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Birational invariants of toric orbifold surfaces

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Abstract
We study birational invariants of toric orbifold surfaces.

Keywords Algebraic orbifolds · Toric surfaces · Birational invariants

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1 Introduction

Let k be an algebraically closed field. Algebraic orbifolds over k are smooth separated irreducible Deligne-Mumford stacks of finite type over k, with trivial generic stabilizer. Birational invariants of algebraic orbifolds were introduced in [7]; they are invariant under birational projective morphisms of algebraic orbifolds. In this paper, we focus on *toric* Deligne-Mumford stacks, introduced in [3], and specialize to dimension 2, i.e., toric orbifold surfaces. The goal of this paper is to study their birational invariants over an algebraically closed field of characteristic 0.

2 Toric varieties

To fix notation, we recall basic definitions of toric varieties [6]. We fix a base field k, which will later be taken to be algebraically closed of characteristic 0, and a dimension n, which will later be taken to be 2. There are lattices

$$M := \mathbb{Z}^n \qquad \text{and} \qquad N := \mathrm{Hom}(M, \mathbb{Z}).$$

The elements of M are characters on the algebraic torus

$$T := \mathrm{Spec}(k[M]).$$

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We view the lattice N as sitting inside its extension over the real numbers

$$N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R};$$

we employ analogous notation, e.g., for an extension over $\mathbb{Q}$.

Cones in $N_{\mathbb{R}}$ are always strongly convex rational polyhedral cones: A *fan* in N is a finite collection of cones in $N_{\mathbb{R}}$, such that for every cone in the fan, its faces are also in the fan, and for every pair of cones in the fan, the intersection is a face of each. The *support* of a fan is the union of the cones.

A cone σ determines an affine toric variety

$$U_\sigma := \operatorname{Spec}(k[\sigma^\vee \cap M]),$$

where $\sigma^\vee$ denotes the dual cone in $M_{\mathbb{R}}$.

The *toric variety* determined by a fan Σ is the union of U_σ over all $\sigma \in \Sigma$, glued by identifying a common open subvariety $U_{\sigma\cap\tau}$ of U_σ and U_τ, for all $\sigma, \tau \in \Sigma$. This yields a normal algebraic vareity $X(\Sigma)$ over k. Every normal algebraic variety over k with T-action and equivariant open immersion of T is the toric variety of a uniquely determined fan in N.

Each U_σ is T-invariant and has a unique closed T-orbit, the image of

$$\operatorname{Spec}(k[\sigma^\perp \cap M]) \to \operatorname{Spec}(k[\sigma^\vee \cap M]),$$

where $\sigma^\perp \subset M_{\mathbb{R}}$ is defined by pairing to 0 with all elements of σ. The closure is a closed subvariety of $X(\Sigma)$, determined by σ. This is a point when σ is an n-dimensional cone, and has codimension 1 when σ is a ray (1-dimensional cone).

A cone is *smooth* if it is generated by part of a $\mathbb{Z}$-basis of N. We have $X(\Sigma)$ smooth if and only if all of the cones of Σ are smooth.

For $X(\Sigma)$ to be projective, it is necessary and sufficient for the support of Σ to be equal to $N_{\mathbb{R}}$ and Σ to admit a piecewise linear convex support function. When $n = 2$, the second condition is automatic, so projective toric surfaces correspond to fans with support equal to $N_{\mathbb{R}}$.

As well, when $n = 2$ we profit from the elementary structure of low-dimensional cones. The cones are 0, rays $\mathbb{R}_{\ge 0}v$, and 2-dimensional cones $\mathbb{R}_{\ge 0}v + \mathbb{R}_{\ge 0}w$. The generator $v \in N_{\mathbb{R}}$ of a ray may be uniquely specified by the requirements to lie in N and not be a multiple of another generator in N; we call such $v \in N$ the *primitive* ray generator. For a 2-dimensional cone, primitive generators $v, w \in N$ are uniquely determined up to swapping.

3 Orbifold toric surfaces

Toric Deligne-Mumford stacks were introduced in [3]. The idea is to consider only fans whose cones are simplicial, and in this case to glue the stack quotients of smooth affine toric varieties by finite groups to obtain a smooth separated irreducible Deligne-Mumford stack with trivial generic stabilizer, i.e., an orbifold. The construction requires the given fan Σ, with simplicial cones, to be supplemented by the additional data of a generator in N of every ray of Σ, to make a so-called *stacky fan* $\mathbf{\Sigma}$.

We continue to work over a base field k, take $n = 2$, and fix a prime number p, not equal to the characteristic of k. Let Σ be a fan in N, such that every 2-dimensional cone is of the form $\sigma = \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w$ with primitive generators $v, w \in N$ such that the (well-defined) absolute value of the determinant of the 2×2 integer matrix of coordinates of v and w, with respect to a $\mathbb{Z}$-basis of N, is equal to 1 or p. We consider a stacky fan $\mathbf{\Sigma}$, where the additional data of a choice of generator of each ray $\varrho \in \Sigma$ is subject to the following conditions:

- The chosen generator is equal to or is p times the primitive generator of ϱ.
- When the generator is p times the primitive generator, the 2-dimensional cones containing ϱ are smooth.
- No pair of rays, for which the chosen generator is p times the primitive generator, generates a cone of Σ.

The construction of the toric orbifold $\mathcal{X}(\mathbf{\Sigma})$ of a stacky fan $\mathbf{\Sigma}$ proceeds by suitably modifying the construction of $X(\Sigma)$; we only describe the construction in the present setting, where the conditions have been chosen to correspond precisely to toric orbifold surfaces $\mathcal{X}(\mathbf{\Sigma})$, whose nontrivial stabilizer groups all have order p. For every 2-dimensional cone $\sigma = \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w$ with primitive vectors $v, w \in N$ and corresponding 2×2 integer matrix with determinant of absolute value p, we have an index p sublattice of N generated by v and w, and by duality, a lattice $M' \subset M_{\mathbb{Q}}$ with $M \subset M'$ and $|M'/M| = p$. Then the image of

$$\mathrm{Spec}(k[M'/M]) \to \mathrm{Spec}(k[M'])$$

is a subgroup, isomorphic to μ_p, of $T' := \mathrm{Spec}(k[M'])$, which is the kernel of $T' \to T$. The stack quotient

$$[\,\mathrm{Spec}(k[\sigma^\vee \cap M'])/\mu_p]$$

is used in place of U_σ in the construction; just with this modification, the result would be a smooth orbifold surface with finitely many points with μ_p-stabilizer. This is further modified by passing to the pth root stack [4, Sect. 2] [1, App. B] of the union of divisors, that correspond to rays where p times the primitive ray generator is given in $\mathbf{\Sigma}$. The resulting orbifold $\mathcal{X}(\mathbf{\Sigma})$ has locus with μ_p-stabilizer equal to the union of finitely many points and finitely many pairwise disjoint rational curves.

The smooth Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ has coarse moduli space $X(\Sigma)$. We describe an orbifold as projective, when it has projective coarse moduli space. But $n = 2$, so $\mathcal{X}(\mathbf{\Sigma})$ is projective if and only if the support of Σ is equal to $N_{\mathbb{R}}$.

4 Orbifold Burnside group

The *Burnside group of stacks* $\overline{\mathrm{Burn}}_n$ and its *weight module* $\overline{\mathcal{B}} = \bigoplus_{j\geq 0} \overline{\mathcal{B}}_j$ were introduced in [7]. The weight module is an abelian group, defined by explicit generators and relations.

We suppose that the base field k is algebraically closed of characteristic 0. Then a projective orbifold $\mathcal{X}$ of dimension n determines a class in an abelian group $\overline{\mathrm{Burn}}_n$, generated by pairs (K, α) with K a finitely generated field over k, of some transcendence degree $d \le n$, and $\alpha \in \overline{\mathcal{B}}_{n-d}$. With suitable relations, this class is invariant under birational projective morphisms of projective orbifolds. In other words, $[\mathcal{X}] = [\mathcal{X}']$ in $\overline{\mathrm{Burn}}_n$ if there exists an orbifold $\mathcal{Y}$ with birational projective morphisms

$$\mathcal{Y} \to \mathcal{X} \quad \text{and} \quad \mathcal{Y} \to \mathcal{X}'.$$

Symbols with $K = k$ generate a subgroup

$$\overline{\mathcal{B}}_n \subset \overline{\mathrm{Burn}}_n. \tag{1}$$

We describe the group $\overline{\mathcal{B}}_2$, or rather a particular subgroup $\overline{\mathcal{B}}_2^{[p]}$, for which we have

$$[\mathcal{X}(\boldsymbol{\Sigma})] \in \overline{\mathcal{B}}_2^{[p]},$$

in the subgroup (1). The locus with μ_p-stabilizer of $\mathcal{X}(\boldsymbol{\Sigma})$, as mentioned above, consists of finitely many divisors and isolated points, which we call μ_p-divisors and μ_p-points, respectively. Their complement is a quasi-projective variety.

The abelian group $\overline{\mathcal{B}}_2^{[p]}$ is generated by symbols (triv, (0, 0)), $(\mathbb{Z}/p\mathbb{Z}, (0, 1))$, and $(\mathbb{Z}/p\mathbb{Z}, (1, a))$ for $a \in (\mathbb{Z}/p\mathbb{Z})^\times$. These are subject to the relations

$$\begin{aligned}
(\mathbb{Z}/p\mathbb{Z}, (0, 1)) &= (\mathbb{Z}/p\mathbb{Z}, (1, 1)),\\
(\mathrm{triv}, (0, 0)) &= (\mathbb{Z}/p\mathbb{Z}, (1, p-1)),\\
(\mathbb{Z}/p\mathbb{Z}, (1, a)) &= (\mathbb{Z}/p\mathbb{Z}, (1, a^{-1})),\\
(\mathbb{Z}/p\mathbb{Z}, (1, a)) &= (\mathbb{Z}/p\mathbb{Z}, (1, a-1)) + (\mathbb{Z}/p\mathbb{Z}, (1, a^{-1}-1)) - (\mathrm{triv}, (0, 0))\\
&\quad \text{for} \quad a \in \{2, \dots, p-1\},
\end{aligned}$$

where a^{-1} denotes the multiplicative inverse to a mod p. The class $[\mathcal{X}(\boldsymbol{\Sigma})]$ is the sum of contributions from the μ_p-divisors, the μ_p-points, and the quasi-projective variety that is the complement of the μ_p-divisors and μ_p-points. We let n_1 denote the number of μ_p-divisors, and n_2, the number of μ_p-points.

The quasi-projective variety contributes

$$(1 - n_1 - n_2)(\mathrm{triv}, (0, 0)).$$

The μ_p-divisors contribute

$$n_1(\mathbb{Z}/p\mathbb{Z}, (0, 1)).$$

The contribution from the μ_p-points is the sum of individual contributions

$$(\mathbb{Z}/p\mathbb{Z}, (1, a))$$

from each μ_p-point. At a μ_p-point, an identification of the stabilizer with μ_p is made, so that one of the weights of the action on the tangent space is 1; then a is the second weight. But this involves a choice: if we reverse the roles of the two weights then the resulting weight would be a^{-1}. Thanks to the corresponding relation in $\overline{\mathcal{B}}_2^{[p]}$, the contribution $(\mathbb{Z}/p\mathbb{Z}, (1, a)) \in \overline{\mathcal{B}}_2^{[p]}$ of a μ_p-point is well-defined.

The quotient group

$$\overline{\mathcal{B}}_2^{[p]}/\langle(\mathrm{triv}, (0, 0)), (\mathbb{Z}/p\mathbb{Z}, (0, 1))\rangle \tag{2}$$

is the group denoted by $\overline{\mathcal{B}}_2^{[p]}/(\mathcal{C}\cap\overline{\mathcal{B}}_2^{[p]})$ in [7, Sect. 4]. In [7, Lemma 5.3], an explicit isomorphism is given from this group to a certain quotient of the group $H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z})$, the first orbifold homology of the modular curve $X_0(p)$. This group (2) is also mentioned as providing an obstruction to being linked by blow-ups of points to an orbifold surface without μ_p-points. For the group (2) the following isomorphism types were found:

$$p = 2\colon\ 0, \quad p = 3\colon\ 0, \quad p = 5\colon\ \mathbb{Z}/2\mathbb{Z}, \quad p = 7\colon\ 0, \quad p = 11\colon\ \mathbb{Z}.$$

As mentioned in [7, Sect. 4], the vanishing for $p = 2$ and $p = 3$ is the $n = 2$ case of a more general phenomenon, related to destackification [2]. By [7, Prop. 4.4], the group (2) is nonzero for all $p \geq 5$, except $p = 7$.

5 Refined invariant

We continue to work over an algebraically closed base field k of characteristic 0. A natural class of birational projective morphisms of projective toric Deligne-Mumford stacks is the class of morphisms associated with the *stacky star subdivisions* (of toric Deligne-Mumford stacks) of [5]. All toric stacks will be projective toric orbifold surfaces with μ_p as the only possible nontrivial stabilizer group. We emphasize that the torus T is regarded as fixed. All stacky fans will satisfy the corresponding conditions, stated in Sect. 3, and have support equal to $N_{\mathbb{R}}$.

A stacky star subdivision is an operation that, given a stacky fan $\mathbf{\Sigma}$ and a chosen 2-dimensional cone $\sigma \in \Sigma$, yields a new stacky fan $\mathbf{\Sigma}_\sigma$ and a T-equivariant birational projective morphism

$$f_\sigma\colon \mathcal{X}(\mathbf{\Sigma}_\sigma) \to \mathcal{X}(\mathbf{\Sigma}).$$

Let us write $\sigma = \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w$ with primitive vectors $v, w \in N$; then the stacky fan $\mathbf{\Sigma}_\sigma$ is constructed by deleting the cone σ, adding the cones

$$\mathbb{R}_{\geq 0}(v + w), \quad \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}(v + w), \quad \mathbb{R}_{\geq 0}(v + w) + \mathbb{R}_{\geq 0}w,$$

and endowing the ray $\mathbb{R}_{\geq 0}(v + w)$ with the choice of generator $v + w$. Geometrically, f_σ is the blow-up of $\mathcal{X}(\mathbf{\Sigma})$ at the point, corresponding to σ; this is a point of the scheme locus when σ is a smooth cone whose rays come with integer multiple 1, a μ_p-point when σ is not smooth, a point on a μ_p-divisor when σ is smooth and one of its rays comes with multiple p.

By the classical factorization of birational projective morphisms of smooth surfaces as compositions of blow-ups, for any T-equivariant birational projective morphism

$$f\colon \mathcal{X}(\boldsymbol{\Sigma}') \to \mathcal{X}(\boldsymbol{\Sigma}),$$

we obtain $\boldsymbol{\Sigma}'$ from $\boldsymbol{\Sigma}$ by performing a finite sequence of stacky star subdivisions, and f is the composite of the corresponding T-equivariant birational projective morphisms.

Inspired by [2, Example 4.2] and [7, Section 4], we might ask: For a given pair of toric orbifold surfaces $\mathcal{X}(\boldsymbol{\Sigma})$ and $\mathcal{X}(\boldsymbol{\Sigma}')$, does there exist a toric orbifold surface $\mathcal{X}(\boldsymbol{\Sigma}'')$ with T-equivariant birational projective morphisms

$$\mathcal{X}(\boldsymbol{\Sigma}'') \to \mathcal{X}(\boldsymbol{\Sigma}) \quad \text{and} \quad \mathcal{X}(\boldsymbol{\Sigma}'') \to \mathcal{X}(\boldsymbol{\Sigma}')\,?$$

When the answer is affirmative, we say that $\mathcal{X}(\boldsymbol{\Sigma})$ and $\mathcal{X}(\boldsymbol{\Sigma}')$ are equivalent under T-equivariant birational projective morphisms.

As mentioned in Sect. 4, the group $\overline{\mathcal{B}}_2^{[p]}/\langle(\mathrm{triv}, (0,0)), (\mathbb{Z}/p\mathbb{Z}, (0,1))\rangle$ supplies an obstruction to the equivalence under T-equivariant birational projective with a toric orbifold surface whose coarse moduli space is smooth.

We record the observation, that in the toric setting the abelian group $\overline{\mathcal{B}}_2^{[p]}$ admits a refinement.

Definition 1 The *refined toric orbifold surface weight group*, for μ_p-stabilizers, is the abelian group

$$\overline{\mathcal{T}}_2^{[p]},$$

generated by $(\mathrm{triv}, (0,0))$, $(\mathbb{Z}/p\mathbb{Z}, (0,1))$, and $(\mathbb{Z}/p\mathbb{Z}, (1,a))$ for $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, with relations

$$\begin{aligned}(\mathbb{Z}/p\mathbb{Z}, (0,1)) &= (\mathbb{Z}/p\mathbb{Z}, (1,1)),\\ (\mathrm{triv}, (0,0)) &= (\mathbb{Z}/p\mathbb{Z}, (1,p-1)),\\ (\mathbb{Z}/p\mathbb{Z}, (1,a)) &= (\mathbb{Z}/p\mathbb{Z}, (1,a-1)) + (\mathbb{Z}/p\mathbb{Z}, (1,(a^{-1}-1)^{-1})) - (\mathrm{triv}, (0,0))\\ &\qquad \text{for} \quad a \in \{2,\dots,p-1\}.\end{aligned}$$

With the additional relations $(\mathbb{Z}/p\mathbb{Z}, (1,a)) = (\mathbb{Z}/p\mathbb{Z}, (1,a^{-1}))$ for $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, we recover the presentation of $\overline{\mathcal{B}}_2^{[p]}$. So, we have a canonical surjective homomorphism

$$\overline{\mathcal{T}}_2^{[p]} \to \overline{\mathcal{B}}_2^{[p]}. \tag{3}$$

Proposition 2 *The assignment, to $\mathcal{X}(\boldsymbol{\Sigma})$, of the class*

$$(1-n_1-n_2)(\mathrm{triv},(0,0))+n_1(\mathbb{Z}/p\mathbb{Z},(0,1))+\sum_{\substack{\text{non-smooth }\sigma\in\Sigma\\ \sigma=\mathbb{R}_{\ge 0}v+\mathbb{R}_{\ge 0}w\\ v_1w_2-w_1v_2=p}}(\mathbb{Z}/p\mathbb{Z},(1,-w_i^{-1}v_i))$$

in $\overline{\mathcal{T}}_2^{[p]}$, where in the sum we have $v=(v_1,v_2)$, $w=(w_1,w_2)\in N$, and $i\in\{1,2\}$ is chosen so that $p\nmid w_i$, gives rise to an invariant of toric orbifold surfaces with μ_p as only possible nontrivial stabilizer group, under torus-equivariant birational projective morphisms. The assignment is compatible with the homomorphism (3).

Proof From the explicit description of stacky star subdivision, it is easy to check that $\boldsymbol{\Sigma}$ and any stacky star subdivision $\boldsymbol{\Sigma}_\sigma$ have the same class in $\overline{\mathcal{T}}_2^{[p]}$. □

Example We may form a quotient

$$\overline{\mathcal{T}}_2^{[p]}/\langle(\mathrm{triv},(0,0)),(\mathbb{Z}/p\mathbb{Z},(0,1))\rangle,$$

analogous to (2). Isomorphism types for some p are:

$$p=2\colon\ 0,\quad p=3\colon\ 0,\quad p=5\colon\ \mathbb{Z}/2\mathbb{Z},\quad p=7\colon\ \mathbb{Z}/3\mathbb{Z},\quad p=11\colon\ \mathbb{Z}^2.$$

In particular, we obtain a nontrivial refinement for $p=7$ of the trivial obstruction group mentioned in Sect. 4.

Comparison with the isomorphism types of $H_1(X_0(p)_{\mathrm{orb}},\mathbb{Z})$ from [7, Sect. 5] suggests that this refinement reproduces the full first orbifold homology of which (2) is a quotient[1]. As recalled in [7, Lemma 5.2], the group $H_1(X_0(p)_{\mathrm{orb}},\mathbb{Z})$ is generated by Manin symbols

$$\left\{0,\frac{1}{a}\right\},\quad 2\le a\le p-2,$$

with relations

$$\left\{0,\frac{1}{a}\right\}+\left\{0,\frac{1}{p-a^{-1}}\right\}=0, \tag{4}$$

$$\left\{0,\frac{1}{(p-1)/2}\right\}+\left\{0,\frac{1}{p-2}\right\}=0, \tag{5}$$

$$\left\{0,\frac{1}{a}\right\}+\left\{0,\frac{1}{a'}\right\}+\left\{0,\frac{1}{a''}\right\}=0\quad (a\notin\{p-2,(p-1)/2\}), \tag{6}$$

where

$$a'\equiv -a^{-1}-1 \bmod p,\quad a''\equiv -(a+1)^{-1} \bmod p.$$

[1] We thank the referee for this obesrvation.

Theorem 3 *There is an isomorphism*

$$\overline{\mathcal{T}}_2^{[p]}/\langle(\mathrm{triv}, (0, 0)), (\mathbb{Z}/p\mathbb{Z}, (0, 1))\rangle \to H_1(X_0(p)_{\mathrm{orb}}, \mathbb{Z})$$

sending $(\mathbb{Z}/p\mathbb{Z}, (1, a))$ *to the Manin symbol* $\{0, 1/a\}$ *for* $2 \leq a \leq p - 2$.

Proof We are working in the quotient $\overline{\mathcal{T}}_2^{[p]}/\langle(\mathrm{triv}, (0, 0)), (\mathbb{Z}/p\mathbb{Z}, (0, 1))\rangle$. We start by obtaining the relation

$$(\mathbb{Z}/p\mathbb{Z}, (1, -b)) = -(\mathbb{Z}/p\mathbb{Z}, (1, b^{-1})). \tag{7}$$

The relation follows by induction on b, starting with the base case $b = 2$, which is a consequence of the last relation from Definition 1 for $a = 2^{-1}$ and $a = p - 1$. The inductive step uses the same relation with $a = b^{-1}$ and $a = p + 1 - b$. This recovers (4). Relation (5) is recovered by once again using the last relation from Definition 1 with $a = p - 1$. Using (7) we may rewrite the last relation from Definition 1 in the form

$$0 = (\mathbb{Z}/p\mathbb{Z}, (1, -a^{-1})) + (\mathbb{Z}/p\mathbb{Z}, (1, a - 1)) + (\mathbb{Z}/p\mathbb{Z}, (1, ((a^{-1} - 1)^{-1}))).$$

Replacing a by $a + 1$ we reproduce (6). □

The isomorphism of Theorem 3 is compatible with that of [7, Lemma 5.3].

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Declarations

Conflict of interest The author has no conflicts of interest to declare that are relevant to the content of this article.

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