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Global and blow-up results for a quasilinear parabolic equation with variable sources and memory terms

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Abstract
The paper presents a general model of quasi-linear parabolic equations with variable exponents for the source and dissipative term types

$$\mathrm{L}\,(t)\,|u_t|^{m(x)-2}\,u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s = |u|^{p(x)-2}\,u.$$

When $p(x) \geq m(x) \geq 2$, the matrix $\mathrm{L}(t)$ is both positive definite and bounded, while the function g is continuously differentiable and decays over time. The paper shows that the blow-up result occurs at two different finite times and provides an upper bound for the blow-up time. Finally, it establishes that the energy function decays globally for solutions, with both positive and negative initial energy.

Keywords Parabolic · Memory · General decay · Viscoelastic · Blow-up · Critical exponents · Variable nonlinearity

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1 Introduction

Natural heat conduction in materials with memory is one of the most active areas of heat transfer research today. The system with viscoelastic and source-term effects has

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seen significant growth in the last few decades.

$$\mathrm{L}\,(t)\,|u_t|^{m-2}\,u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s = |u|^{p-2}\,u \quad (x,t) \in \Omega \times [0,T)\,, \tag{1.1}$$

has attracted many researchers and has been studied extensively on solutions, existence, nonexistence, stability, and blow-up, where $p \geq 2$ and Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$, with a smooth boundary $\partial\Omega$, and $\mathrm{L} \in C\,(\mathbb{R}^n)$ is a bounded square matrix satisfying

$$c_0\,|v|^2 \leq (\mathrm{L}\,(t)\,v, v) \leq c_1\,|v|^2 \;\text{ for } t \in \mathbb{R}^+,\; v \in \mathbb{R}^n, \tag{1.2}$$

$(.,.)$ is the inner product in $\mathbb{R}^n$ and $c_1 \geq c_0 > 0$. In the mathematical explanation of how heat spreads through materials with memory [1], researchers have found that the global existence and blow-up of the equation depend roughly on m, p, the relaxation g, and the initial datum, replacing the classic Fourier law with the following form (cf. [2]).

$$q = -d\nabla u - \int_{-\infty}^t g(x, t-s)\nabla u(x,s)\mathrm{d}s, \tag{1.3}$$

where q is proportional to the temperature differences per unit length, u is the temperature, d is the diffusion coefficient, and the integral term represents the memory effect in the material, here (1.6) means that q does not depend linearly on ∇u. If we then substitute Fourier's law (1.3) into the law of heat law, we can conclude that

$$u_t + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s - d\Delta u = 0 \text{ in } \Omega \times (0,T).$$

Researchers have extensively studied damned viscoelastic operators, collecting many facts about the existence and regularity of both the weak and classical solutions [3]. Viscoelasticity often leads to problems of this type from a physical perspective. In 1970, Dafermos [4] was the first to consider the issue of general decay, which has since been the subject of much research attention over the last two decades, leading to various results on the solutions' existence and long-term behavior [5–9, 21]. We are interested in the finite-time blow-up property, so we use some inequality methods together with energy technique to study some properties of local solutions of damped viscoelastic type second-order nonlinear parabolic equations that involve variable source nonlinearities concerning the solution and its solution spatial derivatives. We determine properties of local weak solutions such as the finite propagation speed of the initial perturbations, the global localization and the blow-up time phenomenon. The conditions that provide these effects are formulated in terms of local assumptions on the data and the non-linear nature of the problem

$$\begin{cases} \mathrm{L}\,(t)\,|u_t|^{m(x)-2}\,u_t - \Delta u + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s = |u|^{p(x)-2}\,u & (x,t) \in \Omega \times [0,T)\,, \\ u = 0 \quad (x,t) \in \partial\Omega \times [0,T)\,, \\ u(x,0) = u_0(x) \quad x \in \Omega, \end{cases} \tag{1.4}$$

where Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $T > 0$, Δ represents the Laplacian with respect to the spatial variables and the initial value functions. We prove the blow-up in a finite time of weak solutions and get a new blow-up criterion. In the meantime, the lifespan and upper and lower bound for the blow-up time are also derived. The exponents $m(.)$ and $p(.)$ are given measurable functions on $\overline{\Omega}$ such that:

$$\begin{aligned} 2 < p_1 = ess \inf_{x\in\Omega} p\,(x) \leq p\,(x) \leq p_2 = ess \sup_{x\in\Omega} p\,(x) < \infty, \\ 2 < m_1 = ess \inf_{x\in\Omega} m\,(x) \leq m\,(x) \leq m_2 = ess \sup_{x\in\Omega} m\,(x) < \infty. \end{aligned} \tag{1.5}$$

We also assume that $p(.)$ and $m(.)$ satisfies the log-Holder continuity condition

$$\begin{aligned} \max\left(|p\,(x) - p\,(y)|\,, |m\,(x) - m\,(y)|\right) \leq \frac{M}{|\log|x-y||}, \\ \text{for a.e. } x, y \text{ in } \Omega, \text{ with } 0 < |x-y| < \delta, \end{aligned} \tag{1.6}$$

$M > 0$, $1 < \delta < 1$, and $M(r)$ satisfies

$$\limsup_{r\to 0^+} M(r)\ln\left(\frac{1}{r}\right) = c < \infty.$$

$$u_0 \in H_0^1\,(\Omega) \cap W_0^{1,p(.)}(\Omega). \tag{1.7}$$

The significance of the viscoelastic impacts of materials has been realized because of the rapid results in the rubber and plastics industries. Many passages in the examinations of constitutive concerns, failure theories, and life projection of viscoelastic materials and structures were notified and studied in the last two decades [10]. Equations with variable exponents of nonlinearity have recently been employed to model various physical phenomena, such as the flow of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through porous media, and image processing. You can refer to the sources listed in [11–17] for further information on these topics. Analysis of the long-term behavior of the variable-exponent viscoelastic wave equation has been the subject of active studies and mathematical efforts. In the present work we will proceed in the direction of the previous quasilinear investigations by considering the source and damping terms (1.4) appearing as variable exponents that we discuss in a bounded domain of $\mathbb{R}^n$, the global existence and the explosion results when the initial data exist at different energy levels $\mathrm{E}(u_0) < 0$ and $\mathrm{E}(u_0) > 0$.

2 Preliminaries

Let $p : \Omega \to [1, \infty]$ be a measurable function. $L^{p(.)}(\Omega)$ denotes the set of the real measurable functions u on Ω such that

$$\int_{\Omega} |\lambda u(x)|^{p(x)} \mathrm{d}x < \infty \text{ for some } \lambda > 0.$$

The variable-exponent space $L^{p(.)}(\Omega)$ equipped with the Luxemburg-type norm

$$\|u\|_{p(.)} = \inf \left\{ \lambda > 0, \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \leq 1 \right\},$$

is a Banach space. Throughout the paper, we use $\|.\|_q$ to indicate the L^q-norm for $1 \leq q \leq +\infty$. $H_0^1(\Omega)$ is the closure of $C_0^{\infty}(\Omega)$ to the following norm:

$$\|u\|_{H_0^1(\Omega)} = \left(\|u\|_2^2 + \|\nabla u\|_2^2 \right)^{\frac{1}{2}}.$$

It is known that for the elements of $H_0^1(\Omega)$ the Poincaré inequality holds,

$$\|u\|_2 \leq C^* \|\nabla u\|_2 \text{ for all } u \in H_0^1(\Omega).$$

Throughout the paper, we use $\|.\|_q$ to indicate the L^q-norm for $1 \leq q \leq +\infty$.

For the relaxation function g and the number $m(.)$ and $p(.)$, we assume that:

(H1) g is a positive function that represents the kernel of the memory term, and satisfies the following:

$$g(0) > 0, \; 1 - \int_0^{\infty} g(s)\mathrm{d}s = \kappa > 0, \tag{2.1}$$

and

$$\int_0^{\infty} g(s)\,\mathrm{d}s < \frac{(q-2)q}{(q-1)^2}, \tag{2.2}$$

where q is any fixed number such that $2 < q < p_1$.

(H2) There exists a nonincreasing function

$$\zeta : \mathbb{R}^+ \to \mathbb{R}^+,$$

such that

$$g'(t) \leq -\zeta(t)\, g(t), \; t \geq 0.$$

(H3) The exponents $m(.)$ and $p(.)$ are given measurable functions on Ω satisfying

$$2 < p_{1,2} < \infty,\ n \leq 2,$$
$$2 < p_1 \leq p(x) \leq p_2 < \frac{2n}{n-2},\ n \geq 3,$$

and

$$2 < m_{1,2} < \infty,\ n \leq 2,$$
$$2 < m_1 \leq m(x) \leq m_2 < \frac{2n-2}{n-2},\ n \geq 3.$$

The following lemma is used in the proof of the main results.

Lemma 1 (Sobolev-Poincaré inequality) *If* $p\,(.)$ *satisfy* (H1) *For all* $u \in H_0^1(\Omega)$*, then the following embedding*

$$H_0^1(\Omega) \hookrightarrow L^{p_2}(\Omega) \hookrightarrow L^{p(.)}(\Omega) \hookrightarrow L^{p_1}(\Omega) \hookrightarrow L^2(\Omega),$$

are continuous, and we get

$$\|u\|_{p(.)} \leq B\|\nabla u\|_2,\ \|u\|_{\frac{2n}{n-2}} \leq \bar{B}\|\nabla u\|_2\ \ (n \geq 3),\ \|u\|_{p_2} \leq \hat{B}\|\nabla u\|_2, \tag{2.3}$$

where B, $\bar{B}$, $\hat{B}$ *are the optimal constant of the Sobolev embedding and* $\|.\|_{p(.)}$ *denotes the norm of* $L^{p(.)}(\Omega)$*, with the following propriety*

$$\min\left(\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right) \leq \varrho(u) = \int_\Omega |u\,(x)|^{p(x)}\,\mathrm{d}x \leq \max\left(\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right),$$

for any $u \in L^{p(.)}(\Omega)$.

We denote $\|.\|_q$ and $\|.\|_{H^1(\Omega)}$ to the usual $L^q(\Omega)$ norm and $H^1(\Omega)$ norm, respectively.

To examine our main results, we define

$$I(t) = \left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) - \int_\Omega |u\,(x,t)|^{p(x)}\,\mathrm{d}x, \tag{2.4}$$

and the energy functional $E : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ by

$$\mathrm{E(t)} = \frac{1}{2}\,(g \diamond \nabla u) + \frac{1}{2}\left(1 - \int_0^t g\,(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 - \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x, \tag{2.5}$$

then, testing (1.4) by u_{t}, we have $E(t)$ is nonincreasing, i.e.,

$$\frac{d}{dt}\mathrm{E(t)} = -\frac{1}{2}g\,(t)\int_\Omega |\nabla u\,(t)|^2\,\mathrm{d}x + \frac{1}{2}\left(g' \diamond \nabla u\right) - \int_\Omega \mathrm{L}\,(t)\,|u_t|^{m(x)}\,\mathrm{d}x \leq 0, \tag{2.6}$$

and

$$\mathrm{E}(t) = \int_0^t \left(\frac{1}{2} g\,(s) \int_\Omega |\nabla u\,(s)|^2 \,\mathrm{d}x + \frac{1}{2} \left(g' \diamond \nabla u \right) + \mathrm{L}\,(s) \int_\Omega |u_t|^{m(x)} \,\mathrm{d}x \right) \mathrm{d}s$$
$$\leq \mathrm{E}(0), \tag{2.7}$$

where

$$(\mathrm{g} \diamond \nabla u)(t) = \int_0^t g(t-s) \|\nabla u(t) - \nabla u(s)\|_2^2 \mathrm{d}s.$$

3 Existence of weak solutions

In this section, we aim to prove the local existence of solutions for system equations (1.4). To achieve this, we will examine a related initial-boundary value problem and use the well-known contraction mapping theorem to prove the existence of solutions. The Galerkin method, as used in [18, 19] and Lions [20], can be employed to establish the desired theorem. We have now to state the following existence result of the local solution to the problem (1.4).

Theorem 1 (Local existence) *Suppose that* (H1)*−*(H3) *are satisfied. Then for any given* $u_0 \in H_0^1(\Omega)$*, the problem* (1.4) *admits a unique local solution satisfying* $u \in C\left([0, T]; H_0^1(\Omega)\right)$, $u_t \in L^{m(.)}(\Omega \times (0, T))$ *for some* $T > 0$.

The first step in proving Theorem 1 is to consider the following initial boundary value problem for a given f:

$$\begin{cases} \mathrm{L}\,(t)\, |u_t|^{m(x)-2}\, u_t - \Delta u + \displaystyle\int_0^t g(t-s) \Delta u(x,s) \mathrm{d}s = f(x,t) \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{3.1}$$

where $f \in L^2(\Omega \times (0, T))$, $u_0 \in H_0^1(\Omega)$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $m(.)$ is a given measurable function satisfying (1.6) and (1.5).

Lemma 2 *Under the conditions of Theorem 1, problem* (3.1) *has a unique local solution*

$$u \in C\left([0, T]\,;\, H_0^1(\Omega)\right) \cap C^1\left((0, T)\,;\, L^{m(.)}(\Omega)\right).$$

Proof **Uniqueness:** To prove the uniqueness of the solution, let u and v u and v be two solutions of (3.1). Then, $w = u - v$ satisfies

$$\mathrm{L}\,(t) \left(|u_t|^{m(x)-2}\, u_t - |v_t|^{m(x)-2}\, v_t \right) - \Delta w + \int_0^t g(t-s) \Delta w(x,s) \mathrm{d}s = 0 \text{ in } \Omega \times (0,T),$$

$$u = 0 \text{ on } \partial\Omega \times (0, T),$$
$$u(x, 0) = u_0(x) \text{ in } \Omega.$$

Multiply by w_t and integrate over Ω, to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(1-\int_0^t g(s)\mathrm{d}s\right)\int_\Omega |\nabla w(t)|^2\,\mathrm{d}x + (g\circ\nabla w)(t)\right\}\\&+\int_\Omega \mathrm{L}(t)\left(|u_t|^{m(.)-2}u_t - |v_t|^{m(.)-2}v_t\right)w_t\mathrm{d}x\\&+\frac{1}{2}g(t)\int_\Omega |\nabla w(t)|^2\,\mathrm{d}x - \frac{1}{2}\left(g'\circ\nabla w\right)(t) = 0.\end{aligned}$$

Integrate over $(0, t)$, takin into account that

$$\int_\Omega \mathrm{L}(t)\left(|u_t|^{m(.)-2}u_t - |v_t|^{m(.)-2}v_t\right)w_t\mathrm{d}x \geq c_0\int_\Omega |w_t|^{m(x)}\,\mathrm{d}x \geq 0 \ \text{ a.e. } x \in \Omega,$$

to get

$$\kappa\,\|\nabla w\|_2^2 \leq 0,$$

which implies that $w = w(0) = 0$. Hence, the uniqueness of the solution.

Existence:

Using the Galerkin method the straightforward proof of the existence result is due to the linearity of the principal part of the problem (3.1). Let $\{\varphi_i\}_{i=1}^{\infty}$ be an orthonormal basis of $H_0^1(\Omega)$, with

$$-\Delta\varphi_i = \lambda_i\varphi_i \text{ in } \Omega,\ \ \varphi_i = 0 \text{ on } \partial\Omega,$$

and define the finite-dimensional subspace $\Phi_k = \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}$ with $\|\varphi_i\| = 1$. We start with

$$u^k(x, t) = \sum_{i=1}^{k} c_i(t)\varphi_i,$$

solution of the following approximate problems

$$\begin{aligned}&\int_\Omega \mathrm{L}(t)\left|u_t^k\right|^{m(x)-2}u_t^k(x,t)\varphi_i(x)\mathrm{d}x + \int_\Omega \nabla u^k(x,t)\nabla\varphi_i(x)\mathrm{d}x\\&-\int_\Omega\int_0^t g(t-s)\nabla u^k(x,s)\nabla\varphi_i(x)\mathrm{d}s\mathrm{d}x = \int_\Omega f(x,t)\varphi_i(x)\mathrm{d}x\\&\qquad u^k(x,0) = u_0^k,\ \forall i = 1,\ldots,k,\\&\qquad u_0^k = \sum_{i=1}^{k}(u_0,\varphi_i)\,\varphi_i \to u_0 \text{ in } H_0^1(\Omega).\end{aligned} \tag{3.2}$$

which generates the system of k ordinary differential equations. System (3.2) has a local solution in $[0, t_k)$, where $0 < t_k < T 0 < t_k < T$ for any $T > 0$. Our next step is to prove that $t_k = T$, $\forall k \geq 1$. We can do this by multiplying $(3.2)_1$ by $c_i'(t)$ and summing up the products for i. This leads us to conclude that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(1-\int_0^t g(s)\mathrm{d}s\right)\int_\Omega \left|\nabla u^k(t)\right|^2 \mathrm{d}x + \left(g \circ \nabla u^k\right)(t)\right\} + \int_\Omega \mathrm{L}\,(t)\left|u_t^k(x,t)\right|^{m(x)} \mathrm{d}x$$
$$= -\frac{1}{2}g(t)\int_\Omega \left|\nabla u^k(t)\right|^2 \mathrm{d}x + \frac{1}{2}\left(g' \circ \nabla u^k\right)(t) + \int_\Omega f(x,t)u_t^k(x,t)\mathrm{d}x.$$

Since $m(x) > 2$, the following embedding is continuous:

$$H_0^1(\Omega) \hookrightarrow L^{m_2}(\Omega) \hookrightarrow L^{m(.)}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{m'(.)}(\Omega).$$

Specifically, we have

$$\|u\|_{m'(.)} \leq \mathrm{c}'\|u\|_2 \leq \mathrm{c}\|u\|_{m(.)},\ \ \frac{1}{m\,(x)} + \frac{1}{m'\,(x)} = 1, \tag{3.3}$$

where c and c′ are the optimal constants of the Sobolev embedding. By using the hypotheses on g and the boundedness of L, we can integrate over the interval $(0, t)$ to get

$$\begin{aligned}
&\frac{1}{2}\left\{\int_\Omega\left(1-\int_0^t g(s)\mathrm{d}s\right)\int_\Omega \left|\nabla u^k(t)\right|^2 \mathrm{d}x + \left(g \circ \nabla u^k\right)(t)\right\} + c_0\int_0^t\int_\Omega \left|u_t^k(x,s)\right|^{m(x)} \mathrm{d}x\mathrm{d}s\\
&\leq \frac{1}{2}\int_\Omega \left|\nabla u_0^k\right|^2 \mathrm{d}x + \int_0^t\int_\Omega f(x,s)u_t^k(x,s)\mathrm{d}x\mathrm{d}s\\
&\leq \frac{1}{2}\int_\Omega \left|\nabla u_0^k\right|^2 \mathrm{d}x + \varepsilon\int_0^t\int_\Omega \left|u_t^k(x,s)\right|^{m(x)} \mathrm{d}x\mathrm{d}s + c\,(\varepsilon)\int_0^t\int_\Omega |f(x,s)|^{m'(x)}\mathrm{d}x\mathrm{d}s\\
&\leq \frac{1}{2}\int_\Omega \left|\nabla u_0^k\right|^2 \mathrm{d}x + \varepsilon\int_0^t\int_\Omega \left|u_t^k(x,s)\right|^{m(x)} \mathrm{d}x\mathrm{d}s + c'c\,(\varepsilon)\int_0^T\int_\Omega |f(x,s)|^2\mathrm{d}x\mathrm{d}s\\
&\leq C + \varepsilon\int_0^t\int_\Omega \left|u_t^k(x,s)\right|^{m(x)} \mathrm{d}x\mathrm{d}s,\quad \forall t \in \left[0, t_k\right).
\end{aligned}$$

So, for $(0, t_k)$, choosing $\varepsilon = \frac{c_0}{2}$, we get

$$\sup_{(0,t_k)} \kappa\int_\Omega \left|\nabla u^k(t)\right|^2 \mathrm{d}x + \frac{c_0}{2}\int_0^{t_k}\int_\Omega \left|u_t^k(x,s)\right|^{m(x)} \mathrm{d}x\mathrm{d}s \leq C.$$

Then the solution can be extended to $[0, T)$ and we obtain

$$\left(u^k\right) \text{ is a bounded sequence in } L^\infty\left((0,T); H_0^1(\Omega)\right)$$
$$\left(u_t^k\right) \text{ is a bounded sequence in } L^{m(.)}(\Omega \times (0,T)).$$

Hence, there exists a subsequence (u^{μ}) of $\left(u^k\right)$ such that

$$u^{\mu} \to u \quad \text{weak star in } L^{\infty}\left((0,T);\, H_0^1(\Omega)\right)$$
$$u_t^{\mu} \to u_t \text{ weakly in } \; L^{m(.)}(\Omega \times (0,T)).$$

According to Lion's lemma [18, Lemme 1.2.], we can conclude that $u \in C\left([0,T];\, L^2(\Omega)\right)$. Also, as $\left(u_t^{\mu}\right)$ is bounded in $L^{m(.)}(\Omega \times (0,T))$, $\mathrm{L}(t)\left|u_t^{\mu}\right|^{m(x)-2} u_t^{\mu}$ is bounded in $L^{\frac{m(.)}{m(.)-1}}(\Omega \times (0,T))$,

$$\mathrm{L}(t)\left|u_t^{\mu}\right|^{m(.)-2} u_t^{\mu} \to \mathrm{L}(t)\left|u_t\right|^{m(.)-2} u_t \text{ weakly in } \; L^{\frac{m(-)}{m(.)-1}}(\Omega \times (0,T)).$$

By utilizing Lion's Lemma [18, Lemme 1.3.] and the boundedness of L, we deduce that the above statement is true. To obtain the desired result, we can use the limit in equation (3.2) and incorporate the convergence mentioned above, to get:

$$\int_0^T \int_{\Omega} \mathrm{L}(t)\left|u_t\right|^{m(x)-2} u_t(x,t)\varphi(x)\sigma \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\Omega} \nabla u(x,t)\nabla\varphi(x)\sigma \,\mathrm{d}x\mathrm{d}t$$
$$- \int_0^T \int_{\Omega} \int_0^t g(t-s)\nabla u(x,s)\nabla\varphi(x)\sigma \,\mathrm{d}s\mathrm{d}x\mathrm{d}t = \int_0^T \int_{\Omega} f(x,t)\varphi(x)\sigma \,\mathrm{d}x\mathrm{d}t,$$

for all $\sigma \in D(0,T)$ and for all $\varphi \in L^{m(.)}\left((0,T) \times H_0^1(\Omega)\right)$. From the above identity, we have

$$\mathrm{L}(t)\left|u_t\right|^{m(x)-2} u_t(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s$$
$$= f(x,t) \text{ in } L^{m(.)}\left((0,T) \times H_0^1(\Omega)\right). \tag{3.4}$$

We will provide a brief overview of the local solutions for problem (1.4). □

Proof of Theorem 1 **Existence:** To clarify, similar to the case in [20, Theorem 5.], we have for any $v \in L^{\infty}\left((0,T);\, H_0^1(\Omega)\right)$

$$\left\| \left| v \right|^{p(.)-2} v \right\|_2^2 \le \int_{\Omega} |v|^{2p_1-2}\,\mathrm{d}x + \int_{\Omega} |v|^{2p_2-2}\,\mathrm{d}x < \infty.$$

For the given

$$1 < p_1 \le p(x) \le p_2 < \frac{2n}{n-2}, \tag{3.5}$$

we have

$$|v|^{p(.)-2} v \in L^{\infty}\left((0,T),\, L^2(\Omega)\right) \subset L^2(\Omega \times (0,T).$$

Hence, there exists a unique

$$u \in C\left([0, T]; H_0^1(\Omega)\right) \cap C^1\left((0, T); L^{m(.)}(\Omega)\right),$$

satisfying the nonlinear problem

$$\begin{cases} \mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t(x,t) - \Delta u(x,t) \\ +\displaystyle\int_0^t g(t-s)\Delta u(x,s)\mathrm{d}s = |v|^{p(.)-2}\,v \quad \text{in } \Omega \times (0,T), \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{3.6}$$

Let R_0 be a positive real number such that

$$R_0 = \|\nabla u_0\|_2 .$$

For a sufficiently small time $T > 0$, we define the space X_{T,R_0} as follows:

$$\mathrm{X}_{T,R_0} = \left\{ \begin{array}{l} v\,(t) \in L^{\infty}\left((0,T), H_0^1(\Omega)\right), \\ v_t\,(t) \in L^{m(.)}(\Omega \times (0,T)), \\ \dfrac{c_0}{2}\displaystyle\int_0^t \int_\Omega |v_t(x,s)|^{m(x)}\,\mathrm{d}x\mathrm{d}s + \frac{1}{2}\kappa\,\|\nabla v\,(t)\|_2^2 \le R_0^2 \text{ on } [0,T], \\ v\,(0) = v_0. \end{array} \right\}$$

which is a complete metric space with the distance

$$\mathrm{d}\,(u,v) = \kappa \sup_{0\le t\le T} \|\nabla\,(u\,(t) - v\,(t))\|_2^2 \text{ for } u,\ v \in \mathrm{X}_{T,R_0}. \tag{3.7}$$

We define the nonlinear mapping $\mathrm{B}\,(v) = u$, and then, we shall show that there exists $T > 0$ and $R_0 > 0$ such that

(i) $\mathrm{B} : \mathrm{X}_{T,R_0} \to \mathrm{X}_{T,R_0}$
(ii) In the space X_{T,R_0}, the mapping B is a contraction according to the metric given in (3.7).

After multiplication by u_t in the equation (3.6), and integration over Ω, we find

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega \left(1 - \int_0^t g(s)\mathrm{d}s\right)\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x + (g \circ \nabla u)\,(t)\right\} + \int_\Omega \mathrm{L}\,(t)\,|u_t(x,s)|^{m(x)}\,\mathrm{d}x \\ &\quad = -\frac{1}{2}g(t)\int_\Omega |\nabla u(t)|^2\,\mathrm{d}x + \frac{1}{2}\left(g' \circ \nabla u\right)(t) + \int_\Omega |v|^{p(x)-2}v(x,t)u_t(x,t)\mathrm{d}x. \end{aligned} \tag{3.8}$$

Using Young's inequality and the boundness of L, then for all $\varepsilon > 0$, we have

$$\begin{aligned}\left|\int_{\Omega} |v|^{p(x)-2} v u_t \mathrm{d}x\right| &\le \frac{1}{2}\varepsilon \int_{\Omega} u_t^2 \mathrm{d}x + \frac{1}{2} c(\varepsilon) \int_{\Omega} |v|^{2p(x)-2} \mathrm{d}x \\ &\le \frac{1}{2} \mathrm{c}\varepsilon \int_{\Omega} |u_t(x,s)|^{m(x)} \mathrm{d}x + \frac{1}{2} c(\varepsilon) \left[\int_{\Omega} |v|^{2p_2-2} \mathrm{d}x + \int_{\Omega} |v|^{2p_1-2} \mathrm{d}x\right] \\ &\le \frac{1}{2} \mathrm{c}\varepsilon \int_{\Omega} |u_t(x,s)|^{m(x)} \mathrm{d}x + \frac{\mathrm{c_e}}{2} c(\varepsilon) \left[\|\nabla v\|_2^{2p_2-2} + \|\nabla v\|_2^{2p_1-2}\right].\end{aligned}$$

Thus, for ε sufficiently small, (3.8) give

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\Omega}\left(1-\int_0^t g(s)\mathrm{d}s\right)\int_{\Omega}|\nabla u(t)|^2 \mathrm{d}x + (g \circ \nabla u)(t)\right\} + \frac{c_0}{2}\int_{\Omega}|u_t(x,s)|^{m(x)} \mathrm{d}x \\ \le \frac{\mathrm{c_e}}{2} c(\varepsilon)\left[R_0^{2p_2-2} + R_0^{2p_1-2}\right].\end{aligned}$$

Integrating from 0 to t we have

$$\frac{c_0}{2}\int_0^t \int_{\Omega} |u_t(x,s)|^{m(x)} \mathrm{d}x\mathrm{d}s + \frac{1}{2}\kappa \|\nabla u\|_2^2 \le \frac{1}{2}\kappa R_0^2 + \lambda_0 T,$$

where $\lambda_0 = \frac{\mathrm{c_e}}{2} c(\varepsilon)\left(R_0^{2p_2-2} + R_0^{2p_1-2}\right)$, $\mathrm{c_e}$ is the Sobolev embedding constant. Therefore, if the parameters T and R_0 satisfy $\frac{1}{2}\kappa R_0^2 + \lambda_0 T < R_0^2$ (remembering that $\kappa < 1$), we obtain

$$\frac{c_0}{2}\int_0^t \int_{\Omega} |u_t(x,s)|^{m(x)} \mathrm{d}x\mathrm{d}s + \frac{1}{2}\kappa \|\nabla u\|_2^2 \le R_0^2. \tag{3.9}$$

Hence, it implies that B maps X_{T,R_0} into itself.

Let us now prove (ii). To demonstrate that B is a contraction mapping with respect to the metric $\mathrm{d}(u,v)$ given above, we consider $u^1 = \mathrm{B}(v_1)$, $u^2 = \mathrm{B}(v_2)$ with v_1, $v_2 \in \mathrm{X}_{T,R_0}$, then $w(t) = \left(u^1 - u^2\right)(t)$ satisfies for any $T \le T_0$, the following system:

$$\begin{aligned}\mathrm{L}(t)\left|u_t^1(t)\right|^{m(x)-2} u_t^1(t) - \mathrm{L}(t)\left|u_t^2(t)\right|^{m(x)-2} u_t^2(t) + \Delta w \\ - \int_{\Omega}\int_0^t g(t-s)\Delta w(x,s)\mathrm{d}s\mathrm{d}x \\ = |v_1|^{p(x)-2} v_1 - |v_2|^{p(x)-2} v_2 \text{ in } L^2\left(0,T;L^2(\Omega)\right),\end{aligned} \tag{3.10}$$

with initial conditions $w(0) = 0$ in Ω, and boundary condition $w(x,t) = 0$ on $\partial\Omega$. Multiplying (3.10) by w_t and integrating it over Ω, taking into account that

$$\left(\mathrm{L}(t)\left|u_t^1(t)\right|^{m(x)-2} u_t^1(t) - \mathrm{L}(t)\left|u_t^2(t)\right|^{m(x)-2} u_t^2(t), u_t^1(t) - u_t^2(t)\right)$$

$$\geq c_0 \int_\Omega |w_t|^{m(x)}\, \mathrm{d}x \geq 0, \text{ a.e. } x \in \Omega,$$

we find

$$c_0 \int_\Omega |w_t|^{m(x)}\, \mathrm{d}x + \frac{1}{2}\kappa \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla w\|_2^2 \leq \int_\Omega \left(|v_1|^{p(.)-2}\, v_1 - |v_2|^{p(.)-2}\, v_2\right) w_t \mathrm{d}x. \tag{3.11}$$

Using the fact that, for any $x \in \Omega$ fixed, we have

$$|v_1|^{p(.)-2}\, v_1 - |v_2|^{p(.)-2}\, v_2 = (p(x)-1)\, \zeta^{p(x)-2} v,$$

with $v = v_1 - v_2$, and $\zeta = sv_1 + (1-s)v_2$, $s \in (0,1)$. Young's inequality implies

$$\begin{aligned}
I &= \left| \int_\Omega \left(|v_1(s)|^{p(.)-2}\, v_1(s) - |v_2(s)|^{p(.)-2}\, v_2(s)\right) w_t \mathrm{d}x \right| \\
&\leq \frac{1}{2}\varepsilon \int_\Omega w_t^2 \mathrm{d}x + \frac{1}{2} c(\varepsilon) \int_\Omega \left|(p(x)-1)\, \zeta^{p(x)-2}\right|^2 |v|^2 \mathrm{d}x \\
&\leq \frac{1}{2}\varepsilon \mathrm{c} \int_\Omega |w_t(t)|^{m(x)}\, \mathrm{d}x + \frac{p_2^2}{2} c(\varepsilon) \int_\Omega |sv_1 + (1-s)v_2|^{2(p(x)-2)}\, |v|^2 \mathrm{d}x \\
&\leq \frac{1}{2}\varepsilon \mathrm{c} \int_\Omega |w_t(t)|^{m(x)}\, \mathrm{d}x \\
&+ \frac{p_2^2}{2} c(\varepsilon) \left(\int_\Omega |v|^{\frac{2n}{n-2}} \mathrm{d}x\right)^{\frac{n-2}{n}} \left[\left(\left(\int_\Omega |sv_1 + (1-s)v_2|^{n(p_2-2)}\right)^{\frac{2}{n}} \mathrm{d}x\right.\right. \\
&+ \left.\left(\int_\Omega |sv_1 + (1-s)v_2|^{n(p_1-2)}\right)^{\frac{2}{n}} \mathrm{d}x\right].
\end{aligned}$$

Picking $\varepsilon = \frac{c_0}{\mathrm{c}}$ and recalling (3.5), we arrive at

$$\begin{aligned}
I &\leq \frac{c_0}{2} \int_\Omega |w_t(t)|^{m(x)}\, \mathrm{d}x \\
&+ c_e \frac{p_2^2}{2} c(\varepsilon)\, \|\nabla v\|_2^2 \left[\|\nabla v_1\|_2^{2(p_2-2)} + \|\nabla v_1\|_2^{2(p_1-2)} + \|\nabla v_2\|_2^{2(p_2-2)} + \|\nabla v_2\|_2^{2(p_1-2)}\right] \\
&\leq \frac{c_0}{2} \int_\Omega |w_t(t)|^{m(x)}\, \mathrm{d}x + 2p_2^2 \mathrm{c_e} c(\varepsilon)\, R_0^{2(p_2-2)} \mathrm{d}(v_1, v_2).
\end{aligned}$$

Therefore, (3.11) takes the form

$$\frac{1}{2}\kappa \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla w\|_2^2 \leq 2T \mathrm{c_e} c(\varepsilon)\, p_2^2 R_0^{2(p_2-2)} \mathrm{d}(v_1, v_2).$$

By (3.7), we have

$$\mathrm{d}\,(u_1, u_2) \leq C\,(T, R_0)\,\mathrm{d}\,(v_1, v_2)\,. \tag{3.12}$$

where $C\,(T, R_0) = 2T\mathrm{c_e}c\,(\varepsilon)\,p_2^2 R_0^{2(p_2-2)}$. Therefore, if $C\,(T, R_0) < 1$, B is a contraction mapping according to inequality (3.9). To satisfy both conditions (3.9) and (3.12), we select R_0 to be adequately large and T to be sufficiently small. By utilizing the contraction mapping theorem, we can obtain the result for local existence.

Uniqueness: Suppose we have two solutions u and v. Then $U = u - v$ satisfies

$$\begin{aligned}
&\mathrm{L}\,(t)\,|u_t\,(t)|^{m(x)-2}\,u_t\,(t) - \mathrm{L}\,(t)\,|v_t\,(t)|^{m(x)-2}\,v_t\,(t) - \Delta U \\
&+ \int_0^t g(t-s)\Delta U(x,s)\mathrm{d}s = |u|^{p(x)-2}u - |v|^{p(x)-2}v \quad in \quad \Omega \times (0,T), \\
&U(x,t) = 0 \quad on \quad \partial\Omega \times (0,T), \\
&U(x,0) = 0 \quad in \quad \Omega.
\end{aligned}$$

Multiply by U_t and integrate over $\Omega \times (0, t)$ to obtain

$$c_0 \int_0^t \int_\Omega |U_t|^{m(x)}\,\mathrm{d}x\mathrm{d}s + \frac{1}{2}\kappa \int_\Omega |\nabla U|^2 \mathrm{d}x \leq \int_0^t \int_\Omega \left(|u|^{p(.)-2}\,u - |v|^{p(.)-2}\,v\right) U_t \mathrm{d}x\mathrm{d}s.$$

By repeating the same estimates as in above, we arrive at

$$\frac{c_0}{2} \int_0^t \int_\Omega |U_t|^{m(x)}\,\mathrm{d}x\mathrm{d}s + \frac{1}{2}\kappa \int_\Omega |\nabla U|^2 \mathrm{d}x \leq C \int_0^t \int_\Omega |\nabla U(x,s)|^2 \mathrm{d}x\mathrm{d}s.$$

Gronwall's inequality yields

$$\int_\Omega |\nabla U|^2 \mathrm{d}x = 0.$$

Thus, $U \equiv 0$. This shows the uniqueness. The proof of Theorem 1 is completed.

4 Blow-up and bounds of blow-up time

In this section, we get new bounds for the blow-up time to problem (1.4) if the variable exponents $m(.)$, $p(.)$ and the initial data satisfy some conditions. We prefer to state the following theorem of existence, uniqueness, and regularity before stating our key conclusions without providing evidence

Definition 1 A function $u(x, t)$ is said to be a weak solution of problem (1.4) defined on the time interval $[0, T)$, provide that $u(x, t) \in C\left([0, T)\,, H_0^1(\Omega)\right) \cap$

$C^1\left([0,T), L^{m(.)}(\Omega)\right)$, if for every test-function $\eta \in C\left([0,T), H_0^1(\Omega)\right)$ and a.e. $t \in [0,T)$, the following identity holds:

$$\begin{aligned}&\int_0^t\int_\Omega \mathrm{L}(s)\,|u_t|^{m(x)-2}u_t(s)\,\eta(s)\,\mathrm{d}x\mathrm{d}s\\&\quad+\int_0^t\int_\Omega\left(\nabla u(s)-\int_0^s g(t-\tau)\nabla u(x,\tau)\mathrm{d}\tau\right)\nabla\eta(x,s)\mathrm{d}x\mathrm{d}s\\&\quad-\int_0^t\int_\Omega |u(s)|^{p(x)-2}u(s)\eta(s)\,\mathrm{d}x\mathrm{d}s=0.\end{aligned} \tag{4.1}$$

Without proof, we give the local existence of a solution of (1.4) that can be derived from the fixed point theorem in Banach spaces and the Faedo-Galerkin arguments.

Theorem 2 *Assume that* (1.5)–(1.6) *hold. Then the problem* (1.4) *for given* $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ *admits a unique local solution*

$$u\in C\left([0,T_{\max});H_0^1(\Omega)\right),\ u_t\in C\left([0,T_{\max}), L^{m(.)}(\Omega)\right),$$

where $T_{\max}>0$ *is the maximal existence time of* $u(t)$.

5 First blow-up properties

For our result, we want to consider the following characteristics

$$\alpha(t)=\left[\kappa\|\nabla u(t)\|_2^2+(g\circ\nabla u)(t)\right]^{\frac{1}{2}}, \tag{5.1}$$

and for ε (positive small) and N precise positive constants to be chosen later,

$$\mathrm{A}(t):=H^{1-\sigma}(t)-\varepsilon\int_0^t\int_\Omega|u(s)|^{p(x)}\,\mathrm{d}x\mathrm{d}s+\varepsilon N\mathrm{E}_1 t,\ t\in[0,T). \tag{5.2}$$

The values B, α_1, α_0, E_1 and $\widetilde{\mathrm{E}}_1$ are positive constants given by

$$\begin{aligned}B_1=\left(3\frac{p_2}{p_1}\right)^{\frac{1}{p_2}}\hat{B}/\sqrt{\kappa},\ \alpha_1=B_1^{\frac{-P_2}{P_2-2}},\ \alpha(0)=\alpha_0=\kappa^{\frac{1}{2}}\|\nabla u_0\|_2,\\ \mathrm{E}_1=\left(\frac{1}{2}-\frac{1}{p_2}\right)\alpha_1^2,\ \widetilde{\mathrm{E}}_1=\left(\frac{1}{q}-\frac{1}{p_1}\right)\alpha_1^2.\end{aligned} \tag{5.3}$$

The first result of the blow-up is as follows

Theorem 3 *Supposing that g, $m(.)$, and $p(.)$ fulfill various conditions* (H1) − (H3) *with* $p_1>m_2$. *Then the local solution of problem* (1.1) *under boundary conditions* *satisfying* $\mathrm{E}(0) < \mathrm{E}_1$, $\kappa^{\frac{1}{2}} \|\nabla u_0\| > \alpha_1$ *blows up in finite time* T*, which equips the following estimates*

$$T \leq \frac{1-\sigma}{\sigma \frac{\delta_1}{\delta_2} \mathrm{A}^{\frac{\alpha}{1-\alpha}}(0)},$$

where

$$0 < \sigma \leq \min\left\{\frac{p_1-2}{2p_1}, \frac{p_1-m_2}{p_1(m_2-1)}\right\}, \tag{5.4}$$

and δ_1, δ_2 *are defined in* (5.33), (5.37)*, respectively.*

For our result, the following lemmas must be taken into account:

Lemma 3 *Let* $h : [0, +\infty) \to \mathbb{R}$ *be defined by*

$$h(t) := h(\alpha) = \frac{1}{2}\alpha^2 - \frac{B_1^{p_2}}{P_2}\alpha^{p_2}, \tag{5.5}$$

then h has the following results:

(i) *h is increasing for* $0 < \alpha \leq \alpha_1$ *and decreasing for* $\alpha \geq \alpha_1$,
(ii) $\lim_{\alpha \to +\infty} h(\alpha) = -\infty$ *and* $h(\alpha_1) = \mathrm{E}_1$,
(iii) $\mathrm{E}(t) \geq h(\alpha(t))$,

where $\alpha(t)$ *is given in* (5.1)*,* α_1 *and* E_1 *are given in* (5.3).

Proof $h(\alpha)$ is continuous and differentiable in $[0, +\infty)$,

$$h'(\alpha) = \alpha\left(1 - B_1^{p_2}\alpha^{p_2-2}(t)\right)\begin{cases} > 0, & \alpha \in (0, \alpha_1) \\ < 0, & \alpha \in (\alpha_1, +\infty), \end{cases}$$

Consequently

$$\begin{aligned} h(\alpha) &\text{ is strictly increasing in } (0, \alpha_1), \\ h(\alpha) &\text{ is strictly decreasing in } (\alpha_1, +\infty). \end{aligned} \tag{5.6}$$

Then **(i)** follows. Since $p_2 - 2 > 0$, we have $\lim_{\alpha \to +\infty} h(\alpha) = -\infty$. An easy calculation yields $h(\alpha_1) = \mathrm{E}_1$. Then **(ii)** is correct. By Lemma 1:

$$\begin{aligned} \int_\Omega |u|^{p_1}\mathrm{d}x &= \int_{\{x\in\Omega:|u(x,t)|\geq 1\}} |u|^{p_1}\mathrm{d}x + \int_{\{x\in\Omega:|u(x,t)|<1\}} |u|^{p_1}\mathrm{d}x \\ &\leq 2\int_{\{x\in\Omega:|u(x,t)|\geq 1\}} |u|^{p_1}\mathrm{d}x \leq 2\int_{\{x\in\Omega:|u(x,t)|\geq 1\}} |u|^{p_2}\mathrm{d}x \leq 2\int_\Omega |u|^{p_2}\mathrm{d}x, \end{aligned}$$

which means

$$\int_\Omega |u|^{p(x)}\,\mathrm{d}x \le \int_\Omega |u|^{p_1}\mathrm{d}x + \int_\Omega |u|^{p_2}\mathrm{d}x$$
$$\le 3\int_\Omega |u|^{p_2}\mathrm{d}x \le 3\hat{B}^{p_2}\left(\int_\Omega |\nabla u(t)|^2\mathrm{d}x\right)^{\frac{p_2}{2}}. \tag{5.7}$$

Using (H1), (2.5) and Lemma 1, we have

$$\begin{aligned}
\mathrm{E}(t) &\ge \frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}\left(g\circ\nabla u\right)(t) - \frac{1}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x\\
&\ge \frac{1}{2}\kappa\|\nabla u(t)\|_2^2 + \frac{1}{2}\left(g\circ\nabla u\right)(t) - \frac{3\hat{B}^{p_2}}{p_1}\|\nabla u(t)\|_2^{p_2}\\
&\ge \frac{1}{2}\left[\kappa\|\nabla u(t)\|_2^2 + \left(g\circ\nabla u\right)(t)\right] - \frac{B_1^{p_2}}{p_2}\left[\kappa\|\nabla u(t)\|_2^2 + \left(g\circ\nabla u\right)(t)\right]^{\frac{p_2}{2}}\\
&= \frac{1}{2}\alpha^2(t) - \frac{B_1^{p_2}}{p_2}\alpha^{p_2}(t) = h(\alpha(t)).
\end{aligned} \tag{5.8}$$

Then **(iii)** holds.

Lemma 4 *Supposing the conditions* $0 \le \mathrm{E}(0) < \mathrm{E}_1$. *Then we have*

1. *If* $\kappa^{\frac{1}{2}}\|\nabla u_0\| < \alpha_1$, *there is a positive constant* $0 \le \alpha_2' < \alpha_1$ *such that*

$$\alpha(t) < \alpha_2',\ t \ge 0. \tag{5.9}$$

2. *If* $\kappa^{\frac{1}{2}}\|\nabla u_0\| > \alpha_1$, *there is a positive constant* $\alpha_2 > \alpha_1$ *such that*

$$\alpha(t) \ge \alpha_2 > \alpha_1,\ t \ge 0, \tag{5.10}$$
$$\varrho(u) \ge B_1^{p_2}\alpha_2^{p_2}, \tag{5.11}$$

where α_1, B_1 *and* E_1 *are given in* (5.3).

Proof Because $0 \le \mathrm{E}(0) < \mathrm{E}_1$ and $h(\alpha)$ is a continuous function, there exist α_2' and α_2 with $\alpha_2' < \alpha_1 < \alpha_2$ such that

$$h\left(\alpha_2'\right) = h\left(\alpha_2\right) = \mathrm{E}(0). \tag{5.12}$$

1. When $\kappa^{\frac{1}{2}}\|\nabla u_0\| < \alpha_1$, from (2.6) and (5.8), we have

$$h(\alpha_0) \le \mathrm{E}(0) = h\left(\alpha_2'\right),$$

which means $\kappa^{\frac{1}{2}}\|\nabla u_0\| \le \alpha_2'$. We claim that $\alpha(t) \le \alpha_2'$ for $0 < t < T$. If not, then there exist $t_0 \in (0, T)$ such that $\alpha(t_0) > \alpha_2'$. If $\alpha_2' < \alpha(t_0) < \alpha_2$, then

$$h\left(\alpha\left(t_0\right)\right) > \mathrm{E}(0) \ge \mathrm{E}\left(t_0\right),$$

which contradicts to (5.8). If $\alpha(t_0) \geq \alpha_2$ then by the continuity of $\alpha(t)$, there exists $t_1 \in (0, t_0)$ such that

$$h(\alpha(t_1)) > \mathrm{E}(0) \geq \mathrm{E}(t_1).$$

This is also a contradiction.

2. When $\kappa^{\frac{1}{2}} \|\nabla u_0\| > \alpha_1$, joins (5.12) with Lemma 3 show

$$h(\alpha_0) \leq \mathrm{E}(0) = h(\alpha_2). \tag{5.13}$$

From Lemma 3**(i)**, we deduce that

$$\alpha_0 \geq \alpha_2, \tag{5.14}$$

so (5.10) holds for $t = 0$.
Assuming that there is $t^* > 0$ such that $\alpha(t^*) < \alpha_2$, we proceed to prove (5.10) by contradiction, separating two cases,
Case 1: If $\alpha_2' < \alpha(t^*) < \alpha_2$, we can infer from Lemma 3 and (5.6) that

$$h\left(\alpha\left(t^*\right)\right) > \mathrm{E}(0) \geq \mathrm{E}\left(t^*\right),$$

which contradicts Lemma 3**(iii)**.
Case 2. If $\alpha(t^*) \leq \alpha_2'$, then $\alpha(t^*) \leq \alpha_2' < \alpha_2$. Set $\lambda(t) = \alpha(t) - \frac{\alpha_2+\alpha_2'}{2}$, then $\lambda(t)$ is a continuous function, $\lambda(t^*) < 0$ and by using (5.14) $\lambda(0) > 0$. Thus, there exists $t_0 \in (0, t^*)$ such that $\lambda(t_0) = 0$, which signifies $\alpha(t_0) = \frac{\alpha_2+\alpha_2'}{2}$, that leads

$$h(\alpha(t_0)) > \mathrm{E}(0) \geq \mathrm{E}(t_0).$$

This contradicts to Lemma 3**(iii)**, hence (5.10) follows.
By (2.5), we have

$$\frac{1}{2}\left[\left(1 - \int_0^t g(s)\mathrm{d}s\right) \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right] \leq \mathrm{E}(t) + \frac{1}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x,$$

which imply

$$\begin{aligned}\frac{1}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x &\geq \frac{1}{2}\left[\left(1 - \int_0^t g(s)\mathrm{d}s\right) \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right] - \mathrm{E}(t) \\ &\geq \frac{1}{2}\left[\kappa \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right] - \mathrm{E}(0) \\ &\geq \frac{1}{2}\alpha_2^2 - h(\alpha_2) = \frac{B_1^{p_2}}{P_2}\alpha_2^{p_2},\end{aligned}$$

then the second inequality in (5.11) holds.

□

Let

$$H(t) = \mathrm{E}_1 - \mathrm{E}(t) \text{ for } t \geq 0. \tag{5.15}$$

The following lemma holds

Lemma 5 *Under the assumptions of Theorem 3, if* $0 \leq \mathrm{E}(0) < \mathrm{E}_1$, *the functional* $H(t)$ *defined in* (5.15) *satisfies the following estimates:*

$$0 < H(0) \leq H(t) \leq \int_\Omega \frac{1}{p(x)} |u(t)|^{p(x)} \mathrm{d}x \leq \frac{1}{p_1} \varrho(u), \ t \geq 0. \tag{5.16}$$

Proof Lemma 1 provides that $H(t)$ is nondecreasing in t. Thus

$$H(t) \geq H(0) = \mathrm{E}_1 - \mathrm{E}(0) > 0, \ t \geq 0. \tag{5.17}$$

By (5.3) and Lemma 4, we have

$$\begin{aligned}
&\mathrm{E}_1 - \left[\frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right) \|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t)\right] \\
&\leq \mathrm{E}_1 - \frac{1}{2}\left[\left(\kappa \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right)\right] \\
&= \mathrm{E}_1 - \frac{1}{2}\alpha^2(t) \leq \mathrm{E}_1 - \frac{1}{2}\alpha^2(t) \leq \mathrm{E}_1 - \frac{1}{2}\alpha_1^2 = -\frac{1}{p_2}\alpha_1^2 < 0,
\end{aligned}$$

for all $t \in [0, T)$, which imply

$$\begin{aligned}
H(t) = \mathrm{E}_1 - &\left[\frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right) \|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t)\right] \\
&+ \int_\Omega \frac{1}{p(x)} |u(t)|^{p(x)} \mathrm{d}x \leq \int_\Omega \frac{1}{p(x)} |u(t)|^{p(x)} \mathrm{d}x \leq \frac{1}{p_1}\varrho(u).
\end{aligned} \tag{5.18}$$

(5.16) follows from (5.17) and (5.18).

Lemma 6 *Assuming the conditions in Theorem 3 hold, then there exists a positive constant* C *such that*

$$\|\nabla u(t)\|_2^2 \leq C\varrho(u). \tag{5.19}$$

for all $t \in [0, T)$.

Proof By Lemma 4 and $\alpha_2 > \alpha_1$, we have

$$\varrho(u) \geq B_1^{p_2}\alpha_2^{p_2} > B_1^{p_2}\alpha_1^{p_2-2}\alpha_1^2 = \alpha_1^2,$$

which combined with (5.3) imply

$$\mathrm{E}_1 \leq \left(\frac{1}{2} - \frac{1}{p_2}\right) \varrho(u). \tag{5.20}$$

combining with the definition of $H(t)$, (5.15), and (5.20), we have

$$\begin{aligned}
\frac{1}{2}\kappa \|\nabla u(t)\|_2^2 &\leq \frac{1}{2}\left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2 \\
&= \mathrm{E}(t) - \frac{1}{2}(g \circ \nabla u)(t) + \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x \\
&\leq \left(\frac{1}{2} - \frac{1}{p_2}\right)\varrho(u) - H(t) - \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{p_1}\varrho(u) \\
&= \left(\left(\frac{1}{2} - \frac{1}{p_2}\right) + \frac{1}{p_1}\right)\varrho(u) - H(t) - \frac{1}{2}(g \circ \nabla u)(t) \\
&\leq \left(\left(\frac{1}{2} - \frac{1}{p_2}\right) + \frac{1}{p_1}\right)\varrho(u).
\end{aligned} \tag{5.21}$$

Then the desired result, with $C = \frac{\left(1 - \frac{2}{p_2}\right) + \frac{2}{p_1}}{\kappa}$. □

A proof of the following theorem Theorem 3 based on the above lemmas

Proof of Theorem 3 **Case 1.** If $0 \leq \mathrm{E}(0) < \mathrm{E}_1$, then by differentiating (5.2), we get

$$\mathrm{A}'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon \int_\Omega |u(s)|^{p(x)}\,\mathrm{d}x + N\mathrm{E}_1. \tag{5.22}$$

Integrating by parts on Ω, recalling Eq (1.4), we obtain

$$\begin{aligned}
\mathrm{A}'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon \int_\Omega \mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t u\mathrm{d}x - \varepsilon\|\nabla u(t)\|_2^2 \\
&\quad + \varepsilon \int_0^t g(t-s)\int_\Omega \nabla u(t)\nabla u(s)\mathrm{d}x\mathrm{d}s + \varepsilon N\mathrm{E}_1 \\
&= (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\|\nabla u(t)\|_2^2 \\
&\quad + \varepsilon \int_0^t g(t-s)\int_\Omega \nabla u(t)(\nabla u(s) - \nabla u(t))\mathrm{d}x\mathrm{d}s \\
&\quad + \varepsilon \int_0^t g(t-s)\int_\Omega |\nabla u(t)|^2\mathrm{d}x\mathrm{d}s - \varepsilon\int_\Omega \mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t u\mathrm{d}x + \varepsilon N\mathrm{E}_1.
\end{aligned} \tag{5.23}$$

Taking advantage of Young's inequality, we have

$$\begin{aligned} &\left|\int_0^t g(t-s)\int_\Omega \nabla u(t)(\nabla u(s)-\nabla u(t))\mathrm{d}x\mathrm{d}s\right| \\ &\leq \tau\int_0^t g(t-s)\|\nabla u(s)-\nabla u(t)\|_2^2\mathrm{d}s+\frac{1}{4\tau}\int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2 \\ &= \tau(g\circ\nabla u)(t)+\frac{1}{4\tau}\int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2 \end{aligned} \tag{5.24}$$

for any $\tau > 0$.

Replacing (5.24) in (5.23), and using (2.5), picking $\tau > 0$ such that $0 < \tau < \frac{p_1}{2}$, we infer

$$\begin{aligned} \mathrm{A}'(t) \geq\ & (1-\sigma)H^{-\sigma}(t)H'(t)-\varepsilon\|\nabla u(t)\|_2^2+\int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2 \\ & -\varepsilon\int_\Omega \mathrm{L}\,(t)\,|u_t|^{m(x)-2}\,u_t u\mathrm{d}x-\tau\varepsilon(g\circ\nabla u)(t) \\ & -\frac{1}{4\tau}\varepsilon\int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2+\varepsilon p_2\,(H(t)-\mathrm{E}_1)+\frac{p_2}{2}\varepsilon(g\circ\nabla u)(t)+\varepsilon N\mathrm{E}_1 \\ & +\frac{p_2}{2}\varepsilon\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2-\varepsilon p_2\int_\Omega\frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x \\ \geq\ & (1-\sigma)H^{-\sigma}(t)H'(t)+\varepsilon\left(\frac{p_2}{2}-\tau\right)(g\circ\nabla u)\,(t) \\ & +\varepsilon(N-p_2)\mathrm{E}_1+\varepsilon p_2 H(t)-\varepsilon\int_\Omega|u_t|^{m(x)-2}\,u_t u\mathrm{d}x-\varepsilon\frac{p_2}{p_1}\int_\Omega|u(t)|^{p(x)}\mathrm{d}x \\ & +\varepsilon\left[\left(\frac{p_2}{2}-1\right)-\left(\frac{p_2}{2}-1+\frac{1}{4\tau}\right)\int_0^\infty g(s)\mathrm{d}s\right]\|\nabla u(t)\|_2^2. \end{aligned} \tag{5.25}$$

By combining (2.2) and (5.25), we get

$$\begin{aligned} \mathrm{A}'(t) \geq\ & (1-\sigma)H^{-\sigma}(t)H'(t)+a_1\varepsilon\,(g\circ\nabla u)\,(t) \\ & +a_2\varepsilon\|\nabla u(t)\|_2^2-\varepsilon\frac{p_2}{p_1}\int_\Omega|u(t)|^{p(x)}\mathrm{d}x+\varepsilon(N-p_2)\mathrm{E}_1+\varepsilon p_2 H(t) \\ & -\varepsilon\int_\Omega|u_t|^{m(x)-2}\,u_t u\mathrm{d}x, \end{aligned} \tag{5.26}$$

where

$$a_1=\left(\frac{p_1}{2}-\tau\right)>0,\ a_2=\left(\frac{p_1}{2}-1\right)-\left(\frac{p_1}{2}-1+\frac{1}{4\tau}\right)\int_0^\infty g(s)\mathrm{d}s>0.$$

For a large enough constant $\sigma > 0$ to be determined later, the last term on the right-hand side of (5.26) can be estimated from the Hölder inequality as follows:

$$\int_\Omega |u_t|^{m(x)-1}|u|\mathrm{d}x \le \frac{1}{\lambda^{m_1}}\int_\Omega H^{\sigma(m(x)-1)}(t)|u|^{m(x)}\mathrm{d}x + \lambda^{\frac{m_1}{m_1-1}}H^{-\sigma}(t)\int_\Omega |u_t|^{m(x)}\,\mathrm{d}x. \tag{5.27}$$

Combining (5.26) and (5.27) results in

$$\begin{aligned} \mathrm{A}'(t) \ge & \left[(1-\sigma) - \varepsilon\lambda^{\frac{m_1}{m_1-1}}\right]H^{-\sigma}(t)H'(t) + \varepsilon a_1\,(g\circ\nabla u)\,(t) + \varepsilon a_2\|\nabla u(t)\|_2^2 \\ & - \varepsilon\frac{p_2}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x + \varepsilon(N-p_2)\mathrm{E}_1 + \varepsilon p_2 H(t) \\ & - \varepsilon\lambda^{-m_1}\int_\Omega H^{\sigma(m(x)-1)}|u|^{m(x)}\mathrm{d}x. \end{aligned} \tag{5.28}$$

When $0 < H(t) \le 1$, according to (5.18), we have

$$\begin{aligned} & \int_\Omega |u|^{m(x)}H^{\sigma(m(x)-1)}(t)\mathrm{d}x \le \int_\Omega |u|^{m(x)}\mathrm{d}x \le \max\left(\|u\|_{m(.)}^{m_1}, \|u\|_{m(.)}^{m_2}\right) \\ & \le c_1\max\left(\|u\|_{p(.)}^{m_1}, \|u\|_{p(.)}^{m_2}\right) \\ & \le c_1\max\left(\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{m_1}{p_2}}, \left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{m_2}{p_1}}\right) \\ & \le c_1\max\left(\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{m_1-p_2}{p_2}}, \left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{m_2-p_1}{p_1}}\right)\int_\Omega |u|^{p(x)}\mathrm{d}x \\ & \le c_1\max\left((p_1H(0))^{\frac{m_1-p_2}{p_2}}, (p_1H(0))^{\frac{m_2-p_1}{p_1}}\right)\int_\Omega |u|^{p(x)}\mathrm{d}x \\ & = c_2\int_\Omega |u|^{p(x)}\mathrm{d}x. \end{aligned}$$

When $H(t) > 1$, we have

$$\begin{aligned} & \int_\Omega |u|^{m(x)}H^{\sigma(m(x)-1)}(t)\mathrm{d}x \le H^{\sigma(m_2-1)}(t)\int_\Omega |u|^{m(x)}\mathrm{d}x \\ & \le c_1H^{\sigma(m_2-1)}(t)\max\left(\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{m_1}{p_1}}, \left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{m_2}{p_1}}\right) \\ & \le c_1\left(\frac{1}{p_1}\right)^{\sigma(m_2-1)}\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\sigma(m_2-1)}\max\left(\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{m_1}{p_2}},\right. \\ & \left.\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{m_2}{p_1}}\right) \end{aligned}$$

$$\leq c_1 \left(\frac{1}{p_1}\right)^{\sigma(m_2-1)} \max\left(\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{m_1-p_2}{p_2}+\sigma(m_2-1)},\right.$$

$$\left.\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{m_2-p_1}{p_1}+\sigma(m_2-1)}\right)\int_\Omega |u|^{p(x)}\mathrm{d}x$$

$$\leq c_2 H^{\sigma(m_2-1)}(0)\int_\Omega |u|^{p(x)}\mathrm{d}x.$$

By combining the two cases we have

$$H^{\sigma(m(x)-1)}(t)\int_\Omega |u|^{m(x)}\mathrm{d}x \leq c_3\int_\Omega |u|^{p(x)}\mathrm{d}x, \tag{5.29}$$

where

$$c_1 = \left(1+|\Omega|^{\frac{p_2-m_1}{p_2}\frac{m_2}{m_1}}\right),$$

$$c_2 = c_1 \max\left((p_1H(0))^{\frac{m_1-p_2}{p_2}}, (p_1H(0))^{\frac{m_2-p_1}{p_1}}\right),$$

$$c_3 = c_2\left(1+H^{\sigma(m_2-1)}(0)\right).$$

Combining (5.28) and (5.29) result in

$$\begin{aligned}\mathrm{A}'(t) \geq \left[(1-\sigma)-\varepsilon\lambda^{\frac{m_1}{m_1-1}}\right]H^{-\sigma}(t)H'(t)+\varepsilon a_1(g\circ\nabla u)(t)+\varepsilon a_2\|\nabla u(t)\|_2^2 \\ +\varepsilon(N-p_2)\mathrm{E}_1+\varepsilon p_2H(t)-\varepsilon\lambda^{-m_1}c_3\int_\Omega |u|^{p(x)}\mathrm{d}x-\varepsilon\frac{p_2}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x,\end{aligned} \tag{5.30}$$

clearly

$$H(t) \geq \mathrm{E}_1 - \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{2}(g\circ\nabla u)(t)+\frac{1}{p_2}\varrho(u). \tag{5.31}$$

Making use (5.31) in (5.30) and rewriting proceeds as $p_2 = p_2 - 2a_3 + 2a_3$, with $\frac{p_2}{2p_1} < a_3 < \min\left(a_1, a_2, \frac{p_2}{2}\right)$ yield

$$\begin{aligned}\mathrm{A}'(t) \geq \left[(1-\sigma)-\varepsilon\lambda^{\frac{m_1}{m_1-1}}\right]H^{-\sigma}(t)H'(t) \\ +\varepsilon(a_1-a_3)(g\circ\nabla u)(t)+\varepsilon(a_2-a_3)\|\nabla u(t)\|_2^2 \\ +\varepsilon(N-(p_2-2a_3))\mathrm{E}_1+\varepsilon(p_2-2a_3)H(t) \\ +\varepsilon\left(\left(2a_3-\frac{p_2}{p_1}\right)-\lambda^{-m_1}c_3\right)\varrho(u).\end{aligned}$$

At this end, we choose λ and N large enough so that

$$\gamma_1 = 2a_3 - \frac{p_2}{p_1} - \lambda^{-m_1} c_3 > 0,$$
$$N - (p_2 - 2a_3) > 0.$$

Once N and λ are fixed (i.e. γ_1), we choose ε small enough so that

$$(1-\sigma) - \varepsilon \lambda^{\frac{m_1}{m_1-1}} > 0, \text{ and } \mathrm{A}(0) = H^{1-\sigma}(0) > 0, \text{ since } H(0) > 0. \quad (5.32)$$

Then a constant δ_1 satisfaction

$$0 < \delta_1 \le \min\left\{ \frac{p_2}{2} - a_3, a_1 - a_3, \gamma_1, p_2 - 2a_3 \right\}, \quad (5.33)$$

and

$$\mathrm{A}'(t) \ge \delta_1 \varepsilon \left[(g \circ \nabla u)(t) + \|\nabla u(t)\|_2^2 + H(t) + \varrho(u) \right], \quad (5.34)$$

which combined with (5.32) infer

$$\mathrm{A}(t) \ge \mathrm{A}(0) > 0, \ \forall t \in [0, T).$$

Choose $\varepsilon > 0$ to ensure that $0 < \varepsilon < \frac{1}{T}\left(\frac{\alpha_2}{\alpha_1}\right)^{(1-\alpha)p_2} (N\mathrm{E}_1)^{\alpha}$, and remember Lemma 4 and then, we have

$$|\varepsilon N \mathrm{E}_1 T|^{\frac{1}{1-\alpha}} \le \left(\frac{\alpha_2}{\alpha_1}\right)^{p_2} N\mathrm{E}_1 \le \frac{N\mathrm{E}_1}{B_1^{p_2}\alpha_1^{p_2}} \varrho(u). \quad (5.35)$$

Exploiting the algebraic inequality and (5.2), (5.35), we have

$$\begin{aligned} \mathrm{A}^{\frac{1}{1-\sigma}}(t) \le \left(H^{1-\sigma}(t) + \varepsilon N \mathrm{E}_1 T\right)^{\frac{1}{1-\sigma}} &\le 2^{\frac{\sigma}{1-\sigma}} \left(H(t) + \varepsilon^{\frac{1}{1-\sigma}} (N\mathrm{E}_1 T)^{\frac{1}{1-\sigma}}\right) \\ &\le \delta_2 \left(H(t) + \frac{N\mathrm{E}_1}{B_1^{p_2}\alpha_1^{p_2}} \varrho(u)\right), \end{aligned} \quad (5.36)$$

where δ_2 and ε are positive constants such that

$$\delta_2 = 2^{\frac{\sigma}{1-\sigma}} \max\left(1, \frac{N\mathrm{E}_1}{B_1^{p_2}\alpha_1^{p_2}}\right), \quad (5.37)$$

joining (5.34), with (5.36), results in

$$\mathrm{A}'(t) \ge \frac{\varepsilon\delta_1}{\delta_2} \mathrm{A}^{\frac{1}{1-\sigma}}(t), \text{ for all } t \ge 0. \quad (5.38)$$

Simply integrating (5.24) over $(0, t)$ yields the conclusion that

$$\mathrm{A}^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\mathrm{A}^{\frac{\sigma}{1-\sigma}}(0) - \frac{\sigma}{1-\sigma}\frac{\varepsilon\delta_1}{\delta_2}t}. \tag{5.39}$$

As a result, $\mathrm{A}(t)$ explodes in a finite time $\widehat{T}$

$$\widehat{T} \leq \frac{1-\sigma}{\sigma\frac{\varepsilon\delta_1}{\delta_2}\mathrm{A}^{\frac{\sigma}{1-\sigma}}(0)}.$$

Since $\mathrm{A}(0) > 0$, (5.39) demonstrates that $\lim_{t\to T}\mathrm{A}(t) = \infty$, where $T = \frac{1-\sigma}{\sigma\frac{\varepsilon\delta_1}{\delta_2}\mathrm{A}^{\frac{\sigma}{1-\sigma}}(0)}$. This completes the proof.

Case 2. In the case $\mathrm{E}(0) < 0$. Setting $H(t) = -\mathrm{E}(t)$ in Lemma 6, one can obtain a similar result as Lemma 6. Previously $0 < -\mathrm{E}(0) = H(0) \leq H(t)$ and $H(t) \leq \frac{1}{p_1}\varrho(u)$. Making $N = 0$ in (5.2) and by applying the same reason as in part **Case 1.**, we can gain our result. □

6 Second blow-up properties

The blow-up property for system (1.4) is examined in this section, and the following Theorem 4 is proved. Because of the existence of the nonlinear term $\mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t$, our method is different.

Theorem 4 *Suppose g, $m(.)$, and $p(.)$ satisfy the conditions* (H1) − (H2) *with $p_1 > m_2 \geq 2$. Then, under one of the following boundary conditions:*

(i) $\mathrm{E}(0) < 0$

(ii) $\mathrm{E}(0) < \widetilde{\mathrm{E}}_1$ *and* $\kappa^{\frac{1}{2}}\|\nabla u_0\| > \alpha_1$*, the local solution to problem* (1.4) *blows up in finite time T^*.*

Proof Looking at the case $\kappa^{\frac{1}{2}}\|\nabla u_0\|_2 > \alpha_1$ and $0 \leq E(0) < \widetilde{\mathrm{E}}_1$. Let's decide

$$H(t) = \widetilde{\mathrm{E}}_1 - E(t). \tag{6.1}$$

From Lemma 4 (ii), by combining (2.5), (2.6), (5.1) and (6.1), we get

$$\begin{aligned}0 < H(0) \leq H(t) = \widetilde{\mathrm{E}}_1 - E(t) \leq \widetilde{\mathrm{E}}_1 - \frac{\alpha_2^2}{2} + \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x \\ < \left(\frac{1}{q} - \frac{1}{p_1}\right)\alpha_1^2 - \frac{1}{2}\alpha_1^2 + \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x \leq -\frac{1}{p_1}\alpha_1^2 + \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x.\end{aligned}$$

Hence,

$$\int_\Omega |u(t)|^{p(x)}\mathrm{d}x > p_1 H(t) + \alpha_1^2. \tag{6.2}$$

Multiply by u and integrate over Ω, add and subtract $qE(t)$ from the system (1.4) to get

$$
\begin{aligned}
0 &= \int_\Omega |u(t)|^{p(x)}\mathrm{d}x - \|\nabla u\|_2^2 + \int_\Omega\int_0^t g(t-s)\nabla u(t)\nabla u(s)\mathrm{d}s\mathrm{d}x - \left(\mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t, u\right)\\
&= \int_\Omega |u(t)|^{p(x)}\mathrm{d}x - \|\nabla u\|_2^2 + \int_\Omega\int_0^t g(t-s)\nabla u(t)(\nabla u(s)-\nabla u(t))\mathrm{d}s\mathrm{d}x\\
&\quad + \int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2 - \left(\mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t, u\right) + qE(t) - qE(t)\\
&\geq \left(1-\frac{q}{p_1}\right)\int_\Omega |u(t)|^{p(x)}\mathrm{d}x + \left(\frac{q}{2}-1\right)\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2 - \left(\mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t, u\right)\\
&\quad + \frac{\varepsilon(1-\eta)p_1}{2}(g\circ\nabla u)(t) - \left(\delta(g\circ\nabla u)(t) + \frac{1}{4\delta}\int_0^t g(s)\mathrm{d}s\|\nabla u(t)\|_2^2\right) + qH(t) - q\widetilde{\mathrm{E}}_1\\
&\geq a_1(g\circ\nabla u)(t) + a_2\|\nabla u\|_2^2 - \left(\mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t, u\right) + \frac{p_1-q}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x - q\widetilde{\mathrm{E}}_1,
\end{aligned}
\tag{6.3}
$$

regarding a number δ, such that $0 < \delta < \frac{q}{2}$,

$$
a_1 = \frac{q}{2} - \delta > 0,\ a_2 = \left(\frac{q}{2}-1\right) - \left(\frac{q}{2} - 1 + \frac{1}{4\delta}\right)\int_0^t g(s)\mathrm{d}s > 0,
$$

which is feasible from (2.2).

On the other hand, we use Lemma 2.6 to get

$$
\frac{1}{2}\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) \leq E(0) + \int_\Omega \frac{1}{p(x)}|u(t)|^{p(x)}\mathrm{d}x.
$$

Hence, Lemma 4 (ii) generates

$$
\begin{aligned}
\frac{1}{p_1}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x &\geq \frac{1}{2}\left[\left(1-\int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t)\right] - \mathrm{E}(t)\\
&\geq \frac{1}{2}\left[\kappa\|\nabla u(t)\|_2^2 + (g\circ\nabla u)(t)\right] - \mathrm{E}(0)\\
&\geq \frac{1}{2}\alpha_2^2 - h(\alpha_2) = \frac{B_1^{p_2}}{P_1}\alpha_2^{p_2},
\end{aligned}
$$

That's why we get

$$
\begin{aligned}
\left(1-\frac{q}{p_1}\right)\int_\Omega |u(t)|^{p(x)}\mathrm{d}x - q\widetilde{\mathrm{E}}_1 &\geq \left(1-\frac{q}{p_1}\right)\int_\Omega |u(t)|^{p(x)}\mathrm{d}x\\
&\quad - \frac{q}{B_1^{p_2}\alpha_2^{p_2}}\widetilde{\mathrm{E}}_1\int_\Omega |u(t)|^{p(x)}\mathrm{d}x \geq \widetilde{c}\int_\Omega |u(t)|^{p(x)}\mathrm{d}x,
\end{aligned}
\tag{6.4}
$$

where $\widetilde{c} > 0$ due to (5.3).

On the other hand

$$\int_{\Omega} \mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t u \mathrm{d}x \leq c \max\left(\|u_t\|_{m(.)}^{m_1-1}, \|u_t\|_{m(.)}^{m_2-1}\right) \max\left(\|u\|_{m(.)}, \|u\|_{m(.)}\right)$$
$$\leq c_1 \max\left(\left(\int_{\Omega} |u_t|^{m(x)}\,\mathrm{d}x\right)^{\frac{m_1-1}{m_2}}, \left(\int_{\Omega} |u_t|^{m(x)}\,\mathrm{d}x\right)^{\frac{m_2-1}{m_1}}\right)$$
$$\times \max\left(\left(\int_{\Omega} |u|^{p(x)}\mathrm{d}x\right)^{\frac{1}{p_2}}, \left(\int_{\Omega} |u|^{p(x)}\mathrm{d}x\right)^{\frac{1}{p_1}}\right). \tag{6.5}$$

According to (5.18), we have

$$\max\left(\left(\int_{\Omega} |u|^{p(x)}\mathrm{d}x\right)^{\frac{1}{p_2}}, \left(\int_{\Omega} |u|^{p(x)}\mathrm{d}x\right)^{\frac{1}{p_1}}\right)$$
$$\leq \max\left(1, \left(\int_{\Omega} |u|^{p(x)}\mathrm{d}x\right)^{\frac{1}{p_2}-\frac{1}{p_1}}\right)\left(\int_{\Omega} |u|^{p(x)}\mathrm{d}x\right)^{\frac{1}{p_1}} \leq c_2 \left(\int_{\Omega} |u|^{p(x)}\mathrm{d}x\right)^{\frac{1}{p_1}},$$

where

$$c_2 = \max\left((p_1 H(0))^{\frac{p_1-p_2}{p_2}}, 1\right).$$

By combining (1.2), (6.3), (6.4) and (6.5), we have

$$0 \geq \widetilde{c} \int_{\Omega} |u(t)|^{p(x)}\mathrm{d}x - \left(\mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t, u\right)$$
$$\geq \widetilde{c} \int_{\Omega} |u(t)|^{p(x)}\mathrm{d}x - c_2 c_1 \max\begin{pmatrix} \left(\int_{\Omega} |u_t|^{m(x)}\,\mathrm{d}x\right)^{\frac{m_1-1}{m_2}}, \\ \left(\int_{\Omega} |u_t|^{m(x)}\,\mathrm{d}x\right)^{\frac{m_2-1}{m_1}} \end{pmatrix} \left(\int_{\Omega} |u|^{p(x)}\mathrm{d}x\right)^{\frac{1}{p_1}},$$

that is

$$\max\left(\left(\int_{\Omega} |u_t|^{m(x)}\,\mathrm{d}x\right)^{\frac{m_1-1}{m_2}}, \left(\int_{\Omega} |u_t|^{m(x)}\,\mathrm{d}x\right)^{\frac{m_2-1}{m_1}}\right) \geq \frac{\widetilde{c}}{c_2 c_1}\left(\int_{\Omega} |u|^{p(x)}\mathrm{d}x\right)^{\frac{p_1-1}{p_1}},$$

either

$$\left(\int_{\Omega} |u_t|^{m(x)}\,\mathrm{d}x\right)^{\frac{m_1-1}{m_2}} \geq \frac{\widetilde{c}}{c_2 c_1}\left(\int_{\Omega} |u|^{p(x)}\mathrm{d}x\right)^{\frac{p_1-1}{p_1}},$$

or

$$\left(\int_\Omega |u_t|^{m(x)}\,\mathrm{d}x\right)^{\frac{m_2-1}{m_1}} \geq \frac{\widetilde{c}}{c_2c_1}\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{p_1-1}{p_1}},$$

which does mean

$$\begin{aligned}
&\int_\Omega |u_t|^{m(x)}\,\mathrm{d}x \\
&\geq \min\begin{pmatrix} \left(\frac{\widetilde{c}}{c_2c_1}\right)^{\frac{m_2}{m_1-1}}\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{p_1-1}{p_1}\frac{m_2}{m_1-1}}, \\ \left(\frac{\widetilde{c}}{c_2c_1}\right)^{\frac{m_1}{m_2-1}}\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{p_1-1}{p_1}\frac{m_1}{m_2-1}} \end{pmatrix} \\
&= \left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{p_1-1}{p_1}\frac{m_2}{m_1-1}}\min\begin{pmatrix} \left(\frac{\widetilde{c}}{c_2c_1}\right)^{\frac{m_2}{m_1-1}}, \\ \left(\frac{\widetilde{c}}{c_2c_1}\right)^{\frac{m_1}{m_2-1}}\left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{p_1-1}{p_1}\left(\frac{m_1}{m_2-1}-\frac{m_2}{m_1-1}\right)} \end{pmatrix} \\
&\geq \left(\int_\Omega |u|^{p(x)}\mathrm{d}x\right)^{\frac{p_1-1}{p_1}\frac{m_2}{m_1-1}}\min\begin{pmatrix} \left(\frac{\widetilde{c}}{c_2c_1}\right)^{\frac{m_2}{m_1-1}}, \\ \left(\frac{\widetilde{c}}{c_2c_1}\right)^{\frac{m_1}{m_2-1}}(p_1H(0))^{\frac{p_1-1}{p_1}\left(\frac{m_1}{m_2-1}-\frac{m_2}{m_1-1}\right)} \end{pmatrix}.
\end{aligned} \tag{6.6}$$

By combining the embedding theorem, (2.6), (6.2) and (6.6), we get to

$$H'(t) \geq c_0\int_\Omega |u_t|^{m(x)}\,\mathrm{d}x \geq c_3\left(p_1H(t)+\alpha_1^2\right)^{\frac{p_1-1}{p_1}\frac{m_2}{m_1-1}},$$

where

$$c_3 = c_0\min\left(\left(\frac{\widetilde{c}}{c_2c_1}\right)^{\frac{m_2}{m_1-1}}, \left(\frac{\widetilde{c}}{c_2c_1}\right)^{\frac{m_1}{m_2-1}}(p_1H(0))^{\frac{p_1-1}{p_1}\left(\frac{m_1}{m_2-1}-\frac{m_2}{m_1-1}\right)}\right).$$

Because $2 \leq m_1 < p_1$

$$\frac{p_1-1}{p_1}\frac{m_2}{m_1-1} - 1 = \frac{(p_1-1)\,m_2 - p_1\,(m_1-1)}{p_1\,(m_1-1)} > 0,$$

then, for $\gamma = \frac{(p_1-1)m_2-p_1(m_1-1)}{p_1(m_1-1)} > 0$, we have

$$\frac{\left(p_1H(t)+\alpha_1^2\right)'}{\left(p_1H(t)+\alpha_1^2\right)^{1+\gamma}} \geq c_3,$$

and by integrating, considering $H(t) \geq H(0) = \widetilde{\mathrm{E}}_1 - \mathrm{E}(0) \in \left(0, \widetilde{\mathrm{E}}_1\right]$, we have

$$\frac{1}{\left(p_1H(t)+\alpha_1^2\right)^{\gamma}} \geq \frac{1}{\left(p_1H(t)+\alpha_1^2\right)^{\gamma}} + \gamma c_3 t.$$

This is impossible because the right-hand goes to $+\infty$ as t goes to $+\infty$, and the left hand is finite.

By setting $H(t) = -\mathrm{E}(t)$ in (6.1), the proof for the case $\mathrm{E}(0) < 0$ is analogous. Then follows the second result of the blow-up.

7 Global existence and energy decay

By considering the global existence and energy decay of solutions associated with system (1.4). This section is devoted to the proof of the theorem 5.

We start with the well-known lemma

Lemma 7 *Let* $\mathrm{E} : \mathbb{R}^+ \to \mathbb{R}^+$ *be a nonincreasing function and* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a* C^2 *increasing function such that* $\varphi(0) = 0$ *and* $\lim\limits_{t\to+\infty} \varphi(t) = +\infty$. *Assume there is* $c > 0$ *for that*

$$\int_S^{+\infty} \mathrm{E}(t)\varphi'(t)\,\mathrm{d}t \le c\mathrm{E}(S),\ \textit{for any } S \ge 0. \tag{7.1}$$

Then

$$\mathrm{E}(t) \le \lambda \mathrm{E}(0)\, e^{-\varpi\varphi(t)} \ \textit{on}\ [0, +\infty).$$

where λ *and* ϖ *are two positive constants.*

Theorem 5 (Global existence and energy decay) *Suppose* $0 < \sqrt{\kappa}\,\|\nabla u_0\|_2 < \alpha_1$, $0 < \mathrm{E}(0) < \mathrm{E}_1$ *and* (H1) − (H3) *hold. Then the solution is* $u(t)$ *of the system* (1.4) *is globally available, and we can estimate its energy decay as*

$$\mathrm{E}(t) \le k e^{-\varpi \int_0^t \xi(s)\mathrm{d}s} \ \textit{on}\ [0, +\infty). \tag{7.2}$$

Remark 1 Lemma 4 and the hypotheses (H1), and (H2) give us

$$\begin{aligned}\kappa\|\nabla u(t)\|_2^2 &\le \left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) = \alpha(t)\\ &< \alpha_1^2 = \kappa^{\frac{p_2}{p_2-2}}\left(3\frac{p_2}{p_1}\right)^{\frac{-2}{p_2-2}} \hat{B}^{-\frac{2p_2}{p_2-2}} = B_1^{-\frac{2p_2}{p_2-2}},\end{aligned}$$

what that means

$$\begin{aligned}I(t) &= \left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) - \int_\Omega |u(x,t)|^{p(x)}\,\mathrm{d}x\\ &\ge \kappa\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) - \int_\Omega |u(x,t)|^{p(x)}\,\mathrm{d}x\\ &\ge \kappa\|\nabla u(t)\|_2^2 - 3\hat{B}^{p_2}\|\nabla u(t)\|_2^{p_2} \ge 0.\end{aligned}$$

We can also infer this from (2.4) and (2.5)

$$\mathrm{E}(t) \geq \frac{p_1 - 2}{2p_1}\left(\left(1 - \int_0^t g\,(s)\,\mathrm{d}s\right)\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)\right) + \frac{1}{p_1} I(u).$$

Based on the assumptions (H1), (H2), and $\mathrm{E}(t) \leq \mathrm{E}(0)$ this leads us to conclude that

$$\kappa \|\nabla u(t)\|_2^2 \leq \left(1 - \int_0^t g(s)\mathrm{d}s\right)\|\nabla u(t)\|_2^2 \leq \frac{2p_1}{p_1 - 2}\mathrm{E}(t) \leq \frac{2p_1}{p_1 - 2}\mathrm{E}(0). \quad (7.3)$$

Lemma 4 and (7.3) also means

$$\begin{aligned}\int_\Omega |u\,(x,t)|^{p(x)}\,\mathrm{d}x &\leq 3\hat{B}^{p_2}\|\nabla u(t)\|_2^{p_2} \leq 3\hat{B}^{p_2}\|\nabla u(t)\|_2^2\|\nabla u(t)\|_2^{p_2-2} \\ &\leq \frac{3\hat{B}^{p_2}}{\kappa}\left(\frac{2p_1}{(p_1-2)\,\kappa}\mathrm{E}(0)\right)^{\frac{p_2-2}{2}}\left(\kappa\|\nabla u(t)\|_2^2\right) \\ &\leq \varrho\frac{2p_1}{p_1-2}E(t) \text{ for } t \in [0,T)\,, \end{aligned} \quad (7.4)$$

with $\varrho = \frac{3\hat{B}^{p_2}}{\kappa}\left(\frac{2p_1}{(p_1-2)\kappa}\mathrm{E}(0)\right)^{\frac{p_2-2}{2}}$.

Remark 2 Here, from the description of E_1 in (5.3), we also derive that $\mathrm{E}\,(0) < \mathrm{E}_1$ if and only if

$$\varrho = \frac{3\hat{B}^{p_2}}{\kappa}\left(\frac{2p_1}{(p_1-2)\,\kappa}\mathrm{E}(0)\right)^{\frac{p_2-2}{2}} < 1.$$

We can now proceed to prove the Theorem 5.

Proof of Theorem 5 The global existence conclusion follows directly from Remark 1. The decay estimate (7.2) just needs to be proved. If we multiply the equation (1.4) by $\xi(t)u$ and then integrate it over $\Omega \times (S,T)$, we get

$$\begin{aligned}&\int_S^T\int_\Omega \xi(t)\mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t.u\mathrm{d}x\mathrm{d}t + \int_S^T\int_\Omega \xi(t)|\nabla u|^2\mathrm{d}x\mathrm{d}t \\ &-\int_S^T\int_\Omega \xi(t)\nabla u(t).\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\mathrm{d}x\mathrm{d}t = \int_S^T\int_\Omega \xi(t)|u|^{p(x)}\mathrm{d}x\mathrm{d}t.\end{aligned} \quad (7.5)$$

The last term on the left is estimated as follows:

$$
\begin{aligned}
&-\int_s^T \int_\Omega \xi(t)\nabla u(t).\int_0^t g(t-s)\nabla u(s)\mathrm{d}s\mathrm{d}x\mathrm{d}t\\
&=\int_s^T \int_\Omega \xi(t)\nabla u(t).\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\ \mathrm{d}x\ \mathrm{d}t\\
&\quad-\int_s^T \int_0^t g(s)\int_\Omega \xi(t)|\nabla u|^2\ \mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{7.6}
$$

Combine (7.6) and (7.5) from the previous equation

$$
\begin{aligned}
2\int_S^T \xi(t)\mathrm{E(t)} \le &-\int_S^T \int_\Omega \xi(t)\mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t u\ \mathrm{d}x\mathrm{d}t+\int_S^T \xi(t)(g\circ\nabla u)(t)\mathrm{d}t\\
&-\int_S^T \int_\Omega \xi(t)\nabla u(t).\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\mathrm{d}x\mathrm{d}t\\
&+\frac{p_2-2}{p_2}\int_S^T \xi(t)\int_\Omega |u(t)|^{p(x)}\mathrm{d}x,
\end{aligned}
\tag{7.7}
$$

by combining (2.5), (2.6), (7.3) the boundedness of L and the condition (H3) we get

$$
\begin{aligned}
&\int_\Omega \xi(t)\mathrm{L}(t)\,|u_t|^{m(x)-2}\,u_t u\mathrm{d}x \le \delta\int_\Omega |u|^{m(x)}\mathrm{d}x + c_\delta\int_\Omega |u_t|^{m(x)}\,\mathrm{d}x\\
&\le 3\delta\hat{B}^{p_2}\max\left(\|\nabla u\|_2^{m_1},\|\nabla u\|_2^{m_2}\right)+c_\delta\int_\Omega |u_t|^{m(x)}\,\mathrm{d}x\\
&\le 3\delta\hat{B}^{p_2}\max\left(\|\nabla u\|_2^{m_1-2},\|\nabla u\|_2^{m_2-2}\right)\|\nabla u\|_2^2+c_\delta\int_\Omega |u_t|^{m(x)}\,\mathrm{d}x\\
&\le 3\delta\hat{B}^{p_2}\max\left(\frac{1}{\kappa}\left(\frac{2p_1}{\kappa\,(p_1-2)}\mathrm{E}(0)\right)^{\frac{m_1-2}{2}},\frac{1}{\kappa}\left(\frac{2p_1}{\kappa\,(p_1-2)}\mathrm{E}(0)\right)^{\frac{m_2-2}{2}}\right)\kappa\|\nabla u\|_2^2\\
&+c_\delta\int_\Omega |u_t|^{m(x)}\,\mathrm{d}x\\
&\le \delta c_1\mathrm{E}(t)-\frac{c_\delta}{c_0}\mathrm{E}'(t),\qquad \text{for any } \delta>0,
\end{aligned}
\tag{7.8}
$$

where $c_1=3\hat{B}^{p_2}\max\left(\left(\sqrt{\kappa}\right)^{-m_1}\left(\frac{2p_1}{p_1-2}\mathrm{E}(0)\right)^{\frac{m_1-2}{2}},\left(\sqrt{\kappa}\right)^{-m_2}\left(\frac{2p_1}{p_1-2}\mathrm{E}(0)\right)^{\frac{m_2-2}{2}}\right).$

Out of (7.3) also exists

$$
\begin{aligned}
&\int_\Omega \nabla u(t) \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\mathrm{d}s\,\mathrm{d}x \\
&\leq \delta\|\nabla u(t)\|_2^2 + \frac{1}{4\delta}\int_\Omega \left|\int_0^t g(t-s)(\nabla u(s)-\nabla u(t))\mathrm{d}s\right|^2 \mathrm{d}x \\
&\leq \delta\|\nabla u(t)\|_2^2 + \frac{1}{4\delta}\left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega\int_0^t g(t-s)|\nabla u(s)-\nabla u(t)|^2\mathrm{d}s\mathrm{d}x \\
&\leq \frac{2p_1\delta}{(p_1-2)\,\kappa}\mathrm{E}(t) + \frac{1-1}{4\delta}(g\circ\nabla u)(t), \quad \text{for any } \delta>0.
\end{aligned}
\tag{7.9}
$$

From (H2) and (2.6), we can conclude that,

$$
\xi(t)(g\circ\nabla u)(t) \leq -\left(g'\circ\nabla u\right)(t) \leq -2\mathrm{E}'(t). \tag{7.10}
$$

Consequently, by combing (7.4) and (7.7)–(7.10), we conclude

$$
\begin{aligned}
2\int_S^T \xi(t)\mathrm{E}(t) &\leq \left(\delta c_1 + \frac{2p_1\delta}{(p_1-2)\,\kappa}\right)\int_S^T \xi(t)\mathrm{E}(t)\mathrm{d}t - \frac{c_\delta}{c_0}\int_S^T \xi(t)\mathrm{E}'(t)\mathrm{d}t \\
&+ \left(1+\frac{1-\kappa}{4\delta}\right)\int_S^T \xi(t)(g\circ\nabla u)(t)\mathrm{d}t + \frac{2p_1}{p_1-2}\int_S^T \xi(t)\|u(t)\|_p^p\,\mathrm{d}t \\
&\leq \left(2\alpha + \delta c_1 + \frac{2p_1\delta}{(p_1-2)\,\kappa}\right)\int_S^T \xi(t)\mathrm{E}(t)\mathrm{d}t + \frac{c_\delta}{c_0}\xi(0)\mathrm{E}(S) \\
&- \left(2+\frac{1-\kappa}{2\delta}\right)\int_S^T \mathrm{E}'(t)\mathrm{d}t \\
&\leq \left(2\alpha + \delta c_1 + \frac{2p_1\delta}{(p_1-2)\,\kappa}\right)\int_S^T \xi(t)\mathrm{E}(t)\mathrm{d}t + \left(\frac{c_\delta}{c_0}\xi(0) + 2 + \frac{1-\kappa}{2\delta}\right)\mathrm{E}(S).
\end{aligned}
$$

Note that $\alpha < 1$, is chosen δ too small enough for

$$
2 - 2\alpha - \delta c_1 - \frac{2p_1\delta}{(p_1-2)\,\kappa} > 0.
$$

As a result, there is a positive constant $\sigma > 0$ such that

$$
\int_S^T \xi(t)\mathrm{E}(t)\mathrm{d}t \leq \sigma\mathrm{E}(S), \text{ for any } S \geq 0.
$$

In the inequality started earlier, by letting T go to $+\infty$ in the left hand, one can easily conclude that (7.1) is satisfied with $\varphi(t) = \int_0^t \xi(s)\, ds$. Thus, (7.2) is confirmed.

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Conflict of interest The authors have no conflict of interest to report.

References

1. Liu, G., Chen, H.: Global and blow-up of solutions for a quasilinear parabolic system with viscoelastic and source terms. Math. Methods Appl. Sci. **37**(I), 148–156 (2013)
2. John, N.A.: Nonlinear Voltera equations for heat flow in materials with memory, Technical Summary Report, pp. 20–81 (1990)
3. Gao, W.L., Han, Y.Z.: Blow-up of a nonlocal semilinear parabolic equation with positive initial energy. Appl. Math. Lett. **24**, 784–788 (2011)
4. Dafermos, C.M.: Asymptotic stability in viscoelasticity. Arch. Ration. Mech. Anal. **37**, 208–297 (1970)
5. Cavalcanti, M.M., Oquendo, H.P.: Frictional versus viscoelastic damping in a semilinear wave equation. SIAM J. Control Optim. **42**, 1310–1324 (2003)
6. Ferreira, J., Cavalcanti, M.M., Cavalcanti, D.: Existence and uniform decay for nonlinear viscoelastic equation with strong damping. Math. Methods Appl. Sci. **24**, 1043–1053 (2002)
7. Pata, V.: Exponential stability in viscoelasticity. Q. Appl. Math. **3**, 499–513 (2006)
8. Wang, Y.: A global nonexistence theorem for viscoelastic equations with arbitrary positive initial energy. Appl. Math. Lett. **22**, 1394–1400 (2009)
9. Song, H., Zhong, C.: Blow-up of solutions of a nonlinear viscoelastic wave equation. Nonlinear Anal. Real World Appl. **11**, 3877–3883 (2010)
10. Christensen, R.: Theory of Viscoelasticity: An Introduction. Elsevier, Amsterdam (1982)
11. Aboulaicha, R., Meskinea, D., Souissia, A.: New diffusion models in image processing. Comput. Math. Appl. **56**, 874–882 (2008)
12. Levine, S., Chen, Y., Stanich, J.: Image restoration via nonstandard diffusion, Technical Report 04-01, Dept. of Mathematics and Computer Science, Duquesne University (2004)
13. Chen, Y., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. **66**, 1383–1406 (2006)
14. Stanislav, A.N., José, R.F.: On stationary thermo-rheological viscous, flows. Ann. Univ. Ferrara **52**, 19–36 (2006)
15. Antontsev, S., Shmarev, S.: Blow-up of solutions to parabolic equations with nonstandard growth conditions. J. Comput. Appl. Math. **234**(9), 2633–2645 (2010)
16. Antontsev, S., Zhikov, V.: Higher integrability for parabolic equations of $p(x, t)$-Laplacian type. Adv. Differ. Equ. **10**(9), 1053–1080 (2005)
17. Růžička, M.: Electrorheological Fluids: Modeling and Mathematical Theory. In: Lecture Notes in Mathematics, vol. 1748. Springer, Berlin (2000)
18. Lions, J.L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Paris (1966)
19. Abita, R.: Existence and asymptotic stability for the semilinear wave equation with variable-exponent nonlinearities. J. Math. Phys. **60**, 122701 (2019)
20. Abita, R.: Existence and asymptotic behavior of solutions for degenerate nonlinear Kirchhoff strings with variableexponent nonlinearities. Acta Math. Vietnam **46**, 613–643 (2021)
21. Abita, R.: Lower and upper bounds for the blow-up time to a viscoelastic Petrovsky wave equation with variable sources and memory term. Appl. Anal. **102**(12), 3503–3531 (2023)

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