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Hyperstability of cubic functional equation in banach space

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Abstract
In this paper, we prove some hyperstability results of the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

on a restricted domain.

Keywords Hyperstability · Banach space · Cubic functional equation

Mathematics Subject Classification Primary 39B82 · Secondary 39B52

1 Introduction

The starting pont for studying the stability of functional equations seems to be the famous talk of Ulam [16] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group.

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Given $\delta > 0$ does there exists a $\epsilon > 0$, such that if a mapping $f : G_1 \to G_2$ satisfies

$$d(f(x.y), f(x)f(y)) \leq \delta$$

for all $x, y \in G_1$, then there exists an homomorphism $\phi : G_1 \to G_2$ such that

$$d(f(x), \phi(x)) \leq \epsilon$$

for all $x \in G_1$?

The first partial answer to Ulam question was presented by Hyers [10] in the case of Cauchy functional equation in Banach spaces.

Later, the result of Hyers was significantly generalized by Rassias [15] in 1978 and Găvruţa [8] in 1994. Since then, the stability problems of several functional equations have been extensively investigated.

In 2014, Brzdek [1] responded to a problem formulated by Th. M. Rassias in 1991 concerning the stability of the Cauchy equation; in which he presents a new method to prove the stability results of the functional equations.

The next definition describes the notion of hyperstability that we apply here (B^A to mean " the family of all functions mapping from a nonempty set A into a nonempty set B ").

Definition 1.1 Let X be a nonempty set, (Y, d) be a metric space, $\varepsilon \in \mathbb{R}_0^{X^n}$ and $\mathcal{F}_1$, $\mathcal{F}_2$ be operators mapping from a nonempty set $\mathcal{D} \subset Y^X$ into Y^{X^n}. We say that the operator equation

$$\mathcal{F}_1\varphi(x_1, \ldots, x_n) = \mathcal{F}_2\varphi(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X) \tag{1.1}$$

is ε-hyperstable provided that every $\varphi_0 \in \mathcal{D}$ which satisfies

$$d\left(\mathcal{F}_1\varphi_0(x_1, \ldots, x_n), \mathcal{F}_2\varphi_0(x_1, \ldots, x_n)\right) \leq \varepsilon(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n \in X)$$

fulfills the Eq. (1.1).

From which we deduce a functional equation is hyperstable if any function f satisfying the equation approximately (in some sense) must be actually solution to it.

It seems that the first hyperstability result was published in [9]. However, The term *hyperstability* has been used for the first time in [13]. Let X be a nonempty subset symmetric with respect to 0 and Y be a Banach space.

The method of the proof of the main theorem is motivated by an idea used by Brzdęk in [2] and further by Piszczek in [14]. It is based on a fixed point theorem for functional spaces obtained by Brzdęk et al in [4]. Some generalizations of their result were proved by Cădariu et al. in [5]. The case of fixed point theorem for non-Archimedean metric spaces was also studied by Brzdęk and Ciepliński in [3]. It is worth mentioning that using fixed point theorem is now one of the most popular methods of investigating the stability of functional equations in single as well as in several variables.

In 2014, Brzdęk et al. in [3] discussed the fixed point method, namely the second most popular way to stabilize functional equations.

Let us recall a few recent approaches of Jung in [11], Lee and Jung in [12]. More information on the application of the fixed point method was collected by Ciepliński in [7]. First, we take the following three hypotheses (all notations come from [6]).

Now, we present some results in Banach spaces using the fixed point method. Before proceeding to the main results, we state Theorem 1.2 which is useful for our purpose. To present it, we introduce the following three hypotheses:

$(H_1) X$ is a nonempty set, Y a Banach spaces, and $f_1, \ldots, f_k : X \to X$ and $L_1, \ldots, L_K : X \to \mathbb{R}_+$ are given.

$(H_2) \mathcal{F} : Y^X \to Y^X$ is an operator satisfying the inequality

$$\|\mathcal{F}\xi(x) - \mathcal{F}\mu(x)\| \le \sum_{i=1}^{k} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\| \quad \xi, \mu \in Y^X, x \in X$$

$(H_3) \Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ is defined by:

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \quad \delta \in \mathbb{R}_+^X, \quad x \in X.$$

The mentioned fixed point theorem is stated as follows.

Theorem 1.2 [4] *Let hypotheses $(H_1) - (H_2)$ be valid, functions $\varepsilon : X \to \mathbb{R}_+$ and let $\varphi : X \to Y$ fulfill the following two conditions:*

$$i) \ \ \|\mathcal{F}\varphi(x) - \varphi(x)\| \le \varepsilon(x), \quad x \in X$$

$$ii) \ \ \varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \quad x \in X.$$

Then there exists a unique fixed point ψ of $\mathcal{F}$ with $\|\varphi(x) - \psi(x)\| \le \varepsilon^*(x), x \in X$.
Moreover $\psi(x) = \lim_{n\to\infty} \mathcal{F}^n\varphi(x), \quad x \in X$.

Throughout the paper, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{N}_{m_0}$ denote the set of all positive integers, the set of all nonnegative integers and the set of all integers greater than or equal to m_0, respectively, the set of real numbers by $\mathbb{R}$, $\mathbb{R}_+ := [0, \infty)$, and we use the notation X_0 for the set $X \backslash \{0\}$.

We say that a function $f : X \to Y$ satisfies the cubic functional equation on X if

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.2}$$

for all $x, y \in X$ such that $x + y, x - y \in X$.

2 Main results

In this section, we use Theorem 1.2 as a basic tool to prove the hyperstability results of the cubic functional equation in Banach spaces.

Theorem 2.1 *Assume that X is a nonempty symmetric with respect to 0 subset of a normed space such that $0 \notin X$ and there exist $n_0 \in \mathbb{N}$ with $nx \in X$ for $x \in X$ and $n \in \mathbb{N}_{n_0}$. Let Y be a Banach space, $c \geq 0$, and $p+q<0$. If $f : X \to Y$ satisfies*

$$\left\| \frac{1}{12} f(2x+y) + \frac{1}{12} f(2x-y) - \frac{1}{6} f(x+y) - \frac{1}{6} f(x-y) - f(x) \right\| \leq \frac{c}{12} \|x\|^p \|y\|^q \tag{2.1}$$

for all $x, y \in X$ such that $x+y$, $x-y \in X$, then f satisfies the cubic equation on X.

Proof First observe that there exists $m_0 \in \mathbb{N}_{n_0}$ such that

$$\alpha_m = \frac{1}{12}(2+m)^{p+q} + \frac{1}{12}(m-2)^{p+q} + \frac{1}{6}(1+m)^{p+q} + \frac{1}{6}(m-1)^{p+q} < 1.$$

Assume that $q<0$ and replacing y with mx in (2.1) we get:

$$\left\| \frac{1}{12} f((2+m)x) + \frac{1}{12} f((2-m)x) - \frac{1}{6} f((1+m)x) - \frac{1}{6} f((1-m)x) - f(x) \right\| \leq \frac{c}{12} m^q \|x\|^{p+q}$$

if $x \in X$.

Further put

$$\mathcal{F}_m \xi(x) := \frac{1}{12}\xi((2+m)x) + \frac{1}{12}\xi((2-m)x) - \frac{1}{6}\xi((1+m)x) - \frac{1}{6}\xi((1-m)x),$$

$x \in X$, $\xi \in Y^X$ and $\epsilon_m(x) := \frac{c}{12} m^q \|x\|^{p+q}$.

Then the inequality (2.1) takes the form $\|\mathcal{F}_m f(x) - f(x)\| \leq \epsilon_m(x)$.

The operator

$$\Delta\delta(x) := \frac{1}{12}\delta((2+m)x) + \frac{1}{12}\delta((m-2)x) + \frac{1}{6}\delta((1+m)x) + \frac{1}{6}\delta((m-1)x), \quad \delta \in \mathbb{R}_+^X, \quad x \in X$$

has the form described in (H_3) with $k=4$ and

$$f_1(x) = (m+2)x,\ f_2(x) = (2-m)x,\ f_3(x) = (1+m)x,\ f_4(x) = (1-m)x,$$

$$L_1(x) = L_2(x) = \frac{1}{12}, L_3(x) = L_4(x) = \frac{1}{6}.$$

Moreover, for every $\xi, \mu \in X^X$ and $x \in X$

$$\|\mathcal{F}_m\xi(x) - \mathcal{F}_m\mu(x)\| \leq \sum_{i=1}^{4} L_i(x)\|(\xi - \mu)(f_i(x))\|.$$

So (H_2) is valid.

Next we can find $m_0 \in \mathbb{N}$ such that

$$\alpha_m = \frac{1}{12}(2+m)^{p+q} + \frac{1}{12}(m-2)^{p+q} + \frac{1}{6}(1+m)^{p+q} + \frac{1}{6}(m-1)^{p+q} < 1.$$

Therefore we obtain that

$$\begin{aligned}\epsilon^*(x) :&= \sum_{n=0}^{\infty} \triangle^n \epsilon(x) \\ &= \frac{c}{12}m^q\|x\|^{p+q}\sum_{n=0}^{\infty}\left(\frac{1}{12}(2+m)^{p+q} + \frac{1}{12}(m-2)^{p+q}\right. \\ &\quad \left.+\frac{1}{6}(1+m)^{p+q} + \frac{1}{6}(m-1)^{p+q}\right)^n \\ &= \frac{cm^q\|x\|^{p+q}}{12(1-\alpha_m)}.\end{aligned}$$

Thus according to theorem (1.2) there exists a unique solution $F : X \to Y$ of the equation

$$F(x) = \frac{1}{12}F((2+m)x) + \frac{1}{12}F((2-m)x) - \frac{1}{6}F((1+m)x) - \frac{1}{6}F((1-m)x)$$

such that

$$\|f(x) - F(x)\| \leq \frac{cm^q\|x\|^{p+q}}{12(1-\alpha_m)}.$$

Moreover: $F(x) = \lim_{n\to\infty} \mathcal{F}^n f(x)$. To prove that F satisfies the cubic equation on $\mathbb{X}$, observe that

$$\begin{aligned}&\left\|\frac{1}{12}\mathcal{F}^n f(2x+y)+\frac{1}{12}\mathcal{F}^n f(2x-y)-\frac{1}{6}\mathcal{F}^n f(x+y)-\frac{1}{6}\mathcal{F}^n f(x-y)-\mathcal{F}^n f(x)\right\| \\ &\leq \frac{c}{12}(\alpha_m)^n\|x\|^p\|y\|^q \qquad (2.2)\end{aligned}$$

for every $x, y \in X$, such that $x + y \in X$, $x - y \in X$.

Indeed: if $n = 0$ then, (2.2) is simple. So, fix $n \in \mathbb{N}_0$ and suppose that (2.2) holds for n and $x, y \in X$ such that $x + y, x - y \in X$.

Then

$$\begin{aligned}
&\bigg\| \frac{1}{12}\mathcal{F}^{n+1} f(2x+y) + \frac{1}{12}\mathcal{F}^{n+1} f(2x-y) - \frac{1}{6}\mathcal{F}^{n+1} f(x+y) \\
&\quad -\frac{1}{6}\mathcal{F}^{n+1} f(x-y) - \mathcal{F}^{n+1} f(x)\bigg\| \\
&= \bigg\| \frac{1}{12}\left(\frac{1}{12}\mathcal{F}^n f((2+m)(2x+y)) + \frac{1}{12}\mathcal{F}^n f((2-m)(2x+y))\right. \\
&\quad - \frac{1}{6}\mathcal{F}^n f((1+m)(2x+y)) \\
&\quad \left. -\frac{1}{6}\mathcal{F}^n f((1-m)(2x+y))\right) + \frac{1}{12}\left(\frac{1}{12}\mathcal{F}^n f((2+m)(2x-y))\right. \\
&\quad + \frac{1}{12}\mathcal{F}^n f((2-m)(2x-y)) \\
&\quad \left. -\frac{1}{6}\mathcal{F}^n f((1+m)(2x-y)) - \frac{1}{6}\mathcal{F}^n f((1-m)(2x-y))\right) \\
&\quad - \frac{1}{6}\left(\frac{1}{12}\mathcal{F}^n f((2+m)(x+y))\right. \\
&\quad + \frac{1}{12}\mathcal{F}^n f((2-m)(x+y)) - \frac{1}{6}\mathcal{F}^n f((1+m)(x+y)) \\
&\quad \left. -\frac{1}{6}\mathcal{F}^n f((1-m)(x+y))\right) \\
&\quad - \frac{1}{6}\left(\frac{1}{12}\mathcal{F}^n f((2+m)(x-y)) + \frac{1}{12}\mathcal{F}^n f((2-m)(x-y))\right. \\
&\quad - \frac{1}{6}\mathcal{F}^n f((1+m)(x-y)) \\
&\quad \left. -\frac{1}{6}\mathcal{F}^n f((1-m)(x-y))\right) - \frac{1}{12}\mathcal{F}^n f((2+m)(x)) \\
&\quad - \frac{1}{12}\mathcal{F}^n f((2-m)(x)) + \frac{1}{6}\mathcal{F}^n f((1+m)(x)) \\
&\quad +\frac{1}{6}\mathcal{F}^n (f((1-m)(x))\bigg\| \\
&\le \bigg\| \frac{1}{12}\left(\frac{1}{12}\mathcal{F}^n f((2+m)(2x+y)) + \frac{1}{12}(\mathcal{F}^n f((2+m)(2x-y))\right. \\
&\quad - \frac{1}{6}\mathcal{F}^n f((2+m)(x+y)) \\
&\quad \left. -\frac{1}{6}\mathcal{F}^n f((2+m)(x-y)) - \mathcal{F}^n f((2+m)x)\right)\bigg\| \\
&\quad + \bigg\| \frac{1}{12}\left(\frac{1}{12}\mathcal{F}^n f((2-m)(2x+y))\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{12}(\mathcal{F}^n f((2-m)(2x-y)) - \frac{1}{6}(\mathcal{F}^n f((2-m)(x+y))) \\
&-\frac{1}{6}(\mathcal{F}^n f((2-m)(x-y))) \\
&-\mathcal{F}^n f((2-m)x)\Big)\Big\| + \Big\|\frac{1}{12}\frac{1}{6}\mathcal{F}^n f((1+m)(2x+y)) \\
&+\frac{1}{12}\left(\frac{1}{6}\mathcal{F}^n f((1+m)(2x-y))\right. \\
&-\frac{1}{6}\left(\frac{1}{6}\mathcal{F}^n f((1+m)(x+y))\right) - \frac{1}{6}\left(\frac{1}{6}\mathcal{F}^n f((1+m)(x-y))\right) \\
&\left.-\frac{1}{6}\mathcal{F}^n f(1+m)x\right)\Big\| \\
&+\Big\|\frac{1}{12}\frac{1}{6}\mathcal{F}^n f((1-m)(2x+y)) + \frac{1}{12}\left(\frac{1}{6}\mathcal{F}^n f((1-m)(2x-y))\right. \\
&-\frac{1}{6}(\frac{1}{6}\mathcal{F}^n f((1-m)(x+y))) \\
&\left.-\frac{1}{6}\left(\frac{1}{6}\mathcal{F}^n f((1-m)(x-y))\right) - \frac{1}{6}\mathcal{F}^n f(1-m)x\right)\Big\| \\
\le\ & \frac{c}{12}\left(\frac{1}{12}(2+m)^{p+q} + \frac{1}{12}(m-2)^{p+q} + \frac{1}{6}(1+m)^{p+q}\right. \\
&\left.+\frac{1}{6}(m-1)^{p+q}\right)^n \|x\|^p\|y\|^q\left(\frac{(2+m)^{p+q}}{12}\right. \\
&\left.+\frac{(m-2)^{p+q}}{12} + \frac{(m+1)^{p+q}}{6} + \frac{(m-1)^{p+q}}{6}\right) \\
=\ & \frac{c}{12}\left(\frac{1}{12}(2+m)^{p+q} + \frac{1}{12}(m-2)^{p+q} + \frac{1}{6}(1+m)^{p+q}\right. \\
&\left.+\frac{1}{6}(m-1)^{p+q}\right)^{(n+1)} \|x\|^p\|y\|^q.
\end{aligned}
$$

By induction, we have shown that (2.2) holds. Letting $n \to +\infty$ in (2.2) we obtain $F(2x+y)+F(2x-y) = 2F(x+y)+2F(x-y)+12F(x)$. Thus, we have proved that for every $m \in \mathbb{N}_{m_0}$ there exists a function $F_m : X \to Y$ such that F_m is a solution of the cubic equation on X and

$$\|f(x) - F_m(x)\| \le \frac{c}{12}(\alpha_m)^n\|x\|^p\|y\|^q.$$

Since $p+q < 0$ with $q < 0$, the sequence

$$\left\{\frac{cm^q\|x\|^{p+q}}{12(1-(\alpha_m)})\right\}_{m\in N_{m_0}}$$

tends to zero. Consequently f satisfies the cubic equation on X as the pointwise limit of $(F_m)_{m\in\mathbb{N}_{m_0}}$. □

Theorem 2.2 *If $f : X \to Y$ satisfies*

$$\left\| \frac{1}{12} f(2x+y) + \frac{1}{12} f(2x-y) - \frac{1}{6} f(x+y) - \frac{1}{6} f(x-y) - f(x) \right\| \leq \frac{c}{12} \|x\|^p \|y\|^q \tag{2.3}$$

for all $x, y \in X$ such that $x+y, x-y \in X$, and $0 < p+q < 1$. Then f satisfies the cubic functional equation on X.

Proof Assume that $q > 0$.

Replacing y by $\frac{x}{m}$ we get:

$$\left\| \frac{1}{12} f\left(\left(2+\frac{1}{m}\right)x\right) + \frac{1}{12} f\left(\left(2-\frac{1}{m}\right)x\right) - \frac{1}{6} f\left(\left(1+\frac{1}{m}\right)x\right) - \frac{1}{6} f\left(1-\frac{1}{m}\right)x) - f(x) \right\| \leq \frac{c}{12}\frac{1}{m^q}\|x\|^{p+q} = \epsilon_m(x)$$

such that $x \in X$.

Similarly as previously we define

$$\mathcal{F}_m\xi(x) := \frac{1}{12}\xi\left(\left(2+\frac{1}{m}\right)x\right) + \frac{1}{12}\xi\left(\left(2-\frac{1}{m}\right)x\right) - \frac{1}{6}\xi\left(\left(1+\frac{1}{6}\right)x\right) - \frac{1}{6}\xi\left(\left(1-\frac{1}{m}\right)x\right), \quad x \in X, \quad \xi \in Y^X$$

and

$$\Delta_m\delta(x) := \frac{1}{12}\delta\left(\left(2+\frac{1}{m}\right)x\right) + \frac{1}{12}\delta\left(\left(\frac{1}{m}-2\right)x\right)\frac{1}{6}\delta\left(\left(1+\frac{1}{m}\right)x\right) + \frac{1}{6}\delta\left(\left(\frac{1}{m}-1\right)x\right)$$

$\delta \in \mathbb{R}_+^X$, $x \in X$ and see that (2.2) is

$$\|\mathcal{F}_m f(x) - f(x)\| \leq \epsilon_m(x), \quad x \in X.$$

Obviously Δ_m has the form described in (H_3) with $k = 4$ and $f_1(x) = (2+\frac{1}{m})x$ $f_2(x) = (2-\frac{1}{m})x$, $f_3(x) = (1+\frac{1}{m})x$, $f_4(x) = (1-\frac{1}{m})x$, $L_1(x) = L_2(x) = \frac{1}{12}$, $L_3(x) = L_4(x) = \frac{1}{6}$

$$\|\mathcal{F}_m\xi(x) - \mathcal{F}_m\mu(x)\| \leq \sum_{i=1}^{4} L_i(x)\|(\xi-\mu)(f_i(x))\|.$$

So (H_2) is valid.

Next we can find $m_0 \in \mathbb{N}_{n_0}$ such that

$$\beta_m = \frac{1}{12}\left(2+\frac{1}{m}\right)^{p+q} + \frac{1}{12}\left(2-\frac{1}{m}\right)^{p+q} + \frac{1}{6}\left(1+\frac{1}{m}\right)^{p+q} + \frac{1}{6}\left(1-\frac{1}{m}\right)^{p+q} < 1,$$

for all $m \geq m_0$.

Therefore we obtain that

$$\begin{aligned}\epsilon^*(x) :&= \sum_{n=0}^{\infty} \triangle^n \epsilon(x)\\ &= cm^q \|x\|^{p+q} \sum_{n=0}^{\infty} (\beta_m)^n\\ &= \frac{cm^q \|x\|^{p+q}}{1-\beta_m}.\end{aligned}$$

Thus, according to theorem (1.2) there exists a unique solution $F : X \to Y$ of the equation

$$\begin{aligned}F_m(x) = &\frac{1}{12}F_m\left(\left(2+\frac{1}{m}\right)x\right) + \frac{1}{12}F_m\left(\left(2-\frac{1}{m}\right)x\right)\\ &-\frac{1}{6}F_m\left(\left(1+\frac{1}{m}\right)x\right) - \frac{1}{6}F_m\left(\left(1-\frac{1}{m}\right)x\right)\end{aligned}$$

such that:

$$\|f(x) - F_m(x)\| \leq \frac{cm^q \|x\|^{p+q}}{12(1-\beta_m)}$$

and $F_m(2x+y) + F_m(2x-y) = 2F_m(x+y) + 2F_m(x-y) + 12F_m(x)$, $x \in X$, $y \in X$, $x+y \in X$, $x-y \in X$.

In this way we obtain a sequence $(F_m)_{m\in N_{m_0}}$ of cubic functions on X such that

$$\|f(x) - F_m(x)\| \leq \frac{cm^q \|x\|^{p+q}}{12(1-\beta_m)},$$

it follows, with $m \to \infty$, that f is cubic on X. □

Remark 2.3 : In the case $p > 1$ and $q > 1$, the considered cubic equation is not hyperstable.

Example 2.4 Let $X = \mathbb{R} - \{[-\sqrt{14}; \sqrt{14}]\}$ and $f : X \to \mathbb{R}$ be a constant $f(x) = c$, $x \in X$ for some $c > 0$.

Then f satisfies the inequality

$$\left\| \frac{1}{12}f(2x+y) + \frac{1}{12}f(2x-y) - \frac{1}{6}f(x+y) - \frac{1}{6}f(x+y) - f(x) \right\| \leq \frac{c}{12}\|x\|^p \|y\|^q$$

for all $x, y \in X$ such that $x + y, x - y \in X$, with $p > 1$ and $q > 1$, but is not a solution of the cubic equation on X.

Theorem 2.5 *Assume that X is a nonempty, symmetric with respect to 0 subset of a normed space such that $0 \notin X$ and there exists $n_0 \in \mathbb{N}$ with $nx \in X$ for $x \in X$ and $n \in \mathbb{N}_{n_0}$. Let Y be a Banach space, $c \geq 0$, and $p < 0$. If $f : X \to Y$ satisfies*

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - 6f(x+y) \\ &\quad -6f(x-y) - 12f(x)\| \leq c(\|x\|^p + \|y\|^p) \end{aligned} \tag{2.4}$$

for all $x, y \in X$ such that $x + y$, $x - y \in X$, then f satisfies the cubic equation on X.

Proof Replacing (x, y) by $(mx, (2m-1)x)$, where $m \in \mathbb{N}^* - \{1; 2\}$ in (2.4), we get

$$\begin{aligned} &\|f(x) + f((4m-1)x) - 6f((3m-1)x) - 6f((1-m)x) - 12f(mx)\| \\ &\quad \leq c(m^p + (2m-1)^p)\|x\|^p \end{aligned} \tag{2.5}$$

for all $x \in X$.

Further put

$$\mathcal{F}_m\xi(x) := 12\xi((m)x) + 6\xi((3m+1)x) + 6\xi((1-m)x) - \xi((4m-1)x)$$

$x \in X$, $\xi \in Y^X$ and $\epsilon_m(x) := c(m^p + (2m-1)^p)\|x\|^p$.

Then the inequality (2.5) takes the form $\|\mathcal{F}_m f(x) - f(x)\| \leq \epsilon_m(x)$. $x \in X$.

The operator $\Delta_m\delta(x) := 12\delta(mx)+6\delta((3m+1)x)+6\delta((1-m)x)+\delta((4m-1)x)$ $\delta \in \mathbb{R}_+^X$, $x \in X$ has the form described in (H_3) with $k = 4$ and $f_1(x) = mx$, $f_2(x) = (3m+1)x$, $f_3(x) = (1-m)x$, $f_4(x) = (4m-1)x$, $L_1(x) = 12$, $L_3(x) = L_4(x) = 6$, $L_4(x) = 1$ for all $x \in X$.

Moreover, for every $\xi, \mu \in Y^X$ and $x \in X$, we have

$$\|\mathcal{F}_m\xi(x) - \mathcal{F}_m\mu(x)\| \leq \sum_{i=1}^{4} L_i(x)\|(\xi - \mu)(f_i(x))\|.$$

So, H_2 is valid. Now, we can find $m \in N^* - \{1; 2\}$ such that

$$12m^p + 6(3m+1)^p + 6(m-1)^p + (4m-1)^p < 1$$

for all $m \leq m_0$.

Therefore, we obtain that

$$\begin{aligned} \epsilon^*(x) := \sum_{n=0}^{\infty} \triangle^n \epsilon(x) = c(m^p + (2m-1)^p)\|x\|^p \\ \sum_{n=0}^{\infty}(12m^p + 6(3m+1)^p + 6(m-1)^p + (4m-1)^p)^n \end{aligned}$$

$$= \frac{c(m^p + (2m-1)^p)}{1 - (12m^p + 6(3m+1)^p + 6(m-1)^p + (4m-1)^p)}$$

for all $x \in X$ and $m \geq m_0$. The rest of the proof is similar to the proof of theorem (2.1.) □

Corollary 2.6 *Assume that X is that a nonempty symmetric with respect to 0 subset of a normed space such that $0 \notin X$ and Y be a Banach space. Let $F : X^2 \to Y$ be a mapping such that $F(x_0, y_0) \neq 0$ for some $x_0, y_0 \in X$ and*

$$\|F(x, y)\| \leq c\|x\|^p\|y\|^q, \tag{2.6}$$

or

$$\|F(x, y)\| \leq c(\|x\|^p + \|y\|^p) \tag{2.7}$$

for all $x, y \in X$, where $c \geq 0$ and $p, q \in \mathbb{R}$. Assume that the numbers $p; q$ satisfy $p + q < 1$ and $p + q \neq 0$. In the case (2.6) and $p < 0$ in the case (2.7), then the functional equation:

$$h(2x + y) + h(2x - y) + F(x, y) = 2h(x + y) + 2h(x - y) + 12h(x) \tag{2.8}$$

$x, y \in X$ has no solution in the class of functions $h : X \to Y$.

Proof Suppose that $h : X \to Y$ is a solution to (2.6), Then (2.1) or (2.7) holds, and consequently, according to above theorems, h is cubic on X, which means that $F(x_0, y_0) = 0$. This is contradiction. □

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