mathcal{R}$ such that

$$x \in \text{comm}(a), xax = x \text{ and } a - ax \in \mathcal{R}^{nil}.$$

If such x exists, it is unique and is called the strong Drazin inverse (called also s-Drazin inverse) of a, denoted by a^{sD}. The least non-negative integer k, for which $(a - ax)^k = 0$ holds, is called the *Drazin index* of a, denoted by $\text{i}_{\text{sD}}(a)$. The set of all elements in $\mathcal{R}$ that have a s-Drazin inverse is denoted by $\mathcal{R}^{sD}$. An element $a \in \mathcal{R}$ is called Drazin invertible if we replace the condition $a - ax \in \mathcal{R}^{\text{nil}}$ in the definition of the s-Drazin invertible element with $a(1 - ax) \in \mathcal{R}^{\text{nil}}$ (see [10]). In this case, x is called a Drazin inverse of a and denoted by $x = a^D$. By $\mathcal{R}^D$ we represent the set of all Drazin invertible elements of $\mathcal{R}$. As observed in [5], $\mathcal{R}^{sD} \subsetneq \mathcal{R}^D$. Thus, the s-Drazin inverse of a is unique if it exists, and it belongs to the double commutant of a. However, invertible elements may not be s-Drazin invertible in general. For example (see [29]), 2 is invertible but not s-Drazin invertible in complex number field $\mathbb{C}$.

In [5], Chen and Abdolyousefi introduced the *Hirano inverse*. An element $a \in \mathcal{R}$ is called Hirano invertible provided that there exists $x \in \mathcal{R}$ such that

$$x \in \text{comm}(a), xax = x \text{ and } a^2 - ax \in \mathcal{R}^{\text{nil}}.$$

If such x exists, it is unique and is called the Hirano inverse of a denoted by a^H and the least non-negative integer k for which $\left(a^2 - ax\right)^k = 0$ holds is called the *Hirano index* of a, and is denoted by $\text{i}_{\text{H}}(a)$. The set of all elements in $\mathcal{R}$ that have a Hirano inverse is denoted by $\mathcal{R}^H$. Also by [5], $\mathcal{R}^{sD} \subsetneq \mathcal{R}^H \subsetneq \mathcal{R}^D$. However, the indices $i_{sD}(a)$ and $i_H(a)$ need not be the same in general (see for instance [29]).

Mosić [22] in a complex Banach algebra and Gürgün [11] in a ring have generalized the notion of strong Drazin inverse. Recall that an element $a \in \mathcal{R}$ is *generalized strongly Drazin invertible* (or, *gs-Drazin invertible*) if there exists $x \in \mathcal{R}$ such that

$$x \in \text{comm}^2(a), xax = x \text{ and } a - ax \in \mathcal{R}^{\text{qnil}}.$$

If such x exists, it is unique (see [17, Theorem 4.2]) and is called the *generalized strong Drazin inverse* of a and is denoted by a^{gSD}. The set of all elements in $\mathcal{R}$ that have a generalized strong Drazin inverse is denoted by $\mathcal{R}^{gSD}$. If we replace the third condition $a-ax \in \mathcal{R}^{\text{qnil}}$ in the above definition by of $a(1-ax) \in \mathcal{R}^{\text{qnil}}$, then $a \in \mathcal{R}$ is called *generalized Drazin invertible* (see [17]). In this case, x is called a generalized Drazin inverse of a and denoted by $x = a^{gD}$. By $\mathcal{R}^{gD}$ we represent the set of all generalized Drazin invertible elements of $\mathcal{R}$. By [11, Corollary 3.3], $\mathcal{R}^{gsD} \subsetneq \mathcal{R}^{gD}$.

Recently, Abdolyousefi and Chen [1] introduced a concept that extended the notion of generalized Drazin inverse called *generalized Hirano inverse*. An element $a \in \mathcal{R}$ is generalized Hirano inverse if there exists an $x \in \text{comm}^2(a)$ such that

$$x \in \text{comm}^2(a), xax = x \text{ and } a^2 - ax \in \mathcal{R}^{\text{qnil}}.$$

The element x above is unique, if it exists, it is called the generalized Hirano inverse of a, denoted by $x = a^{gH}$. We denote by $\mathcal{R}^{gH}$ the set of all generalized Hirano invertible elements in $\mathcal{R}$.

Furthermore, Cline [6], proved that the product ab is Drazin invertible if and only if ba is Drazin invertible. Moreover, the Drazin inverse of ba can be expressed as:

$$(ba)^D = b\left((ab)^D\right)^2 a.$$

This equality is known as Cline's fromula, which is very useful in finding the Drazin inverse for the sum of two elements and also for the Drazin inverse of block matrices. Since this formula is widely used in mentioned topics and there are plenty significant results (we refer the reader to see [9, 15, 21, 23, 26, 30, 32]).

In order to extend the Cline's formula, the authors in [5, 11] studied the strong Drazin invertibility, generalized strong Drazin invertibility, Hirano invertibility and generalized Hirano invertibility of products ac and ba under condition:

$$aba = aca. \tag{1}$$

In [29] and [25], the authors extended the Cline's formula for strong Drazin inverse, generalized strong Drazin inverse, Hirano inverse and generalized Hirano inverse, under other conditions.

Recently, in [12, 16, 20, 25, 31], authors investigated the Drazin inverse for operators AC and BD.

Several works have been devoted to extend Cline's formula for generalized inverses, when the following condition is satisfied:

$$a(ba)^2 = abaca = acaba = (ac)^2 a. \tag{2}$$

In this paper, we extend Cline's formula for new generalized inverses, related to the Drazin inverse, such as (generalized) strong Drazin inverse and (generalized) Hirano inverse. Precisely, when the condition (2) is satisfied, we prove that

$$ac \in \mathcal{R}^\bullet \Longleftrightarrow ba \in \mathcal{R}^\bullet,$$

and we have

$$(ac)^\bullet = a\left((ba)^\bullet\right)^2 c \text{ and } (ba)^\bullet = b\left((ac)^\bullet\right)^2 a,$$

where $\bullet \in \{sD, gsD, H, gH\}$. As a special case, we recover some recent results in [1, 5, 11, 27].

2 Results

Wang established Cline's formula for s-Drazin inverse (see [27, Theorem 3.1]). In the following, we generalize it to the case when (2) holds.

Theorem 1 *Suppose that* $a, b, c \in \mathcal{R}$ *satisfy* $a(ba)^2 = abaca = acaba = (ac)^2a$. *Then*

$$ac \in \mathcal{R}^{sD} \text{ if and only if } ba \in \mathcal{R}^{sD}.$$

In this case, we have

$$(ac)^{sD} = a\left((ba)^{sD}\right)^2 c,\ (ba)^{sD} = b\left((ac)^{sD}\right)^2 a \text{ and } |\mathrm{i}_{sD}(ac) - \mathrm{i}_{sD}(ba)| \le 2.$$

Proof Let $ba \in \mathcal{R}^{sD}$ and let $s \in \mathcal{R}$ be s-Drazin inverse of ba. Then, for some non-negative integer k, we have:

$$s(ba) = (ba)s, \quad s(ba)s = s \quad \text{and} \quad (ba - sba)^k = 0.$$

Take $t = as^2c$. By [4, Theorem 2.7], t is a Drazin inverse of ac, so we have

$$t(ac) = (ac)t \text{ and } t(ac)t = t.$$

Thus, to prove that $ac \in \mathcal{R}^{sD}$, we need to show that $(ac - act)^n = 0$, for some non-negative integer n. Since $(1-s)ba = ba(1-s)$, we have that $(1-s)^k(ba)^k = 0$. By the hypothesis of the theorem, we get

$$ac - act = ac - acas^2c = ac - acabas^3c = ac - a(ba)^2s^3c = ac - asc = a(1-s)c.$$

Let us prove that, for every non-negative integer m, the following equality holds

$$[a(1-s)c]^{m+2} = a(1-s)ca(ba - sba)^m(1-s)c. \tag{3}$$

Using the induction by m and by the hypothesis of the theorem, we have the following. For $m = 1$, we get

$$\begin{aligned}
[a(1-s)c]^3 &= a(1-s)(caca - casca)(1-s)c \\
&= (a - as)(caca - casca)(1-s)c \\
&= (a - as^2ba)(caca - cas^3babaca)(1-s)c \\
&= (1 - as^2b)(acaca - acas^3bababa)(1-s)c \\
&= (1 - as^2b)(acaba - acasba)(1-s)c \\
&= (1 - as^2b)aca(ba - sba)(1-s)c \\
&= a(1-s)ca(ba - sba)(1-s)c.
\end{aligned}$$

Therefore, the statement (3) holds for $m = 1$. Now, let us suppose that the statement (3) holds for some $m \ge 2$ and let us prove that, in that case, (3) holds for $m + 1$. We have

$$
\begin{aligned}
[a(1-s)c]^{m+3} &= a(1-s)c[a(1-s)c]^{m+2} = a(1-s)ca(1-s)ca(ba-sba)^m(1-s)\\
&= (a-as^2ba)(caca-casca)(ba-sba)^m(1-s)c\\
&= (1-as^2b)(acaca-acasca)(ba-sba)^m(1-s)c\\
&= (1-as^2b)(acaba-acas^3babaca)(ba-sba)^m(1-s)c\\
&= (1-as^2b)aca(ba-sba)(ba-sba)^m(1-s)c\\
&= a(1-s)ca(ba-sba)(ba-sba)^m(1-s)c\\
&= a(1-s)ca(ba-sba)^{m+1}(1-s)c.
\end{aligned}
$$

Hence, the statement (3) holds for every m. Since $[a(1-s)c]^{k+2} = a(1-s)ca(ba-sba)^k(1-s)c$ and $(ba-sba)^k = 0$, we have that $(ac-act)^{k+2} = 0$.

Conversely, suppose that $ac \in \mathcal{R}^{sD}$ and let u be the strong Drazin inverse of ac and m its index. Set $v = bu^2a$. By [4, Theorem 2.7] again, we know that v is a Drazin inverse of ba. Therefore, it remains to show that $(ba-bav)^n = 0$, for some non-negative integer n. By the hypothesis of the theorem, we get

$$
ba - bav = ba - babu^2a = ba - babacacu^4a = ba - bacacacu^4a = b(1-u)a.
$$

Now, let us prove that

$$
[b(1-u)a]^{m+2} = b(1-u)(ac-acu)^m ab(1-u)a \tag{4}
$$

holds for every m. Using the induction by m and by the hypothesis of the theorem, we have the following. For $m = 1$, we get

$$
\begin{aligned}
[b(1-u)a]^3 &= b(1-u)(abab-abuab)(a-ua)\\
&= b(1-u)(abab-abuab)\left(a-acu^2a\right)\\
&= b(1-u)(ababa-abuaba)\left(1-cu^2a\right)\\
&= b(1-u)\left(ababa-abacacu^3aba\right)\left(1-cu^2a\right)\\
&= b(1-u)\left(acaba-acacacu^3aba\right)\left(1-cu^2a\right)\\
&= b(1-u)(ac-acu)aba\left(1-cu^2a\right)\\
&= b(1-u)(ac-acu)ab(1-u)a.
\end{aligned}
$$

Therefore, the statement (4) holds for $m = 1$. Now, let us suppose that the statement (4) holds for some $m \geq 2$ and let us prove that, in that case, (4) holds for $m+1$. We have

$$\begin{aligned}
[b(1-u)a]^{m+3} &= [b(1-u)a]^{m+2} b(1-u)a = b(1-u)(ac-acu)^m ab(1-u)ab(1-u)a \\
&= b(1-u)(ac-acu)^m (ab-abu)\left(aba-abacu^2a\right) \\
&= b(1-u)(ac-acu)^m (ababa-abuaba)\left(1-cu^2a\right) \\
&= b(1-u)(ac-acu)^m \left(ababa-abacacu^3aba\right)\left(1-cu^2a\right) \\
&= b(1-u)(ac-acu)^m \left(acaba-acacacu^3aba\right)\left(1-cu^2a\right) \\
&= b(1-u)(ac-acu)^m (ac-acu)aba\left(1-cu^2a\right) \\
&= b(1-u)(ac-acu)^{m+1} ab(1-u)a.
\end{aligned}$$

Hence, the statement (4) holds for every m. Since $(ba-bav)^{m+2} = [b(1-u)a]^{m+2} = b(1-u)(ac-acu)^m ab(1-u)a$ and $(ac-acu)^m = 0$, the proof is complete. □

In the sequel, we use $\mathcal{B}(X, Y)$ to denote the set of all bounded linear operators from Banach space X to Banach space Y. Using the technique of block matrices, we obtain the operator case of Theorem 1.

Corollary 1 *Suppose that $A, B, C \in \mathcal{B}(X, Y)$ satisfy $A(BA)^2 = ABACA = ACABA = (AC)^2A$. Then AC is s-Drazin invertible if and only if BA is s-Drazin invertible.*

In this case, we have

$$(AC)^{sD} = A\left((BA)^{sD}\right)^2 C \text{ and } (BA)^{sD} = B\left((AC)^{sD}\right)^2 A.$$

Moreover, we have the following corollary.

Corollary 2 *Suppose that $a, b, c \in \mathcal{R}$ satisfy $a(ba)^2 = abaca = acaba = (ac)^2a$ and let $n \geq 2$ be an integer. Then*

$$(ac)^n \in \mathcal{R}^{sD} \text{ if and only if } (ba)^n \in \mathcal{R}^{sD}.$$

In this case,

$$\left((ba)^n\right)^{sD} = b\left[\left((ac)^n\right)^{sD}\right]^2 a(ba)^{n-1} \text{ and } \left((ac)^n\right)^{sD} = a\left[\left((ba)^n\right)^{sD}\right]^2 b(ac)^{n-1}.$$

Proof Let $b' = b(ab)^{n-1}$ and $c' = c(ac)^{n-1}$. Then $a(b'a)^2 = ab'ac'a = ac'ab'a = (ac')^2a$. From Theorem 1, it follows that $(ac)^n = ac' \in \mathcal{R}^{sD}$ if and only if $(ba)^n = b'a \in \mathcal{R}^{sD}$, and we have

$$\left((ba)^n\right)^{sD} = \left(b'a\right)^{sD} = b'\left(\left(ac'\right)^{sD}\right)^2 a = b(ab)^{n-1}\left[\left((ac)^n\right)^{sD}\right]^2 a$$

and

$$\left((ac)^n\right)^{sD} = \left(ac'\right)^{sD} = a\left(\left(b'a\right)^{sD}\right)^2 c' = a\left[\left((bd)^n\right)^{sD}\right]^2 c(ac)^{n-1},$$

as required. □

Jacobson's lemma states that $1-ab \in \mathcal{R}^{-1}$ if and only if $1-ba \in \mathcal{R}^{-1}$. In recent years, numerous mathematicians paid much attention to Jacobson's lemma for (generalized) Drazin inverse (see [3, 8, 12, 13, 18, 24, 35]). The next goal of this paper is finding proper counterparts of Jacobson's lemma for s-Drazin inverse under the condition (2).

In [27, Lemma 3.3], Wang proved that if $a \in \mathcal{R}^{sD}$ with $\mathrm{i}_{sD}(a) = k$, then $1 - a \in \mathcal{R}^{sD}$ with $\mathrm{i}_{sD}(1-a) = k$ and $(1-a)^{sD} = \sum_{i=0}^{k-1} a^i \left(1 - aa^{sD}\right)$. This result establishes a bridge from Cline's formula for s-Drazin inverse to Jacobson's Lemma for s-Drazin inverse.

Corollary 3 *Suppose that $a, b, c \in \mathcal{R}$ satisfy $a(ba)^2 = abaca = acaba = (ac)^2 a$.*

(1) If $1 - ac \in \mathcal{R}^{sD}$ with $\mathrm{i}_{sD}(1 - ac) = k$, then $1 - ba \in \mathcal{R}^{sD}$ and

$$(1-ba)^{sD} = \sum_{i=0}^{k+1}(ba)^i - b\left[\sum_{i=0}^{k+1}\sum_{j=0}^{k-1}(ab)^i(1-ac)^j\right]\left[1-(1-ac)(1-ac)^{sD}\right]a.$$

(2) If $1 - ba \in \mathcal{R}^{sD}$ with $\mathrm{i}_{sD}(1 - ba) = k$, then $1 - ac \in \mathcal{R}^{sD}$ and

$$(1-ac)^{sD} = \sum_{i=0}^{k+1}(ac)^i - a\left[\sum_{i=0}^{k+1}(ca)^i\right]\left[\sum_{j=0}^{k-1}(1-ba)^j\right]^2\left[1-(1-ba)(1-ba)^{sD}\right]c.$$

Proof (1) Let $1-ac \in \mathcal{R}^{sD}$. By [27, Lemma 3.3] it follows that $ac \in \mathcal{R}^{sD}$, $i_{sD}(ac) = k$ and

$$(ac)^{sD} = \left[\sum_{j=0}^{k-1}(1-ac)^j\right]\left[1-(1-ac)(1-ac)^{sD}\right].$$

Applying Theorem 1, we can get $ba \in \mathcal{R}^{sD}$ and $(ba)^{sD} = b\left((ac)^{sD}\right)^2 a$ and $\mathrm{i}_{sD}(ba) \leq k+2$. According to [27, Lemma 3.3] again, it follows that $1-ba \in \mathcal{R}^{sD}$ and

$$\begin{aligned}(1-ba)^{sD} &= \left[\sum_{i=0}^{k+1}(ba)^i\right]\left[1-(ba)(ba)^{sD}\right] = \left[\sum_{i=0}^{k+1}(ba)^i\right]\left[1-b(ac)^{sD}a\right]\\ &= \left[\sum_{i=0}^{k+1}(ba)^i\right]\left[1-b\left[\sum_{j=0}^{k-1}(1-ac)^j\right]\left[1-(1-ac)(1-ac)^{sD}\right]a\right]\\ &= \sum_{i=0}^{k+1}(ba)^i - \left[\sum_{i=0}^{k+1}(ba)^i b\sum_{j=0}^{k-1}(1-ac)^j\right]\left[1-(1-ac)(1-ac)^{sD}\right]a\\ &= \sum_{i=0}^{k+1}(ba)^i - b\left[\sum_{i=0}^{k+1}\sum_{j=0}^{k-1}(ab)^i(1-ac)^j\right]\left[1-(1-ac)(1-ac)^{sD}\right]a.\end{aligned}$$

(2) The proof is similar to that of (1).

□

Corollary 4 *Suppose that $a, b, c \in \mathcal{R}$ satisfy $a(ba)^2 = abaca = acaba = (ac)^2a$, let $n \geq 2$ be an integer. Then*

$$(1 - ac)^n \in \mathcal{R}^{sD} \text{ if and only if } (1 - ba)^n \in \mathcal{R}^{sD}.$$

Proof Let

$$c_n = \sum_{k=1}^{n} (-1)^{k-1}(ca)^{k-1}c,$$
$$b_n = \sum_{k=1}^{n} (-1)^{k-1}(ba)^{k-1}b.$$

Then we have $a(b_na)^2 = ab_nac_na = ac_nab_na = (ac_n)^2a$. Note that $1 - ac_n = (1-ac)^n$ and $1-b_na = (1-ba)^n$. From Corollary 3, we can get that $(1-ba)^n \in \mathcal{R}^{sD}$.
The proof of the converse is similar. □

Cline's formula for gs-Drazin inverse was established in [11, Theorem 4.14] under the assumption (1). Next, we generalize it to the case when (2) holds. The following auxiliary lemma is needed.

Lemma 1 *([4, Lemma 2.1]) Suppose that $a, b, c \in \mathcal{R}$ satisfy $a(ba)^2 = abaca = acaba = (ac)^2a$. Then*

$$ac \in \mathcal{R}^{qnil} \text{ if and only if } ba \in \mathcal{R}^{qnil}.$$

Theorem 2 *Suppose that $a, b, c \in \mathcal{R}$ satisfy $a(ba)^2 = abaca = acaba = (ac)^2a$. Then $ac \in \mathcal{R}^{gsD}$ if and only if $ba \in \mathcal{R}^{gsD}$. In this case, we have*

$$(ac)^{gsD} = a\left((ba)^{gsD}\right)^2 c \text{ and } (ba)^{gsD} = b\left((ac)^{gsD}\right)^2 a.$$

Proof Suppose that $ba \in \mathcal{R}^{gsD}$ and let s be the generalized strong Drazin inverse of ba. Take $t = as^2c$. From [4, Theorem 2.2] it follows that, t is a generalized Drazin inverse of ac. Hence, in order to show that $ac \in \mathcal{R}^{gsD}$, it is sufficient to prove $(ac - act) \in \mathcal{R}^{\text{qnil}}$. Let $c' = c(1 - t)$ and $b' = (1 - s)b$. Then $b'a \in \mathcal{R}^{\text{qnil}}$. By the hypothesis of the theorem, we get

$$\begin{aligned}(ac')^2a &= ac\,(1-t)\,ac\,(1-t)\,a\\ &= ac\left(1-as^2c\right)ac\left(1-as^2c\right)a\\ &= \left(acac-acas^2cac\right)\left(1-as^2c\right)a\\ &= \left(acaca-acabas^4bacaca\right)\left(1-s^2ca\right)\\ &= \left(ababa-ababas^4bababa\right)\left(1-s^4babaca\right)\\ &= \left(ababa-ababas^4bababa\right)\left(1-s^4bababa\right)\\ &= a\,(1-s)\,ba\,(1-s)\,ba = a(b'a)^2.\end{aligned}$$

Moreover, a direct calculation shows that $a(b'a)^2 = ab'ac'a = ac'ab'a = (ac')^2a$ and $ac-act = ac'$. Then by Lemma 1, we deduce that $ac-act \in \mathcal{R}^{\mathrm{qnil}}$, which yields that $ac \in \mathcal{R}^{gsD}$. By similar arguments as above, one can show that if $ac \in \mathcal{R}^{gsD}$, then $ba \in \mathcal{R}^{gsD}$ and $(ba)^{gsD} = b\left((ac)^{gsD}\right)^2 a$. □

From the proof of [27, Lemma 3.3], we can also deduce that $a \in \mathcal{R}^{gsD}$ if and only if $1-a \in \mathcal{R}^{gsD}$. Hence, as an immediate consequence of Theorem 2, we arrive at the following result.

Corollary 5 *Suppose that $a, b, c \in R$ such that $a(ba)^2 = abaca = acaba = (ac)^2a$. Then*

$$1-ac \in \mathcal{R}^{gsD} \text{ if and only if } 1-ba \in \mathcal{R}^{gsD}.$$

In the following, as an extension of [5, Theorem 4.1], we establish Cline's formula for Hirano inverse under the condition (2).

Theorem 3 *Suppose that $a, b, c \in \mathcal{R}$ satisfy $a(ba)^2 = abaca = acaba = (ac)^2a$. Then $ac \in \mathcal{R}^H$ if and only if $ba \in \mathcal{R}^H$. In this case, we have*

$$(ac)^H = a\left((ba)^H\right)^2 c,\ (ba)^H = b\left((ac)^H\right)^2 a \text{ and } |\mathrm{i}_H(ac) - \mathrm{i}_H(ba)| \le 2.$$

Proof Suppose that $ba \in \mathcal{R}^H$, i.e. there exists $s \in \mathcal{R}$ and non-negative integer k, such that

$$s(ba) = (ba)s,\ s(ba)s = s \text{ and } \left((ba)^2 - sba\right)^k = 0.$$

By [5, Theorem 2.1], we have that $ba \in \mathcal{R}^D$ and $(ba)^D = s$. Take $t = as^2c$. By [4, Theorem 2.7], t is a Drazin inverse of ac, so we have $t(ac) = (ac)t$ and $t(ac)t = t$. Note that we have $\left((ba^2) - sba\right)^k = 0$ holds and we have to prove that $\left((ac)^2 - act\right)^n = 0$ holds, for some n. Since we have

$$\begin{aligned}\left((ac)^2 - act\right)^2 &= (ac-t)ac\left((ac)^2 - acas^2c\right) = (ac-t)\left(acacac - acacas^2c\right)\\ &= (ac-t)\left(acaba - acabas^2\right)c\\ &= (ac-t)aca\left(ba - bas^2\right)c = (acaca - as^2caca)\,(ba-s)\,c\\ &= (abaca - as^2baca)\,(ba-s)\,c = a(ba-s)ca\,(ba-s)\,c\\ &= [a(ba-s)c]^2,\end{aligned}$$

using the induction by k, we get

$$\left((ac)^2 - act\right)^{k+1} = [a(ba-s)c]^{k+1}, \tag{5}$$

for every integer $k \geq 1$. Next, we will prove that

$$[a(ba-s)c]^{k+2} = a\left[(ba)^2 - bas\right]^{k+1}(ba-s)c, \tag{6}$$

using the induction by k. For $k=1$, we have

$$\begin{aligned}[a(ba-s)c]^3 &= a(ba-s)ca(ba-s)ca(ba-s)c\\ &= a(ba-s)c(abaca - asca)(ba-s)c\\ &= a(ba-s)c(ababa - as^3babaca)(ba-s)c\\ &= a(ba-s)ca(baba - sba)(ba-s)c\\ &= a(baca - sca)(baba - sba)(ba-s)c\\ &= (abaca - as^3babaca)(baba - sba)(ba-s)c\\ &= (ababa - as^3bababa)(baba - sba)(ba-s)c\\ &= a(baba - sba)^2(ba-s)c.\end{aligned}$$

Therefore, (6) holds for $k=1$. Suppose that (6) holds for some $k > 1$ and let us prove that, in that case, (6) holds for $k+1$. We have

$$\begin{aligned}[a(ba-s)c]^{k+3} &= [a(ba-s)c]^{k+2}a(ba-s)c\\ &= a\left[(ba)^2 - bas\right]^{k+1}(ba-s)ca(bac - sc)\\ &= a\left[(ba)^2 - bas\right]^{k+1}(baca - sca)(bac - bas^2c)\\ &= a\left[(ba)^2 - bas\right]^{k+1}(bacaba - s^2bacaba)(c - s^2c)\\ &= a\left[(ba)^2 - bas\right]^{k+1}(bababa - s^2bababa)(c - s^2c)\\ &= a\left[(ba)^2 - bas\right]^{k+1}(baba - sba)ba(c - s^2c)\\ &= a\left[(ba)^2 - bas\right]^{k+2}(ba-s)c.\end{aligned}$$

Hence, we proved that (6) holds for every $k \geq 1$.Therefore, by (6) and (5), we have $\left((ac)^2 - act\right)^{k+2} = [a(ba-s)c]^{k+2} = a\left[(ba)^2 - bas\right]^{k+1}(ba-s)c = 0$.

Conversely, suppose that $ac \in \mathcal{R}^H$ and let $u = (ac)^H$, $i_H(ac) = m$, where m is a non-negative integer. Then $\left((ac)^2 - acu\right)^m = 0$. Moreover, by [5, Theorem 2.1], we have $ac \in \mathcal{R}^D$ and $(ac)^D = s$. Set $v = bu^2a$. By [4, Theorem 2.7] again, we know that v is a Drazin inverse of ba. Therefore, it remains to prove that $\left((ba)^2 - bav\right)^n = 0$, for some non-negative integer n. Since we have

$$\begin{aligned}
\left((ba)^2 - bav\right)^2 &= \left((ba)^3 - bavba\right)(ba-v) = \left(bacaba - babu^2aba\right)(ba-v) \\
&= b\left(ac - abu^2\right)aba(ba-v) = b\left(ac - abacacu^4\right)aba(ba-v) \\
&= b\left(ac - acacacu^4\right)aba(ba-v) = b(ac-u)aba(ba-v) \\
&= b(ac-u)(ababa - abav) = b(ac-u)\left(ababa - ababu^2a\right) \\
&= b(ac-u)\left(abaca - ababacu^3a\right) = b(ac-u)\left(abaca - abacacu^3a\right) \\
&= b(ac-u)(abaca - abua) = [b(ac-u)a]^2,
\end{aligned}$$

using the induction by k, we get

$$\left((ba)^2 - bav\right)^{k+1} = [b(ac-u)a]^{k+1}, \tag{7}$$

for every integer $k \geq 1$. Next, we will prove that

$$[b(ac-u)a]^{k+2} = b(ac-u)\left[(ac)^2 - acu\right]^{k+1}a, \tag{8}$$

using the induction by k. For $k = 1$, we have

$$\begin{aligned}
[b(ac-u)a]^3 &= b(ac-u)ab(ac-u)ab(ac-u)a \\
&= b(ac-u)(abac-abu)(abaca-abua) \\
&= b(ac-u)(abac-abu)\left(abaca - abacu^2a\right) \\
&= b(ac-u)(abacaba - abuaba)\left(ca - cu^2a\right) \\
&= b(ac-u)\left(acacaba - abacacu^3aba\right)\left(ca - cu^2a\right) \\
&= b(ac-u)\left(acacaba - acacacu^3aba\right)\left(ca - cu^2a\right) \\
&= b(ac-u)(acacaba - acuaba)\left(ca - cu^2a\right) \\
&= b(ac-u)(acac - acu)\left(abaca - abacu^2a\right) \\
&= b(ac-u)(acac - acu)\left(acaca - abacacu^3a\right) \\
&= b(ac-u)(acac - acu)\left(acaca - acacacu^3a\right)
\end{aligned}$$

$$= b(ac-u)(acac-acu)^2 a.$$

Therefore, (8) holds for $k=1$. Suppose that (8) holds for some $k>1$ and let us prove that, in that case, (8) holds for $k+1$. We have

$$\begin{aligned}
[b(ac-u)a]^{k+3} &= [b(ac-u)a]^{k+2} b(ac-u)a \\
&= b(ac-u)\left[(ac)^2 - acu\right]^{k+1} ab(ac-u)a \\
&= b(ac-u)\left[(ac)^2 - acu\right]^{k+1} \left(abaca - abacacu^3 a\right) \\
&= b(ac-u)\left[(ac)^2 - acu\right]^{k+1} \left(acaca - acacacu^3 a\right) \\
&= b(ac-u)\left[(ac)^2 - acu\right]^{k+1} (acac-acu)a.
\end{aligned}$$

Hence, we proved that (8) holds for every $k \geq 1$. Therefore, by (8) and (7), we have $\left((ba)^2 - bav\right)^{m+2} = [b(ac-u)a]^{m+2} = b(ac-u)\left[(ac)^2 - acu\right]^{m+1} a = 0$. The proof is complete. □

At last, Cline's formula for generalized Hirano inverse is established under the assumption (2), extending [1, Theorem 4.1].

Theorem 4 *Suppose that $a, b, c \in \mathcal{R}$ satisfy $a(ba)^2 = abaca = acaba = (ac)^2 a$. Then*

$$ac \in \mathcal{R}^{gH} \text{ if and only if } ba \in \mathcal{R}^{gH}.$$

In this case, we have

$$(ac)^{gH} = a\left((ba)^{gH}\right)^2 c \text{ and } (ba)^{gH} = b\left((ac)^{gH}\right)^2 a.$$

Proof Suppose that $ba \in \mathcal{R}^{gH}$ and let s be the generalized Hirano inverse of ba. By [1, Corollary 2.3], it follows that $s = (ba)^{gD}$. Take $t = as^2c$. From [4, Theorem 2.2] it follows that, t is a generalized Drazin inverse of ac. Hence, in order to show that $ac \in \mathcal{R}^{gH}$, it is sufficient to prove $(ac)^2 - act \in \mathcal{R}^{\text{qnil}}$. Let $b' = (ba-s)b$ and $c' = c(ac-t)$. Then $b'a \in \mathcal{R}^{\text{qnil}}$. By the hypothesis of the theorem, we get

$$\begin{aligned}
(ac')^2 a &= ac\,(ac-t)\,ac\,(ac-t)\,a \\
&= ac\left(ac-as^2c\right)ac\left(ac-as^2c\right)a \\
&= \left(acacac-acas^2cac\right)\left(ac-as^2c\right)a \\
&= \left(acacaca-acabas^4bacaca\right)\left(ca-s^2ca\right) \\
&= \left(abababa-ababas^4bababa\right)\left(ca-s^4babaca\right) \\
&= \left(abababa-ababas^4bababa\right)\left(ca-s^4bababa\right) \\
&= \left(abab-ababas^4bab\right)\left(abaca-abas^4bababa\right) \\
&= \left(abab-ababas^4bab\right)\left(ababa-abas^4bababa\right) \\
&= \left(abababa-ababas^4bababa\right)\left(ba-s^4bababa\right) \\
&= a\,(ba-s)\,ba\,(ba-s)\,ba = a(b'a)^2.
\end{aligned}$$

Moreover, a direct calculation shows that $a(b'a)^2 = ab'ac'a = ac'ab'a = (ac')^2a$ and $(ac)^2 - act = ac'$. Then by Lemma 1, we deduce that $(ac)^2 - act \in \mathcal{R}^{\text{qnil}}$, which yields that $ac \in \mathcal{R}^{gH}$.

By similar arguments as above, one can show that if $ac \in \mathcal{R}^{gH}$, then $ba \in \mathcal{R}^{gH}$ and $(ba)^{gH} = b\left((ac)^{gH}\right)^2 a$. $\square$

We remark that analogous results of Corollary 1 and 2 also hold for generalized strong Drazin inverse, Hirano inverse and generalized Hirano inverse. We conclude this note by a numerical example (example taken from [4]) to demonstrate Theorems 1, 2, 3 and 4.

Example 1 Let $\mathcal{R} = M_6(\mathbb{Z}_2)$, $x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in M_3(\mathbb{Z}_2)$. Then $x^2 \neq 0$ and $x^3 = 0$. Choose

$$a = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $a(ba)^2 = abaca = acaba = (ac)^2a$, but $aba \neq aca$. In this case, $ac \in \mathcal{R}^{\bullet}$, where $\bullet \in \{sD, gsD, H, gH\}$.

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Declarations

Conflict of interest The author declare that they have no conflict of interest.

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