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Inertial Halpern-type method for solving split feasibility and fixed point problems via dynamical stepsize in real Banach spaces

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Abstract
In this paper, we introduce a modified Halpern inertial method for approximating solutions of split feasibility problem and fixed point problem of Bregman strongly nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth real Banach spaces. We establish a strong convergence result for the sequence generated by our iterative scheme under some mild conditions without the computation of the operator norm. We state some consequences and present some examples to show the efficiency and implementation of our proposed method. The result discussed in this paper extends and generalizes many recent results in this direction. Our result extends and complements some related results in literature.

Keywords Split feasibility problem · Bregman strongly nonexpansive · Iterative scheme · Inertial method · Fixed point problem

Mathematics Subject Classification 47H06 · 47H09 · 47J05 · 47J25

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1 Introduction

Let X_1 and X_2 be p-uniformly convex and uniformly smooth real Banach spaces, C and Q are nonempty, closed and convex subsets of X_1 and X_2 respectively. The Split Feasibility Problem (SFP) is to find

$$x \in C \text{ such that } y = Fx \in Q, \tag{1.1}$$

where $F : X_1 \to X_2$ is a bounded linear operator. We denote by $\Omega := C \cap F^{-1}(Q)$ the solution set of SFP, then we have that Ω is a closed and convex.

One of the most attractive problem in optimization is the SFP due to its numerous applications to real life problems such as signal processing, image reconstruction and medical care, (see [9, 10, 20]). Many interesting optimization problems such as equilibrium, variational inequality, variational inclusion and convex minimization problems have been defined in terms of SFP, (see [1, 5, 14, 20–22]). Many well known iterative algorithms have been proposed to solve the SFP (see [3, 4, 6, 17, 20]). In 1994, Censor and Elving [10] used the idea of multi-distance to obtain iterative methods for solving SFP. Their iterative methods, as well as others later, involve matrix inverses at each iteration. Bryne [8] introduced a projection method known as the CQ algorithm for approximating the SFP that does not involve matrix inverses, but assumed that the metric projections onto C and Q are easily calculated. However in most cases, it is impossible or needs too much work to compute the metric projections. Therefore if such appears, the efficiency of the projection-type methods including the CQ algorithm will be affected. In 2004, Yang [31] introduced a relaxed CQ for solving the SFP, where he employed two half spaces C_k and Q_k to replace C and Q respectively, at the kth iteration and the metric projections onto C_k and Q_k are easily computed. Recently Lopez et al. [15] introduced a self-adaptive step size to improve the CQ and the relaxed CQ iterative methods. It was noted that all these aforementioned iterative methods only use the current point to get the next iteration, which does not use the previous iteration $x^{k-1}, x^{k-2}, \ldots,$ and affect the flexibility. It is known that using some information of previous iterates will increase the flexibility of the algorithm. The study of SFP has been extended to the framework of 2-uniformly convex and uniformly smooth real Banach spaces. For instance, Ma et al. [17] proposed a shrinking iterative method for SFP and fixed point problem of quasi-ϕ-nonexpansive mappings in Banach spaces. They proved a strong convergence result without imposing any compactness conditions and display a numerical example to show the behavior of their result.

In 2007, Schopfer [25] introduced the following algorithm: $x_1 \in X_1$ and

$$x_{n+1} = \Pi_C J_{X_1^*}\big[J_{X_1}(x_n) - \gamma_n F^* J_{X_2}(Fx_n - P_Q(Fx_n))\big],\ n \geq 1, \tag{1.2}$$

where Π_C denotes the Bregman projection and J is the duality mapping. It is clear that (1.2) contains the CQ algorithm as a special case. In addition, Schopfer [25] obtained a weak convergence result for solving SFP provided the duality mapping J is weak-to-weak continuous and $\gamma_n \in \left(0, \left(\frac{q}{C_q\|F\|^q}\right)^{\frac{1}{q-1}}\right)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and C_q is the uniform smoothness coefficient of X. Readers should consult [3, 4, 6, 10, 20, 22, 24, 31] for more results on SFP and its generalization.

In optimization theory, one of the best ways to fasten up the rate of convergence of iterative method is to combine the iterative method with an inertial term. This term which is represented in its originality as $\theta_n(x_n - x_{n-1})$ is a remarkable tool for improving the performance of iterative methods and it is known to have some nice convergence properties. Polyak [23] was the first to proposed the inertial extrapolation method for solving convex minimization problem. The inertial method is a two-step iterative method, using the first two iterations to define the next iteration. Nestrov [19] proposed a modified method to improve the convergence rate as follows:

$$\begin{cases} v_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = v_n - \lambda_n \nabla f(v_n), \ n \geq 1, \end{cases} \tag{1.3}$$

where $\theta_n \in [0, 1)$ is an extrapolation factor, and $\{\lambda_n\}$ is a positive sequence. Inspired by the inertial extrapolation method, many authors have proposed different inertial iterative methods to solve a number of optimization problems, see [1, 2, 4, 23, 24, 28]. It is worth mentioning that most results involving inertial extrapolation method in Banach spaces requires the modification or relaxation of the inertial term (most especially when Halpern method is employed, see (1.4) below) due to the geometry of the space and convexity problem. To retain its originality (i.e. $\theta_n(x_n - x_{n-1})$) in the aforementioned space, the shrinking or Hybrid iterative methods need to be employed. For instance, Godwin et al. [14] introduced the following inertial Halpern method for solving common solution of split minimization and fixed point problems with finite family of Bregman relatively nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth Banach spaces. Given iterates x_{n-1}, x_n, compute $\{x_n\}$ as follows:

$$\begin{cases} w_n = J^q_{X^*}\big[J^p_X(x_n) + \theta_n(J^p_X(x_{n-1}) - J^p_X(x_n))\big], \\ y_n = J^q_{X^*}\left[\sum_{i=0}^{N} \beta_{i,n}(J^p_X(w_n) - \tau_{i,n} T_i^* J^p_{X_i}(I^{X_i} - prox^{f_i}_{\lambda^i}) T_i(w_n))\right] \\ z_n = J^q_{X^*}\left[\phi_{n,0} J^p_X(y_n) + \sum_{j=1}^{m} \phi_{n,j} J^p_X(S_j y_n)\right] \\ x_{n+1} = J^q_{X^*}\big(\alpha_n J^p_X(u) + (1-\alpha_n) J^p_X(z_n)\big), \end{cases} \tag{1.4}$$

where

$$\tau_{i,n} \in \left(\epsilon, \left(\frac{q\|T_i(w_n) - (prox^{f_i}_{\lambda^i_n})T_i(w_n)\|^p}{C_q\|T_i^* J^p_{X_i}(I^{X_i} - prox^{f_i}_{\lambda^i_n})T_i(w_n)\|^q} - \epsilon\right)^{\frac{1}{q-1}}\right),$$

$\forall\, n \in \Omega$, where the index set $\Omega := \{n \in \mathbb{N} : T_i(w_n) - (prox^{f_i}_{\lambda^i_n} T_i(w_n) \neq 0)\}$, otherwise, $\tau_{i,n} = \tau_i$ is any nonnegative real number for each $i = 0, 1, \ldots, N$. (Readers should consult [14] for definition of terms used in (1.4)). Also see [1, 2, 22] for results on modified inertial methods in Banach spaces.

Very recently, Shehu et al. [24] introduced the following self adaptive projection method with an inertial technique for split feasibility problems in Banach spaces: set $x_0, x_1 \in C$, define a sequence $\{x_n\}$ by the following manner:

$$\begin{cases} w_n = J_{E_1}^q \Big[J_{X_1}^p(x_n) + \alpha_n (J_{X_1}^p(x_n) - J_{X_1}^p(x_{n-1})) \Big] \\ y_n = \Pi_C J_{X_1}^q \Big[J_{X_1}^p(w_n) - \rho_n \frac{f^{p-1}(w_n)}{\|\nabla f(w_n)\|^p} \nabla f(w_n) \Big] \\ C_n = \{u \in X_1 : \Delta_p(y_n, u) \le \Delta_p(w_n, u)\} \\ Q_n = \{u \in X_1 : \langle x_n - u, J_{X_1}^p(x_0) - J_{X_1}^p(x_n) \rangle \ge 0\} \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.5}$$

for all $n \ge 0$ where $f(w_n) := \frac{1}{p}\|(I - P_Q)Aw_n\|^p$, $\{\rho_n\} \subset (0, \infty)$, and $\liminf\limits_{n\to\infty} \rho_n (p - C_q \frac{\rho_n^{q-1}}{q}) > 0$. In (1.5), X_i, $i = 1, 2$ is a p-uniformly convex real Banach space which is also uniformly smooth, C and Q are nonempty, closed and convex subsets of X_1 and X_2.

It cab be seen from (1.4) where the Halpern method is employed that the inertial term is being modified. Also, in (1.5), the inertial term retain its originality as defined by Polyak [23] due to the nature of the algorithm.

Question Can we approximate solution of SFP and fixed point problem in p-uniformly Banach spaces which are also uniformly smooth with an inertial-Halpern method without modifying the inertial term, (see [2])?

In this article, we give an affirmative answer to the above question. We also state our contributions in this article as follows:

Remark 1.1 (i) We consider SFP in p-uniformly convex and uniformly smooth Banach space which generalizes the results of [17].

(ii) The step size ρ_n employed in our main result is generated at each iteration by some computation. Thus our algorithm is easily implemented without prior knowledge of operator norm.

(iii) The inertial term employed in our main result retain its originality as defined by Polyak [23]. It is worth-mentioning that the results on inertial Halpern method in Banach spaces requires the modification or relaxation of the inertial term (see [2, 14, 22]) due to the geometry of the spaces (convexity to be precise). Thus, in our result, we proved a strong convergence result without modifying the inertial term.

(iv) Our algorithm does not require at each step of the iteration process, the computation of subsets of C_n, Q_n and D_n (or C_{n+1}) as in the case in [24] and the computation of the projection of the initial point onto their intersection, which leads to a high computational cost of iteration processes.

The removal of all these restrictions makes our work applicable to more real world problems.

(v) The inertial technique employed in our article is easily implemented since the value of $\|J_E^p(x_n) - J_E^p(x_{n-1})\|$ is a prior known before choosing θ_n.

Motivated by the works of [20, 22, 24] and other related results in literature, we proposed a modified Halpern inertial method for approximating solution of split feasibility problem of Bregman strongly nonexpansive mappings in p-uniformly Banach spaces which are also uniformly smooth. We establish a strong convergence result for solving the solution of the aforementioned problems. It is worth-mentioning that the iterative algorithm employed in this article is designed in such a way that it does not require the computation of operator norm. The result discussed in this article extends and complements many related results in the literature.

2 Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Let X be a re al Banach space with norm $\|.\|$ and X^* be the dual space of E. Let $K(X) := \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. The modulus of convexity is the function $\delta_X : (0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in K(X),\ \|x - y\| \geq \epsilon\right\}.$$

The space X is said to be uniformly convex, if $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. Let $p > 1$, then X is said to be p-uniformly convex (or to have a modulus of convexity of power type p) if there exists $c_p > 0$ such that $\delta_X(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0, 2]$. Note that every p-uniformly convex space is uniformly convex. The modulus of smoothness of X is the function $\rho_X : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x + \tau y + \|x - \tau y\|}{2} - 1 : x, y \in K(X)\right\}.$$

The space X is said to be uniformly smooth, if $\frac{\rho_X(\tau)}{\tau} \to 0$ as $\tau \to 0$. Let $q > 1$, then a Banach space X is said to be q-uniformly smooth if there exists $\kappa_q > 0$ such that $\rho_X(\tau) \leq \kappa_q\tau^q$ for all $\tau > 0$. Moreover, a Banach space X is p-uniformly convex if and only if X^* is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$, (see [12]). Let $p > 1$ be a real number, the generalized duality mapping $J_X^p : X \to 2^{X^*}$ is defined by

$$J_X^p(x) = \{\bar{x} \in X^* : \langle x, \bar{x}\rangle = \|x\|^p, \|\bar{x}\| = \|x\|^{p-1}\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between elements of X and X^*. In particular, $J_X^p = J_X^2$ is called the normalized duality mapping.

If X is p-uniformly convex and uniformly smooth, then X^* is q-uniformly smooth and uniformly convex. In this case, the generalized duality mapping J_X^p is one-to-one, single-valued and satisfies $J_X^p = (J_{X^*}^q)^{-1}$, where $J_{X^*}^q$ is the generalized duality mapping of X^*. Furthermore, if X is uniformly smooth then the duality mapping J_X^p is norm-to-norm uniformly continuous on bounded subsets of X, (see [13] for more details).
Let $f : X \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Frenchel conjugate of f denoted as $f^* : X^* \to (-\infty, +\infty]$ is define as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\},\ x^* \in X^*.$$

Let the domain of f be denoted as $(domf) = \{x \in X : f(x) < +\infty\}$, hence for any $x \in int(domf)$ and $y \in X$, we define the right-hand derivative of f at x in the direction y by

$$f^0(x, y) = \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

Definition 2.1 [7] Let $f : X \to (-\infty, +\infty]$ be a convex and G$\hat{a}$teaux differentiable function. The function $\Delta_f : X \times X \to [0, +\infty)$ defined by

$$\Delta_f(x, y) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle$$

is called the Bregman distance with respect of f.

It is well-known that Bregman distance Δ_f does not satisfy the properties of a metric because Δ_f fail to satisfy the symmetric and triangular inequality property. Moreover, it is well known that the duality mapping J_X^p is the sub-differential of the functional $f_p(.) = \frac{1}{p}\|.\|^p$ for $p > 1$, see [11]. Then, the Bregman distance Δ_p is defined with respect to f_p as follows:

$$\begin{aligned}\Delta_p(x, y) &= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle J_X^p x, y - x\rangle\\ &= \frac{1}{q}\|x\|^p - \langle J_X^p x, y\rangle + \frac{1}{p}\|y\|^p\\ &= \frac{1}{q}\big(\|x\|^p - \frac{1}{q}\|y\|^p\big) - \langle J_X^p x - J_X^p y, y\rangle. \end{aligned} \tag{2.1}$$

The Bregman distance is not symmetric therefore is not a symmetric but it possess the following important properties:

$$\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_X^p x - J_X^p y\rangle,\ \forall\, x, y, z \in X, \tag{2.2}$$

and

$$\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_X^p - J_X^p\rangle,\ \forall\, x, y \in X. \tag{2.3}$$

Let $Fix(T)$ denotes the set of fixed points of a mapping T from C into itself. That is $Fix(T) = \{x \in C : Tx = x\}$. A point $p \in C$ is said to be an asymptotic fixed point of T, if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}ix(T)$, the set of asymptotic fixed points of T. Moreso, a mapping $T : C \to int(domf)$ is said to be

(i) Bregman relatively nonexpansive, if

$$\hat{F}ix(T) = Fix(T) \text{ and } \Delta_p(p, Tx) \le \Delta_p(p, x), \ \forall\, x \in C, \ \ p \in Fix(T).$$

(ii) Bregman quasi-nonexpansive, if

$$Fix(T) \ne \emptyset \text{ and } \Delta_p(p, Tx) \le \Delta_p(p, x), \ \forall\, x \in C, \ \ p \in Fix(T).$$

(iii) Bregman firmly nonexpansive mapping (BFNE) if

$$\langle J_p^X(Tx) - J_p^X(Ty), Tx - Ty\rangle \le \langle J_p^X(x) - J_p^X(y), Tx - Ty\rangle, \ \forall\, x, y \in C,$$

(iv) Bregman strongly nonexpansive mapping (BSNE) [27] with $\hat{F}ix(T) \ne \emptyset$ if

$$\Delta_p(y, Tx) \le \Delta_p(y, x), \ \forall\, y \in \hat{F}ix(T)$$

and for any bounded sequence $\{x_n\}_{n\ge1} \subset C$,

$$\lim_{n\to\infty} (\Delta_p(y, x_n) - \Delta_p(y, Tx_n)) = 0$$

implies

$$\lim_{n\to\infty} \Delta_p(Tx_n, x_n) = 0.$$

Recall that a metric projection P_C from X onto C satisfies the following property:

$$\|x - P_C x\| \le \inf_{y\in C} \|x - y\|, \ \forall\, x \in X.$$

It is well known that $P_C x$ is the unique minimizer of the norm distance. Moreover, $P_C x$ is characterized by the following properties:

$$\langle J_X^p(x - P_C x), y - P_C x\rangle \le 0, \ \forall\, y \in C. \tag{2.4}$$

The Bregman projection from X onto C denoted by Π_C also satisfies the property

$$\Delta_p(x, \Pi_C(x)) = \inf_{y\in C} \Delta_p(x, y), \ \forall\, x \in X. \tag{2.5}$$

Also, if C is a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space X and $x \in X$. Then the following assertions holds:

(i) $z = \Pi_C x$ if and only if

$$\langle J_X^p(x) - J_X^p(z),\ y - z\rangle \leq 0,\ \forall\, y \in C; \tag{2.6}$$

(ii)

$$\Delta_p(\Pi_C x, y) + \Delta_p(x, \Pi_C x) \leq \Delta_p(x, y),\ \forall\, y \in C. \tag{2.7}$$

When considering the p-uniformly convex space, the Bregman distance and the metric distance have the following relation, (see [24]).

$$\pi_p \|x - y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_X^p(x) - J_X^p(y)\rangle, \tag{2.8}$$

where $\pi_p > 0$ is some fixed number. If $\frac{1}{p} + \frac{1}{q} = 1$, by Young's inequality, we have

$$\begin{aligned}\langle J_X^p(x), y\rangle \leq \|J_X^p(x)\|\|y\| &\leq \frac{1}{q}\|J_X^p(x)\|^q + \frac{1}{p}\|y\|^p\\ &= \frac{1}{q}(\|x\|^{p-1})^q + \frac{1}{p}\|y\|^p\\ &= \frac{1}{q}\|x\|^p + \frac{1}{p}\|y\|^p. \end{aligned}\tag{2.9}$$

Lemma 2.2 *[11] Let X be a Banach space and $x, y \in X$. If X is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$\|x - y\|^q \leq \|x\|^q - q\langle J_q^X(x), y\rangle + C_q\|y\|^q.$$

Lemma 2.3 *[26] Let X be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p : X^* \times X \to [0, +\infty)$ be defined by*

$$V_p(x^*, x) = \frac{1}{q}\|x^*\|^q - \langle x^*, x\rangle + \frac{1}{p}\|x\|^p,\ \forall\, x \in X, x^* \in X.$$

Then the following assertions hold:

(i) V_p is nonnegative and convex in the first variable.
(ii) $\Delta_p(J_q^{X^*}(x^*), x) = V_p(x^*, x),\ \forall\, x \in X,\ x^* \in X.$
(iii) $V_p(x^*, x) + \langle y^*, J_q^{X^*}(x^*) - x\rangle \leq V_p(x^* + y^*, x), \forall\, x \in X,\ x^*, y^* \in X.$

Lemma 2.4 *[12] Let X be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in X. Then $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$ implies $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.5 *[30] Assume $\{a_n\}$ is a sequence of nonnegative real sequence such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \sigma_n\delta_n,\ n > 0,$$

where $\{\sigma_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a real sequence such that*

(i) $\sum_{n=1}^{\infty} \sigma_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 *[18] Let* Γ_n *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $\{\Gamma_{n_j}\}_j \geq 0$ *of* $\{\Gamma_{n_j}\}$ *which satisfies* $\Gamma_{n_j} \leq \Gamma_{n_j+1}$ *for all* $j \geq 0$*. Also consider a sequence of integers* $\{\tau(n)\}_n \geq n_0$ *defined by*

$$\tau(u) = \max\{k \leq n | \Gamma_{n_k} \leq \Gamma_{n_k+1}\}.$$

Then $\{\tau(n)\}_n \geq n_0$ *is a nondecreasing sequence satisfying* $\lim_{n\infty} \tau(n) = \infty$.
If it holds that $\Gamma_\tau(n) \leq \Gamma_{\tau(n)+1}$

3 Main result

Theorem 3.1 *Let* X_1 *and* X_2 *be p-uniformly convex and uniformly smooth real Banach spaces and* $F : X_1 \to X_2$ *be a bounded linear operator with its adjoint* $F^* : X_2^* \to X_1^*$*. Let C and Q be nonempty, closed and convex subsets of* X_1 *and* X_2 *respectively, and* $f : X_2 \to \mathbb{R}$ *be a non-negative lower semi-continous convex function. Suppose* $S : X_1 \to X_1$ *is a Bregman strongly nonexpansive mapping with* $\Gamma := \Omega \cap Fix(S)$ *is nonempty. Let* $\{\lambda_n\}$ *be a positive sequence in* $(0, \frac{p\pi_p}{2^{p-1}})$*, where* π_p *is defined in (2.8),* $\lambda_n = \circ(\alpha_n)$*,* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *are sequences in* $(0, 1)$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*,* $\beta_n \in (a, b) \subset (0, 1)$ *and* $\gamma_n \in (c, d) \subset (0, 1)$ *for all* $n \geq 1$*. For fixed* $v, x_0, x_1 \in X_1$*, choose* θ_n *such that* $0 \leq \theta_n \leq \bar{\theta}_n$*, then define a sequence* $\{x_n\}$ *by the following manner:*

$$\begin{cases} u_n = J_{X_1^*}^q \left[J_{X_1}^p(x_n) + \theta_n (J_{X_1}^p(x_n) - J_{X_1}^p(x_{n-1})) \right] \\ y_n = \Pi_C J_{X_1^*}^q \left[J_{X_1}^p(u_n) - \rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p} \nabla f(u_n) \right] \\ x_{n+1} = \Pi_C J_{X_1^*}^q \left[\alpha_n J_{X_1}^p(v) + \beta_n J_{X_1}^p(y_n) + \gamma_n J_{X_1}^p(Sy_n) \right], \ n \geq 1, \end{cases} \tag{3.1}$$

where

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\lambda_n}{\|J_{X_1}^p(x_n) - J_{X_1}^p(x_{n-1})\|}\}, & if \ \ x_n \neq x_{n-1}, \\ \theta, \ \ otherwise, \end{cases} \tag{3.2}$$

$f(u_n) := \frac{1}{p}\|(I-P_Q)Fu_n\|^p$, $\nabla f(u_n) := F^* J_{X_2}^p (I-P_Q)Fu_n$, $\{\rho_n\} \subset (0,\infty)$ *and* $\liminf_{n\to\infty} \rho_n\left(p - C_q \frac{\rho_n^{q-1}}{q}\right) > 0$, *where* C_q *is the uniform smoothness coefficient of* X_1. *Then* $\{x_n\}$ *converges strongly to* $x^* = \Pi_\Gamma v$.

Proof Let $z \in \Gamma$ and $b_n = J_{X_1^*}^q [J_{X_1}^p(u_n) - \rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\ f(u_n)]$ for all $n \geq 1$. We obtain from Lemma 2.2 that

$$
\begin{aligned}
\|b_n\|_{X_1^*}^q &= \|J_{X_1}^p(u_n) - \rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\nabla f(u_n)\|_{X_1^*}^q \\
&\leq \|u_n\|^p - q\rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\langle u_n, \nabla f(u_n)\rangle \\
&\quad + C_q \rho_n^q \frac{f^{(p-1)q}(u_n)}{\|\nabla f(u_n)\|^{pq}}\|\nabla f(u_n)\|^q \\
&= \|u_n\|^p - q\rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\langle u_n, \nabla f(u_n)\rangle + C_q\rho_n^q \frac{f^p(u_n)}{\|\nabla f(u_n)\|^p}. \qquad (3.3)
\end{aligned}
$$

By applying (2.7) and (3.3), we get

$$
\begin{aligned}
\Delta_p(y_n, z) &\leq \Delta_p(J_{X_1}^p(b_n), z) \\
&= \frac{\|z\|^p}{p} + \frac{\|J_{X_1}^p(b_n)\|^p}{q} - \langle z, b_n\rangle \\
&= \frac{\|z\|^p}{p} + \frac{1}{q}\|b_n\|^{(q-1)p} - \langle z, b_n\rangle \\
&= \frac{\|z\|^p}{p} + \frac{1}{q}\|b_n\|^{(q-1)\frac{q}{q-1}} - \langle z, b_n\rangle \\
&= \frac{\|z\|^p}{p} + \frac{1}{q}\|b_n\|^q - \langle z, J_{X_1}^p(u_n)\rangle + \rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\langle z, \nabla f(u_n)\rangle \\
&\leq \frac{\|z\|^p}{p} + \frac{1}{q}\big(\|u_n\|^p - q\rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\langle u_n, \nabla f(u_n)\rangle + C_q\rho_n^q \frac{f^p(u_n)}{\|\nabla f(u_n)\|^p}\big) \\
&\quad - \langle z, J_{X_1}^p(u_n)\rangle + \rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\langle z, \nabla f(u_n)\rangle \\
&= \frac{\|z\|^p}{p} + \frac{\|u_n\|^p}{q} - \langle z, J_{X_1}^p(u_n)\rangle + \frac{C_q\rho_n^q}{q}\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p} \\
&\quad + \rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\langle z - u_n, \nabla f(u_n)\rangle \\
&= \Delta_p(u_n, z) + \frac{C_q\rho_n^q}{q}\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p} + \rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\langle z - u_n, \nabla f(u_n)\rangle. \qquad (3.4)
\end{aligned}
$$

But from (2.4) and that $Fz \in Q$

$$\langle \nabla f(u_n), z - u_n\rangle = \langle F^* J_{X_2}^p (I - P_Q)Fu_n, z - u_n\rangle$$

$$
\begin{aligned}
&= \langle J_{X_2}^p (I - P_Q) F u_n, Fz - F u_n \rangle \\
&= \langle J_{X_2}^p (I - P_Q) F u_n, P_Q F u_n - F u_n \rangle \\
&\quad + \langle J_{X_2}^p (I - P_Q) F u_n, Fz - P_Q F u_n \rangle \\
&\leq -\|(I - P_Q) F u_n\|^p \\
&= -p f(u_n).
\end{aligned} \tag{3.5}
$$

On substituting (3.5) into (3.4), it yields

$$
\Delta_p(y_n, z) \leq \Delta_p(u_n, z) + \Big(\frac{C_q \rho_n^q}{q} - \rho_n p\Big) \frac{f^p(u_n)}{\|\nabla f(u_n)\|^p}. \tag{3.6}
$$

Hence, we conclude that

$$
\Delta_p(y_n, z) \leq \Delta_p(u_n, z). \tag{3.7}
$$

Now, using (2.8), (2.9) and (3.1), we have

$$
\begin{aligned}
\langle J_{X_1}^p u_n - J_{X_1}^p x_n, u_n - z \rangle &\leq \|J_{X_1}^p u_n - J_{X_1}^p x_n\| \, \|u_n - z\| \\
&= \theta_n \|J_{X_1}^p x_n - J_{X_1}^p x_{n-1}\| \, \|u_n - z\| \\
&\leq \theta_n \|J_{X_1}^p x_n - J_{X_1}^p x_{n-1}\| \left[\frac{1}{p}\|u_n - z\|^p + \frac{1}{q}\right] \\
&\leq \theta_n \|J_{X_1}^p x_n - J_{X_1}^p x_{n-1}\| \left[2^{p-1}(\|x_n - u_n\|^p + \|x_n - z\|^p)\right] \\
&\quad + \frac{\theta_n}{q} \|J_{X_1}^p x_n - J_{X_1}^p x_{n-1}\| \\
&\leq \frac{2^{p-1}\lambda_n}{p\pi_p} \Big(\Delta_p(x_n, u_n) + \Delta_p(x_n, z)\Big) + \frac{\lambda_n}{q}.
\end{aligned} \tag{3.8}
$$

Also using (2.3), we get

$$
\Delta_p(u_n, z) = \Delta_p(x_n, z) - \Delta_p(x_n, u_n) + \langle J_{X_1}^p u_n - J_{X_1}^p x_n, u_n - z \rangle. \tag{3.9}
$$

On substituting (3.8) into (3.9), we have

$$\begin{aligned}\Delta_p(u_n, z) &= \Delta_p(x_n, z) - \Delta_p(x_n, u_n) + \frac{2^{p-1}\lambda_n}{p\pi_p}\Big(\Delta_p(x_n, u_n) + \Delta_p(x_n, z)\Big) + \frac{\lambda_n}{q} \\ &\leq \Big(1 + \frac{2^{p-1}\lambda_n}{p\pi_p}\Big)\Delta_p(x_n, z) - \Big(1 - \frac{2^{p-1}\lambda_n}{p\pi_p}\Big)\Delta_p(x_n, u_n) + \frac{\lambda_n}{q}.\end{aligned} \tag{3.10}$$

From (3.1), (3.8) and (3.10), we obtain

$$\begin{aligned}\Delta_p(x_{n+1}, z) &\leq \Delta_p\big(J^q_{X_1}[\alpha_n J^p_{X_1}(v) + \beta_n J^p_{X_1}(y_n) + \gamma_n J^p_{X_1}(Sy_n)], z\big) \\ &\leq \alpha_n\Delta_p(v, z) + \beta_n\Delta_p(y_n, z) + \gamma_n\Delta_p(Sy_n, z) \\ &\leq \alpha_n\Delta_p(v, z) + \beta_n\Delta_p(y_n, z) + \gamma_n(y_n, z) \\ &= \alpha_n\Delta_p(v, z) + (1-\alpha_n)\Delta_p(y_n, z) \\ &= \alpha_n\Delta_p(v, z) + (1-\alpha_n)\Delta_p(u_n, z).\end{aligned} \tag{3.11}$$

From the assumption that $\lim_{n\to\infty}\frac{\lambda_n}{\alpha_n} = 0$, taking $\phi \in (0, \frac{p\pi_p}{2^{p-1}})$. Then there exists $N \in \mathbb{N}$ such that $\lambda_n < \alpha_n$ for all $n \geq \mathbb{N}$.

Hence

$$\frac{\lambda_n 2^{p-1}}{p\pi_p} < \alpha_n\phi < \frac{2^{p-1}}{p\pi_p}\alpha_n, \ \forall\, n \in \mathbb{N}.$$

For some constant $M > 0$, it follows from (3.10) that

$$\Delta_p(u_n, z) \leq (1 + \alpha_n\phi)\Delta_p(x_n, z) - (1 - \alpha_n\phi)\Delta_p(x_n, u_n) + \alpha_n M. \tag{3.12}$$

By substituting (3.12) into (3.11), we get

$$\begin{aligned}\Delta_p(x_{n+1}, z) &\leq \alpha_n\Delta_p(v, z) + (1-\alpha_n)\big[(1+\alpha_n\phi)\Delta_p(x_n, z) + \alpha_n M\big] \\ &\leq (1 - \alpha_n(1-\phi))\Delta_p(x_n, z) + \alpha_n\Delta_p(v, z) + \alpha_n M \\ &= (1 - \alpha_n(1-\phi))\Delta_p(x_n, z) + \alpha_n(1-\phi)\frac{\Delta_p(v, z) + M}{1-\phi} \\ &\leq \max\{\Delta_p(x_n, z), \frac{\Delta_p(v, z) + M}{1-\phi}\} \\ &\ \ \vdots \\ &\leq \max\{\Delta_p(x_1, z), \frac{\Delta_p(v, z) + M}{1-\phi}\}, \ \forall\, n \geq 1.\end{aligned}$$

This implies that $\{\Delta_p(x_n, z)\}$ is bounded. Consequently, $\{\Delta_p(u_n, z)\}$ and $\{\Delta_p(y_n, z)\}$ are bounded. By applying Lemma 2.4, we obtain that $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ are bounded. From (3.1), (3.6) and (3.12), we obtain

$$\begin{aligned}\Delta_p(x_{n+1}, z) &\leq \alpha_n \Delta_p(v, z) + (1-\alpha)\Delta_p(y_n, z) \\ &\leq \alpha_n \Delta_p(v, z) + (1-\alpha_n)\Delta_p(u_n, z) + (1-\alpha_n)\big(\frac{C_q\rho_n^q}{q} - \rho_n p\big)\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p} \\ &\leq \alpha_n \Delta_p(v, z) + (1-\alpha_n\phi)\Delta_p(x_n, z) - (1-\alpha_n\phi)\Delta_p(x_n, u_n) + \alpha_n M \\ &- (1-\alpha_n)\big(\frac{C_q\rho_n^q}{q} - \rho_n p\big)\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p}.\end{aligned} \tag{3.13}$$

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Delta_p(x_n, z)\}$ is non-increasing for all $n \geq n_0$. Then $\{\Delta_p(x_n, z)\}$ converges and

$$\Delta_p(x_{n+1}, z) - \Delta_p(x_n, z) \to 0, \ n \to \infty. \tag{3.14}$$

From (3.13), we get

$$\begin{aligned}&(1-\alpha_n\phi)\Delta_p(x_n, u_n) - (1-\alpha_n)\big(\frac{C_q\rho_n^q}{q} - \rho_n p\big)\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p} \\ &\leq (1-\alpha_n\phi)\Delta_p(x_n, z) - \Delta_p(x_{n+1}, z) \\ &+ \alpha_n M.\end{aligned} \tag{3.15}$$

Hence,

$$\lim_{n\to\infty} \Delta_p(x_n, u_n) = 0 = \lim_{n\to\infty} \rho_n\big(p - \frac{C_q\rho_n^{q-1}}{q}\big)\frac{f^p(u_n)}{\|\nabla f(u_n)\|^p}. \tag{3.16}$$

Since $\liminf_{n\to\infty} \rho_n(p - \frac{C_q\rho_n^{q-1}}{q}) > 0,$ we obtain that

$$\lim_{n\to\infty} \frac{f^p(u_n)}{\|\nabla f(u_n)\|^p} = 0, \tag{3.17}$$

and hence

$$\lim_{n\to\infty} \frac{f(u_n)}{\|\nabla f(u_n)\|} = 0. \tag{3.18}$$

Since $\{\nabla f(u_n)\}$ is bounded, we obtain from (3.18) that

$$\begin{aligned}0 \leq f(u_n) &= \|\nabla f(u_n)\|\frac{f(u_n)}{\|\nabla f(u_n)\|} \\ &\leq N_1\frac{f(u_n)}{\|\nabla f(u_n)\|} \to 0, \ n \to \infty, \text{ for some } N_1 > 0.\end{aligned}$$

Hence,

$$\lim_{n\to\infty} f(u_n) = 0, \tag{3.19}$$

and thus

$$\lim_{n\to\infty} \|Fu_n - P_Q Fu_n\| = 0. \tag{3.20}$$

By applying Lemma 2.4 in (3.16), we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.21}$$

From the definition of b_n, we obtain that

$$\begin{aligned}\lim_{n\to\infty} \|J_{X_1}^p(b_n) - J_{X_1}^p(u_n)\| &= \lim_{n\to\infty} \rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\|\nabla f(u_n)\| \\ &= \lim_{n\to\infty} \rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^{p-1}} \to 0. \end{aligned} \tag{3.22}$$

Since $J_{X_1^*}^q$ is norm-to-norm uniformly continuous subsets on X_1^*, then

$$\lim_{n\to\infty} \|b_n - u_n\| = 0, \tag{3.23}$$

and in view of (2.8), we get

$$\lim_{n\to\infty} \Delta_p(b_n, u_n) = 0. \tag{3.24}$$

By applying (3.20), we have

$$\|F^* J_{X_2}^p (I - P_Q)Fu_n\| \leq \|F\| \, \|(I - P_Q)Fu_n\| \to 0, \; n \to \; \infty. \tag{3.25}$$

Let $h_n = J_{X_1^*}^p\big[\frac{\beta_n}{1-\alpha_n} J_{X_1}^p(y_n) + \frac{\gamma_n}{1-\alpha_n} J_{X_1}^p(Sy_n)\big]$, then

$$\begin{aligned}\Delta_p(h_n, z) &= \Delta_p(J_{X_1^*}^p[\frac{\beta_n}{1-\alpha_n} J_{X_1}^p(y_n) + \frac{\gamma_n}{1-\alpha_n}(Sy_n)], z) \\ &\leq \frac{\beta_n}{1-\alpha_n}\Delta_p(y_n, z) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(Sy_n, z) \\ &\leq \frac{\beta_n}{1-\alpha_n}\Delta_p(y_n, z) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(y_n, z) \\ &= \Delta_p(y_n, z).\end{aligned}$$

Hence from (3.12), we have

$$\begin{aligned}0 &\leq \Delta_p(y_n, z) - \Delta_p(h_n, z) \\ &= \Delta_p(y_n, z) - \Delta_p(x_{n+1}, z) + \Delta_p(x_{n+1}, z) - \Delta_p(h_n, z) \\ &= \Delta_p(u_n, z) - \Delta_p(x_{n+1}, z) + \alpha_n\Delta_p(v, z) + (1-\alpha_n)\Delta_p(h_n, z) - \Delta_p(h_n, z)\end{aligned}$$

$$
\begin{aligned}
&\le (1+\alpha_n\phi)\Delta_p(x_n,z)-\Delta_p(x_{n+1},z)+\alpha_n M+\alpha_n\Delta_p(v,z)-\alpha_n\Delta_p(h_n,z)\\
&= \Delta_p(x_n,z)-\Delta_p(x_{n+1},z)+\alpha_n(\Delta_p(v,z)+\phi\Delta_p(x_n,z)-\Delta_p(h_n,z)+M)\\
&\to 0,\ n\to\infty. \qquad (3.26)
\end{aligned}
$$

Also

$$
\begin{aligned}
\Delta_p(h_n,z) &\le \frac{\beta_n}{1-\alpha_n}\Delta_p(y_n,z)+\frac{\gamma_n}{1-\alpha_n}\Delta_p(Sy_n,z)\\
&= (1-\frac{\gamma_n}{\alpha_n})\Delta_p(y_n,z)+\frac{\gamma_n}{1-\alpha_n}\Delta_p(Sy_n,z) \qquad (3.27)\\
&\le \Delta_p(y_n,z)+\frac{\gamma_n}{1-\alpha_n}\Delta_p\big((Sy_n,z)-\Delta_p(y_n,z)\big).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Delta_p(y_n,z)-\Delta_p(Sy_n,z) &\le \frac{\gamma_n}{1-\alpha_n}\big(\Delta_p(y_n,z)-\Delta_p(Sy_n,z)\big)\\
&\le \Delta_p(y_n,z)-\Delta_p(h_n,z)\to 0,\ n\to\infty. \qquad (3.28)
\end{aligned}
$$

Hence, we conclude that

$$
\lim_{n\to\infty}\Delta_p(y_n,Sy_n)=0, \qquad (3.29)
$$

which implies from Lemma 2.4 that

$$
\lim_{n\to\infty}\|y_n-Sy_n\|=0. \qquad (3.30)
$$

Using (2.7), we get

$$
\begin{aligned}
\Delta_p(y_n,u_n) &\le \Delta_p(b_n,u_n)-\Delta_p(y_n,b_n)\\
&\le \Delta_p(b_n,u_n)\to 0,\ n\to\infty. \qquad (3.31)
\end{aligned}
$$

In view of Lemma 2.4, we obtain that

$$
\lim_{n\to\infty}\|y_n-u_n\|=0. \qquad (3.32)
$$

Let $k_n := J^q_{X_1^*}\big[\alpha_n J^p_{X_1}(v)+\beta_n J^p_{X_1}(y_n)+\gamma_n J^p_{X_1}(Sy_n)\big]$, then from (3.1), (3.29) and Lemma 2.4, we obtain

$$
\begin{aligned}
\Delta_p(k_n,y_n) &= \Delta_p(J^q_{X_1^*}\big[\alpha_n J^p_{X_1}(v)+\beta_n J^p_{X_1}(y_n)+\gamma_n J^p_{X_1}(Sy_n)\big],y_n)\\
&\le \Delta_p(v,y_n)+\beta_n\Delta_p(y_n,y_n)+\gamma_n\Delta_p(Sy_n,y_n)\to 0,\ n\to\infty. \qquad (3.33)
\end{aligned}
$$

Hence,

$$
\lim_{n\to\infty}\|k_n-y_n\|=0. \qquad (3.34)
$$

By applying (2.7), (3.33) and Lemma 2.4, we get

$$\begin{aligned}\Delta_p(x_{n+1}, y_n) &\le \Delta_p(k_n, y_n) - \Delta_p(x_{n+1}, k_n)\\ &\le \Delta_p(k_n, y_n) \to 0,\ n \to \infty,\end{aligned} \tag{3.35}$$

and hence,

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.36}$$

From (3.21)and (3.32), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.37}$$

By applying (3.34) and (3.37), we have

$$\lim_{n\to\infty} \|k_n - x_n\| = 0. \tag{3.38}$$

Consequently, using (3.36) and (3.37), we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.39}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $z \in C$. Using (3.21) and (3.37), there exist susbsequences $\{u_{n_j}\}$ of $\{u_n\}$ and $\{y_{n_j}\}$ of $\{y_n\}$ which converge weakly to z. Using (3.30), it follows that $z \in Fix(S)$ as $Fix(S) = \hat{Fix}(S)$. Next, we show that $Fz \in Q$. Now from (2.4), we obtain

$$\begin{aligned}\|(I - P_Q)Fz\|^p &= \langle J^p_{X_2}(Fz - P_QFz), Fz - P_QFz\rangle\\ &= \langle J^p_{X_2}(Fz - P_QFz), Fz - Fu_{n_j}\rangle\\ &\quad + \langle J^p_{X_2}(Fz - P_QFz), Fu_{n_j} - P_QFu_{n_j}\rangle\\ &\quad + \langle J^p_{X_2}(Fz - P_QFz), P_QFu_{n_j} - P_QFz\rangle\\ &\le \langle J^p_{X_2}(Fz - P_QFz), Fz - Fu_{n_j}\rangle\\ &\quad + \langle J^p_{X_2}(Fz - P_QFz), Fu_{n_j} - P_QFu_{n_j}\rangle.\end{aligned}$$

By the continuity of F and (3.32), we obtain that $Fu_{n_j} \rightharpoonup Fz$ as $j \to \infty$. Hence, if we let $j \to \infty$, we get

$$\|Fz - P_QFz\| = 0.$$

Therefore, $Fz = P_QFz$, which implies that $Fz \in Q$. Hence, we conclude that $z \in Fix(S) \cap \Omega = \Gamma$. Since $x^* = \Pi_\Gamma v$, then applying Lemma 2.3 (ii), (iii) and (3.12), we have

$$\begin{aligned}
&\Delta_p(x_{n+1}, x^*) \le \Delta_p(J^q_{X_1^*}(\alpha_n J^p_{X_1}(v) + \beta_n J^p_{X_1}(y_n) + \gamma_n J^p_{X_1}(Sy_n)), x^*) \\
&= V_p(\alpha_n J^p_{X_1}(v) + \beta_n J^p_{X_1}(y_n) + \gamma_n J^p_{X_1}(Sy_n), x^*) \\
&\le V_p(\alpha_n J^p_{X_1}(v) + \beta_n J^p_{X_1}(y_n) + \gamma_n J^p_{X_1}(Sy_n), x^* - \alpha_n(J^p_{X_1}(v) - J^p_{X_1}(x^*)) \\
&\quad - \langle -\alpha_n J^p_{X_1}(v) - J^p_{X_1}(x^*)), J^q_{X_1^*}\big[\alpha_n J^p_{X_1}(x^*) + \beta_n J^p_{X_1}(y_n) + \gamma_n J^p_{X_1}(Sy_n)\big] - x^*\rangle \\
&= V_p(\alpha_n J^p_{X_1}(x^* + \beta_n J^p_{X_1}(y_n) + \gamma_n J^p_{X_1}(Sy_n)) + \alpha_n\langle J^p_{X_1}(v) - J^p_{X_1}(x^*), k_n - x^*\rangle \\
&\le \alpha_n \Delta_p(x^*, x^*) + \beta_n \Delta_p(y_n, x^*) + \gamma_n \Delta_p(Sy_n, x^*) + \alpha_n\langle J^p_{X_1}(v) - J^p_{X_1}(x^*), k_n - x^*\rangle \\
&\le (1-\alpha_n)\Delta_p(y_n, x^*) + \alpha_n\langle J^p_{X_1}(v) - J^p_{X_1}(x^*), k_n - x^*\rangle \\
&\le (1-\alpha_n(1-\phi))\Delta_p(x_n, x^*) + \alpha_n(1-\phi)\big[(1-\phi)^{-1}(\langle J^p_{X_1}(v) - J^p_{X_1}(x^*), k_n - x^*\rangle) + \frac{\lambda_n}{\alpha_n}\big].
\end{aligned} \tag{3.40}$$

Next, since $x_{n_j} \rightharpoonup x^* \in \Gamma$, then for any $x^* = \Pi_\Gamma u$ we get from (2.6) that

$$\begin{aligned}
\limsup_{n\to\infty}\langle J^p_{X_1}(v) - J^p_{X_1}(x^*), x_n - x^*\rangle &= \lim_{j\to\infty}\langle J^p_{X_1}(v) - J^p_{X_1}(x^*), x_{n_j} - x^*\rangle \\
&= \langle J^p_{X_1}(v) - J^p_{X-1}(x^*), z - x^*\rangle \\
&\le 0.
\end{aligned}$$

Hence, from (3.38), we get

$$\begin{aligned}
\limsup_{n\to\infty}\langle J^p_{X_1}(u) - J^p_{X_1}(x^*), k_n - x^*\rangle &= \langle J^p_{X_1}(u) - J^p_{X_1}(x^*), x_n - x^*\rangle \\
&\le 0.
\end{aligned} \tag{3.41}$$

Therefore, on substituting (3.41) into (3.40) and applying Lemma 2.5, we obtain that $\Delta_p(x_n, x^*) \to 0$ as $n \to \infty$. By (2.7), we know that $\tau_p\|x_n - x^*\| \le \Delta_p(x_n, x^*) \to 0$. Hence $\{x_n\}$ converges strongly to $x^* = \Pi_\Gamma v$.

Case 2: Suppose that there exists a subsequence $\{\eta_j\}$ of $\{\eta\}$ such that $\Delta_p(x_{n_j}, x^*) < \Delta_p(x_{n_{j+1}}, x^*)$ for all $j \in \mathbb{N}$. Then by Lemma 2.6, there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$, and

$$\Delta_p(x_{m_k}, x^*) \le \Delta_p(x_{m_{k+1}}, x^*) \text{ and } \Delta_p(x_k, x^*) \le \Delta_p(x_{k+1}, x^*).$$

Following the same process as in Case 1, we obtain that

$$\begin{cases}
\lim_{k\to\infty}\|u_{n_k} - x_{n_k}\| = 0, \\
\lim_{k\to\infty}\|y_{n_k} - u_{n_k}\| = 0, \\
\lim_{k\to\infty}\|x_{n_{k+1}} - x_{n_k}\| = 0, \\
\limsup_{k\to\infty}\langle J^p_{X_1}(v) - J^p_{X_1}(x^*), k_{n_k} - x^*\rangle \le 0.
\end{cases}$$

Again from (3.40), we have

$$\Delta_p(x_{m_{k+1}}, x^*) \leq (1 - \alpha_{m_k}(1-\phi))\Delta_p(x_{m_k}, x^*)$$
$$+ \alpha_{m_k}(1-\phi)\big[(1-\phi)^{-1}(\langle J^p_{X_1}(v) - J^p_{X_1}(x^*), k_{m_k} - x^*\rangle + \frac{\lambda_{m_k}}{\alpha_{m_k}})\big],$$

that is,

$$(1-\phi)\Delta_p(x_{m_k}, x^*) \leq (1-\phi)\alpha_{m_k}\Delta_p(x_{m_k}, x^*) - \Delta_p(x_{m_{k+1}}, x^*)$$
$$+ \alpha_{m_k}(1-\phi)\Bigg[(1-\phi)^{-1}(\langle J^p_{X_1}(v) - J^p_{X_1}(x^*), k_{m_k} - x^*\rangle + \frac{\lambda_{m_k}}{\alpha_{m_k}})\Bigg],$$

which implies that

$$\Delta_p(x_{m_k}, x^*) \leq \alpha_{m_k}\Bigg[(1-\phi)^{-1}(\langle J^p_{X_1}(v) - J^p_{X_1}(x^*), k_{m_k} - x^*\rangle + \frac{\lambda_{m_k}}{\alpha_{m_k}})\Bigg].$$

Therefore, $\Delta_p(x_{m_k}, x^*) = 0$ and since

$$\Delta_p(x_k, x^*) \leq \Delta_p(x_{k+1}, x^*)\ \forall\, k \in \mathbb{N},$$

we conclude that $x_k \to x^*, k \to \infty$. □

Corollary 3.2 *Let X_1 and X_2 be p-uniformly convex and uniformly smooth Banach spaces and $F : X_1 \to X_2$ be a bounded linear operator with its adjoint $F^* : X_2^* \to X_1^*$. Let C and Q be nonempty, closed and convex subsets of X_1 and X_2 respectively, and $f : X_1 \to \mathbb{R}$ be a non-negative lower semi-continous convex function. Suppose $\Omega \neq \emptyset$ and let $\{\lambda_n\}$ be a positive sequence in $(0, \frac{p\pi_p}{2^{p-1}})$, where π_p is defined in (2.8), $\lambda_n = \circ(\alpha_n)$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0, 1)$ and $\alpha_n + \beta_n + \gamma_n = 1$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\beta_n \in (a, b) \subset (0, 1)$ and $\gamma_n \in (c, d) \subset (0, 1)$ for all $n \geq 1$. For fixed $v, x_0, x_1 \in X_1$, choose θ_n such that $0 \leq \theta_n \leq \overline{\theta_n}$, then define a sequence $\{x_n\}$ by the following manner:*

$$\begin{cases} u_n = J^q_{X_1^*}\Big[J^p_{X_1}(x_n) + \theta_n(J^p_{X_1}(x_n) - J^p_{X_1}(x_{n-1}))\Big] \\ y_n = \Pi_C J^q_{X_1^*}\Big[J^p_{X_1}(u_n) - \rho_n \frac{f^{p-1}(u_n)}{\|\nabla f(u_n)\|^p}\nabla f(u_n)\Big] \\ x_{n+1} = \Pi_C J^q_{X_1^*}\Big[\alpha_n J^p_{X_1}(v) + (1-\alpha_n)J^p_{X_1}(y_n)\Big],\ n \geq 1, \end{cases} \tag{3.42}$$

where

$$\bar{\theta}_n = \begin{cases} \min\{\theta, \frac{\lambda_n}{\|J^p_{X_1}(x_n) - J^p_{X_1}(x_{n-1})\|}\}, & if \ \ x_n \neq x_{n-1}, \\ \\ \theta, \ \ otherwise, \end{cases} \tag{3.43}$$

$f(u_n) := \frac{1}{p}\|(I - P_Q)Fu_n\|^p$, $\{\rho_n\} \subset (0, \infty)$ *and* $\liminf_{n\to\infty} \rho_n(p - C_q \frac{\rho_n^{q-1}}{q}) > 0$, *where* C_q *is the uniform smoothness coefficient of* X_1. *Then* $\{x_n\}$ *converges strongly to* $x^* = \Pi_\Omega v$.

4 Numerical example

Example 4.1 Let $X_1 = X_2 = L_2([0, 1])$ with the inner product given as

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

Let

$$C := \{x \in L_2([0, 1]) : \langle x, a\rangle \geq b\},$$

where $a = 2t^2$ and $b = 0$. Then

$$P_C x = x + \frac{b - \langle a, x\rangle}{\|a\|^2} a.$$

Also, let

$$Q := \{x \in L_2([0, 1]) : \langle x, c\rangle = d\},$$

where $c = \frac{t}{3}$, $d = -1$. Then

$$\Pi_Q(x) = P_Q(x) = x + \max\left\{0, \frac{d - \langle c, x\rangle}{\|c\|^2} c\right\}.$$

Let $F : L_2([0, 1]) \to L_([0, 1])$ be defined by $Fx(t) = \frac{x(t)}{2}$ with adjoint $F^*x(t) = \frac{x(t)}{2}$. Then F is a bounded linear operator. We set $Sx(t) = P_C(x(t))$. Hence by taking $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{2n+5}$, $\gamma_n = 1 - \alpha_n - \beta_n$, $\theta_n = 2$ and $\rho_n = 10^{-7}$, $\forall\, n \geq 1$. We choose the stopping criterion as in Example 4.1, we make a comparison of Algorithm 3.1 with one in which the direction of the momentum $x_n - x_{n-1}$ is altered. The report of this experiment is reported in Fig. 2 for different initial values of x_0 and x_1.

Case i $x_0 = t$ and $x_1 = 2t + 1$;
Case ii $x_0 = \frac{5t^2}{2} - 2t$ and $x_1 = \exp(2t)$;
Case iii $x_0 = 2t$ and $x_1 = \log(2t)$;
Case iv $x_0 = t^{\frac{3}{4}} + 3$ and $x_1 = t^2 + 2t + 1$.

Example 4.2 We give a numerical example in $(\mathbb{R}^3, \|.\|_2)$ of the problem considered in Theorem 3.1.

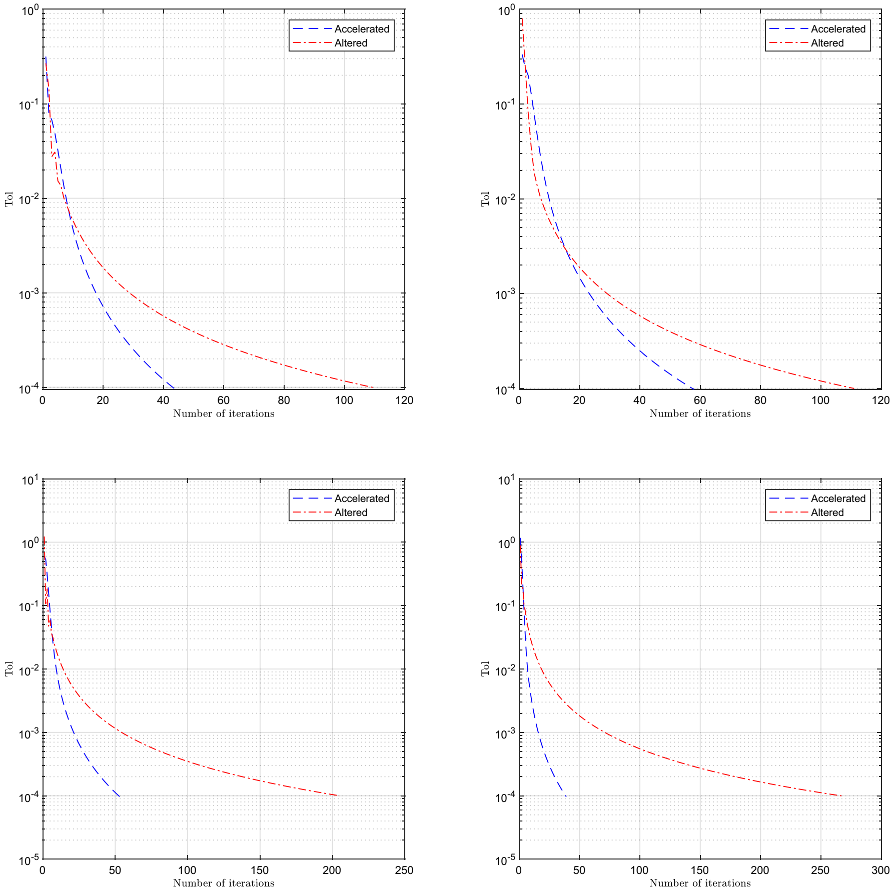

Fig. 1 Example 4.1, Top left: Case (i); Top right: Case (ii); Bottom left: Case (iii); Bottom right: Case (iv)

Let

$$C := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \langle a, x \rangle = b\},$$

where $a = (3, 5, 7)$ and $b = 2$, then

$$\Pi_C(x) = P_C(x) = \max\left\{0, \frac{b-\langle a,x\rangle}{\|a\|_2^2}\right\} a + x.$$

Also, let

$$Q := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \langle a, x \rangle \geq b\},$$

where $a = (2, -1, 5)$ and $b = 1$, then

$$P_Q(x) = \frac{b-\langle a,x\rangle}{\|a\|_2^2} a + x.$$

In addition, let $S = P_C$ and

$$F = \begin{pmatrix} 5 & -5 & -7 \\ -4 & 2 & -4 \\ -7 & -4 & 5 \end{pmatrix}$$

Hence, by taking $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{2n+5}$, $\gamma_n = 1 - \alpha_n - \beta_n$, $\rho_n = 0.1$ and $\theta_n = 1\ \forall\ n \geq 1$. By choosing $\|x_{n+1} - x_n\| = 10^{-4}$ as the stopping criterion, we make a comparison of Algorithm 3.1 with one in which the direction of the momentum $x_n - x_{n-1}$ is altered. The report of this experiment is reported in Fig. 2 for different initial values of x_0 and x_1.

Case i $x_0 = [3, 0, 0]'$ and $x_1 = [2, 3, 2]'$;
Case ii $x_0 = [1, 1, 1]'$ and $x_1 = [2, 1, 2]'$;
Case iii $x_0 = [2, 2, 2]'$ and $x_1 = [1, 0, 2]'$;
Case iv $x_0 = [5, 5, 3]'$ and $x_1 = [4, 4, 4]'$

Remark 4.3 Our proposed method has connections with some recent methods in literature. For instance, the inertial factor θ_n in our iterative algorithm has similar property with the recent papers of Shehu et al. [24] where the inertial factor is bounded. In these articles, In this article, several choices of $\{\theta_n\}$ are considered in numerical implementations and the authors showed that their proposed methods are efficient and implementable.

5 Conclusion

It is well known that the inertial extrapolation method plays a crucial role in the convergence rate of iterative methods in optimization problems. In our article, we proposed an inertial extrapolation method (without modification) together with an Halpern method to approximate solution of split feasibility problem and fixed point problem of Bregman strongly nonexpansive mappings in p-uniformly convex and uniformly smooth real Banach spaces. Some numerical examples were presented to illustrate the performance of our method.

In our future research, we would like to extend this concept to nonlinear spaces due to its numerous applications to real-life problems.

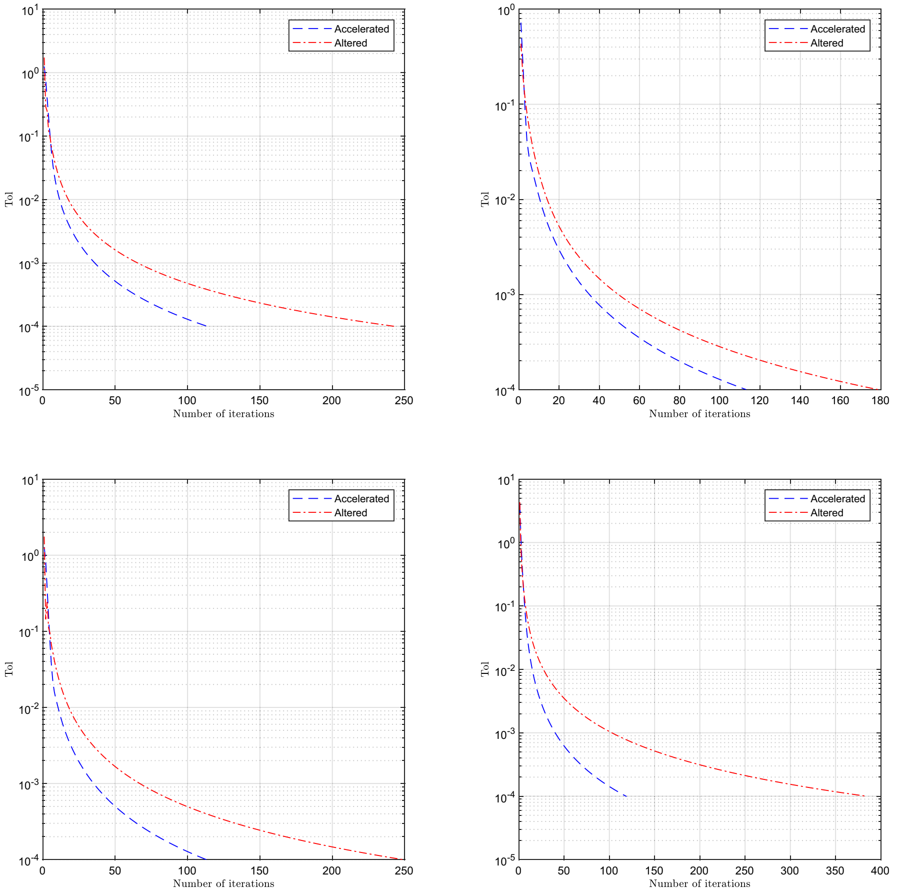

Fig. 2 Example 4.2, Top left: Case (i); Top right: Case (ii); Bottom left: Case (iii); Bottom right: Case (iv)

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Declarations

Conflict of interest The authors declare no competing interests.

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