ANNALI DELL'UNIVERSITA' DI FERRARA (2019) 65:311–326
https://doi.org/10.1007/s11565-019-00326-1

Blow up in a semilinear pseudo-parabolic equation with variable exponents

Salim A. Messaoudi[1] · **Ala A. Talahmeh**[2]

Received: 30 July 2019 / Accepted: 12 September 2019 / Published online: 25 September 2019

Abstract
In this paper, we consider the following pseudo-parabolic equation with variable exponents:

$$u_t - \Delta u - \Delta u_t + \int_0^t g(t-\tau)\Delta u(x,\tau)d\tau = |u|^{p(x)-2}u.$$

Under suitable assumptions on the initial datium u_0, the relaxation function g and the variable exponents p, we prove that any weak solution, with initial data at arbitrary energy level, blows up in finite time.

Keywords Pseudo-parabolic equation · Blow up · Finite time · Weak solutions · Variable-exponents nonlinearity

Mathematics Subject Classification 35B44 · 35D30 · 35L70

1 Introduction

In this paper, we are concerned with the following pseudo-parabolic problem, with variable exponents, of the form

$$\begin{cases} u_t - \Delta u - \Delta u_t + \int_0^t g(t-\tau)\Delta u(x,\tau)d\tau = |u|^{p(x)-2}u, & \text{in } \Omega \times (0,T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \qquad \text{(P)}$$

✉ Salim A. Messaoudi
smessaoudi@sharjah.ac.ae

Ala A. Talahmeh
atalahmeh@birzeit.edu

1 Department of Mathematics, University of Sharjah, P.O. Box 27272, Sharjah, UAE

2 Department of Mathematics, Birzeit University, West Bank, Palestine

where $0 < T < \infty$, $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded regular domain with a smooth boundary $\partial\Omega$, g is a positive nonincreasing function and the exponent $p(\cdot)$ is a given measurable function on Ω satisfying

$$2 \leq p_1 = \text{ess}\ \inf_{x\in\Omega} p(x) \leq p(x) \leq p_2 = \text{ess}\ \sup_{x\in\Omega} p(x) \leq \frac{2n}{n-2}, \quad n \geq 3, \tag{1.1}$$

and the log-Hölder continuity condition:

$$|p(x) - p(y)| \leq -\frac{A}{\log|x-y|}, \text{ for a.e. } x, y \in \Omega, \quad \text{with } |x-y| < \delta, \tag{1.2}$$

$A > 0, \quad 0 < \delta < 1.$

In the case when p is constant, a great deal of mathematical effort has been devoted to the study of existence and uniqueness of solutions, regularity, asymptotic behaviour and blow-up of the solutions for such kind of nonlinear pseudo-parabolic equations. In fact the pseudo-parabolic equation

$$u_t - k\Delta u_t - \Delta u = f(u)$$

is used to describe many interesting physical and biological phenomena, such as the unidirectional propagation of nonlinear dispersive long waves [13], the aggregation of population [35], the heat conduction involving two temperatures [15] and the non-stationary processes in semiconductors [24].

Xu and Su [36] considered

$$\begin{cases} u_t - \Delta u - \Delta u_t = u^p, & \text{in } \Omega \times (0, T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0, T) \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $1 < p < \infty$ if $n = 1, 2$; $1 < p \leq \frac{n+2}{n-2}$ if $n \geq 3$. By exploiting the potential well method and the comparison principle, they obtained global existence and finite-time blow-up results for the solutions with initial data at high energy level.

Fenglong Sun et.al. [34] considered problem (P) with p constant. Under suitable assumptions on the initial datium u_0 and the relaxation function g, they obtained the global existence and finite time blow-up of solutions with initial data at low energy level. They also derived the upper bounds for the blow-up time.

In recent years, a great deal of attention has been paid to the study of mathematical nonlinear models with variable-exponent nonlinearity. For instance, modeling of physical phenomena such as flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filteration processes through porous media and image processing. More details on these problems can be found in [2,6,7,26]. Regarding parabolic problems with nonlinearities of variable-exponent type, many works have appeared. Let us mention some of them. For instance, Pinasco [31] studied the following problem

$$\begin{cases} u_t - \Delta u = f(u), & \text{in } \Omega \times [0, T) \\ u(x,t) = 0, & \text{on } \partial\Omega \times [0, T) \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$, and the source term is of the form

$$f(u) = a(x)u^{p(x)} \quad \text{or} \quad f(u) = a(x)\int_\Omega u^{q(y)}(y,t)dy, \tag{1.5}$$

with $p(x),\ q(x) : \Omega \to (1, \infty)$ and the continuous function $a(x) : \Omega \to \mathbb{R}$ are given functions satisfying specific conditions. He established the local existence of positive solutions and proved that solutions with sufficiently large initial data blow up in finite time. Parabolic problems with sources of the form (1.5) appear in several branches of applied mathematics and have been used to model chemical reactions, heat transfer or population dynamics. Antontsev, Chipot and Shmarev [9] studied the homogeneous Dirichlet problem for the doubly nonlinear parabolic equation with anisotropic variable exponent:

$$u_t = \operatorname{div}\Big(a(x,t,u)|u|^{\alpha(x,t)}|\nabla u|^{p(x,t)}\nabla u\Big) + f(x,t), \quad (x,t) \in \Omega \times (0,T),$$

and established conditions on the data which guarantee the comparison principle and uniqueness of bounded weak solutions in suitable Orlicz–Sobolev spaces subject to some additional restrictions. The uniqueness was proved in a narrower class of functions than that of the existence of solutions. Alaoui et al. [11] considered the following nonlinear heat equation

$$u_t(x,t) - \operatorname{div}\big(|\nabla u|^{m(x)-2}\nabla u\big) = u|u|^{p(x)-2},$$

in a bounded domain in $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$. Under appropriate conditions on the exponent functions m and p, they showed that any solution with a nontrivial initial datum blows up in finite time. They also gave a two-dimensional numerical example to illustrate their result. Guo, Lie and Gao in [23] considered the following $p(x)$—Laplacian equation

$$u_t = \operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u\Big) + |u|^{r-2}u, \quad (x,t) \in \Omega \times (0,T),$$

subject to homogeneous Dirichlet boundary condition, where $r > 1$ is a constant. The authors improved the regularity of weak solutions, and then proved that the weak solutions blow up in finite time for some positive initial energy or vanish in finite time by using energy estimate method. In [30], the authors considered the nonlinear parabolic problem with nonstandard growth:

$$u_t = \operatorname{div}\Big(a(u)|\nabla u|^{p(x)-2}\nabla u\Big) + f(x,t). \tag{1.6}$$

By using the method of parabolic regularization, they proved the existence and uniqueness of weak solutions. Also, they studied the localization property of weak solutions for the Eq. (1.6).

The following nonlinear diffusion equation

$$u_t - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,t), \tag{1.7}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, has been used to study image restoration and electrorheological fluids (see [3,4,14,16,19,22,25,32]). In particular, Bendahmane et al. [12] proved the well-posedness of a solution, for L^1-data. Akagi and Matsuura [5] gave the well-posedness for L^2 initial datum and discussed the long-time behaviour of the solution using the subdifferential calculus approach. Al-Smail et al. [10] gave an alternative proof of the well-posedness to (1.7) and, in addition, they gave a two-dimensional numerical example to illustrate the decay result obtained in [5].

Recently, Shangerganesh et al. [33] studied the following fourth-order degenerate parabolic equation

$$u_t + \operatorname{div}\big(|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\big) = f - \operatorname{div} g, \tag{1.8}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, with a smooth boundary $\partial\Omega$, and proved the existence and uniqueness of weak solutions of (1.8) by using the difference and variation methods under suitable assumptions on f, g and the exponent p. Gao et al. [21] studied the nonlinear diffusion problem:

$$u_t = \operatorname{div}\Big(|\nabla u|^{p(x,t)-2}\nabla u + b(x,t)\nabla u\Big) + f(u), \quad (x,t) \in \Omega \times (0,T),$$

subject to homogeneous Dirichlet boundary condition, where f is a continuous function satisfying

$$|f(u)| \leq a_0|u|^{\alpha-1}, \quad 0 < a_0 = \text{constant}, \quad 1 < \alpha = \text{constant}.$$

They constructed suitable function spaces and used the Galerkin method to obtain the existence of weak solutions. They also obtained the conditions for the existence of finite-time blow-up solutions by using the concavity method and the energy estimates of energy functional. Liu and Dong [27] considered the following nonlinear diffusion problem with $p(x,t)-$Laplacian:

$$u_t = \operatorname{div}\Big(a|\nabla u^m|^{p(x,t)-2}\nabla u^m + b(x,t)\nabla u^m\Big) + u^{q(x,t)}, \quad (x,t) \in \Omega \times (0,T),$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$. Under suitable conditions on a, b, m and the exponents $p(x,t), q(x,t)$, the authors proved the existence of weak solutions and obtained suitable energy estimate of solutions in anisotropic Orlicz–Sobolev spaces. They also established blow-up criteria of solutions by applying the energy functional method and the concavity method, and showed some result on global solutions without assumption on initial data.

Very Recently, Antontsev et al. [1], studied the evolution differential inclusion for a nonlocal operator that involves $p(x)$—Laplacian,

$$u_t - \text{div}\Big(|\nabla u|^{p(x)-2}\nabla u\Big) - \int_0^t g(t-s)\text{div}\Big(|\nabla u(x,s)|^{p(x)-2}\nabla u(x,s)\Big)ds \in \mathbf{F}(u) \quad \in \Omega \times (0,T),$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with Lipschitz-continuous boundary $\partial\Omega$. Under appropriate assumptions on $p(\cdot)$, g and the multivalued function $\mathbf{F}(\cdot)$, they proved that the homogeneous Dirichlet problem has a local in time weak solution. Also they showed that the weak solution possesses the property of finite speed of propagation of disturbances from the initial data and may exhibit the waiting time property.

Our aim in this work is to establish a blow-up result of solutions for problem (P), with initial datium at arbitrary high energy level, under suitable conditions on the exponent p, g and the initial datuim. Our technique of proof is similar to the one in [34] with some necessary modifications due the nature of the problem treated here. This result extends that in [34] to problems with variable-exponent nonlinearities. This paper consists of two sections in addition to the introduction. In Sect. 2, we recall the definitions of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$, the Sobolev spaces $W^{1,p(\cdot)}(\Omega)$, as well as some of their properties. We also state, without proof, an existence result. In Sect. 3, we state and prove our blow-up result.

2 Preliminaries

In this section, we present some preliminary facts about Lebesgue and Sobolev spaces with variable-exponents (see [17,18,20]). Let $q : \Omega \to [1,\infty]$ be a measurable function, where Ω is a domain of $\mathbb{R}^n$ with $n \geq 1$. We define the Lebesgue space with a variable exponent $q(\cdot)$ by

$$L^{q(\cdot)}(\Omega) := \Big\{u : \Omega \to \mathbb{R};\ \text{measurable in } \Omega :\ \varrho_{q(\cdot)}(\lambda u) < \infty,\ \text{ for some } \lambda > 0\Big\},$$

where

$$\varrho_{q(\cdot)}(u) = \int_\Omega |u(x)|^{q(x)}dx.$$

Equipped with the following Luxembourg-type norm

$$\|u\|_{q(\cdot)} := \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{q(x)}dx \leq 1\Big\},$$

$L^{q(\cdot)}(\Omega)$ is a Banach space (see [25]).

We, next, define the variable-exponent Lebesgue Sobolev space $W^{1,q(\cdot)}(\Omega)$ as follows:

$$W^{1,q(\cdot)}(\Omega) = \Big\{u \in L^{q(\cdot)}(\Omega) \ \text{ such that } \nabla u \text{ exists and } |\nabla u| \in L^{q(\cdot)}(\Omega)\Big\}.$$

This space is a Banach space with respect to the norm $\|u\|_{W^{1,q(\cdot)}(\Omega)} = \|u\|_{q(\cdot)} + \|\nabla u\|_{q(\cdot)}$. Furthermore, we set $W_0^{1,q(\cdot)}(\Omega)$ to be the closure of $C_0^{\infty}(\Omega)$ in $W^{1,q(\cdot)}(\Omega)$. Here we note that the space $W_0^{1,q(\cdot)}(\Omega)$ is usually defined in a different way for the variable exponent case. However, both definitions are equivalent under (1.2) (see [25]). The dual of $W_0^{1,q(\cdot)}(\Omega)$ is defined as $W^{-1,q'(\cdot)}(\Omega)$, in the same way as the usual Sobolev spaces, where $\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = 1$.

Lemma 2.1 [25] *Let Ω be a bounded domain of $\mathbb{R}^n$ and $q(\cdot)$ satisfies (1.2), then*

$$\|u\|_{q(\cdot)} \le C\|\nabla u\|_{q(\cdot)}, \quad \textit{for all } \ u \in W_0^{1,q(\cdot)}(\Omega),$$

where the positive constant C depends on $q(\cdot)$ and Ω. In particular, the space $W_0^{1,q(\cdot)}(\Omega)$ has an equivalent norm given by $\|u\|_{W^{1,q(\cdot)}(\Omega)} = \|\nabla u\|_{q(\cdot)}$.

Lemma 2.2 [25] *If $q : \Omega \to [1,\infty)$ is a continuous function and*

$$2 \le q_1 \le q(x) \le q_2 < \frac{2n}{n-2}, \quad n \ge 3.$$

Then the embedding $H_0^1(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.

Lemma 2.3 [25] *If $q : \Omega \to [1,\infty)$ is a measurable function with $q_2 < \infty$, then $C_0^{\infty}(\Omega)$ is dense in $L^{q(\cdot)}(\Omega)$.*

Lemma 2.4 (Hölder's Inequality) [25] *Let $r, q, s \ge 1$ be measurable functions defined on Ω such that*

$$\frac{1}{s(y)} = \frac{1}{r(y)} + \frac{1}{q(y)}, \quad \textit{for a.e} \ \ y \in \Omega.$$

If $f \in L^{r(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then $fg \in L^{s(\cdot)}(\Omega)$ and

$$\|fg\|_{s(\cdot)} \le \ 2\ \|f\|_{r(\cdot)}\|g\|_{q(\cdot)}.$$

Lemma 2.5 [25] *If q is a measurable function on Ω satisfing (1.1), then*

$$\min\left\{\|u\|_{q(\cdot)}^{q_1}, \|u\|_{q(\cdot)}^{q_2}\right\} \le \varrho_{q(\cdot)}(u) \le \max\left\{\|u\|_{q(\cdot)}^{q_1}, \|u\|_{q(\cdot)}^{q_2}\right\},$$

for any $u \in L^{q(\cdot)}(\Omega)$.

We state the local existence theorem that can be established by combing arguments from [28] and [34].

Theorem 2.6 (Existence Theorem) *Assume that* (1.1) *and* (1.2) *hold. Let* $u_0 \in H_0^1(\Omega)$ *be given. Assume further that* g *satisfies*

$$g \in C^1\big(\mathbb{R}^+, \mathbb{R}^+\big), \quad g'(s) \leq 0, \quad 1 - \int_0^\infty g(s)ds := \ell > 0. \tag{2.1}$$

Then problem (P) *has a unique local solution*

$$u \in C\big([0, T], H_0^1(\Omega)\big), \quad u_t \in C\big([0, T], L^2(\Omega)\big) \cap L^2\big([0, T], H_0^1(\Omega)\big). \tag{2.2}$$

for some $T > 0$ *depending on* $\|u_0\|_{H_0^1(\Omega)}$. *Moreover, denoting by* T^* *the maximal existence time of solution, then we have*

$$\limsup_{t \to (T^*)^-} \|u(t)\|_{H_0^1(\Omega)} = +\infty$$

if $T^* < +\infty$.

3 Finite-time blow–up

In this section, we prove a finite-time blow-up result of solutions with initial datium at high level energy. First, we define the energy functional for problem (P) by

$$E(u(t)) = \frac{1}{2}(1 - G(t))\|\nabla u\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \int_\Omega \frac{|u|^{p(x)}}{p(x)}dx, \tag{3.1}$$

where

$$(g \circ \nabla u)(t) := \int_0^t g(t - \tau)\|u(t) - u(\tau)\|_2^2 d\tau,$$

and

$$G(t) := \int_0^t g(s)ds.$$

By multiplying the equation in (P) by u_t and performing routine calculations, one gets

$$\frac{d}{dt}E(u(t)) = -\|u_t\|_{H_0^1}^2 + \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|_2^2 \leq -\|u_t\|_{H_0^1}^2 \leq 0, \tag{3.2}$$

and, consequently, we have

$$E(u(0)) - E(u(t)) = \int_0^t \Big[\|u_t(\tau)\|_{H_0^1}^2 - \frac{1}{2}(g' \circ \nabla u)(\tau) + \frac{1}{2}g(\tau)\|\nabla u(\tau)\|_2^2\Big]d\tau. \tag{3.3}$$

By applying Levine's concavity method, we obtain the following blow-up result.

Theorem 3.1 *Assume that* (1.1), (1.2), *and* (2.1) *hold and*

$$\ell = 1 - \int_0^\infty g(s)ds > \frac{1}{(p_1 - 1)^2},$$

and

$$0 \leq \hat{E}(u_0) = \frac{1}{2}\|\nabla u_0\|_2^2 - \int_\Omega \frac{|u_0|^{p(x)}}{p(x)} dx < \frac{C}{2p_1}\|u_0\|^2_{H_0^1(\Omega)}, \tag{3.4}$$

where

$$C = \frac{(p_1 - 1)^2\ell - 1}{p_1}\Big(\frac{\lambda_1}{1 + \lambda_1}\Big) > 0$$

and λ_1 *is the principle eigenvalue of* $-\Delta$ *on* $H_0^1(\Omega)$. *Then the maximal existence time* T^* *of the solution* u *of problem* (P) *is finite with*

$$\lim_{t \to (T^*)^-} \|u(t)\|_{H_0^1(\Omega)} = +\infty$$

and

$$0 < T^* \leq \frac{8(p_1 - 1)\|u_0\|^2_{H_0^1(\Omega)}}{(p_1 - 2)^2\Big(C\|u_0\|^2_{H_0^1(\Omega)} - 2p_1\hat{E}(u_0)\Big)}.$$

Proof Let u be the solution of problem (P) with initial datuim $u_0 \in H_0^1(\Omega)$ satisfying (3.4). Multiplying (P) by u and integrating over Ω, we have

$$\int_\Omega uu_t dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega \nabla u \cdot \nabla u_t dx - \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t) dx d\tau$$
$$= \int_\Omega |u|^{p(x)} dx.$$

By using (3.1) and Young's inequality, we get for some $\rho > 0$,

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{1}{2}\|u\|^2_{H_0^1(\Omega)}\Big) &= -\|\nabla u\|_2^2 + \int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t) dx d\tau + \int_\Omega |u|^{p(x)} dx \\
&= -\|\nabla u\|_2^2 + \int_0^t g(t-\tau)\int_\Omega (\nabla u(\tau) - \nabla u(t))\cdot\nabla u(t) dx d\tau \\
&\quad + \int_0^t g(t-\tau)\int_\Omega \nabla u(t)\cdot\nabla u(t) dx d\tau + \int_\Omega |u|^{p(x)} dx. \\
&= -\big(1 - G(t)\big)\|\nabla u(t)\|_2^2 + \int_\Omega |u|^{p(x)} dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t g(t-\tau)\int_\Omega (\nabla u(\tau)-\nabla u(t))\cdot \nabla u(t)dxd\tau\\
&\geq -\big(1-G(t)\big)\|\nabla u(t)\|_2^2+\int_\Omega |u|^{p(x)}dx\\
&-\frac{1}{4\rho}\int_0^t g(t-\tau)\|\nabla u(\tau)-\nabla u(t)\|_2^2 d\tau\\
&-\rho\int_0^t g(t-\tau)\|\nabla u(t)\|_2^2 d\tau\\
&=-\big(1-G(t)+\rho G(t)\big)\|\nabla u(t)\|_2^2+\int_\Omega |u|^{p(x)}dx\\
&-\frac{1}{4\rho}\big(g\circ\nabla u\big)(t)\\
&=-\big(1-G(t)+\rho G(t)\big)\|\nabla u(t)\|_2^2+\int_\Omega |u|^{p(x)}dx\\
&-\frac{1}{4\rho}\Big[2E(u(t))-\big(1-G(t)\big)\|\nabla u\|_2^2+2\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx\Big]\\
&\geq \Big[\Big(\frac{1}{4\rho}-1\Big)+\Big(1-\rho-\frac{1}{4\rho}\Big)G(t)\Big]\|\nabla u\|_2^2\\
&+\Big(1-\frac{1}{2\rho p_1}\Big)\int_\Omega |u|^{p(x)}dx\\
&-\frac{1}{2\rho}E(u(t)).
\end{aligned}
$$

Taking $\rho=\frac{1}{2p_1}$ and recalling the assumption that $\ell>\frac{1}{(p_1-1)^2}$, we arrive at

$$
\begin{aligned}
\frac{d}{dt}\Big(\frac{1}{2}\|u\|^2_{H_0^1(\Omega)}\Big) &\geq \Big(\frac{p_1-2}{2}-\frac{(p_1-1)^2}{2p_1}G(t)\Big)\|\nabla u\|_2^2-p_1E(u(t))\\
&\geq \Big(\frac{p_1-2}{2}-\frac{(p_1-1)^2}{2p_1}(1-\ell)\Big)\|\nabla u\|_2^2-p_1E(u(t))\\
&\geq \frac{(p_1-1)^2\ell-1}{2p_1}\Big(\frac{\lambda_1}{\lambda_1+1}\Big)\|u(t)\|^2_{H_0^1(\Omega)}-p_1E(u(t)). \qquad (3.5)
\end{aligned}
$$

Since $\frac{d}{dt}E(u(t))\leq 0$, then for any $\alpha>0$, we obtain

$$
\begin{aligned}
\frac{d}{dt}\Big(\frac{1}{2}\|u\|^2_{H_0^1(\Omega)}-\alpha E(u(t))\Big) &\geq \frac{d}{dt}\Big(\frac{1}{2}\|u\|^2_{H_0^1(\Omega)}\Big)\\
&\geq \frac{(p_1-1)^2\ell-1}{2p_1}\Big(\frac{\lambda_1}{\lambda_1+1}\Big)\|u(t)\|^2_{H_0^1(\Omega)}-p_1E(u(t))\\
&=C\Big(\frac{1}{2}\|u\|^2_{H_0^1(\Omega)}-\frac{p_1}{C}E(u(t)),\Big) \qquad (3.6)
\end{aligned}
$$

where

$$C = \frac{(p_1 - 1)^2 \ell - 1}{p_1} \Big(\frac{\lambda_1}{\lambda_1 + 1} \Big) > 0.$$

Let

$$\alpha = \frac{p_1}{C} \quad \text{and} \quad H(t) = \Big(\frac{1}{2} \|u\|^2_{H^1_0(\Omega)} - \frac{p_1}{C} E(u(t)) \Big),$$

then

$$\frac{d}{dt} H(t) \geq \frac{d}{dt} \Big(\frac{1}{2} \|u(t)\|^2_{H^1_0(\Omega)} \Big) \geq C H(t).$$

Therefore, Gronwall's inequality and the assumption (3.4) lead to

$$\frac{d}{dt} H(t) \geq \frac{d}{dt} \Big(\frac{1}{2} \|u(t)\|^2_{H^1_0(\Omega)} \Big) \geq C H(t) \geq C e^{Ct} H(0) > 0,$$

for any $t \in [0, T^*)$. So $H(t)$ and $\|u(t)\|^2_{H^1_0(\Omega)}$ are both strictly increasing on $[0, T^*)$.

For any $T \in (0, T^*)$, we define the positive function $M(t)$ on $[0, T]$ by

$$M(t) = \int_0^t \|u(\tau)\|^2_{H^1_0(\Omega)} d\tau + (T - t) \|u_0\|^2_{H^1_0(\Omega)} + \gamma (t + \sigma)^2, \qquad (3.7)$$

where $\gamma, \sigma > 0$ are positive parameters. Through simple calculations, we easily get, for $t \in [0, T]$

$$\begin{aligned} M'(t) &= \|u(t)\|^2_{H^1_0(\Omega)} - \|u_0\|^2_{H^1_0(\Omega)} + 2\gamma (t + \sigma) \\ &= 2 \int_0^t \int_\Omega u(\tau) u_t(\tau) dx d\tau + 2 \int_0^t \int_\Omega \nabla u(\tau) \cdot \nabla u_t(\tau) dx d\tau + 2\gamma (t + \sigma) \end{aligned}$$

and

$$\begin{aligned} M''(t) &= 2 \int_\Omega u(t) u_t(t) dx + 2 \int_\Omega \nabla u(t) \cdot \nabla u_t(t) dx + 2\gamma \\ &= -2 \|\nabla u(t)\|_2^2 + 2 \int_0^t g(t - \tau) \int_\Omega \nabla u(\tau) \cdot \nabla u(t) dx d\tau \\ &\quad + 2 \int_\Omega |u|^{p(x)} dx + 2\gamma. \end{aligned}$$

By using Cauchy–Schwartz inequality and Young's inequality, we obtain, $\forall t \in [0, T]$

$$\eta(t) := \Big(\int_0^t \|u\|^2_{H^1_0(\Omega)} + \gamma (t + \sigma)^2 \Big) \Big(\int_0^t \|u_t\|^2_{H^1_0(\Omega)} + \gamma \Big)$$

$$
\begin{aligned}
&- \Big(\int_0^t \int_\Omega u u_t dx d\tau + \int_0^t \int_\Omega \nabla u \cdot \nabla u_t dx d\tau + \gamma (t+\sigma)^2 \Big) \\
&= \Big[\int_0^t \|\nabla u_t\|_2^2 d\tau \int_0^t \|\nabla u\|_2^2 d\tau - \Big(\int_0^t \int_\Omega \nabla u \cdot \nabla u_t dx d\tau \Big)^2 \Big] \\
&+ \Big[\int_0^t \|u\|_2^2 d\tau \int_0^t \|u_t\|_2^2 d\tau - \Big(\int_0^t \int_\Omega u_t u dx d\tau \Big)^2 \Big] \\
&+ \Big[\int_0^t \|\nabla u\|_2^2 d\tau \int_0^t \|u_t\|_2^2 d\tau + \int_0^t \|\nabla u_t\|_2^2 d\tau \int_0^t \|u\|_2^2 d\tau \\
&- 2\Big(\int_0^t \int_\Omega \nabla u_t \cdot \nabla u dx d\tau \Big)\Big(\int_0^t \int_\Omega u_t u dx d\tau \Big) \Big] \\
&+ \Big[\gamma \int_0^t \|\nabla u\|_2^2 d\tau + \gamma (t+\sigma)^2 \int_0^t \|\nabla u_t\|_2^2 d\tau \\
&- 2\gamma (t+\sigma) \int_0^t \int_\Omega \nabla u_t \cdot \nabla u dx d\tau \Big] \\
&+ \Big[\gamma \int_0^t \|u\|_2^2 d\tau + \gamma (t+\sigma)^2 \int_0^t \|u_t\|_2^2 d\tau \\
&- 2\gamma (t+\sigma) \int_0^t \int_\Omega u u_t dx d\tau \Big] \geq 0.
\end{aligned}
$$

So, for any arbitrary $\alpha > 0$, it follows that

$$
\begin{aligned}
MM'' - \frac{\alpha}{4}\big(M'\big)^2 &= MM'' - \alpha \Big(\int_0^t \int_\Omega u u_t dx d\tau + \int_0^t \int_\Omega \nabla u \cdot \nabla u_t dx d\tau \\
&\quad + \gamma (t+\sigma) \Big)^2 \\
&= MM'' + \alpha \Big[- \Big(\int_0^t \int_\Omega u u_t dx d\tau + \int_0^t \int_\Omega \nabla u \cdot \nabla u_t dx d\tau \\
&\quad + \gamma (t+\sigma) \Big)^2 \\
&\quad + \Big(\int_0^t \|u\|_{H_0^1(\Omega)}^2 d\tau + \gamma (t+\sigma)^2 \Big) \Big(\int_0^t \|u_t\|_{H_0^1(\Omega)}^2 d\tau + \gamma \Big) \\
&\quad - \Big(M(t) - (T-t)\|u_0\|_{H_0^1(\Omega)}^2 \Big) \Big(\int_0^t \|u_t\|_{H_0^1(\Omega)}^2 d\tau + \gamma \Big) \Big] \\
&= MM'' - \alpha M \Big(\int_0^t \|u_t\|_{H_0^1(\Omega)}^2 d\tau + \gamma \Big) + \alpha \eta(t) \\
&\quad + \alpha (T-t) \|u_0\|_{H_0^1(\Omega)}^2 \Big(\int_0^t \|u_t\|_{H_0^1(\Omega)}^2 d\tau + \gamma \Big) \\
&> M\psi(t), \forall t \in [0, T],
\end{aligned} \tag{3.8}
$$

where

$$
\begin{aligned}
\psi(t) &= M'' - \alpha\Big(\int_0^t \|u_t\|^2_{H_0^1(\Omega)} d\tau + \gamma\Big) \\
&= -\alpha \int_0^t \|u_t(\tau)\|^2_{H_0^1(\Omega)} d\tau - 2\|\nabla u(t)\|_2^2 + 2\int_0^t g(t-\tau)\int_\Omega \nabla u(\tau)\cdot\nabla u(t) dx d\tau \\
&\quad +2\int_\Omega |u(t)|^{p(x)} dx - (\alpha-2)\gamma.
\end{aligned}
$$

By recalling (3.1), (3.3) and using Young's inequality, we get, for some $\beta > 0$,

$$
\begin{aligned}
\psi(t) &= -\alpha \int_0^t \|u_t(\tau)\|^2_{H_0^1(\Omega)} d\tau - 2\|\nabla u(t)\|_2^2 + 2\int_0^t g(t-\tau)\int_\Omega \nabla u(t)\cdot\nabla u(t) dx d\tau \\
&\quad + 2\int_0^t g(t-\tau)\int_\Omega (\nabla u(\tau) - \nabla u(t))\cdot\nabla u(t) dx d\tau \\
&\quad + 2\int_\Omega |u(t)|^{p(x)} dx - (\alpha-2)\gamma \\
&\geq -\alpha \int_0^t \|u_t(\tau)\|^2_{H_0^1(\Omega)} d\tau - 2(1-G(t))\|\nabla u(t)\|_2^2 \\
&\quad + 2\int_\Omega |u(t)|^{p(x)} dx - (\alpha-2)\gamma \\
&\quad - 2\Big(\beta\int_0^t g(t-\tau)\|\nabla u(t)\|_2^2 d\tau + \frac{1}{4\beta}\int_0^t g(t-\tau)\|\nabla u(\tau) - \nabla u(t)\|_2^2 d\tau\Big) \\
&\geq -\alpha \int_0^t \|u_t(\tau)\|^2_{H_0^1(\Omega)} d\tau - 2(1-G(t))\|\nabla u(t)\|_2^2 + p_1(1-G(t))\|\nabla u(t)\|_2^2 \\
&\quad + p_1(g\circ\nabla u)(t) - 2p_1 E(u(t)) - 2\beta G(t)\|\nabla u(t)\|_2^2 \\
&\quad - \frac{1}{2\beta}(g\circ\nabla u)(t) - (\alpha-2)\gamma \\
&= -\alpha \int_0^t \|u_t(\tau)\|^2_{H_0^1(\Omega)} d\tau + \Big((p_1-2)(1-G(t)) - 2\beta G(t)\Big)\|\nabla u(t)\|_2^2 \\
&\quad + \Big(p_1 - \frac{1}{2\beta}\Big)(g\circ\nabla u)(t) - 2p_1 E(u(0)) \\
&\quad + 2p_1\int_0^t \|u_t(\tau)\|^2_{H_0^1(\Omega)} d\tau - (\alpha-2)\gamma \\
&\quad + 2p_1\int_0^t \Big(\frac{1}{2}g(\tau)\|\nabla u(\tau)\|_2^2 - \frac{1}{2}(g'\circ\nabla u)(\tau)\Big) d\tau \\
&= (2p_1-\alpha)\int_0^t \|u_t(\tau)\|^2_{H_0^1(\Omega)} d\tau + \Big((p_1-2)(1-G(t)) - 2\beta G(t)\Big)\|\nabla u(t)\|_2^2 \\
&\quad - (\alpha-2)\gamma \\
&\quad + \Big(p_1 - \frac{1}{2\beta}\Big)(g\circ\nabla u)(t) - 2p_1 E(u(0)) \\
&\quad + p_1\int_0^t \Big(g(\tau)\|\nabla u(\tau)\|_2^2 - (g'\circ\nabla u)(\tau)\Big) d\tau.
\end{aligned}
$$

By taking $\alpha = 2p_1$, and recalling (2.1), we get

$$\psi(t) \geq \Big((p_1-2)(1-G(t)) - 2\beta G(t)\Big)\|\nabla u(t)\|_2^2 + \Big(p_1 - \frac{1}{2\beta}\Big)(g \circ \nabla u)(t) - 2p_1 E(u(0)) - 2(p_1-1)\gamma, \ \forall t \in [0,T]. \quad (3.9)$$

Combining (3.8), (3.9) and letting $\beta = \frac{1}{2p_1}$, we have

$$MM'' - \frac{p_1}{2}(M')^2 > M\psi(t), \ \forall t \in [0,T]$$

with

$$\psi(t) \geq \Big((p_1-2)(1-G(t)) - \frac{1}{p_1}G(t)\Big)\|\nabla u(t)\|_2^2 - 2p_1 E(u(0)) - 2(p_1-1)\gamma.$$

From (3.5), (3.6) and considering the monotonicity of $\|u(t)\|^2_{H_0^1(\Omega)}$, we obtain

$$\begin{aligned}\psi(t) &\geq \Big((p_1-2)(1-G(t)) - \frac{1}{p_1}G(t)\Big)\|\nabla u(t)\|_2^2 - 2p_1 E(u(0)) - 2(p_1-1)\gamma \\ &\geq C\|u(t)\|^2_{H_0^1(\Omega)} - 2p_1 E(u(0)) - 2(p_1-1)\gamma \\ &\geq C\|u_0\|^2_{H_0^1(\Omega)} - 2p_1 E(u(0)) - 2(p_1-1)\gamma \\ &= 2CH(0) - 2(p_1-1)\gamma \geq 0, \ \forall t \in [0,T],\end{aligned}$$

where $\gamma \in \Big(0, \frac{CH(0)}{p_1-1}\Big)$. Therefore, by (3.8), we get

$$MM'' - \frac{p_1}{2}(M')^2 > 0, \ \forall t \in [0,T]. \quad (3.10)$$

We set $F(t) = M^{1-\frac{p_1}{2}}(t)$. Then, simple calculations yield

$$\begin{aligned}F'(t) &= \Big(1 - \frac{p_1}{2}\Big)M^{-\frac{p_1}{2}}(t)M'(t), \\ F''(t) &= \Big(1 - \frac{p_1}{2}\Big)M^{-\frac{p_1}{2}-1}(t)\Big(M(t)M''(t) - \frac{p_1}{2}(M'(t))^2\Big).\end{aligned}$$

Using (3.10) and since $M(t) > 0$, we have $F''(t) < 0$ for all $t \in [0,T]$, which means that $F(t)$ is strictly concave on $[0,T]$. Therefore, we have

$$0 < F(T) < F(0) + F'(0)T. \quad (3.11)$$

Noting that

$$F'(0) = \Big(1 - \frac{p_1}{2}\Big)M^{-\frac{p_1}{2}}(0)M'(0) = (2-p_1)M^{-\frac{p_1}{2}}(0)\gamma\sigma < 0,$$

then (3.11) yields

$$0 < T < \frac{F(0)}{-F'(0)} = \frac{2M(0)}{(p_1-2)M'(0)} = \frac{\|u_0\|^2_{H_0^1(\Omega)}}{(p_1-2)\gamma\sigma}T + \frac{\sigma}{p_1-2}.$$

Fixing $\gamma \in \left(0, \frac{CH(0)}{p_1-1}\right)$ and choosing $\sigma > \frac{\|u_0\|^2_{H_0^1(\Omega)}}{(p_1-2)\gamma}$, we have

$$0 < \frac{\|u_0\|^2_{H_0^1(\Omega)}}{(p_1-2)\gamma\sigma} < 1, \quad 0 < T < \frac{\gamma\sigma^2}{(p_1-2)\gamma\sigma - \|u_0\|^2_{H_0^1(\Omega)}}. \tag{3.12}$$

Let

$$T_\gamma(\sigma) = \frac{\gamma\sigma^2}{(p_1-2)\gamma\sigma - \|u_0\|^2_{H_0^1(\Omega)}}, \quad \sigma \in \left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(p_1-2)\gamma}, \infty\right).$$

One can easily verify that $T_\gamma(\sigma)$ takes its minimum at $\sigma = \frac{2\|u_0\|^2_{H_0^1(\Omega)}}{(p_1-2)\gamma}$ and

$$\begin{aligned} 0 < T &\le \inf_{\sigma \in \left(\frac{\|u_0\|^2_{H_0^1(\Omega)}}{(p_1-2)\gamma}, \infty\right)} T_\gamma(\sigma) = \frac{4\|u_0\|^2_{H_0^1(\Omega)}}{(p_1-2)^2\gamma} \\ &\le \frac{8(p_1-1)\|u_0\|^2_{H_0^1(\Omega)}}{(p_1-2)^2\left(C\|u_0\|^2_{H_0^1(\Omega)} - 2p_1\hat{J}(u_0)\right)}. \end{aligned}$$

Since $T \in (0, T^*)$ is arbitrary, we finally obtain that

$$0 < T^* \le \frac{8(p_1-1)\|u_0\|^2_{H_0^1(\Omega)}}{(p_1-2)^2\left(C\|u_0\|^2_{H_0^1(\Omega)} - 2p_1\hat{J}(u_0)\right)} < +\infty.$$

By Theorem 2.6, we have

$$\limsup_{t\to(T^*)^-} \|u(t)\|_{H_0^1(\Omega)} = +\infty.$$

By considering the monotonicity of $\|u(t)\|_{H_0^1(\Omega)}$, we get

$$\lim_{t\to(T^*)^-} \|u(t)\|_{H_0^1(\Omega)} = +\infty.$$

This completes the proof. □

Acknowledgements The authors thank University of Sharjah and Birzeit University for their support. This work was partially supported by MASEP Research Group in the Research Institute of Sciences and Engineering at University of Sharjah.

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