ANNALI DELL'UNIVERSITA' DI FERRARA (2024) 70:1083–1092
https://doi.org/10.1007/s11565-024-00525-5

Automorphisms of Fano threefolds of rank 2 and degree 28

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Received: 21 March 2024 / Accepted: 29 April 2024 / Published online: 1 June 2024

Abstract
We describe the automorphism groups of smooth Fano threefolds of rank 2 and degree 28 in the cases where they are finite.

Keywords Fano varieties · Automorphism groups · Birational geometry

1 Introduction

A smooth Fano threefold of Picard rank 2 and degree 28 is the blow-up of a smooth quadric threefold $Q \subset \mathbb{P}^4$ in a smooth rational quartic curve $C_4 \subset Q$. Isomorphism classes of such threefolds form an irreducible two-dimensional family, which according to the Mori-Mukai classification corresponds to family 2.21. Let $\pi : X \to Q$ be such a threefold. Then the action of $\mathrm{Aut}(Q, C_4)$ on Q lifts to an action on X, so that we may identify it with a subgroup of $\mathrm{Aut}(X)$.

By a result of Cheltsov-Przyjalkowski-Shramov ([3, Lemma 9.2]), we have that either $\mathrm{Aut}(X)$ is finite, or $\mathrm{Aut}(X) \cong \mathrm{Aut}(Q, C_4) \times \mathbb{Z}_2$, where upto isomorphism $\mathrm{Aut}(Q, C_4)$ is described as follows:

(1) There is a unique smooth threefold in family 2.21, unique upto isomorphism, such that $\mathrm{Aut}(Q, C_4) \cong \mathrm{PGL}_2(\mathbb{C})$,
(2) There is a one-dimensional family of non-isomorphic smooth threefolds in family 2.21 such that $\mathrm{Aut}(Q, C_4) \cong \mathbb{G}_m \rtimes \mathbb{Z}_2$,
(3) There is a unique smooth threefold in family 2.21 such that $\mathrm{Aut}(Q, C_4) \cong \mathbb{G}_a \rtimes \mathbb{Z}_2$.

The goal of this paper is to describe $\mathrm{Aut}(X)$ when it is finite, where X is a smooth threefold in family 2.21. Our main result is the following:

Theorem 1.1 *Let X be a smooth Fano threefold of rank 2 and degree 28. Then* $\mathrm{Aut}(X) \cong \mathrm{Aut}(Q, C_4) \times \mathbb{Z}_2$. *Furthermore, if* $\mathrm{Aut}(Q, C_4)$ *is finite then it is isomorphic to* $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2$ *or* 0.

We prove this theorem in two parts: Theorem 2.1 and Theorem 3.1.

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Remark 1.2 The factor of $\mathbb{Z}_2$ appearing in the factorisation $\mathrm{Aut}(X) \cong \mathrm{Aut}(Q, C_4) \times \mathbb{Z}_2$ is generated by an involution g, which may be described as follows:

Let $\mathfrak{d}$ denote the restriction to Q of the linear system of quadric hypersurfaces in $\mathbb{P}^4$ which contain C_4, and let $\phi\colon Q \dashrightarrow \mathbb{P}^4$ be the corresponding rational map. The image of ϕ is a smooth quadric threefold, and ϕ contracts the intersection of the secant variety of C_4 with Q, V, onto a smooth rational curve $C_4' \subset Q'$. The base locus of ϕ is equal to C_4, and there is a birational morphism $\pi'\colon X \to Q'$, where X is the blow-up of Q along C_4. This morphism contracts the strict transform, E', of V onto the curve C_4'. Thus, there is a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\pi}\swarrow & & \searrow^{\pi'} \\ Q & \overset{\phi}{\dashrightarrow} & Q'. \end{array}$$

In [3], it is shown that in cases (1) and (2) of the above classification, there exists a basis of $\mathfrak{d}$ such that $Q' = Q$ and $C_4' = C_4$, so that ϕ lifts to an involution $g \in \mathrm{Aut}(X)$. We will show in Theorem 3.1 that this is always the case.

We can explicitly describe the threefolds appearing in [3, Lemma 9.2]. Let us fix some notation. Observe that after a projective transformation C_4 is the image of the Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^4$:

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^4 \\ [u : v] &\mapsto [u^4 : u^3v : u^2v^2 : uv^3 : v^4]. \end{aligned}$$

The space of global sections of $\mathcal{I}_{C_4}(2)$ is generated by the following quadratic forms:

$$\begin{aligned} f_0 &= x_3^2 - x_2x_4, \\ f_1 &= x_2x_3 - x_1x_4, \\ f_2 &= x_2^2 - x_0x_4, \\ f_3 &= x_1x_2 - x_0x_3, \\ f_4 &= x_1^2 - x_0x_2, \\ f_5 &= 3x_2^2 - 4x_1x_3 + x_0x_4, \end{aligned}$$

where $\mathcal{I}_{C_4}$ is the ideal sheaf of C_4 in $\mathbb{P}^4$, and x_0, x_1, x_2, x_3, x_4 are homogeneous coordinates on $\mathbb{P}^4$. Observe that the standard $\mathrm{PGL}_2(\mathbb{C})$-action on C_4 lifts to an action on $\mathbb{P}^4$ such that C_4 is invariant. We fix the following subgroups of $\mathrm{PGL}_2(\mathbb{C})$:

$$\mathbb{Z}_2, \text{ generated by } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\mathbb{G}_m, \text{ consisting of matrices } \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \text{ for every } t \in \mathbb{G}_m,$$
$$\mathbb{G}_a, \text{ consisting of matrices } \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ for every } t \in \mathbb{G}_a.$$

Now we can describe $\mathrm{Aut}(X)$ for the threefolds listed before:

Example 1.3 ([1, Section 5.9]). Let Q be the quadric given by the equation

$$(1 - 4s^2) f_2 + f_5 = 0,$$

for some $s \in \mathbb{C} \setminus \{-1, 0, 1\}$. Then Q is $\mathbb{G}_m$-invariant and $\mathbb{Z}_2$-invariant, and conversely any smooth quadric admitting a faithful $\mathbb{G}_m$-action is isomorphic, via an element of $\mathrm{PGL}_2(\mathbb{C})$, to a quadric given by an equation of this form. Moreover, we have the following:

$$\mathrm{Aut}(Q, C_4) \cong \begin{cases} \mathbb{G}_m \rtimes \mathbb{Z}_2, & s \neq \pm\frac{1}{2}, \\ \mathrm{PGL}_2(\mathbb{C}), & s = \pm\frac{1}{2}. \end{cases}$$

The involution g described before is given by:

$$\tau\colon [x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [f_4 : s f_3 : s^2 f_2 : s f_1 : f_0].$$

See [1, Remark 5.52] for an explanation of why $\tau \circ \tau\colon Q \dashrightarrow Q$ is the identity map on $Q \setminus C_4$.

Example 1.4 Suppose that the quadric Q is given by the equation

$$f_0 + f_5 = 0.$$

Then Q is $\mathbb{G}_a$-invariant and $\mathbb{Z}_2$-invariant, and $\mathrm{Aut}(Q, C_4) \cong \mathbb{G}_a \rtimes \mathbb{Z}_2$. We will prove in case (2) of Theorem 3.1 that the blow-up of Q in C_4 admits an action of the involution g.

Remark 1.5 Recall that for a finite subgroup $G \subset \mathrm{Aut}(Y)$, a variety Y is called *G-Fano* if it has terminal singularities, $-K_Y$ is ample and $\mathrm{Cl}(Y)^G$ is rank 1. It is proven in [4] that the Hilbert scheme of conics on a smooth threefold X from the family No2.21 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, with the degenerate conics being parameterised by a smooth curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 2)$. If X is G-Fano for the group $G = \langle g \rangle \cong \mathbb{Z}_2$, then this curve must be invariant upon swapping the two factors of $\mathbb{P}^1$.

An informal conjecture of Y. Prokhorov is that the invariance of this curve is a sufficient condition for X to be G-Fano. It is proven in [2] that every smooth curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, 2)$ is invariant. As a corollary to Theorem 1.1, we have that every smooth threefold X in the family No2.21 is G-Fano, so that Prokhorov's informal conjecture is true. For a detailed discussion of G-Fano threefolds, see [6].

Remark 1.6 Smooth threefolds in the family No2.21 are parametrised by $\mathbb{P}^5 \setminus \Delta$, where $\Delta \subset \mathbb{P}^5$ is the discriminant locus of singular quadrics. The group $\mathrm{PGL}_2(\mathbb{C})$ acts on this space, and it follows from Theorem 1.1 that any two threefolds in family 2.21 are isomorphic if and only if their corresponding points in the parameter space $\mathbb{P}^5 \setminus \Delta$ lie in the same $\mathrm{PGL}_2(\mathbb{C})$-orbit. Moreover, the moduli space of smooth GIT-polystable threefolds in family 2.21 is given by the GIT quotient

$$(\mathbb{P}^5 \setminus \Delta)//\mathrm{PGL}_2(\mathbb{C}).$$

2 Computation of Aut(Q, C_4)

The first half of proving Theorem 1.1 is the computation of $\mathrm{Aut}(Q, C_4)$, which we will do in this section. The result we will prove is:

Theorem 2.1 *Let Q be a smooth quadric threefold containing the quartic curve C_4. If $\mathrm{Aut}(Q, C_4)$ is finite, then it is isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2$ or 0.*

The following lemma will be useful:

Lemma 2.2 *Let $Q \subset \mathbb{P}^4$ be any quadric hypersurface containing the curve C_4. Suppose that $\mathrm{Aut}(Q, C_4)$ is finite, and contains an element of finite order $n > 2$. Then Q is singular.*

Proof Since $\mathrm{Aut}(Q, C_4) \subseteq \mathrm{Aut}(\mathbb{P}^4, C_4) \cong \mathrm{PGL}_2(\mathbb{C})$, we may identify $\mathrm{Aut}(Q, C_4)$ with a subgroup of $\mathrm{PGL}_2(\mathbb{C})$. Moreover, by considering the action of $\mathrm{PGL}_2(\mathbb{C})$ on the parameter space $\mathbb{P}^5$, we identify $\mathrm{Aut}(Q, C_4)$ with the stabiliser of the point of $\mathbb{P}^5$ corresponding to Q. Fix a finite cyclic subgroup $G \subset \mathrm{PGL}_2(\mathbb{C})$ of order n, and let $g_1 \in G$ be a generator. Then g_1 fixes precisely two distinct points of $\mathbb{P}^1$, which upto projective transformation are $[0 : 1]$ and $[1 : 0]$. Thus

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix},$$

for some primitive n^{th} root of unity ζ. Then g_1 acts on $\mathbb{P}^5$ by:

$$[a_0 : a_1 : a_2 : a_3 : a_4 : a_5] \mapsto [\zeta^6 a_0 : \zeta^5 a_1 : \zeta^4 a_2 : \zeta^3 a_3 : \zeta^2 a_4 : \zeta^4 a_5],$$

and we can read off the points of $\mathbb{P}^5$ whose stabiliser contains G:

- $n = 2$: $(\mathbb{P}^5)^G = \{[a_0 : 0 : a_2 : 0 : a_4 : a_5], [0 : a_1 : 0 : a_3 : 0 : 0]\}$,
- $n = 3$: $(\mathbb{P}^5)^G = \{[a_0 : 0 : 0 : a_3 : 0 : 0], [0 : a_1 : 0 : 0 : a_4 : 0]\}$,
- $n = 4$: $(\mathbb{P}^5)^G = \{[a_0 : 0 : 0 : 0 : a_4 : 0]\}$,
- $n > 4$: $(\mathbb{P}^5)^G = \varnothing$,

where the numbers $a_0, a_1, a_2, a_3, a_4, a_5$ are all arbitrary complex numbers. One checks that for $n > 2$, the corresponding threefolds which have finite $\mathrm{Aut}(Q, C_4)$ are all singular.

Now let us recall the following classification theorem for quadric threefolds which contain the curve C_4:

Theorem 2.3 ([5]) *Let $Q \subset \mathbb{P}^4$ be a smooth quadric containing C_4. Then there exists an automorphism $\phi \in \mathrm{PGL}_2(\mathbb{C})$ such that $\phi(Q)$ is given by one of the following equations:*

(1) $\mu(f_0 + f_4) + \lambda f_2 + f_5 = 0$, *for some* $\lambda \in \mathbb{C} \setminus \{1, -3\}$ *and* $\mu \in \mathbb{C} \setminus \{2, -2\}$ *such that* $\mu^2 \neq -\lambda^2 - 2\lambda + 3$,
(2) $f_0 + \lambda f_2 + f_5 = 0$, *for some* $\lambda \in \mathbb{C} \setminus \{1, -3\}$,
(3) $f_1 + f_5 = 0$.

Let us find $\mathrm{Aut}(Q, C_4)$ in each of these cases.

Proof of Theorem 2.1. We may assume that $\mu \neq 0$ in case (1), and $\lambda \neq 0$ in case (2), as otherwise the threefolds are isomorphic to those which are described in Example 1.3 and Example 1.4. Then $\mathrm{Aut}(Q, C_4)$ is finite, and since $\mathrm{Aut}(Q, C_4)$ is isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C})$, it must be isomorphic to one of the following groups:

$$0, \mathbb{Z}_n, \mathbb{Z}_2 \times \mathbb{Z}_2, D_{2n}, \mathfrak{A}_4, \mathfrak{S}_4, \mathfrak{A}_5,$$

where $\mathfrak{S}_n$ (resp. $\mathfrak{A}_n$) is the symmetric (resp. alternating) group on n letters. Then by Lemma 2.2 the only possibilities are that $\mathrm{Aut}(Q, C_4)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2$ or 0.

Suppose Q is in case (1). Then Q admits an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by $g_1, g_2 \in \mathrm{PGL}_2(\mathbb{C})$, which are given by:

$$g_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence, $\mathrm{Aut}(Q, C_4) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose that Q is in case (2). Then Q admits an action of the group $\mathbb{Z}_2$, generated by the element g_1. Suppose that $\mathrm{Aut}(Q, C_4) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and let $g \in \mathrm{Aut}(Q, C_4)$ be a non-trivial element distinct from g_1. Considering the standard action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{P}^1$, observe that g_1 fixes the points $[0 : 1]$ and $[1 : 0]$, and since $gg_1 = g_1 g$, we see that g must swap these points. Since g has order 2, it must be equal to either g_2 or $g_1 g_2$. The threefold Q is not invariant under either of these.

Finally suppose that Q is in case (3), and suppose that $\mathrm{Aut}(Q, C_4)$ is non-trivial. Then it contains an element, g, of order 2. Since g fixes two distinct points of $\mathbb{P}^1$, it must be equal to g_2, $g_1 g_2$, or be given by a matrix of the form

$$\begin{pmatrix} 1 & a \\ b & -1 \end{pmatrix}, \text{ for some } a, b \in \mathbb{C} \text{ such that } \quad ab \neq -1.$$

One checks that g_2 nor $g_1 g_2$ leave Q invariant, and if g is given by a matrix of the above form then $g(Q)$ is given by the equation:

$$\begin{aligned} &4bf_0 + 2(1 - 3bc)f_1 - 3c(1 - bc)f_2 + 2c^2(3 - bc)f_3 - 4c^3 f_4 \\ &\quad +(bc^2 - 2b^2c^2 - 4bc - c - 2)f_5 = 0 \end{aligned}$$

Clearly $g(Q) \neq Q$, so that $\mathrm{Aut}(Q, C_4)$ has to be trivial. □

3 Existence of the additional involution

The second half of proving Theorem 1.1 is the assertion that $\mathrm{Aut}(X) \cong \mathrm{Aut}(Q, C_4) \times \mathbb{Z}_2$, which we will do in this section. The result is:

Theorem 3.1 *Let X be a smooth Fano threefold in family No2.21. Then there exists an involution $g \in \mathrm{Aut}(X)$ such that $\mathrm{Aut}(X) \cong \mathrm{Aut}(Q, C_4) \times \langle g \rangle$.*

Proof We proceed case-by-case, according to the classification in Theorem 2.3.

Case (1): Q is given by $\mu(f_0 + f_4) + \lambda f_2 + f_5 = 0$

Observe that the linear system of quadrics which contain C_4 is 5-dimensional, so it is more natural to express members of family No2.21 in terms of fourfolds. Let us show how to do this. Fix the Veronese surface $S_4 \subset \mathbb{P}^5$ given by the embedding:

$$\upsilon \colon \mathbb{P}^2 \to \mathbb{P}^5$$
$$[x : y : z] \mapsto [x^2 : xy : y^2 : yz : z^2 : xz].$$

The space of global sections of $\mathcal{I}_{S_4}(2)$ is generated by the quadratic forms:

$$\begin{aligned} g_0 &= x_3^2 - x_2x_4, \\ g_1 &= x_3x_5 - x_1x_4, \\ g_2 &= x_5^2 - x_0x_4, \\ g_3 &= x_1x_5 - x_0x_3, \\ g_4 &= x_1^2 - x_0x_2, \\ g_5 &= x_1x_3 - x_2x_5, \end{aligned}$$

where $x_0, x_1, x_2, x_3, x_4, x_5$ are homogeneous coordinates on $\mathbb{P}^5$.

Consider the following rational map:

$$\phi \colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$$
$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [g_0 : g_1 : g_2 : g_3 : g_4 : g_5].$$

I claim that ϕ is a birational involution. The following observation is due to I. Dolgachev: we can identify $\mathbb{P}^5$ with the space of symmetric 3×3 matrices, upto scaling. Then under this identification, the rational map above is:

$$\phi\colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$$
$$\begin{pmatrix} x_0 & x_1 & x_5 \\ x_1 & x_2 & x_3 \\ x_5 & x_3 & x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_3^2 - x_2x_4 & x_3x_5 - x_1x_4 & x_1x_3 - x_2x_5 \\ x_3x_5 - x_1x_4 & x_5^2 - x_0x_4 & x_1x_5 - x_0x_3 \\ x_1x_3 - x_2x_5 & x_1x_5 - x_0x_3 & x_1^2 - x_0x_2 \end{pmatrix}$$

But this is the same map as taking a matrix M to its adjoint $\operatorname{adj}(M)$. Thus it follows from the relation $\operatorname{adj}(\operatorname{adj}(A)) = \det(A)^{n-2}A$ for any $n \times n$ matrix A that ϕ is a birational involution.

Let $\sigma : \widetilde{\mathbb{P}}^5 \to \mathbb{P}^5$ be the blow-up of $\mathbb{P}^5$ in S_4, and let E be the exceptional divisor. Observe that for general divisors $\widetilde{H} \in |\sigma^*\mathcal{O}_{\mathbb{P}^5}(1)|$ and $\widetilde{Q} \in |\sigma^*\mathcal{O}_{\mathbb{P}^5}(2) - E|$, we have that $\widetilde{H} \cap \widetilde{Q}$ is a smooth element of family 2.21.

Since ϕ has base locus equal to S_4, it lifts to a biregular involution $g \in \operatorname{Aut}(\widetilde{\mathbb{P}}^5)$ which swaps the linear systems $|\widetilde{H}|$ and $|\widetilde{Q}|$. Thus, the intersection $\widetilde{H} \cap g(\widetilde{H})$ is $\langle g\rangle$-invariant, for any $\widetilde{H}$. We will now show that every smooth element X of family 2.21 which is in case (1) of Theorem 2.3 is isomorphic to a subvariety of $\widetilde{\mathbb{P}}^5$ of the form $\widetilde{H} \cap g(\widetilde{H})$, for some hyperplane $H \subset \mathbb{P}^5$, and therefore possesses an involution not coming from $\operatorname{Aut}(Q, C_4)$.

So fix such a threefold X. Then the quadric Q is given by the equation

$$\mu(f_0 + f_4) + \lambda f_2 + f_5 = 0,$$

for some $\lambda \in \mathbb{C} \setminus \{1, -3\}$ and $\mu \in \mathbb{C} \setminus \{2, -2\}$ such that $\mu^2 \neq -\lambda^2 - 2\lambda + 3$. Let us choose roots a, b of the equations

$$(\mu + 2)x^4 + 2\lambda - 2 = 0,$$
$$(\mu + 2)x^4 + \mu - 2 = 0,$$

respectively, so that the equation of Q becomes:

$$\frac{2 - 2b^4}{1 + b^4}(f_0 + f_4) + \frac{1 - 2a^4 + b^4}{1 + b^4} f_2 + f_5 = 0. \tag{3.2}$$

Now consider the following hypersurfaces in $\mathbb{P}^5$:

$$H = \{x_0 = a^2x_2 + b^2x_4\},$$
$$Q_2 = \{g_0 = a^2g_2 + b^2g_4\}.$$

We have that the intersections $H \cap S_4$ and $Q_2 \cap S_4$ are smooth, so that the intersection of their strict transforms, $\widetilde{H} \cap \widetilde{Q}_2 \subset \widetilde{\mathbb{P}}^5$, is a smooth member of family 2.21. Moreover, $Q_2 = \phi(H)$, so that $\widetilde{H} \cap \widetilde{Q}_2$ is $\langle g\rangle$-invariant. Consider the projective transformation

$\psi: \mathbb{P}^5 \to \mathbb{P}^5$ given by the matrix[1]

$$\begin{pmatrix} 1 & 0 & 0 & 0 & b^2 & -2b \\ 0 & -a & 0 & ab & 0 & 0 \\ 0 & 0 & a^2 & 0 & 0 & 0 \\ 0 & -a & 0 & -ab & 0 & 0 \\ 1 & 0 & 0 & 0 & b^2 & 2b \\ 1 & 0 & 0 & 0 & -b^2 & 0 \end{pmatrix}.$$

Then the intersection of the fourfolds

$$\psi(H) = \{x_2 - x_5 = 0\}$$
$$\psi(Q_2) = \{(1-b^4)(g_0 + g_4) - a^4 g_2 + 2(1+b^4)g_5 = 0\}$$

is given as a subvariety of $\mathbb{P}^4$ by Eq. 3.2, which defines X. Thus $X \cong \widetilde{H} \cap \widetilde{Q}_2$.

It remains to show that the birational involution g commutes with the action of $\mathrm{Aut}(Q, C_4)$.

Consider the subgroup $G \subset \mathrm{PGL}_6(\mathbb{C})$ generated by the commuting involutions

$$\alpha: [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_0 : x_1 : x_2 : -x_3 : x_4 : -x_5]$$
$$\beta: [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_0 : -x_1 : x_2 : -x_3 : x_4 : x_5].$$

Then α and β commute with the birational involution described previously:

$$\phi: [x_0 : x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [x_3^2 - x_2x_4 : x_3x_5 - x_1x_4 : x_5^2 - x_0x_4 : \\ : x_1x_5 - x_0x_3 : x_1^2 - x_0x_2 : x_1x_3 - x_2x_5],$$

The hypersurfaces H and Q_2 are G-invariant. Moreover S_4 is G-invariant, so that G is isomorphic to a subgroup of $\mathrm{Aut}(Q, C_4)$. Thus since $\mathrm{Aut}(Q, C_4) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by Theorem 2.1, we have that $\mathrm{Aut}(Q, C_4) \cong G$. So we see that $\mathrm{Aut}(Q, C_4)$ commutes with the involution g.

For the remaining cases, we will compute bases for the linear system $\mathfrak{d}$ of quadrics sections of $Q \subset \mathbb{P}^4$ containing the curve C_4 such that the corresponding rational map is an involution, and commutes with the action of $\mathrm{Aut}(Q, C_4)$.

[1] The matrix defining ψ comes from the embedding $\mathrm{PGL}_3(\mathbb{C}) \hookrightarrow \mathrm{PGL}_6(\mathbb{C})$, which is given by the projectivisation of the symmetric square, $\mathbb{P}(\mathrm{Sym}^2(\mathbb{C}^3)) \cong \mathbb{P}^5$, of the standard $\mathrm{GL}_3(\mathbb{C})$ action. The Veronese surface S_4 is invariant under this action, so that $\psi(Q_2)$ contains S_4.

Case (2): Q is given by $f_0 + \lambda f_2 + f_5 = 0$

Let us make the substitution $\lambda = 1 - 4s^2$, for some $s \in \mathbb{C} \setminus \{-1, 0, 1\}$. Consider the rational map:

$$\iota\colon Q \dashrightarrow \mathbb{P}^4$$
$$[x_0 : x_1 : x_2 : x_3 : x_4] \mapsto \left[f_4 + \frac{s^2}{2} f_2 - \frac{1}{16} f_0 : \frac{s}{4} f_1 + s f_3 : s^2 f_2 : s f_1 : f_0\right].$$

Observe that it has base locus equal to C_4, so indeed corresponds to the linear system $\mathfrak{d}$. To see that the map ι is a birational involution, consider the following rational parametrisation of Q,

$$p\colon \mathbb{P}^3 \dashrightarrow Q$$
$$[x_0 : x_2 : x_3 : x_4] \mapsto \left[x_0x_3 : s^2x_0x_4 - s^2x_2^2 + x_2^2 - \frac{1}{4}x_2x_4 + \frac{1}{4}x_3^2 : x_2x_3 : x_3^2 : x_3x_4\right].$$

This is a rational inverse to the projection $Q \dashrightarrow \mathbb{P}^3$ from the point $[0 : 1 : 0 : 0 : 0]$. Moreover, it is an isomorphism between the open subsets $\mathbb{P}^3 \setminus \Pi$ and $Q \setminus V$, where $\Pi \subset \mathbb{P}^3$ is the plane given by $x_3 = 0$, and $V \subset Q$ is the singular quadric surface given by the intersection of Q with the plane $x_3 = 0$, this latter variety being the closure of the union of lines through $[0 : 1 : 0 : 0 : 0]$. Let Z be the curve $p^{-1}(C_4)$, which is a quartic rational curve in $\mathbb{P}^3$.

Then $\iota(p(\mathbb{P}^3 \setminus (\Pi \cup Z))$ lies in Q, and since $p(\mathbb{P}^3 \setminus (\Pi \cup Z)) = Q \setminus (V \cup C_4)$ is dense in $Q \setminus C_4$, it follows that ι is a rational self-map of Q. To see that ι is an involution on $Q \setminus C_4$, observe that $\iota \circ \iota \circ p$ is equal to the map

$$\mathbb{P}^3 \dashrightarrow Q$$
$$[x_0 : x_1 : x_2 : x_3] \mapsto \left[x_0 : \frac{(-4s^2+4)x_2^2 - x_2x_4 + 4s^2x_0x_4 + x_3^2}{4x_3} : x_2 : x_3 : x_4\right],$$

which is equal to the identity morphism on $\mathbb{P}^3 \setminus \Pi$. Thus $\iota \circ \iota$ is equal to the identity morphism on $Q \setminus (V \cup C_4)$, so that it is equal to the identity morphism on $Q \setminus C_4$.

Let us prove that ι commutes with the action of $\mathrm{Aut}(Q, C_4)$. If $s \neq \pm\frac{1}{2}$ then by Theorem 2.1, $\mathrm{Aut}(Q, C_4) = \langle g_1 \rangle$, where g_1 is the linear transformation

$$[x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [x_0 : -x_1 : x_2 : -x_3 : x_4].$$

Then it is plain that ι commutes with g_1. If $s = \pm\frac{1}{2}$, then Q is the quadric described in Example 1.4, and $\mathrm{Aut}(Q, C_4)$ contains subgroup isomorphic to $\mathbb{G}_a$ consisting of automorphisms of the form

$$[x_0 : x_1 : x_2 : x_3 : x_4] \mapsto [x_0 + 4tx_1 + 6t^2x_2 + 4t^3x_3 + t^4x_4 : x_1 + 3tx_2 + 3t^2x_3 + x_4 : \\ : x_2 + 2tx_3 + t^2x_4 : x_3 + tx_4 : x_4],$$

for every $t \in \mathbb{C}$. One can easily see that ι commutes with each of these automorphisms.

Case (3): Q is given by $f_1 + f_5 = 0$

Let σ be the rational map

$$\begin{aligned} &\sigma : Q \dashrightarrow \mathbb{P}^4 \\ [x_0 : x_1 : x_2 : x_3 : x_4] \mapsto &\left[\frac{1}{64} f_0 - \frac{1}{4} f_1 + \frac{3}{2} f_2 + 16 f_3 + 64 f_4 : \right. \\ &\left. : -\frac{1}{8} f_0 + \frac{3}{2} f_1 + 2 f_2 + 32 f_3 : f_0 - 8 f_1 + 16 f_2 : -8 f_0 + 32 f_1 : 64 f_0 \right]. \end{aligned}$$

By replacing the map ι with σ, the map p by the rational parametrisation

$$\begin{aligned} &\mathbb{P}^3 \dashrightarrow Q \\ [x_1 : x_2 : x_3 : x_4] \mapsto &\left[\frac{4x_1x_3 + x_1x_4 - 3x_2^2 - x_2x_3}{x_4} : x_1 : x_2 : x_3 : x_4 \right], \end{aligned}$$

and the varieties Π and V with $\{x_4 = 0\}$, it follows verbatim from the proof of case (2) that σ is a birational involution of Q with base locus equal to C_4. By Theorem 2.1, $\mathrm{Aut}(Q, C_4)$ is trivial in this case, so there is nothing left to prove. □

Acknowledgements I am grateful to I. Cheltsov for introducing me to this topic, and for his attentive guidance, support and insight with regards to the creation of this document. I would also like to thank I. Dolgachev for his useful observation in the proof of case (1) in Theorem 3.1.

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