ANNALI DELL'UNIVERSITA' DI FERRARA (2024) 70:381–403
https://doi.org/10.1007/s11565-023-00477-2

Refinements of some inequalities involving Berezin norms and Berezin number and related questions

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Received: 12 July 2023 / Accepted: 25 September 2023 / Published online: 24 October 2023

Abstract
In this paper, we give several refinements of Berezin norm and Berezin number inequalities of bounded linear operators defined on a reproducing kernel Hilbert space. In particular, we present some refinements of the triangle inequality for the Berezin norm of operators. In addition, we derive new upper bounds for the sum and poduct of Berezin number for two bounded operators. Moreover, we prove some new upper bounds for the Davis–Wielandt–Berezin radius of operators. Some applications of the newly obtained inequalities are also provided.

Keywords Berezin number · Berezin norm · Reproducing kernel Hilbert space · Davis–Wielandt–Berezin radius · Inequality

1 Introduction and preliminaries

Throughout this paper, $\mathcal{B}(\mathcal{H})$ denotes the C^*- algebra of all bounded linear operators acting on a non trivial complex Hilbert space $\mathcal{H}$ with inner product $\langle ., . \rangle$ and associated norm $\|.\|$. Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$. The real and imaginary parts of A have been defined as follows $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$ where A^* denotes the adjoint of A.

Let Ω be a nonempty set. A functional Hilbert space $\mathcal{H}(\Omega)$ is a Hilbert space of complex valued functions, which has the property that point evaluations are continuous

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i.e., for each $\lambda \in \Omega$ the map $f \longmapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. The Riesz representation theorem ensues that for each $\lambda \in \Omega$ there exists a unique element $k_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The set $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of the space $\mathcal{H}$. If $\{e_n\}_{n \geq 0}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$ is given by $k_\lambda(z) = \sum_{n=0}^{+\infty} \overline{e_n(\lambda)} e_n(z)$ (see [15]). For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}$. Let A a bounded linear operator on $\mathcal{H}$, the Berezin symbol of A, which firstly have been introduced by Berezin [3, 4] is the function $\tilde{A}$ on Ω defined by

$$\tilde{A}(\lambda) := \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle.$$

The Berezin set and the Berezin number of the operator A are defined respectively by:

$$\mathbf{Ber}(A) := \left\{ \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle : \lambda \in \Omega \right\},$$

and

$$\mathbf{ber}(A) := \sup \left\{ \left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| : \lambda \in \Omega \right\}.$$

It is clear that the Berezin symbol $\tilde{A}$ is the bounded function on Ω whose value lies in the numerical range of the operator A and hence for any $A \in \mathcal{B}(\mathcal{H}(\Omega))$,

$$\mathbf{Ber}(A) \subset W(A) \text{ and } \mathbf{ber}(A) \leq \omega(A),$$

where

$$W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, \ \|x\| = 1\},$$

is the numerical range of the operator A and

$$\omega(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathcal{H}, \ \|x\| = 1\},$$

is the numerical radius of A. For some results about the numerical radius inequalities and their applications, we refer to see [6, 9, 19, 20, 29].

Moreover, the Berezin number of an operator A satisfies the following properties:

(i) $\mathbf{ber}(A) = \mathbf{ber}(A^*)$.
(ii) $\mathbf{ber}(A) \leq \|A\|$.
(iii) $\mathbf{ber}(\alpha A) = |\alpha| \, \mathbf{ber}(A)$ for all $\alpha \in \mathbb{C}$.
(iv) $\mathbf{ber}(A + B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$ for all $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$.

Notice that, in general, the Berezin number does not define a norm. However, if $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions, (for instance on the unit disc $D = \{z \in \mathbb{C} : |z < 1|\}$), then $\mathbf{ber}(.)$ defines a norm on $\mathcal{B}(\mathcal{H}(D))$ (see [16, 17]).

The Berezin symbol has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces. A nice property of the Berezin symbol is mentioned next. If $\tilde{A}(\lambda) = \tilde{B}(\lambda)$ for all $\lambda \in \Omega$, then $A = B$. Therefore, the Berezin symbol uniquely determines the operator. The Berezin symbol and Berezin number have been studied by many mathematicians over the years, a few of them are [1, 5, 12, 14, 26, 30–32].

Now, for any operator $A \in \mathcal{B}(\mathcal{H}(\Omega))$, the Berezin norm of A denoted as $\|A\|_{ber}$ is defined by

$$\|A\|_{ber} := \sup\left\{\left|\left\langle A\hat{k}_\lambda, \hat{k}_\mu\right\rangle\right| : \lambda, \mu \in \Omega\right\},$$

where $\hat{k}_\lambda, \hat{k}_\mu$ are normalized reproducing kernels for λ, μ, respectively.

For $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ it is clear from the definition of the Berezin norm that the following properties hold:

(i) $\|\lambda A\|_{ber} = |\lambda| \, \|A\|_{ber}$ for all $\lambda \in \mathbb{C}$,
(ii) $\|A + B\|_{ber} \leq \|A\|_{ber} + \|B\|_{ber}$,
(iii) $\|A\|_{ber} = \|A^*\|_{ber}$.

Also, it is clear that for $A \in \mathcal{B}(\mathcal{H}(\Omega))$,

$$\mathbf{ber}(A) \leq \|A\|_{ber} \tag{1.1}$$

For futher results about the Berezin norm inequalities and their applications, we refer to see [2, 5, 7] and references therein.

In this paper, several refinements of Berezin norms and Berezin number inequalities of bounded linear operators defined on a reproducing kernel Hilbert space are established. This work is organized as follows: In Sect. 2, we collect a few lemmas that are required to state and prove the results in the subsequent sections. In Sect. 3, we establish some refinements of the triangle inequality for the Berezin norm of operators. In addition, we derive some new upper bounds for the sum and poduct of Berezin number for two bounded operators. In Sect. 4, by applying the continous functional calculus we give a new Berezin number inequality. In Sect. 5, we prove some new upper bounds for the Davis–Wielandt–Berezin radius of resproducing kernel Hibert space operators.

2 Prerequisites

In this section, we present the following lemmas that will be used to develop new results in this paper.

Lemma 2.1 [2] *Let* $A \in B(\mathcal{H}(\Omega))$. *Then*

$$\mathbf{ber}(A) = \sup_{\theta \in \mathbb{R}} \mathbf{ber}\left(\operatorname{Re}\left(e^{i\theta} A\right)\right).$$

Lemma 2.2 [7] *Let $A \in B(\mathcal{H}(\Omega))$ be positive operator. Then*

$$\|A\|_{ber} = \mathbf{ber}(A).$$

Lemma 2.3 [8] *Let $x, y, z \in \mathcal{H}$ with $\|z\| = 1$. Then*

$$|\langle x, z\rangle \langle z, y\rangle| \leq \frac{1}{2}(\|x\| \|y\| + |\langle x, y\rangle|).$$

Lemma 2.4 [25] *Let $A \in B(\mathcal{H})$ be a positive operator and let $x \in \mathcal{H}$ with $\|x\| = 1$. Then*

(i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for $r \geq 1$.
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for $r \leq 1$.

Lemma 2.5 [19] *Let $A \in B(\mathcal{H})$ and let f and g be non-negative continuous functions on $[0, +\infty)$ such that $f(t)g(t) = t$ for all $t \in [0, +\infty)$. Then*

$$|\langle Ax, y\rangle|^2 \leq \left\langle f^2(|A|)x, x\right\rangle\left\langle g^2\left(\left|A^*\right|\right)y, y\right\rangle,$$

for all $x, y \in \mathcal{H}$.
In particular, if $f(t) = g(t) = \sqrt{t}$, then we have

$$|\langle Ax, y\rangle|^2 \leq \langle |A|x, x\rangle\left\langle\left|A^*\right|y, y\right\rangle.$$

Lemma 2.6 [23] *If f is a convex function on a real interval J containing the spectrum of the self-adjoint operator A, then for any unit vector $x \in \mathcal{H}$,*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle.$$

Lemma 2.7 [19] *Let $A, B \in B(\mathcal{H})$ such that $|A|B = B^*|A|$. and g be non-negative continuous functions on $[0, +\infty)$ such that $f(t)g(t) = t$ for all $t \in [0, +\infty)$, then*

$$|\langle ABx, y\rangle| \leq r(B)\|f(|A|)x\|\left\|g\left(\left|A^*\right|\right)y\right\|.$$

Lemma 2.8 [21] *Let f be a twice differentiable convex function such that $\alpha \leq f''$ and $\alpha \in \mathbb{R}$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2} - \frac{1}{8}\alpha(a-b)^2.$$

Lemma 2.9 [24] *If $a, b \geq 0$, $0 \leq \alpha \leq 1$ and $r > 0$, then*

$$a^{2r\alpha}b^{2r(1-\alpha)} + r_0\left(a^r - b^r\right)^2 \leq \alpha a^{2r} + (1-\alpha)b^{2r},$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

3 Inequalities involving Berezin norm and Berezin number

First, we start with the following theorem which is a refinement of the triangle inequality for the Berezin norm of operators.

Theorem 3.1 *Let* $A, B \in B(\mathcal{H}(\Omega))$. *Then*

$$\|A+B\|_{ber} \leq 2\int_0^1 \|tA+(1-t)B\|_{ber}\,dt \leq \|A\|_{ber}+\|B\|_{ber}. \tag{3.1}$$

Proof We put $f:\mathbb{R}\to\mathbb{R}$, $f(t):=\|tA+(1-t)B\|_{ber}$ for $t\in\mathbb{R}$. It is not diffuclt to verify that the function f is convex. Using Hermite-Hadamard inequality (see, e.g., [22, p. 137]), we can see that

$$f\left(\frac{0+1}{2}\right) \leq \int_0^1 f(t)\,dt \leq \frac{f(0)+f(1)}{2}.$$

Therefore, we infer that

$$\left\|\frac{1}{2}A+\frac{1}{2}B\right\|_{ber} \leq \int_0^1 \|tA+(1-t)B\|_{ber}\,dt \leq \frac{\|A\|_{ber}+\|B\|_{ber}}{2}.$$

Thus,

$$\|A+B\|_{ber} \leq 2\int_0^1 \|tA+(1-t)B\|_{ber}\,dt \leq \|A\|_{ber}+\|B\|_{ber},$$

as required. □

In the following theorem, we give an improvement of the inequality in (1.1).

Theorem 3.2 *Let* $A\in B(\mathcal{H}(\Omega))$. *Then*

$$\mathbf{ber}(A) \leq \sup_{\theta\in\mathbb{R}}\int_0^1 \left\|te^{i\theta}A+(1-t)A^*\right\|_{ber}dt \leq \|A\|_{ber}.$$

Proof Let $\theta\in\mathbb{R}$. Replacing A by $\frac{1}{2}e^{i\frac{\theta}{2}}A$ and B by $\frac{1}{2}e^{-i\frac{\theta}{2}}A^*$ in (3.1), we obtain that

$$\left\|\frac{1}{2}e^{i\frac{\theta}{2}}A+\frac{1}{2}e^{-i\frac{\theta}{2}}A^*\right\|_{ber} \leq 2\int_0^1 \left\|\frac{1}{2}te^{i\frac{\theta}{2}}A+\frac{1}{2}(1-t)e^{-i\frac{\theta}{2}}A^*\right\|_{ber}dt$$

$$\leq \left\| \frac{1}{2} e^{i\frac{\theta}{2}} A \right\|_{ber} + \left\| \frac{1}{2} e^{-i\frac{\theta}{2}} A^* \right\|_{ber}.$$

Since $\|\alpha X\|_{ber} = |\alpha| \, \|X\|_{ber}$ for all $X \in B(\mathcal{H}(\Omega))$ and $\alpha \in \mathbb{C}$, it can observe that $\left\| t e^{i\frac{\theta}{2}} A + (1-t) e^{-i\frac{\theta}{2}} A^* \right\|_{ber} = \left\| t e^{i\theta} A + (1-t) A^* \right\|_{ber}$, $\left\| e^{-i\frac{\theta}{2}} A \right\|_{ber} = \|A\|_{ber}$ and $\left\| e^{-i\frac{\theta}{2}} A^* \right\|_{ber} = \|A^*\|_{ber} = \|A\|_{ber}$. Therefore, we get

$$\left\| \frac{1}{2} e^{i\frac{\theta}{2}} A + \frac{1}{2} e^{-i\frac{\theta}{2}} A^* \right\|_{ber} \leq \int_0^1 \left\| t e^{i\theta} A + (1-t) A^* \right\|_{ber} dt \leq \|A\|_{ber}.$$

Since $\mathbf{ber}(X) \leq \|X\|_{ber}$ for all $X \in B(\mathcal{H}(\Omega))$, then

$$\mathbf{ber}\left(\Re\left(e^{i\frac{\theta}{2}} A\right)\right) = \mathbf{ber}\left(\frac{1}{2} e^{i\frac{\theta}{2}} A + \frac{1}{2} e^{-i\frac{\theta}{2}} A^*\right) \leq \int_0^1 \left\| t e^{i\theta} A + (1-t) A^* \right\|_{ber} dt \leq \|A\|_{ber}.$$

Taking the supremum over $\theta \in \mathbb{R}$ in the above inequality, we obtain

$$\sup_{\theta \in \mathbb{R}} \mathbf{ber}\left(\Re\left(e^{i\frac{\theta}{2}} A\right)\right) \leq \sup_{\theta \in \mathbb{R}} \int_0^1 \left\| t e^{i\theta} A + (1-t) A^* \right\|_{ber} dt \leq \|A\|_{ber}.$$

Now, by using Lemma 2.1, we deduce the desired result. □

Next, we present the following theorem.

Theorem 3.3 *Let $A \in \mathcal{B}(\mathcal{H}(\Omega))$ and let f be a twice differentiable nonnegative non-decreasing convex function on $[0, \infty)$ such that $\alpha \leq f''$ and $\alpha \in \mathbb{R}$. Then*

$$f(\mathbf{ber}(A)) \leq \frac{1}{2} \left\| f(|A|) + f\left(\left|A^*\right|\right) \right\|_{ber} - \inf_{\lambda \in \Omega} \delta\left(\hat{k}_\lambda\right),$$

where $\delta\left(\hat{k}_\lambda\right) = \frac{1}{8} \alpha \left\langle (|A| - |A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^2$.

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\begin{aligned} & f\left(\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|\right) \\ & \leq f\left(\left\langle |A| \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{1}{2}} \left\langle |A^*| \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{1}{2}}\right) \\ & \leq f\left(\frac{\left\langle |A| \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \left\langle |A^*| \hat{k}_\lambda, \hat{k}_\lambda \right\rangle}{2}\right). \end{aligned}$$

(by the arithmetic - geometric mean inequality)

$$\leq \frac{f\left(\left\langle |A|\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) + f\left(\left\langle |A^*|\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right)}{2}$$
$$-\frac{1}{8}\alpha\left(\left\langle |A|\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle - \left\langle |A^*|\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right)^2$$
(by Lemma 2.8)
$$\leq \frac{1}{2}\left\langle \left(f\left(|A|\right) + f\left(|A^*|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle - \frac{1}{8}\alpha\left\langle \left(|A| - |A^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^2$$
(by Lemma 2.6)

Thus

$$f\left(\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right) \leq \frac{1}{2}\left\langle \left(f\left(|A|\right) + f\left(|A^*|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle - \frac{1}{8}\alpha\left\langle \left(|A| - |A^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^2$$

Taking supremum over $\lambda \in \Omega$ in the above inequality, we get

$$f\left(\mathbf{ber}\left(A\right)\right) \leq \frac{1}{2}\mathbf{ber}\left(f\left(|A|\right) + f\left(|A^*|\right)\right) - \inf_{\lambda\in\Omega}\delta\left(\hat{k}_\lambda\right),$$

where $\delta\left(\hat{k}_\lambda\right) = \frac{1}{8}\alpha\left\langle \left(|A| - |A^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^2$.

Since $f\left(|A|\right) + f\left(|A^*|\right)$ is positive operator, then by using Lemma 2.2, we get the desired inequality. □

For $f(t) = t^2$ in Theorem 3.3, we get $\alpha \leq 2$ and we have the following remark which is a refinement of [26, Corollary 3.5 (i)].

Remark 3.4 Let $A \in \mathcal{B}\left(\mathcal{H}\left(\Omega\right)\right)$. Then

$$\mathbf{ber}^2\left(A\right) \leq \frac{1}{2}\left\| |A|^2 + \left|A^*\right|^2 \right\|_{ber} - \inf_{\lambda\in\Omega}\delta\left(\hat{k}_\lambda\right),$$

where $\delta\left(\hat{k}_\lambda\right) = \frac{1}{4}\left\langle \left(|A| - |A^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle$.

We now obtain another refinement of the triangle inequality for the Berezin norm.

Theorem 3.5 *Let $A, B \in \mathcal{B}(\mathcal{H}\left(\Omega\right))$ be two positive operators. Then*

$$\|A + B\|_{ber} \leq \sqrt{\mathbf{ber}^2\left(A + iB\right) + 2\|A\|_{ber}\|B\|_{ber}} \leq \|A\|_{ber} + \|B\|_{ber}.$$

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\begin{aligned}\left|\left\langle (A + B)\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &\leq \left(\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| + \left|\left\langle B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right)^2\\ &= \left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + 2\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left|\left\langle B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\\ &= \left|\left\langle (A + iB)\,\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + 2\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left|\left\langle B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|.\end{aligned}$$

Now, by taking supremum over $\lambda \in \Omega$ in the above inequality, we get

$$\begin{aligned}\mathbf{ber}^2(A+B) &\le \mathbf{ber}^2(A+iB) + 2\mathbf{ber}(A)\,\mathbf{ber}(B)\\ &\le \mathbf{ber}^2(A+iB) + 2\|A\|_{ber}\|B\|_{ber}.\\ (\text{since } \mathbf{ber}(X) &\le \|X\|_{ber} \text{ for all } X \in \mathcal{B}(\mathcal{H}(\Omega)))\end{aligned}$$

On the other hand, it can be checked that if A and B are positive operators. Then, $A+B$ is positive operator. So, by Lemma 2.1 we have

$$\mathbf{ber}(A+B) = \|A+B\|_{ber}.$$

Consequently, we get

$$\|A+B\|_{ber}^2 \le \mathbf{ber}^2(A+iB) + 2\|A\|_{ber}\|B\|_{ber}.$$

Therefore, we get the first inequality of the theorem.
Now, we prove the second inequality. We have

$$\begin{aligned}\left|\left\langle (A+iB)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &= \left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + i\left\langle B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\\ &= \left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2\\ &\le \mathbf{ber}^2(A) + \mathbf{ber}^2(B)\\ &\le \|A\|_{ber}^2 + \|B\|_{ber}^2.\\ (\text{since } \mathbf{ber}(X) &\le \|X\|_{ber} \text{ for all } X \in \mathcal{B}(\mathcal{H}(\Omega)))\end{aligned}$$

Thus,

$$\left|\left\langle (A+iB)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \le \|A\|_{ber}^2 + \|B\|_{ber}^2.$$

By taking supremum over $\lambda \in \Omega$ in the above inequality, we obtain

$$\mathbf{ber}^2(A+iB) \le \|A\|_{ber}^2 + \|B\|_{ber}^2.$$

This implies that

$$\begin{aligned}\mathbf{ber}^2(A+iB) + 2\|A\|_{ber}\|B\|_{ber} &\le \|A\|_{ber}^2 + \|B\|_{ber}^2 + 2\|A\|_{ber}\|B\|_{ber}\\ &= \left(\|A\|_{ber} + \|B\|_{ber}\right)^2.\end{aligned}$$

Therefore, we infer that

$$\|A+B\|_{ber} \le \sqrt{\mathbf{ber}^2(A+iB) + 2\|A\|_{ber}\|B\|_{ber}} \le \|A\|_{ber} + \|B\|_{ber}.$$

Thus, we obtain the second inequality and this completes the proof. ◻

In the following theorem we obtain an upper bound for the Berezin number for sum of two operators.

Theorem 3.6 *Let* $A, B \in B(\mathcal{H}(\Omega))$. *Then*

$$\begin{aligned}\mathbf{ber}^2(A+B) &\le \mathbf{ber}^2(A) + \frac{1}{4}\left\| |B|^2 + |B^*|^2 \right\|_{ber} \\ &+ \mathbf{ber}(A)\left\| |B| + |B^*| \right\|_{ber} + \frac{1}{2}\mathbf{ber}\left(B^2\right).\end{aligned}$$

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\begin{aligned}\left|\left\langle (A+B)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &= \left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\ &\le \left(\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| + \left|\left\langle B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right)^2 \\ &= \left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + 2\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left|\left\langle B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| + \left|\left\langle B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 \\ &\le \left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + 2\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\sqrt{\left\langle |B|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\left\langle |B^*|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle} \\ &\quad + \frac{1}{2}\left(\sqrt{\left\langle |B|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\left\langle |B^*|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle} + \left|\left\langle B\hat{k}_\lambda, B^*\hat{k}_\lambda\right\rangle\right|\right) \\ &\quad \text{(by Lemma 2.3 and Lemma 2.5)} \\ &\le \left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left(\left\langle |B|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle |B^*|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) \\ &\quad + \frac{1}{4}\left(\left\langle |B|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle |B^*|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) + \frac{1}{2}\left|\left\langle B^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| \\ &\quad \text{(by the arithmetic - geometric mean inequality)}\end{aligned}$$

Thus,

$$\begin{aligned}\left|\left\langle (A+B)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 &\le \left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\left(\left\langle\left(|B| + |B^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) \\ &\quad + \frac{1}{4}\left(\left\langle\left(|B|^2 + |B^*|^2\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right) + \frac{1}{2}\left|\left\langle B^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\end{aligned}$$

Taking supremum over $\lambda \in \Omega$ in the above inequality, we get

$$\begin{aligned}\mathbf{ber}^2(A+B) &\le \mathbf{ber}^2(A) + \mathbf{ber}(A)\,\mathbf{ber}\left(|B| + |B^*|\right) \\ &+ \frac{1}{4}\mathbf{ber}\left(|B|^2 + |B^*|^2\right) + \frac{1}{2}\mathbf{ber}\left(B^2\right).\end{aligned}$$

Now, by using Lemma 2.2, we get the desired inequality. □

As an immediate consequence of Theorem 3.6, we have the following result.

Corollary 3.7 *Let* $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$. *Then*

$$\max\left\{\mathbf{ber}^2(A+B), \mathbf{ber}^2(A-B)\right\} \\ \leq \mathbf{ber}^2(A) + \mathbf{ber}(A)\,\mathbf{ber}\left(|B| + \left|B^*\right|\right) + \frac{1}{4}\mathbf{ber}\left(|B|^2 + \left|B^*\right|^2\right) \\ + \frac{1}{2}\mathbf{ber}\left(B^2\right).$$

Proof In view of Theorem 3.6, we have

$$\mathbf{ber}^2(A+B) \leq \mathbf{ber}^2(A) + \mathbf{ber}(A)\,\mathbf{ber}\left(|B| + \left|B^*\right|\right) \\ + \frac{1}{4}\mathbf{ber}\left(|B|^2 + \left|B^*\right|^2\right) + \frac{1}{2}\mathbf{ber}\left(B^2\right).$$

Replacing B by $-B$ in above inequality, we get

$$\mathbf{ber}^2(A-B) \leq \mathbf{ber}^2(A) + \mathbf{ber}(A)\,\mathbf{ber}\left(|B| + \left|B^*\right|\right) \\ + \frac{1}{4}\mathbf{ber}\left(|B|^2 + \left|B^*\right|^2\right) + \frac{1}{2}\mathbf{ber}\left(B^2\right).$$

Therefore, we infer that the desired inequality. □

If $A = 0$ in Theorem 3.6, then we get the following corollary.

Corollary 3.8 *Let* $B \in \mathcal{B}(\mathcal{H}(\Omega))$. *Then*

$$\mathbf{ber}^2(B) \leq \frac{1}{4}\left\||B|^2 + \left|B^*\right|^2\right\|_{ber} + \frac{1}{2}\mathbf{ber}\left(B^2\right).$$

Remark 3.9 Since $t \longmapsto t^r, r \geq 1$ is a convex increasing function on $[0, +\infty)$ and by using Corollary 3.8, it is not difficult to see that

$$\mathbf{ber}^{2r}(B) \leq \frac{1}{4}\left\||B|^{2r} + \left|B^*\right|^{2r}\right\|_{ber} + \frac{1}{2}\mathbf{ber}^r\left(B^2\right),$$

this inequality proved recently in [5, Corollary 2.11].

If $A = B$ in Theorem 3.6, then we get the following corollary.

Corollary 3.10 *Let* $A \in \mathcal{B}(\mathcal{H}(\Omega))$. *Then*

$$\mathbf{ber}^2(A) \leq \frac{1}{12}\left\||A|^2 + \left|A^*\right|^2\right\|_{ber} + \frac{1}{3}\mathbf{ber}(A)\left\||A| + \left|A^*\right|\right\|_{ber} + \frac{1}{6}\mathbf{ber}\left(A^2\right).$$

Remark 3.11 Using the fact $\mathbf{ber}(X) \leq \|X\|_{ber} \leq \|X\|$ for every $X \in B(\mathcal{H}(\Omega))$, it follows that

$$\mathbf{ber}^2(A) \leq \frac{1}{12}\left\||A|^2 + \left|A^*\right|^2\right\|_{ber} + \frac{1}{3}\mathbf{ber}(A)\left\||A| + \left|A^*\right|\right\|_{ber} + \frac{1}{6}\mathbf{ber}\left(A^2\right)$$

$$
\begin{aligned}
&\leq \frac{1}{12}\left\||A|^2+\left|A^*\right|^2\right\|+\frac{1}{3}\|A\|\left\||A|+\left|A^*\right|\right\|+\frac{1}{6}\left\|A^2\right\| \\
&\leq \frac{1}{12}\left(2\|A\|^2\right)+\frac{1}{3}\|A\|(2\|A\|)+\frac{1}{6}\|A\|^2 \\
&= \|A\|^2 .
\end{aligned}
$$

Hence,

$$
\mathbf{ber}(A) \leq \sqrt{\frac{1}{12}\left\||A|^2+|A^*|^2\right\|_{ber}+\frac{1}{3}\mathbf{ber}(A)\,\||A|+|A^*|\|_{ber}+\frac{1}{6}\mathbf{ber}\left(A^2\right)} \leq \|A\|,
$$

this is a non-trivial improvement of inequality $\mathbf{ber}(A) \leq \|A\|$.

In the next theorem, we give a new upper bound for the Berezin number of product of operators.

Theorem 3.12 *Let $A, B \in B(\mathcal{H}(\Omega))$. Then*

$$
\mathbf{ber}\left(B^*A\right) \leq \frac{1}{2\sqrt{2}}\left\|A^*A+B^*B\right\|_{ber}^{\frac{1}{2}}\left(\left\|A^*A\right\|_{ber}^{\frac{1}{2}}+\left\|B^*B\right\|_{ber}^{\frac{1}{2}}\right).
$$

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$
\begin{aligned}
&\left|\left\langle B^*A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| \\
&\quad= \left|\left\langle A\hat{k}_\lambda, B\hat{k}_\lambda\right\rangle\right| \\
&\quad\leq \left\|A\hat{k}_\lambda\right\|\left\|B\hat{k}_\lambda\right\| \\
&\quad= \sqrt{\left\langle A\hat{k}_\lambda, A\hat{k}_\lambda\right\rangle\left\langle B\hat{k}_\lambda, B\hat{k}_\lambda\right\rangle} \\
&\quad= \sqrt{\left\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\left\langle B^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle} \\
&\quad= \sqrt{\left\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left(\left\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\left\langle B^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right)^{\frac{1}{4}}\left\langle B^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left(\left\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\left\langle B^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right)^{\frac{1}{4}}} \\
&\quad\leq \frac{1}{2}\left\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left(\left\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\left\langle B^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right)^{\frac{1}{4}} \\
&\qquad+\frac{1}{2}\left\langle B^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left(\left\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\left\langle B^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right)^{\frac{1}{4}} \\
&\qquad\qquad\text{(by the arithmetic-geometric mean inequality)} \\
&\quad\leq \frac{1}{2}\left[\left\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left(\frac{1}{2}\left\langle\left(A^*A+B^*B\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right)^{\frac{1}{2}}\right. \\
&\qquad\left.+\left\langle B^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left(\frac{1}{2}\left\langle\left(A^*A+B^*B\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right)^{\frac{1}{2}}\right] \\
&\quad= \frac{1}{2}\left[\left(\frac{1}{2}\left\langle\left(A^*A+B^*B\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right)^{\frac{1}{2}}\left(\left\langle A^*A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}+\left\langle B^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)\right].
\end{aligned}
$$

Thus,

$$\left|\left\langle B^*A\hat{k}_\lambda,\hat{k}_\lambda\right\rangle_A\right|\le\frac{1}{2\sqrt{2}}\left[\left(\left\langle\left(A^*A+B^*B\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right)^{\frac{1}{2}}\left(\left\langle A^*A\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}+\left\langle B^*B\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)\right].$$

Taking the supermum in over $\lambda\in\Omega$, we get

$$\mathbf{ber}\left(B^*A\right)\le\frac{1}{2\sqrt{2}}\left\|A^*A+B^*B\right\|_{ber}^{\frac{1}{2}}\left(\mathbf{ber}^{\frac{1}{2}}\left(A^*A\right)+\mathbf{ber}^{\frac{1}{2}}\left(B^*B\right)\right).$$

Using Lemma 2.2, we get

$$\mathbf{ber}\left(B^*A\right)\le\frac{1}{2\sqrt{2}}\left\|A^*A+B^*B\right\|_{ber}^{\frac{1}{2}}\left(\left\|A^*A\right\|_{ber}^{\frac{1}{2}}+\left\|B^*B\right\|_{ber}^{\frac{1}{2}}\right),$$

as required. □

We next prove the following theorem.

Theorem 3.13 *Let* $A,B\in B(\mathcal{H}(\Omega))$ *such that* $|A|B=B^*|A|$. *If* f *and* g *are nonnegative continuous functions on* $[0,+\infty)$ *satisfying* $f(t)g(t)=t$ $(t\ge 0)$, *then for all* $s\ge 1$, *we have*

$$\mathbf{ber}^{2s}(AB)\le r^{2s}(B)\left(\frac{1}{4}\left\|f^{4s}(|A|)+g^{4s}\left(\left|A^*\right|\right)\right\|_{ber}+\frac{1}{2}\left\|g^{2s}\left(\left|A^*\right|\right)f^{2s}(|A|)\right\|_{ber}\right).$$

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, we have

$$\begin{aligned}
\left|\left\langle AB\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^{2s}&\le r^{2s}(B)\left\|f(|A|)\hat{k}_\lambda\right\|^{2s}\left\|g\left(\left|A^*\right|\right)\hat{k}_\lambda\right\|^{2s}\\
&\quad\text{(by Lemma 2.7)}\\
&=r^{2s}(B)\left\langle f(|A|)\hat{k}_\lambda,f(|A|)\hat{k}_\lambda\right\rangle^{s}\left\langle g\left(\left|A^*\right|\right)\hat{k}_\lambda,g\left(\left|A^*\right|\right)\hat{k}_\lambda\right\rangle^{s}\\
&=r^{2s}(B)\left\langle f^{2}(|A|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{s}\left\langle g^{2}\left(\left|A^*\right|\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{s}\\
&\le r^{2s}(B)\left\langle f^{2s}(|A|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\left\langle g^{2s}\left(\left|A^*\right|\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\\
&\quad\text{(by Lemma 2.4)}\\
&=r^{2s}(B)\left\langle f^{2s}(|A|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\left\langle\hat{k}_\lambda,g^{2s}\left(\left|A^*\right|\right)\hat{k}_\lambda\right\rangle\\
&\le\frac{1}{2}r^{2s}(B)\left(\left\|f^{2s}(|A|)\hat{k}_\lambda\right\|\left\|g^{2s}\left(\left|A^*\right|\right)\hat{k}_\lambda\right\|+\left\langle f^{2s}(|A|)\hat{k}_\lambda,g^{2s}\left(\left|A^*\right|\right)\hat{k}_\lambda\right\rangle\right)\\
&\quad\text{(by Lemma 2.3)}\\
&\le\frac{1}{2}r^{2s}(B)\left(\frac{\left\|f^{2s}(|A|)\hat{k}_\lambda\right\|^{2}+\left\|g^{2s}(|A^*|)\hat{k}_\lambda\right\|^{2}}{2}\right)\\
&\quad+\frac{1}{2}r^{2s}(B)\left\langle g^{2s}\left(\left|A^*\right|\right)f^{2s}(|A|)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\\
&\quad\text{(by the arithmetic - geometric mean inequality)}
\end{aligned}$$

$$= \frac{1}{4} r^{2s}(B) \left(\left\langle f^{4s}(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \left\langle g^{4s}(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right) + \frac{1}{2} r^{2s}(B) \left\langle g^{2s}(|A^*|) f^{2s}(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle$$

$$= \frac{1}{4} r^{2s}(B) \left(\left\langle \left(f^{4s}(|A|) + g^{4s}(|A^*|) \right) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right) + \frac{1}{2} r^{2s}(B) \left\langle g^{2s}(|A^*|) f^{2s}(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle.$$

Taking the supermum in over $\lambda \in \Omega$, we get

$$\mathbf{ber}^{2s}(AB) \leq r^{2s}(B) \left(\frac{1}{4} \mathbf{ber} \left(f^{4s}(|A|) + g^{4s}(|A^*|) \right) + \frac{1}{2} \mathbf{ber} \left(g^{2s}(|A^*|) f^{2s}(|A|) \right) \right).$$

Now, by using Lemma 2.2, we get the desired inequality. □

Corollary 3.14 *Let $A, B \in B(\mathcal{H}(\Omega))$ such that $|A| B = B^* |A|$ and let $0 \leq p \leq 1$, then for all $s \geq 1$, we have*

$$\mathbf{ber}^{2s}(AB) \leq r^{2s}(B) \left(\frac{1}{4} \left\| |A|^{4ps} + |A^*|^{4(1-p)s} \right\|_{ber} + \frac{1}{2} \left\| |A|^{2ps} |A^*|^{2(1-p)s} \right\|_{ber} \right).$$

Proof The result follows immediately from Theorem 3.1 for $f(t) = t^p$ and $g(t) = t^{1-p}$ $(0 \leq p \leq 1)$. □

For $B = I$ in Theorem 3.1 we get the following result.

Corollary 3.15 *Let $A \in B(\mathcal{H}(\Omega))$ and let f and g as in Theorem 3.1. Then*

$$\mathbf{ber}^{2s}(A) \leq \frac{1}{4} \left\| f^{4s}(|A|) + g^{4s}(|A^*|) \right\|_{ber} + \frac{1}{2} \left\| g^{2s}(|A^*|) f^{2s}(|A|) \right\|_{ber},$$

for all $r \geq 1$.

Remark 3.16 If we take $f(t) = t^p$ and $g(t) = t^{1-p}$ $(0 \leq p \leq 1)$ in Corollary 3.15, then

$$\mathbf{ber}^{2s}(A) \leq \frac{1}{4} \left\| |A|^{4ps} + |A^*|^{4(1-p)s} \right\|_{ber} + \frac{1}{2} \left\| |A|^{2ps} |A^*|^{2(1-p)s} \right\|_{ber}.$$

for all $r \geq 1$.

(2) Taking $f(t) = g(t) = t^{\frac{1}{2}}$ $(t \in [0, +\infty))$ and $r = 1$in Corollary 3.15, we get

$$\mathbf{ber}^{2s}(A) \leq \frac{1}{4} \left\| |A|^{2s} + |A^*|^{2s} \right\|_{ber} + \frac{1}{2} \left\| |A|^s |A^*|^s \right\|_{ber},$$

which proved in [5, Theorem 2.15].
Next, we conclude this section with the following theorem.

Theorem 3.17 *Let $A, B \in B(\mathcal{H}(\Omega))$ such that $|A| B = B^* |A|$ and let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are nonnegative continuous functions on $[0, +\infty)$ satisfying $f(t) g(t) = t$ $(t \geq 0)$, then for all $s \geq 1$, we have*

$$\mathbf{ber}^{2s}(AB) \leq r^{2s}(B) \left\| \frac{1}{p} f^{2ps}(|A|) + \frac{1}{q} g^{2qs}(|A^*|) \right\|_{ber} - r_0 r^{2s}(B) \inf_{\lambda \in \Omega} \delta\left(\hat{k}_\lambda\right),$$

where $\delta\left(\hat{k}_\lambda\right) = \left(\left\langle f^{2p}(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{s}{2}} - \left\langle g^{2q}(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{s}{2}}\right)^2$ and $r_0 = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Then, in view nn we have

$$\begin{aligned}
\left|\left\langle AB\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|^{2s} &\leq r^{2s}(B) \left\| f(|A|) \hat{k}_\lambda \right\|^{2s} \left\| g(|A^*|) \hat{k}_\lambda \right\|^{2s} \\
&\quad \text{(by Lemma 2.7)} \\
&= r^{2s}(B) \left\langle f(|A|) \hat{k}_\lambda, f(|A|) \hat{k}_\lambda \right\rangle^s \left\langle g(|A^*|) \hat{k}_\lambda, g(|A|) \hat{k}_\lambda \right\rangle^s \\
&= r^{2s}(B) \left\langle f^2(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^s \left\langle g^2(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^s \\
&= r^{2s}(B) \left\langle f^{p\frac{2}{p}}(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^s \left\langle g^{q\frac{2}{q}}(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^s \\
&\leq r^{2s}(B) \left(\left\langle f^{2p}(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{1}{p}} \left\langle g^{2q}(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{1}{q}}\right)^s \\
&\quad \text{(by Lemma 2.4)} \\
&\leq r^{2s}(B) \left(\frac{1}{p} \left\langle f^{2p}(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^s + \frac{1}{q} \left\langle g^{2q}(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^s\right) \\
&\quad - r_0 r^{2s}(B) \left(\left\langle f^{2p}(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{s}{2}} - \left\langle g^{2q}(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{s}{2}}\right)^2 \\
&\quad \text{(by Lemma 2.9)} \\
&\leq r^{2s}(B) \left(\frac{1}{p} \left\langle f^{2ps}(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \frac{1}{q} \left\langle g^{2qs}(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right) \\
&\quad - r_0 r^{2s}(B) \left(\left\langle f^{2p}(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{s}{2}} - \left\langle g^{2q}(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{s}{2}}\right)^2 \\
&\quad \text{(by Lemma 2.4)}
\end{aligned}$$

Taking the supermum in over $\lambda \in \Omega$, we get

$$\mathbf{ber}^{2s}(AB) \leq r^{2s}(B) \, \mathbf{ber}\left(\frac{1}{p} f^{2ps}(|A|) + \frac{1}{q} g^{2qs}(|A^*|)\right) - r_0 r^{2s}(B) \inf_{\lambda \in \Omega} \delta\left(\hat{k}_\lambda\right),$$

where $\delta\left(\hat{k}_\lambda\right) = \left(\left\langle f^{2p}(|A|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{s}{2}} - \left\langle g^{2q}(|A^*|) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle^{\frac{s}{2}}\right)^2$.

Now, by using Lemma 2.2, we get the desired inequality. □

Letting $s = 1$ and $p = q = 2$ in Theorem 3.17, we have the following corollary.

Corollary 3.18 *Let $A, B \in B(\mathcal{H}(\Omega))$ such that $|A|B = B^*|A|$. If f and g are nonnegative continuous functions on $[0, +\infty)$ satisfying $f(t)g(t) = t$ $(t \geq 0)$, then*

$$\mathbf{ber}^2(AB) \leq \frac{1}{2}r^2(B)\left\|f^4(|A|) + g^4(|A^*|)\right\|_{ber} - \frac{1}{2}r^2(B)\inf_{\lambda\in\Omega}\delta\left(\hat{k}_\lambda\right),$$

where $\delta\left(\hat{k}_\lambda\right) = \left(\left\langle f^4(|A|)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} - \left\langle g^4(|A^*|)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2$.

Considering $B = I$ and $f(t) = g(t) = \sqrt{t}$ and $2s = r$ in Corollay, we get the following inequality.

Corollary 3.19 *If $A \in B(\mathcal{H}(\Omega))$, then*

$$\mathbf{ber}^r(A) \leq \frac{1}{2}\left\||A|^r + |A^*|^r\right\|_{ber} - \frac{1}{2}\inf_{\lambda\in\Omega}\delta\left(\hat{k}_\lambda\right),$$

where $\delta\left(\hat{k}_\lambda\right) = \left(\left\langle |A|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} - \left\langle |A^*|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2$.

Remark 3.20 We note that the inequality in above corollary refines the inequality

$$\mathbf{ber}^r(A) \leq \frac{1}{2}\mathbf{ber}\left(|A|^r + |A^*|^r\right) \text{ for } r \geq 1,$$

obtained in [26].

4 Functional calculus and a Berezin number inequality

One of the applicable inequalities in analysis and differential equations is the classical Hardy inequality with says that if $p > 1$ and $\{a_n\}_{n=1}^{\infty}$ are positive real numbers such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p. \tag{4.1}$$

The inequality (4.1) is sharp, i,e., the constant $\left(\frac{p}{p-1}\right)^p$ is the smallet number such that the inequality holds. A developed inequality, the so-called Hardy-Hilbert inequaity reads as follows: if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_n, b_n \geq 0$ such that $0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^p < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{n+m}<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{4.2}$$

The are many refinements and reformulations of the above inequality. In particular, Yang [33] proved the following generalization of (4.2):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{(n+m)^s}<L_1\left(\sum_{m=1}^{\infty}m^{1-s}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{1-s}b_n^q\right)^{\frac{1}{p}}, \tag{4.3}$$

in which $2-\min\{p,q\}<s\leq 2$ and $L_1:=B\left(\frac{p+s-2}{p},\frac{q+s-2}{q}\right)$, where $B(\cdot,\cdot)$ is β-function.

In this section, by applying the continous functional calculus we give some inequalities analogue to (4.3) for operators in the real space $B(\mathcal{H})$ of all self-adjoint operators on $\mathcal{H}$. Application obtained inequalities give a new Berezin number inequality. For the related results, see instance, [13, 18, 27, 28, 34].

Now, we state the following theorem.

Theorem 4.1 *Let f,g be continous functions defined on an interval $J\subset[0,+\infty)$ and $f,g\geq 0$. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}&\frac{1}{2^s}(f(A)g(A))^{\sim}(\lambda)+\frac{1}{3^s}\left(\widetilde{g(B)}(\mu)\widetilde{f(A)}(\lambda)\right)\\&\quad+\frac{1}{3^s}\left(\widetilde{f(B)}(\mu)\widetilde{g(A)}(\lambda)\right)+\frac{1}{4^s}(f(B)g(B))^{\sim}(\mu)\\&\leq L_1\left[\left(f(A)^p+2^{1-s}f(B)^p\right)^{\frac{1}{p}}\left(g(A)^q+2^{1-s}g(B)^q\right)^{\frac{1}{q}}\right]^{\sim}(\lambda),\end{aligned}$$

for all operators $A,B\in B(\mathcal{H})_h$ with spectra contained in J and all $\lambda,\mu\in\Omega$.

Proof Let a_1,a_2,b_1,b_2 be positive numbers. Let A be a self-adjoint linear operator on a complex Hilbert space $(\mathcal{H};\langle\cdot,\cdot\rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set $C(Sp(A))$ of all continous functions defined on the spectrum of A, denoted $Sp(A)$, and the C^* -algebra $C^*(A)$ generated by A and the identity operator I on $\mathcal{H}$ as follows (see for instance [11, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$, we have

(1) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;
(2) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi\left(\overline{f}\right)=\Phi(f)^*$;
(3) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;
(4) $\Phi(f_0)=I$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation, we define

$$f(A):=\Phi(f)\ \text{ for all } f\in C(Sp(A)),$$

and we call it the continous functional calculus for a self-adjoint operator A.

If A is a self-adjoint operator and f is a real valued continous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i,e., $f(A)$ is a positive operator on $\mathcal{H}$. Moreover, if both f and g are real valued functions on $Sp(A)$, then the following important property holds: $f(t) \geq g(t)$ for any $t \in Sp(A)$ implies that $f(A) \geq g(A)$ in the operator order of $B(\mathcal{H})$.

Now, by using (4.3) we have

$$\frac{a_1 b_1}{2^s} + \frac{a_1 b_2}{3^s} + \frac{a_2 b_1}{3^s} + \frac{a_2 b_2}{4^s} \leq L_1 \left(a_1^p + 2^{1-s} a_2^p\right)^{\frac{1}{p}} \left(b_1^q + 2^{1-s} b_2^q\right)^{\frac{1}{q}}. \quad (4.4)$$

Let $x, y \in J$. Considering that $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in J$ and putting $a_1 = f(x), a_2 = f(y), b_1 = g(x)$ and $b_2 = g(y)$ in (4.4), we have

$$\begin{aligned} &\frac{f(x) g(x)}{2^s} + \frac{f(x) g(y)}{3^s} + \frac{f(y) g(x)}{3^s} + \frac{f(y) g(y)}{4^s} \\ &\leq L_1 \left(f(x)^p + 2^{1-s} f(y)^p\right)^{\frac{1}{p}} \left(g(x)^q + 2^{1-s} g(y)^q\right)^{\frac{1}{q}}, \end{aligned} \quad (4.5)$$

for all $x, y \in J$. By applying the functional calculus for A to inequality (4.5), we get

$$\begin{aligned} &\frac{f(A) g(A)}{2^s} + \frac{f(A) g(y)}{3^s} + \frac{f(y) g(A)}{3^s} + \frac{f(y) g(y)}{4^s} \\ &\leq L_1 \left(f(A)^p + 2^{1-s} f(y)^p\right)^{\frac{1}{p}} \left(g(A)^q + 2^{1-s} g(y)^q\right)^{\frac{1}{q}}, \end{aligned}$$

from which

$$\begin{aligned} &\frac{1}{2^s} \left\langle f(A) g(A) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \frac{1}{3^s} g(y) \left\langle f(A) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \frac{1}{3^s} f(y) \left\langle g(A) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \frac{f(y) g(y)}{4^s} \\ &\leq L_1 \left\langle \left(f(A)^p + 2^{1-s} f(y)^p\right)^{\frac{1}{p}} \left(g(A)^q + 2^{1-s} g(y)^q\right)^{\frac{1}{q}} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle, \end{aligned}$$

for all $\lambda \in \Omega$ and $y \in J$. Applying the functional calculus once more to the self-adjoint operator B, we obtain

$$\begin{aligned} &\frac{1}{2^s} \left\langle f(A) g(A) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \frac{1}{3^s} g(B) \left\langle f(A) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \\ &+ \frac{1}{3^s} f(B) \left\langle g(A) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \frac{f(B) g(B)}{4^s} \\ &\quad \frac{1}{2^s} \left\langle f(A) g(A) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \frac{1}{3^s} g(B) \left\langle f(A) \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \\ &\quad \leq L_1 \left\langle \left(f(A)^p + 2^{1-s} f(B)^p\right)^{\frac{1}{p}} \left(g(A)^q + 2^{1-s} g(B)^q\right)^{\frac{1}{q}} \hat{k}_\lambda, \hat{k}_\lambda \right\rangle. \end{aligned} \quad (4.6)$$

If $\mu \in \Omega$, then it follows from inequality (4.6) that

$$\frac{1}{2^s}\left\langle f(A)\,g(A)\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle+\frac{1}{3^s}\left\langle g(B)\,\hat{k}_\mu,\hat{k}_\mu\right\rangle\left\langle f(A)\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle$$
$$+\frac{1}{3^s}\left\langle f(B)\,\hat{k}_\mu,\hat{k}_\mu\right\rangle\left\langle g(A)\,\hat{k}_\lambda,\hat{k}_\lambda\right\rangle+\frac{1}{4^s}\left\langle f(B)\,g(B)\,\hat{k}_\mu,\hat{k}_\mu\right\rangle$$
$$\leq L_1\left\langle\left(f(A)^p+2^{1-s}f(B)^p\right)^{\frac{1}{p}}\left(g(A)^q+2^{1-s}g(B)^q\right)^{\frac{1}{q}}\hat{k}_\lambda,\hat{k}_\lambda\right\rangle.$$

Hence,

$$\frac{1}{2^s}\left(f(A)\,g(A)\right)^{\sim}(\lambda)+\frac{1}{3^s}\left(\widetilde{g(B)}(\mu)\,\widetilde{f(A)}(\lambda)\right)$$
$$+\frac{1}{3^s}\left(\widetilde{f(B)}(\mu)\,\widetilde{g(A)}(\lambda)\right)+\frac{1}{4^s}\left(f(B)\,g(B)\right)^{\sim}(\mu)$$
$$\leq L_1\left[\left(f(A)^p+2^{1-s}f(B)^p\right)^{\frac{1}{p}}\left(g(A)^q+2^{1-s}g(B)^q\right)^{\frac{1}{q}}\right]^{\sim}(\lambda),$$

as desired. □

Replacing B by A and μ by λ in Theorem 4.1 and using that $\frac{1}{p}+\frac{1}{q}=1$, we have the following corollary.

Corollary 4.2 *If f, g are continous functions defined on an interval J and $f, g \geq 0$, then*

$$\widetilde{f(A)}(\lambda)\,\widetilde{g(A)}(\lambda)\leq\frac{3^s}{2}\left[L_1\left(1+2^{1-s}\right)-\frac{2^s+1}{4^s}\right]\widetilde{fg(A)}(\lambda),$$

for any self-adjoint operator A and any point $\lambda \in \Omega$.

Replacing g by f in Corollary 4.2, we get the following.

Corollary 4.3 *If f is a continous function defined on an interval J and $f \geq 0$, then*

$$\widetilde{f(A)}^2(\lambda)\leq\left(\frac{3}{4}\right)^s\left[2^sL_1\left(2^{1-s}+1\right)-\left(2^{1-s}+\frac{1}{2}\right)\right]\widetilde{f^2(A)}(\lambda), \tag{4.7}$$

for any self-adjoint operator A on $\mathcal{H}(\Omega)$ and any point $\lambda \in \Omega$.

An immediate corollary of inequality (4.7) is the following reverse inequality for the Berezin number of operator A.

Corollary 4.4 *If f is a continous function defined on an interval J and $f \geq 0$, then*

$$\mathbf{ber}^2(f(A))\leq\left(\frac{3}{4}\right)^s\left[2^sL_1\left(2^{1-s}+1\right)-\left(2^{1-s}+\frac{1}{2}\right)\right]\mathbf{ber}\left(f^2(A)\right),$$

in which, as before, $2-\min\{p,q\}<s\leq 2$ and $L_1:=B\left(\frac{p+s-2}{p},\frac{q+s-2}{q}\right)$, where B is β-function.

5 Upper bounds for the Davis–Wielandt–Berezin radius

In [27], the authors introduced the Davis–Wielandt–Berezin radius of operators as follows.

Definition 5.1 For any $A \in B(\mathcal{H}(\Omega))$, we define its Davis–Wielandt–Berezin radius by the formula

$$\eta(A) := \sup_{\lambda\in\Omega} \sqrt{\left|\widetilde{A}(\lambda)\right|^2 + \left\|A\hat{k}_\lambda\right\|^4}.$$

For $A, B \in B(\mathcal{H}(\Omega))$ one has:

(1) $\eta(A) \geq 0$ and $\eta(A) = 0$ if and only if $A = 0$;

(2) If $\alpha \in \mathbb{C}$, then $\eta(\alpha A) = \begin{cases} \geq |\alpha|\,\eta(A) & \text{if } |\alpha| > 1 \\ = |\alpha|\,\eta(A) & \text{if } |\alpha| = 1 \\ \leq |\alpha|\,\eta(A) & \text{if } |\alpha| < 1; \end{cases}$

(3) $\eta(A+B) \leq \sqrt{2\left(\eta(A) + \eta(B) + 4(\eta(A) + \eta(B))^2\right)}$;

and therefore $\eta(\cdot)$ can not be a norm on $B(\mathcal{H}(\Omega))$.

The following property of $\eta(\cdot)$ is immediate if we denote by $\|A\|_{Ber}$ another Berezin norm of operator A which is defined by $\|A\|_{Ber} := \sup_{\lambda\in\Omega}\left\|A\hat{k}_\lambda\right\|$ and it is different from the Berezin norm $\|A\|_{ber}$ which we defined in Sect. 2. Clearly, $\|A\|_{ber} \leq \|A\|_{Ber}$ and

$$\max\left\{\mathbf{ber}(A), \|A\|_{Ber}^2\right\} \leq \eta(A) \leq \sqrt{\mathbf{ber}^2(A) + \|A\|_{Ber}^4}.$$

The goal of this section is to establish some new upper bounds for the Davis–Wielandt–Berezin radius of resproducing kernel Hibert space operators.

The following result provides a new bound for $\eta(A)$.

Theorem 5.2 *Let $A \in B(\mathcal{H}(\Omega))$. Then*

$$\eta^2(A) \leq \frac{1}{2}\left(\mathbf{ber}\left(|A|^4 + |A|^2\right) + \mathbf{ber}\left(|A|^4 - |A|^2\right)\right) + \sqrt{2}\mathbf{ber}\left(|A|^2 A\right).$$

Proof Let $\lambda \in \Omega$ be an arbitray point. Let $\mathcal{H}$ be a complex Hilbert space and $a, b, c \in \mathcal{H}$ Dragomir proved in [10] the following extension of Cauchy-Schwarz inequality:

$$|\langle a, b\rangle|^2 + |\langle a, c\rangle|^2 \leq \|a\|^2\left(\max\left\{\|b\|^2, \|c\|^2\right\}\right) + \sqrt{2}\,|\langle b, c\rangle|. \tag{5.1}$$

Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Chosing in (5.1) $a = \hat{k}_\lambda$, $b = A\hat{k}_\lambda$ and $c = |A|^2\hat{k}_\lambda$, we get

$$\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2 + \left\|A\hat{k}_\lambda\right\|^4$$

$$= \left|\left\langle \hat{k}_\lambda, A\hat{k}_\lambda \right\rangle\right|^2 + \left|\left\langle \hat{k}_\lambda, |A|^2 \hat{k}_\lambda \right\rangle\right|^2$$
$$\leq \max\left\{\left\|A\hat{k}_\lambda\right\|^2, \left\||A|^2 \hat{k}_\lambda\right\|^2\right\} + \sqrt{2}\left|\left\langle A\hat{k}_\lambda, |A|^2 \hat{k}_\lambda \right\rangle\right|$$
$$= \frac{1}{2}\left(\left\|A\hat{k}_\lambda\right\|^2 + \left\||A|^2 \hat{k}_\lambda\right\|^2 + \left|\left\|A\hat{k}_\lambda\right\|^2 - \left\||A|^2 \hat{k}_\lambda\right\|^2\right|\right) + \sqrt{2}\left|\left\langle |A|^2 A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|$$
$$= \frac{1}{2}\left(\left|\left\langle\left(|A|^4 + |A|^2\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| + \left|\left\langle\left(|A|^4 - |A|^2\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right) + \sqrt{2}\left|\left\langle |A|^2 A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|$$
$$\leq \frac{1}{2}\left(\mathbf{ber}\left(|A|^4 + |A|^2\right) + \mathbf{ber}\left(|A|^4 - |A|^2\right)\right) + \sqrt{2}\mathbf{ber}\left(|A|^2 A\right).$$

Thus

$$\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|^2 + \left\|A\hat{k}_\lambda\right\|^4 \leq \frac{1}{2}\left(\mathbf{ber}\left(|A|^4 + |A|^2\right) + \mathbf{ber}\left(|A|^4 - |A|^2\right)\right) + \sqrt{2}\mathbf{ber}\left(|A|^2 A\right).$$

Now taking the speremum over $\lambda \in \Omega$ in the latter inequality we deduce the required inequality. □

In the sequel, we need the following lemma due to Dragomir [10, p. 132].

Lemma 5.3 *For any $a, b, c \in \mathcal{H}$, we have:*

$$|\langle a, b\rangle|^2 + |\langle a, c\rangle|^2 \leq \|a\| \left(\max\left\{|\langle a, b\rangle|, |\langle a, c\rangle|\right\}\right)\left(\|b\|^2 + \|c\|^2 + 2\,|\langle b, c\rangle|\right)^{\frac{1}{2}}.$$

Our next result gives another upper bound for the Davis–Wielandt–Berezin radius of operators in $B(\mathcal{H}(\Omega))$.

Theorem 5.4 *Let $A \in B(\mathcal{H}(\Omega))$. Then*

$$\eta^2(A) \leq \max\left\{\mathbf{ber}(A), \mathbf{ber}\left(|A|^2\right)\right\}\left(\mathbf{ber}\left(|A|^4 + |A|^2\right) + 2\mathbf{ber}\left(|A|^2 A\right)\right)^{\frac{1}{2}}.$$

Proof Let $\lambda \in \Omega$ be an arbitray and let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Choosing in Lemma 5.3, $a = \hat{k}_\lambda$, $b = A\hat{k}_\lambda$ and $c = |A|^2 \hat{k}_\lambda$, we obtain

$$\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|^2 + \left\|A\hat{k}_\lambda\right\|^4$$
$$= \left|\left\langle \hat{k}_\lambda, A\hat{k}_\lambda \right\rangle\right|^2 + \left|\left\langle \hat{k}_\lambda, |A|^2 \hat{k}_\lambda \right\rangle\right|^2$$
$$\leq \left\|\hat{k}_\lambda\right\|\left(\max\left\{\left|\left\langle \hat{k}_\lambda, A\hat{k}_\lambda \right\rangle\right|, \left|\left\langle \hat{k}_\lambda, |A|^2 \hat{k}_\lambda \right\rangle\right|\right\}\right)\left(\left\|A\hat{k}_\lambda\right\|^2 + \left\||A|^2 \hat{k}_\lambda\right\|^2\right.$$
$$\left. + 2\left|\left\langle A\hat{k}_\lambda, |A|^2 \hat{k}_\lambda \right\rangle\right|\right)^{\frac{1}{2}}$$
$$\leq \max\left\{\mathbf{ber}(A), \mathbf{ber}\left(|A|^2\right)\right\}\left(\left\langle\left(|A|^4 + |A|^2\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + 2\left|\left\langle |A|^2 A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|\right)^{\frac{1}{2}}$$

$$\leq \max\left\{\mathbf{ber}(A), \mathbf{ber}\left(|A|^2\right)\right\}\left(\mathbf{ber}\left(|A|^4+|A|^2\right)+2\mathbf{ber}\left(|A|^2 A\right)\right)^{\frac{1}{2}}.$$

Thus, we have that

$$\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2+\left\|A\hat{k}_\lambda\right\|^4 \leq \max\left\{\mathbf{ber}(A), \mathbf{ber}\left(|A|^2\right)\right\}\left(\mathbf{ber}\left(|A|^4+|A|^2\right)\right.$$
$$\left.+2\mathbf{ber}\left(|A|^2 A\right)\right)^{\frac{1}{2}}.$$

Now, the result follows by taking the supremum over all points $\lambda \in \Omega$. □

Finally, we derive the folloing result from Lemma 5.3.

Theorem 5.5 *Let* $A \in B(\mathcal{H}(\Omega))$. *Then*

$$\eta^2(A) \leq \|A\|_{Ber}\max\left\{\mathbf{ber}(A), \mathbf{ber}\left(|A|^2\right)\right\}\left(1+\|A\|_{Ber}^2+2\mathbf{ber}(A)\right)^{\frac{1}{2}}.$$

Proof Let $\lambda \in \Omega$ be an arbitray point and let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Choosing in Lemma 5.3, $a = A\hat{k}_\lambda$, $b = \hat{k}_\lambda$ and $c = A\hat{k}_\lambda$, we get

$$\begin{aligned}
&\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^2+\left\|A\hat{k}_\lambda\right\|^4\\
&=\left|\left\langle \hat{k}_\lambda, A\hat{k}_\lambda\right\rangle\right|^2+\left|\left\langle \hat{k}_\lambda, |A|^2\hat{k}_\lambda\right\rangle\right|^2\\
&\leq \left\|A\hat{k}_\lambda\right\|\left(\max\left\{\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|, \left|\left\langle A\hat{k}_\lambda, A\hat{k}_\lambda\right\rangle\right|\right\}\right)\left(1+\left\|A\hat{k}_\lambda\right\|^2+2\left|\left\langle \hat{k}_\lambda, A\hat{k}_\lambda\right\rangle\right|\right)^{\frac{1}{2}}\\
&=\left\|A\hat{k}_\lambda\right\|\left(\max\left\{\left|\left\langle A\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|, \left|\left\langle |A|^2\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right\}\right)\left(1+\left\|A\hat{k}_\lambda\right\|^2+2\left|\left\langle \hat{k}_\lambda, A\hat{k}_\lambda\right\rangle\right|\right)^{\frac{1}{2}}\\
&\leq \|A\|_{Ber}\max\left\{\mathbf{ber}(A), \mathbf{ber}\left(|A|^2\right)\right\}\left(1+\|A\|_{Ber}^2+2\mathbf{ber}(A)\right)^{\frac{1}{2}},
\end{aligned}$$

which obviously implies the desired inequality. □

Funding The first author was supported by the Researchers Supporting Project number RSP2023R1056, King Saud University, Riyadh, Saudi Arabia.

Data availibility statement No data was used for the research described in the article.

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

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