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Characterization of *n*-Jordan homomorphisms and their automatic continuity on Banach algebras

Abbas Zivari-Kazempour[1]

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Abstract
In this paper, we prove that each n-Jordan homomorphism φ from Banach algebra $\mathcal{A}$ into a semisimple commutative Banach algebra $\mathcal{B}$ is automatically continuous. Some useful results about characterization of n-Jordan homomorphisms and interesting examples of them on Banach algebras are given as well.

Keywords n-homomorphism · n-Jordan homomorphism · Automatic continuity · Bounded approximate identity

Mathematics Subject Classification Primary 47B48; Secondary 47C05 · 46H40

1 Introduction and Preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be complex Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then φ is called an n-homomorphism if for all $a_1, a_2, ...a_n \in \mathcal{A}$,

$$\varphi(a_1 a_2 ... a_n) = \varphi(a_1)\varphi(a_2)...\varphi(a_n).$$

The concept of n-homomorphisms was studied for complex algebras by Hejazian *et al*. in [10]. One may refer to [5], for certain properties of 3-homomorphisms.

A linear map φ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called an n-Jordan homomorphism if $\varphi(a^n) = \varphi(a)^n$, for all $a \in \mathcal{A}$. This notion was introduced by Herstein in [11]. For $n = 2$, this concepts coincides the classical definitions of homomorphism and Jordan homomorphism, respectively. Moreover, Jordan homomorphism is equivalent by

✉ Abbas Zivari-Kazempour
zivari@abru.ac.ir; zivari6526@gmail.com

1 Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran

$$\varphi(a \circ b) = \varphi(a) \circ \varphi(b), \quad a, b \in \mathcal{A},$$

where $a \circ b = ab + ba$.

It is obvious that each n-homomorphism is an n-Jordan homomorphism, but the converse is false, in general. In fact, the converse is true under certain conditions. For example, it is shown in [8] that each n-Jordan homomorphism between two commutative algebras is an n-homomorphism for $n \in \{3, 4\}$, and this result extended to $n < 8$, in [3]. Note that for $n = 2$, the proof is clear.

The following more general result is due to Gselmann.

Theorem 1.1 [9, Theorem 2.1] *Let $n \in \mathbb{N}$, $\mathcal{R}$ and $\mathcal{R}'$ be two commutative rings such that char$(\mathcal{R}') > n$ and suppose that $\varphi : \mathcal{R} \longrightarrow \mathcal{R}'$ is an n-Jordan homomorphism. Then φ is an n-homomorphism.*

Moreover, if $\mathcal{R}$ is unital, then $\varphi(e) = \varphi(e)^n$ and the map $\psi : \mathcal{R} \longrightarrow \mathcal{R}'$ defined by $\psi(x) = \varphi(e)^{n-2}\varphi(x)$ is a homomorphism.

In 2018, Bodaghi and İnceboz proved that every additive n-Jordan homomorphism between two commutative algebras is an n-homomorphism [2]. However, their proof is different from that of Gselmann. We remark that since char$(\mathcal{B}) > n$ for each algebra $\mathcal{B}$, so Theorem 1.1 is stronger than the result of Bodaghi and İnceboz.

When the domain is not necessarily commutative, Żelazko in [15] proved the following theorem (see also [13]).

Theorem 1.2 *Suppose that $\mathcal{A}$ is a Banach algebra, which need not be commutative, and suppose that $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism.*

This result has been proved by the author in [16] and [17] for 3-Jordan and 5-Jordan homomorphism with the additional hypothesis that the Banach algebra $\mathcal{A}$ is unital. In other words, he presented the next theorem.

Theorem 1.3 *Let $n \in \{3, 5\}$ be fixed. Let $\mathcal{A}$ be a unital Banach algebra, which need not be commutative, and $\mathcal{B}$ be a semisimple commutative Banach algebra. Then each n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an n-homomorphism.*

Later in 2017, An extended Theorem 1.3 and obtained the next result (for alternative proof see [20, Corollary 2.3]).

Theorem 1.4 [1, Corollary 2.5] *Let $\mathcal{A}$ be a unital Banach algebra and $\mathcal{B}$ be a semisimple commutative Banach algebra. Then each n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an n-homomorphism. Moreover, if φ is surjective, then φ is automatically continuous.*

Recall that a bounded approximate identity for $\mathcal{A}$ is a bounded net $(e_\alpha)_{\alpha\in I}$ in $\mathcal{A}$ such that $ae_\alpha \longrightarrow a$ and $e_\alpha a \longrightarrow a$, for all $a \in \mathcal{A}$. For example, it is known that the group algebra $L^1(G)$, for a locally compact group G, and C^*-algebras have a bounded approximate identity bounded by one [7].

In this paper we prove that each n-Jordan homomorphism from Banach algebra $\mathcal{A}$ into a semisimple commutative Banach algebra $\mathcal{B}$ is automatically continuous, and generalize Theorem 1.4, for non unital Banach algebras which equipped a bounded approximate identity. In the other word, the results presented here are indeed extensions and generalizations of the above mentioned results.

2 Characterization of 3-Jordan homomorphisms

The next example provided that we cannot assert that n-Jordan homomorphisms of rings are always n-homomorphisms.

Example 2.1 Let $\mathcal{A} = K[x, y]$ be the polynomial ring in two independent in determinates over a field K of characteristic not two, and let $\mathcal{B} = K[X, Y, Z]$ be the polynomial ring in the three elements X, Y, Z that satisfy the relations

$$YX = XY + Z, \quad XZ = ZX, \quad YZ = ZY, \quad Z^2 = 0.$$

Then the linear mapping φ that sends $x^m y^n$ into $\frac{1}{2}(X^m \circ Y^n)$, $m, n = 0, 1, 2, ...$ is a Jordan homomorphism as is shown in [12, Example 1], hence it is an n-Jordan homomorphism by [12, Theorem 1], or [14, Lemma 6.3.2]. On the other hand, since

$$\big(\frac{1}{2}(X^{m_1} \circ Y^{n_1})\big)...\big(\frac{1}{2}(X^{m_k} \circ Y^{n_k})\big) \neq \frac{1}{2}\big(X^{m_1+...+m_k} \circ Y^{n_1+...+n_k}\big),$$

thus, φ is not n-homomorphism.

The commutativity of Banach algebra $\mathcal{B}$ in Theorem 1.3 is essential. The following example illustrates this fact.

Example 2.2 Let

$$\mathcal{A} = \left\{\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} : \quad X, Y \in M_2(\mathbb{C})\right\}.$$

Then under the usual matrix operations, $\mathcal{A}$ is a unital and semisimple Banach algebra but it is not commutative. Define a continuous linear map $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$\varphi\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} X & 0 \\ 0 & Y^T \end{bmatrix},$$

where Y^T denote the transpose of Y. Then, for all $A \in \mathcal{A}$ and for each $n \in \mathbb{N}$,

$$\varphi(A^n) = \varphi(A)^n.$$

Thus, φ is an n-Jordan homomorphism, but it is not an n-homomorphism.

Note that in the above example $\mathcal{A}$ is unital. Next we construct an example of n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, such that $\mathcal{A}$ is not unital.

Example 2.3 Let

$$\mathcal{A} = \left\{\begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} : \quad X, Y \in M_2(\mathbb{C})\right\},$$

where X is the form $\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}$, and let

$$\mathcal{B} = \left\{ \begin{bmatrix} z_{11} & z_{12} \\ 0 & z_{22} \end{bmatrix} : \quad z_{ij} \in \mathbb{C} \right\}.$$

Then $\mathcal{A}$ is neither unital nor commutative and $\mathcal{B}$ is a noncommutative semisimple Banach algebra. Define a continuous linear map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ by

$$\varphi\left(\begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix} \right) = X^T.$$

Then,

$$\varphi(A^n) = \varphi(A)^n,$$

for all $A \in \mathcal{A}$. Consequently, φ is a n-Jordan homomorphism, but it is easy to see that φ is not an n-homomorphism.

To achieve our aim in this section, we need the following theorem.

Theorem 2.4 [19, Theorem 2.3] *Suppose that $\mathcal{A}$ is a Banach algebra. Then every 3-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ is automatically continuous.*

Next we generalize Theorem 1.3 for nonunital Banach algebras.

Theorem 2.5 *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. Then each 3-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ is a 3-homomorphism.*

Proof Assume that φ is a 3-Jordan homomorphism, then $\varphi(a^3) = \varphi(a)^3$, for all $a \in \mathcal{A}$. Replacing a by $a + b$, we get

$$\varphi(ab^2 + b^2a + a^2b + ba^2 + aba + bab) = 3\varphi(a)^2\varphi(b) + 3\varphi(a)\varphi(b)^2, \quad (1)$$

and interchanging a by $-a$ in (1), to obtain

$$\varphi(-ab^2 - b^2a + a^2b + ba^2 + aba - bab) = 3\varphi(a)^2\varphi(b) - 3\varphi(a)\varphi(b)^2. \quad (2)$$

By (1) and (2),

$$\varphi(a^2b + ba^2 + aba) = 3\varphi(a)^2\varphi(b), \quad (a, b \in \mathcal{A}). \quad (3)$$

Suppose that $(e_\alpha)_{\alpha \in I}$ is a bounded approximate identity for $\mathcal{A}$, and let

$$E = \{\varphi(e_\alpha) : \quad \alpha \in I\}.$$

Then we may suppose, by passing to a subnet, that $\varphi(e_\alpha) \longrightarrow \beta \in \mathbb{C}$. Replacing b by e_α in (3) and using Theorem 2.4, we obtain

$$\varphi(a^2) = \beta\varphi(a)^2, \tag{4}$$

for all $a \in \mathcal{A}$. Replacing a by $a + e_\alpha$ in (4), we arrive at

$$\varphi(a) = \beta\varphi(a) \lim_\alpha \varphi(e_\alpha) = \beta^2\varphi(a), \tag{5}$$

which proves that $\beta^2 = 1$. Define $\psi : \mathcal{A} \longrightarrow \mathbb{C}$ by $\psi(x) = \beta\varphi(x)$, for all $x \in \mathcal{A}$. Then $\psi(x^2) = \beta\varphi(x^2) = \beta^2\varphi(x)^2 = \psi(x)^2$, therefore ψ is a Jordan homomorphism and hence by Theorem 1.2, ψ is a homomorphism. Thus,

$$\beta\varphi(abc) = \psi(abc) = \psi(a)\psi(b)\psi(c) = \beta^3\varphi(a)\varphi(b)\varphi(c),$$

for all $a, b, c \in \mathcal{A}$. Consequently, φ is a 3-homomorphism. □

Corollary 2.6 *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity and let $\mathcal{B}$ be a semisimple commutative Banach algebra. Then each 3-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a 3-homomorphism.*

Proof Let $\mathfrak{M}(\mathcal{B})$ be the maximal ideal space of $\mathcal{B}$. We associate with each $f \in \mathfrak{M}(\mathcal{B})$ a function $\varphi_f : \mathcal{A} \longrightarrow \mathbb{C}$ defined by

$$\varphi_f(a) := f(\varphi(a)), \qquad (a \in \mathcal{A}).$$

Then φ_f is a 3-Jordan homomorphism, so by Theorem 2.5 it is a 3-homomorphism. Hence, by the definition of φ_f we have

$$f(\varphi(abc)) = f(\varphi(a))f(\varphi(b))f(\varphi(c)) = f(\varphi(a)\varphi(b)\varphi(c)).$$

Since $f \in \mathfrak{M}(\mathcal{B})$ was arbitrary and $\mathcal{B}$ is assumed to be semisimple, we obtain

$$\varphi(abc) = \varphi(a)\varphi(b)\varphi(c),$$

for all $a, b, c \in \mathcal{A}$. This completes the proof. □

For a Banach algebra $\mathcal{A}$ without bounded approximate identity, the next result characterizes the 3-Jordan homomorphisms.

Theorem 2.7 *Let $\mathcal{A}$ be a Banach algebra and φ be a 3-Jordan homomorphism from $\mathcal{A}$ into a commutative semisimple Banach algebra $\mathcal{B}$ such that for all $a, b, c \in \mathcal{A}$,*

$$\varphi(abc - cba) = 0.$$

Then φ is a 3-homomorphism.

Proof By a careful adaption of the methods of Theorem 1.1, the result follows. □

3 Characterization of *n*-Jordan homomorphisms

It is shown in [19] that every n-Jordan homomorphism from unital Banach algebra $\mathcal{A}$ into $\mathbb{C}$ is automatically continuous, and without any extra condition asked the following: Is every n-Jordan homomorphism from $\mathcal{A}$ into $\mathbb{C}$ automatically continuous? ([19, Question 2.12]).

Next we answer this question in the affirmative. This result is the main key to characterize n-Jordan homomorphism. For the case $n = 2$, it is [18, Proposition 2.1], and for $n = 3$ it is Theorem 2.4.

Our main theorem in this section is the following.

Theorem 3.1 *Every n-Jordan homomorphism φ from Banach algebra $\mathcal{A}$ into $\mathbb{C}$ is automatically continuous.*

Proof Let $n \geq 4$ be fixed and let $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be an n-Jordan homomorphism. First we prove that for every $a \in \mathcal{A}$ with $\|a\| < 1$, $\varphi(a) \neq 1$. We argue by contradiction. Suppose that there exist $a \in \mathcal{A}$ with $\|a\| < 1$ and $\varphi(a) = 1$. Thus, $\varphi(a^n) = \varphi(a)^n = 1$. Let $\mathfrak{A}$ be a Banach subalgebra of $\mathcal{A}$ generated by the above element a of norm $\|a\| < 1$. Define $\psi : \mathfrak{A} \longrightarrow \mathbb{C}$ by $\psi(x) = \varphi(x)$. Then ψ is an n-Jordan homomorphism, that is $\psi(x^n) = \psi(x)^n$, for all $x \in \mathfrak{A}$. Since $\mathfrak{A}$ is commutative by Theorem 1.1, we have

$$\psi(x_1 x_2 \dots x_{n-1} x_n) = \psi(x_1)\psi(x_2)\dots\psi(x_{n-1})\psi(x_n), \tag{6}$$

for all $x_1, x_2, \dots, x_n \in \mathfrak{A}$. Replacing x_{n-1} by a^{n-1} and x_i by a for all $i \geq 3$ with $i \neq n-1$, in (6), gives

$$\psi(x_1 x_2 a^{2n-4}) = \psi(x_1 x_2 a \dots a^{n-1} a) = \psi(x_1)\psi(x_2)\psi(a)\dots\psi(a^{n-1})\psi(a), \tag{7}$$

for all $x_1, x_2 \in \mathfrak{A}$. Since $\psi(a) = 1$, by (7), we have

$$\psi(x_1 x_2 a^{2n-4}) = \psi(x_1)\psi(x_2)\psi(a^{n-1}).$$

Let $\lambda = \psi(a^{n-1})$, then

$$\psi(x_1)\psi(a^{n-2}x_2) = \psi(x_1)\psi(a^{n-2}x_2)\psi(a)^{n-2} = \psi(x_1 x_2 a^{2n-4}) = \lambda\psi(x_1)\psi(x_2),$$

and since $\psi \neq 0$ we obtain

$$\psi(a^{n-2}x_2) = \lambda\psi(x_2), \tag{8}$$

for all $x_2 \in \mathfrak{A}$. From (6) and (8) we get

$$\psi(x_1)\psi(x_2) = \psi(x_1)\psi(x_2)\psi(a)^{n-2} = \psi(a^{n-2}x_1 x_2) = \lambda\psi(x_1 x_2),$$

for all $x_1, x_2 \in \mathfrak{A}$. Continuing in this way, we conclude that

$$\psi(x_1 x_2 \dots x_{n-1} x_n) = \lambda^{n-1}\psi(x_1)\psi(x_2)\dots\psi(x_{n-1})\psi(x_n).$$

Therefore $\lambda^{n-1} = 1$ and hence $|\lambda| = 1$. Now define $f : \mathfrak{A} \longrightarrow \mathbb{C}$ by $f(x) = \lambda^{n-2}\psi(x)$. For all $x, y \in \mathfrak{A}$,

$$\begin{aligned} f(x)f(y) &= \lambda^{n-2}\psi(x)\lambda^{n-2}\psi(y) \\ &= \lambda^{2n-4}\big(\lambda\psi(xy)\big) \\ &= \lambda^{n-1}\lambda^{n-2}\psi(xy) \\ &= \lambda^{n-2}\psi(xy) \\ &= f(xy). \end{aligned}$$

Thus, f is a multiplicative linear functional and hence by [4, Proposition 3, § 16], it is continuous and $\|f\| \leq 1$. This implies that $\|\psi\| \leq 1$, and hence ψ is continuous. This is a contradiction with $\|a\| < 1$ and $\psi(a) = 1$. Consequently, for all $a \in \mathcal{A}$ with $\|a\| < 1$, we have $\varphi(a) \neq 1$.

Now it is easy to see that if $x \in \mathcal{A}$ in such that $\|x\| \leq 1$, then $|\varphi(x)| \leq 1$. Therefore φ is norm decreasing and hence it is continuous. This finishes the proof. □

As a consequence of preceding theorem, we get the next result.

Corollary 3.2 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two Banach algebras, where $\mathcal{B}$ is semisimple and commutative. Then each n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is continuous.*

Corollary 3.3 *Every n-homomorphism φ from a Banach algebra $\mathcal{A}$ into a semisimple commutative Banach algebra $\mathcal{B}$ is automatically continuous.*

A well-known result due to Šilov [7, Theorem 2.3.3] or [4, Theorem 8, § 17], states that every homomorphism φ from Banach algebra $\mathcal{A}$ into a semisimple commutative Banach algebra $\mathcal{B}$ is automatically continuous. Corollary 3.3 is an extension of this result for all $n \in \mathbb{N}$.

Lemma 3.4 *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity $(e_\alpha)_{\alpha\in I}$, and $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be an n-Jordan homomorphism. Then for all $a \in \mathcal{A}$,*

(i) $\varphi(a) = \beta^{n-1}\varphi(a)$, *where* $\beta = \lim_\alpha \varphi(e_\alpha)$, *and*
(ii) $\varphi(a^2) = \beta^{n-2}\varphi(a)^2$.

Proof The conclusion is proved for $n = 3$, as is done in the proof of Theorem 2.5. By applying Theorem 3.1 and the same method which has been used in [1, Theorem 2.4], we can prove the result for $n > 3$. □

Theorem 3.5 *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. Then each n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ is an n-homomorphism.*

Proof Define a mapping $\psi : \mathcal{A} \longrightarrow \mathbb{C}$ by

$$\psi(a) = \beta^{n-2}\varphi(a),$$

for all $a \in \mathcal{A}$. It follows from Lemma 3.4, that ψ is a Jordan homomorphism and hence it is a homomorphism by Theorem 1.2. By the definition of ψ and Lemma 3.4, we have

$$\beta\psi(a) = \varphi(a). \tag{9}$$

By Lemma 3.4 and (9), we have

$$\begin{aligned}\varphi(a_1a_2...a_n) &= \beta\psi(a_1a_2...a_n)\\ &= \beta\psi(a_1)\psi(a_2)...\psi(a_n)\\ &= \beta\big(\beta^{n-2}\varphi(a_1)\big)\big(\beta^{n-2}\varphi(a_2)\big)...\big(\beta^{n-2}\varphi(a_n)\big)\\ &= \beta^{(n-1)^2}\varphi(a_1)\varphi(a_2)...\varphi(a_n)\\ &= \varphi(a_1)\varphi(a_2)...\varphi(a_n).\end{aligned}$$

Consequently, φ is an n-homomorphism. □

From Theorem 3.5, we deduce the following result which generalize Corollary 2.5 of [1].

Corollary 3.6 *Suppose that $\mathcal{A}$ is a Banach algebra with a bounded approximate identity, and $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an n-homomorphism.*

As a consequence of Corollary 3.6, we have the following result.

Corollary 3.7 *Let $\mathcal{A}$ be an amenable Banach algebra or a C^*-algebra. Then each n-Jordan homomorphism φ from $\mathcal{A}$ into a semisimple commutative Banach algebra $\mathcal{B}$ is an n-homomorphism.*

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Declarations

Conflict of interest Authors declare that they have no conflict of interest.

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