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Uniform-ultimate boundedness of solutions to vector lienard equation with delay

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Abstract
We establish a set of conditions for the uniform-ultimate boundedness of solutions to a certain system of second order differential equations with variable delay using Lypunov–Krasovskii functional as a basic tool. This result is an addition to the body of literature in many ways. In addition, we provide an example to demonstrate the correctness of our result.

Keywords Second order · Boundedness · Lypunov–Krasovskii functional

Mathematics Subject Classification 34K20

1 Introduction

A second order differential equation is generally referred to as a Lienard equation(named after the French physicist Alfred-Marie Lienard) in dynamical system and differential equation(see, [13, 14, 18, 20]). Analysis of qualitative properties of solutions of ordinary and delay differential equations has received considerable attention of many notable researchers and experts in the last few decades of research (see, [1–34]). In analyzing the qualitative properties, the direct method of Lyapunov or Lypunov–Krasovskii method has been found to be very useful. The method requires construction of a suitable scalar function known as Lyapunov or Lypunov–Krasovskii functional which together with its time derivative satisfies certain conditions. However, to construct such functional is tedious especially when it comes to non-linear differential equations.

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In 2013, Tunc [21] employed the Lypunov–Krasovskii method to establish some necessary conditions for a stable trivial solution (when $P(t) \equiv 0$) and boundedness of solutions (when $P(t) \neq 0$) of the equation:

$$X''(t) + F(X(t), X'(t))X'(t) + H(X(t-\tau)) = P(t),$$

where $\tau > 0$ is a delay constant. Later, Omeike *et al.* [16], studied the asymptotic stability and uniform ultimate boundedness of solutions of the differential equation:

$$X'' + AX' + H(X(t - r(t))) = P(t, X, X'),$$

where A is a real $n \times n$ constant, symmetric, positive definite matrix.

In a recent paper, Tunc and Tunc [22] both established some interesting results on the stability, boundedness and square integrability of solutions to the equation:

$$X'' + F(X, X')X' + H(X(t - r(t))) = P(t, X, X'), \tag{1.1}$$

where $X, Y : \mathbb{R}^+ \to \mathbb{R}^n$,$\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$; $H : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function, $H(0) = 0$; $P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$is a continuous function; F is an $n \times n$ continuous symmetric, positive definite matrix function dependent on the arguments displayed explicitly, and the prime($'$) indicate differentiation with respect to variable t. For any two vectors $X,\ Y$ in $\mathbb{R}^n$, the symbol $\langle X, Y\rangle$ is used to denote the usual scalar product in $\mathbb{R}^n$, i.e. $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$, where $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ are the components of the vectors X and Y respectively; therefore, $\| X \|^2 = \langle X, X\rangle$.

In view of the works of Tunc [21], Tunc and Tunc [22], Omeike *et al.* [16] and some other works in the references, we are motivated to examine certain conditions that guarantee the uniform-ultimate boundedness of solutions to the Eq.(1.1). Based on our understanding of literature, the uniform-ultimate boundedness of solutions of the Eq. (1.1) has not been discussed by any author.

Let $X' = Y$, then Eq. (1.1) can be written as a system of first order differential equations given below:

$$\begin{aligned} X' &= Y, \\ Y' &= -F(X, Y)Y - H(X) + \int_{t-r(t)}^{t} J_H(X(s))Y(s)ds + P(t, X, Y), \end{aligned} \tag{1.2}$$

where the $J_h(X)$ in the system (1.2) stands for the Jacobian matrix of vector $H(X)$ and is defined by

$$J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right), (i, j = 1, 2, 3, ...n),$$

where $(x_1, x_2, ..., x_n)$ and $(h_1, h_2, h_3, ..., h_n)$ are respectively the components of the vectors X and H.

2 Preliminary results

The following algebraic results and definitions are necessary to prove our main result. The proofs of the results are found in the following papers ([7, 8, 11, 12, 16, 22]).

Lemma 2.1 [7, 8, 16, 22] *Let A be any real symmetric positive definite $n \times n$ matrix, then for any X in $\mathbb{R}^n$, we have*

$$\delta_a \parallel X \parallel^2 \leq \langle AX, X\rangle \leq \Delta_a \parallel X \parallel^2,$$

where δ_a and Δ_a are respectively the least and greatest eigenvalues of A.

Lemma 2.2 [7, 8, 16, 22] *Let $H(X)$ be a continuous vector function and that $H(0) = 0$, then*

$$\frac{d}{dt}\int_0^1 \langle H(\sigma X), Y\rangle d\sigma = \langle H(X), Y\rangle.$$

Lemma 2.3 [7, 8, 16, 22] *Let $H(X)$ be a continuous vector function and that $H(0) = 0$, then*

$$\delta_h \parallel X \parallel^2 \leq 2\int_0^1 \langle H(\sigma X), X\rangle d\sigma \leq \Delta_h \parallel X \parallel^2,$$

where δ_h and Δ_h are respectively the least and greatest eigenvalues of $J_h(\sigma X)$.

Consider the following non-autonomous delay differential equation

$$x' = F(t, x_t),\ x_t = x(t+\theta),\ \ -r \leq \theta \leq 0, \tag{2.1}$$

where $F : \mathbb{R} \times C \to \mathbb{R}^n$ is a continuous mapping, $F(t, 0) = 0$, and given that F takes closed bounded sets into bounded sets of $\mathbb{R}^n$, and $C = C([-r, 0],\ \mathbb{R}^n)$ and $\phi \in C$. We assume that $a_0 \geq 0$, $t \geq t_0 \geq 0$ and $x \in C([t_0 - \gamma, t_0 + a_0],\ \mathbb{R}^n)$. Suppose that $x_t = x(t+\theta)$ for $-r \leq \theta \leq 0$ and $x(t) = \phi(t),\ t \in \left[-\gamma, 0\right],\ \gamma > 0$.

Definition 2.1 [16] The matrix A is said to be positive definite when $\langle AX, X\rangle > 0$ for all non-zero X in $\mathbb{R}^n$.

Definition 2.2 [9, 22] A continuous function $W : \mathbb{R}^n \to \mathbb{R}^+$ with $W(0) = 0$, $W(s) > 0$, and W strictly increasing is a wedge. (It is denoted by W or W_j, where j is an integer.)

Definition 2.3 [9, 21] Let D be an open set in $\mathbb{R}^n$ with $0 \in D$. A function $V : [0, \infty) \times D \to [0, \infty)$ is called positive definite if $V(t, 0) = 0$ and if there is a wedge W_1 with $V(t, x) \geq W_1(|x|)$, and is called a decrescent function if there is a wedge W_2 with $V(t, x) \leq W_2(|x|)$.

Definition 2.4 [28] The solutions of equation (2.1) are uniformly ultimately bounded for bound M, if there exists an $M > 0$ and if for any $\alpha > 0$ and $t_0 \in I$ there exists a $T(\alpha) > 0$ such that $X_0 \in S_\alpha$, where $S_\alpha = \{x \in \mathbb{R}^n : \| x \| < \alpha\}$, implies that

$$\|X(t; t_0, X_0)\| < M$$

for all $t \geq t_0 + T(\alpha)$.

Lemma 2.4 [9, 16, 21, 26] *Let $V(t, \phi) : \mathbb{R} \times C \to \mathbb{R}$ be continuous and locally Lipschitz in ϕ. We assume that the following conditions hold: (i) $W(|x(t)|) \leq V(t, x_t) \leq W_1(|x(t)|) + W_2\big(\int_{t-r(t)}^{t} W_3(|x(s)|)ds\big)$ and $(ii) \dot{V}_{(2.1)} \leq -W_3(|x(s)|) + M,$ $for\ some$ $M > 0$, where W_i $(i = 1, 2, 3)$ are wedges and $\dot{V}_{(2.1)}$ represents the derivative of the functional $V(t, \phi)$ with respect to the independent variable t along the solution path of (2.1). Then the solutions of (2.1) are uniformly bounded and uniformly ultimately bounded for bound* **B**.

Remark 2.1 The qualitative properties of the solutions of (2.1) can be studied by means of a scalar functional $V(t, \phi)$ called Lypunov–Krasovskii functional as contained in Lemma 2.5.

3 Main result

Theorem 3.1 *Further to the basic assumptions placed on functions F and G that appear in Eq. (1.1) or system (1.2), we assume there exist some positive constants $D_0, D_1, \delta_f, \delta_h, \Delta_f, \Delta_h, \epsilon, \alpha$ and ξ such that the following conditions hold:*

(i) $H(0) = 0$, $H(X) \neq 0$, $(X \neq 0)$, *the matrix $J_H(X)$ exists, symmetric and positive definite such that for all $X \in \mathbb{R}^n$, $\delta_h \leq \lambda_i(J_H(X)) \leq \Delta_h$; $\lambda_i(J_H(X))$ being the eigenvalues of $J_H(X)$.*

(ii) The eigenvalues $\lambda_i(J_F(X, Y))$ of $F(X, Y)$ satisfies $\delta_f = \alpha - \epsilon \leq \lambda_i(J_F(X, Y)) \leq \alpha$.

(iii) $0 \leq r(t) \leq \gamma$, *γ is a positive constant,* $r'(t) \leq \xi, 0 < \xi < 1$.

(iv) $\| P(t, X, Y) \| \leq D_0 + D_1\{\| X \| + \| Y \|\}$. *Then, the solutions of system (1.2) are uniformly ultimately bounded whenever*

$$0 < \gamma < \min\Big(\frac{2\delta_h - \epsilon}{\Delta_h}, \frac{(1-\xi)(2\alpha - \epsilon(\alpha+4))}{\Delta_h\big(2(2-\xi)+\alpha\big)}\Big).$$

Proof Let a continuously differentiable Lypunov–Krasovskii functional $V(t) = V(X(t), Y(t))$ be defined by

$$\begin{aligned} 2V(t) &= \| \alpha X + Y \|^2 + 4\int_0^1 \langle H(\sigma_1 X), X\rangle d\sigma_1 + \| Y \|^2 \\ &\quad + 2\lambda \int_{-r(t)}^{0} \int_{t+s}^{t} \langle Y(\theta), Y(\theta)\rangle d\theta ds, \end{aligned} \tag{3.1}$$

where $\lambda > 0$ and its value is given later.

Our first concern is to establish that the functional $V(t)$ defined by (3.1) is nonnegative. Obviously, $V(0, 0) = 0$. By Lemma 2.3 and assumption (i) of the theorem, we have

$$2\delta_h \parallel X \parallel^2 \leq 4\int_0^1 \langle H(\sigma_1 X), X\rangle d\sigma_1 \leq 2\Delta_h \parallel X \parallel^2 . \tag{3.2}$$

Also, by using the inequality $2|\langle X, Y\rangle| \leq \parallel X \parallel^2 + \parallel Y \parallel^2$, we have

$$0 \leq \parallel \alpha X + Y \parallel^2 \leq 2\{\alpha^2 \parallel X \parallel^2 + \parallel Y \parallel^2\}. \tag{3.3}$$

Lastly,

$$0 < \lambda \int_{-r(t)}^0 \int_{t+s}^t \langle Y(\theta), Y(\theta)\rangle d\theta ds \tag{3.4}$$

Thus, using the estimates (3.2)–(3.4) in (3.1) we have

$$\begin{aligned} 2V(t) &\geq 2\delta_h \parallel X \parallel^2 + \parallel Y \parallel^2 \\ &= D_2\{\parallel X \parallel^2 + \parallel Y \parallel^2\}, \end{aligned}$$

where $D_2 = \min\{2\delta_h, 1\}$.

Similarly, by the same reasoning as above, we have

$$\begin{aligned} 2V(t) &\leq 2\big(\Delta_h + \alpha^2\big) \parallel X \parallel^2 + 3 \parallel Y \parallel^2 + 2\lambda \int_{-r(t)}^0 \int_{t+s}^t \langle Y(\theta), Y(\theta)\rangle d\theta ds \\ &\leq D_3\{\parallel X \parallel^2 + \parallel Y \parallel^2\} + 2\lambda r(t) \int_{t-r(t)}^t \langle Y(\theta), Y(\theta)\rangle d\theta, \end{aligned}$$

where $D_3 = \max\{2(\Delta_h + \alpha^2), 3\}$. Hence, we can get a continuous function, say $v(s)$, such that

$$v(\parallel \psi(0) \parallel) \leq V(\psi), v(\parallel \psi(0) \parallel) \geq 0.$$

Next, we obtain the derivative $\dot{V}(t)$ of $V(t)$ with respect to the independent variable t along the system (1.2) as follows:

$$\begin{aligned} \frac{d}{dt}V(t) = \dot{V}(t) = \langle &\alpha X + Y, \alpha Y - F(X, Y)Y - H(X) \\ &+ \int_{t-r(t)}^t J_H(X(s))Y(s)ds + P(t, X, Y)\rangle \\ &+ 2\frac{d}{dt}\int_0^1 \langle H(\sigma_1 X), X\rangle d\sigma_1 + \langle Y, -F(X, Y)Y - H(X) \end{aligned}$$

$$+\int_{t-r(t)}^{t} J_H(X(s))Y(s)ds$$
$$+P(t,X,Y)\rangle + \lambda \frac{d}{dt}\int_{-r(t)}^{0}\int_{t+s}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta ds.$$

By Lemma 2.2, we have

$$\frac{d}{dt}\int_0^1 \langle H(\sigma_1 X), X\rangle d\sigma_1 = \langle H(X), Y\rangle.$$

Also,

$$\begin{aligned}&\lambda \frac{d}{dt}\int_{-r(t)}^{0}\int_{t+s}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta ds\\ &= \lambda r(t) \parallel Y(t)\parallel^2 - \lambda(1-r'(t))\int_{t-r(t)}^{t} \parallel Y(\theta)\parallel^2 d\theta\\ &\le \lambda\gamma \parallel Y(t)\parallel^2 - \lambda(1-\xi)\int_{t-r(t)}^{t} \parallel Y(\theta)\parallel^2 d\theta,\end{aligned}$$

after we have applied assumption (iii) of our theorem.

Therefore, after simplification and arranging terms, we obtain

$$\begin{aligned}\dot V(t) =& -\alpha\langle X, H(X)\rangle - 2\langle Y, F(X,Y)Y\rangle + \alpha\langle Y, Y\rangle + \alpha\langle X, \big(\alpha I - F(X,Y)\big)Y\rangle\\ &+\alpha\int_{t-r(t)}^{t}\langle X, J_H(X(s))Y(s)\rangle ds + 2\int_{t-r(t)}^{t}\langle Y, J_H(X(s))Y(s)\rangle ds\\ &+\lambda r(t)\parallel Y(t)\parallel^2 - \lambda(1-r'(t))\int_{t-r(t)}^{t}\parallel Y(\theta)\parallel^2 d\theta + \langle \alpha X + 2Y, P(t,X,Y)\rangle\\ \le& -\alpha\langle X, H(X)\rangle - 2\langle Y, F(X,Y)Y\rangle + \alpha\langle Y, Y\rangle + \alpha\langle X, \big(\alpha I - F(X,Y)\big)Y\rangle\\ &+\alpha\int_{t-r(t)}^{t}\langle X, J_H(X(s))Y(s)\rangle ds + 2\int_{t-r(t)}^{t}\langle Y, J_H(X(s))Y(s)\rangle ds\\ &+\lambda\gamma\parallel Y(t)\parallel^2 - \lambda(1-\xi)\int_{t-r(t)}^{t}\parallel Y(\theta)\parallel^2 d\theta + \langle \alpha X + 2Y, P(t,X,Y)\rangle,\end{aligned}$$

where I is an $n\times n$ identity matrix.

If we apply Lemma 2.1, assumptions (i), (ii) of the theorem and the fact that $2\parallel X\parallel\parallel Y\parallel \le \parallel X\parallel^2 + \parallel Y\parallel^2$ in the above, we obtain

$$\begin{aligned}\dot V(t) \le& -\frac{\alpha}{2}\big(2\delta_h - \epsilon - \Delta_h\gamma\big)\parallel X(t)\parallel^2\\ &-\frac{1}{2}\big(2\alpha - \epsilon(\alpha+4) - 2\gamma(\Delta_h+\lambda)\big)\parallel Y(t)\parallel^2\\ &+\frac{1}{2}\big((2+\alpha)\Delta_h - 2\lambda(1-\xi)\big)\int_{t-r(t)}^{t}\parallel Y(\theta)\parallel^2 d\theta\\ &+\parallel P(t,X,Y)\parallel\big(\alpha\parallel X\parallel + 2\parallel Y\parallel\big).\end{aligned} \tag{3.5}$$

On setting $\lambda = \frac{\Delta_h(\alpha+2)}{2(1-\xi)}$, $\gamma < \min\left(\frac{2\delta_h-\epsilon}{\Delta_h}, \frac{(1-\xi)(2\alpha-\epsilon(\alpha+4))}{\Delta_h\left(2(2-\xi)+\alpha\right)}\right)$ and using assumption (iv) of the theorem in (3.5), we obtain the following inequality for some positive constant K_1,

$$
\begin{aligned}
\dot{V}(t) &\leq -K_1\{\| X \|^2 + \| Y \|^2\} + \big(D_0 + D_1(\| X \| + \| Y \|)\big)\big(\alpha \| X \| +2 \| Y \| \big) \\
&\leq -K_1\{\| X \|^2 + \| Y \|^2\} + D_0\big(\alpha \| X \| +2 \| Y \| \big) \\
&\quad + D_1(\| X \| + \| Y \|)\big(\alpha \| X \| +2 \| Y \| \big).
\end{aligned}
$$

By simplifying further and using the inequality $2 \| X \| \| Y \| \leq \| X \|^2 + \| Y \|^2$, we arrive at

$$
\begin{aligned}
\dot{V}(t) &\leq -K_1\{\| X \|^2 + \| Y \|^2\} + D_0\big(\alpha \| X \| +2 \| Y \| \big) \\
&\quad + D_1\Big(\frac{3\alpha+2}{2}\Big) \| X \|^2 + D_1\Big(\frac{6+\alpha}{2}\Big) \| Y \|^2 \\
&\leq -(K_1 - D_1 K_2)\{\| X \|^2 + \| Y \|^2\} + D_0\big(\alpha \| X \| +2 \| Y \| \big),
\end{aligned}
$$

where $K_2 = \max\{\big(\frac{3\alpha+2}{2}\big), \big(\frac{6+\alpha}{2}\big)\}$. If we now choose $D_1 < K_1 K_2^{-1}$ and follow the same procedure of Omeike et al. [16], then there exists some $\beta > 0$ such that

$$
\begin{aligned}
\dot{V} &\leq -\beta(\|X\|^2 + \|Y\|^2) + k\beta(\|X\| + \|Y\|) \\
&= -\frac{\beta}{2}(\|X\|^2 + \|Y\|^2) - \frac{\beta}{2}\left\{(\|X\| - k)^2 + (\|Y\| - k)^2\right\} + \beta k^2 \\
&\leq -\frac{\beta}{2}(\|X\|^2 + \|Y\|^2) + \beta k^2,
\end{aligned}
$$

for some $k, \beta > 0$.

It is now possible to apply Lemma 2.5 to the solutions of Eq. (1.1) as a consequence of assumption (iii) of Theorem 3.1. Thus, from the proof of Theorem 3.1, we have $W = \frac{D_2}{2}\{\| X \|^2 + \| X \|^2\}$, $W_1 = \big(\Delta_h + \alpha^2\big) \| X \|^2 + \frac{3}{2} \| Y \|^2$, $W_2 = \lambda r(t)$ and $W_3 = \frac{\beta}{2}(\|X\|^2 + \|Y\|^2)$. Hence, by Lemma 2.5, we conclude that all the solutions of Eq. (1.1) or system (1.2) are uniform-ultimately bounded. □

4 Example

We provide the following example as a special case of equation (1.1).

Example 4.1

$$\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} 13 + \exp^{-(x_1^2+x_2^2)} & 0 \\ 0 & 13 + \exp^{-(x_1'^2+x_2'^2)} \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$
$$+ \begin{pmatrix} 2x_1(t-r(t)) + \sin x_1(t-r(t)) \\ 2x_2(t-r(t)) + \sin x_2(t-r(t)) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{x_1+x_1'+1}{1+t^2} \\ \frac{x_2+x_2'+1}{1+t^2} \end{pmatrix},$$

where

$$H(X(t-r(t))) = \begin{pmatrix} 2x_1(t-r(t)) + \sin x_1(t-r(t)) \\ 2x_2(t-r(t)) + \sin x_2(t-r(t)) \end{pmatrix}, \ r(t) = \frac{1}{8}\cos^2 t,$$
$$F(X, X') = \begin{pmatrix} 13 + \exp^{-(x_1^2+x_2^2)} & 0 \\ 0 & 13 + \exp^{-(x_1'^2+x_2'^2)} \end{pmatrix} \text{ and}$$
$$P(t, X, X') = \begin{pmatrix} \frac{x_1+x_1'+1}{1+t^2} \\ \frac{x_2+x_2'+1}{1+t^2} \end{pmatrix}.$$

The variable delay $r(t)$ and it's derivative $r'(t)$, respectively, satisfy $0 \le r(t) = \frac{1}{8}\cos^2 t \le \frac{1}{8} = \gamma$ and $r'(t) = -\frac{1}{4}\sin t \cos t < \frac{1}{4} = \xi$.

The eigenvalues of $F(X, X')$ are

$$\lambda_1(F(X, X')) = 13 + \exp^{-(x_1^2+x_2^2)},$$

and

$$\lambda_2(F(X, X')) = 13 + \exp^{-(x_1'^2+x_2'^2)}.$$

Hence, we have $\delta_f = 13 \le \lambda_i(F(X, X')) \le 14 = \Delta_f$.

Also, the Jacobian matrix $J_H(X(t-r(t)))$ of $H(X(t-r(t)))$ is

$$J_H(X(t-r(t))) = \begin{pmatrix} 2 + \cos x_1(t-r(t)) & 0 \\ 0 & 2 + \cos x_2(t-r(t)) \end{pmatrix},$$

and its eigenvalues satisfy $\delta_h = 1 \le \lambda_i(J_H(X)) \le 3 = \Delta_h$.

From the above calculations, we have, $\delta_f = 13, \Delta_f = 14, \delta_h = 1.1, \Delta_h = 4, \epsilon = 1, \alpha = 14, \gamma = \frac{1}{8}, \xi = \frac{1}{4}$.

Therefore,

$$0 < \gamma = \frac{1}{8} < \min\left(\frac{2\delta_h - \epsilon}{\Delta_h}, \frac{(1-\xi)(2\alpha - \epsilon(\alpha+4))}{\Delta_h\big(2(2-\xi)+\alpha\big)}\right) = \min\left(\frac{1}{3}, \frac{1}{7}\right) = \frac{1}{7}.$$

Lastly,

$$P(t, X, X') = \frac{1}{1+t^2}\begin{pmatrix} x_1 + x_1' + 1 \\ x_2 + x_2' + 1 \end{pmatrix}$$
$$|P(t, X, X')| \leq \big(2 + \sqrt{3}\{\| X \| + \| X' \|\}\big).$$

Hence, the example satisfied all the conditions of the theorem.

5 Conclusion

In this paper, we made use of a suitable Lypunov–Krasovskii functional to establish sufficient conditions for the uniform-ultimate boundedness of solutions to certain second order non-linear vector differential equation. An example is given to demonstrate the correctness of our result.

Author Contributions The authors have equal contributions.

Data availability All of the necessary data and the implementation details have been included in the manuscript.

Declarations

Conflict of interest The authors confirm that this article content has no conflict of interest.

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