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General decay of a nonlinear damping porous-elastic system with past history

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Abstract
In this paper, we consider a one-dimensional porous-elastic system with past history and nonlinear damping term. We established the well-posedness using the semigroup theory and we showed that the dissipation given by this complementary controls guarantees the general stability for the case of equal speed of wave propagation.

Keywords Past history · Nonlinear damping · Porous-elastic system · General decay · Lyapunov method

Mathematics Subject Classification 35L70 · 35B40 · 93D20

1 Introduction

In the present work, we consider the following porous elastic system with past history and nonlinear damping term

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$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, x \in (0,1), t > 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^\infty g(s)\,\phi_{xx}(t-s)\,ds \\ +\chi(t)\,f(\phi_t) = 0, x \in (0,1), t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), x \in (0,1), \\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x), x \in (0,1), \\ u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, t > 0, \end{cases} \tag{1.1}$$

where the functions u and ϕ represent respectively the longitudinal displacement and the volume fraction. The parameter ρ designates the mass density and J equals to the product of the mass density by the equilibrated inertia. The term $\chi(t)\,f(\phi_t)$ is the nonlinear damping term where χ is a positive non-increasing differentiable function and f is specified in the preliminaries, the integral represents the infinite memory term and g is the relaxation function which satisfies

$$g'(t) \leq -\alpha(t)g(t),\ t \geq 0, \tag{1.2}$$

where α is a positive non-increasing differentiable function. The parameters μ, b, ξ, δ are positive constitutive constants such that

$$\mu\xi > b^2. \tag{1.3}$$

The original motivation of this problem was introduced by Goodman and Cowin [7] in 1972 when they proposed the idea of introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. This idea gives the relation between the elasticity theory and the porous media theory, for more details we cite the works of Cowin and Nunziato [5] from 1983 and Cowin [4] from 1985.

The system (1.1) was constructed by considering the following two basic evolution equations of the one-dimensional porous materials theory

$$\rho u_{tt} = T_x,\ J\phi_{tt} = H_x + D, \tag{1.4}$$

where T, H and D represent respectively the stress tensor, the equilibrated stress vector and the equilibrated body force. Consequently, to get the system (1.1) we take the constitutive equations T, H and D in this form

$$\begin{aligned} T &= \mu u_x + b\phi,\ H = \delta\phi_x - \int_0^\infty g(s)\phi_x(t-s)\,ds, \\ D &= -bu_x - \xi\phi - \chi(t)\,f(\phi_t), \end{aligned} \tag{1.5}$$

and by combining (1.5) and (1.4), we obtain (1.1).

In [16], Quintanilla gave a result concerning the slow decay for a one-dimensional porous dissipation elasticity, after Apalara [2] showed an exponential stability of the same system considered in [16] under the hypothesis (1.7).

In [1], Apalara established a general decay result for the energy of the same problem considered in [2,16] where he replaced the porous dissipation by a non linear damping term as follows

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \chi(t) f(\phi_t) = 0. \end{cases} \tag{1.6}$$

Note that, if we consider the problem (1.6) with viscoelasticity ($-\gamma u_{txx}$) that is acting only on the first equation and $\chi(t) = 0$, we come across the work of Magańa and Quintanilla [14] where they showed that viscoelasticity is not strong enough to make the solutions decay in an exponential way. If we consider the same problem (1.6) with $\chi(t) = 1$, we refer to the work of Boussouira [3] in the case of Timoshenko system where the authors established a general semi explicit decay of the system.

The purpose of this paper is to study the well posedness and the asymptotic behavior of the solution of (1.6) with past history term when this last is acting only on the second equation. We prove the general decay of this system for the case of equal speed of wave propagation in both equations of the system, that is

$$\frac{\mu}{\rho} = \frac{\delta}{J}. \tag{1.7}$$

Introducing a function $\chi(t)$ in the nonlinear damping term and $\alpha(t)$ which satisfies (1.2) makes our problem different from those considered so far in the literature.

The importance of the past history term and its influence on the asymptotic behavior of the solution appears in many works for different types of problems. To learn more about this term we refer the readers to [6,8–11,13,17] in the case of Timoshenko system, thermoelastic Laminated Beam and the transmission problem.

The paper is organized as follows. In Sect. 2, we introduced some transformations and assumptions needed to prove the main result. In Sect. 3, we used the semigroup method to prove the well-posedness of problem (1.1). In Sect. 4, we considered several lemmas that help us to construct the Lyapunov functional. In Sect. 5, we proved our general stability result.

2 Preliminaries

In this section we present the backgrounds mathematics needed later to prove our main result. We shall use the following hypothesis

(H1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g(0) > 0, \quad \delta - \int_0^\infty g(s)ds = l > 0, \quad \int_0^\infty g(s)ds = g_0. \tag{2.1}$$

(H2) $f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing C^0-function such that there exist the positive constants ν_1, ν_2, ϵ and a strictly increasing function $G \in C^1([0,\infty))$, with $G(0) = 0$. Moreover, G is linear or strictly convex C^2-function on $(0, \epsilon]$ such that

$$\begin{cases} s^2 + f^2(s) \le G^{-1}(sf(s)), \forall |s| \le \epsilon, \\ \nu_1 |s| \le |f(s)| \le \nu_2 |s|, \forall |s| \ge \epsilon, \end{cases} \tag{2.2}$$

which implies that $sf(s) > 0$, for all $s \neq 0$.

(H3) The function f satisfies the following property

$$\left|f(\psi_2)-f(\psi_1)\right| \le k_0(|\psi_1|^{\varrho}+|\psi_2|^{\varrho})|\psi_1-\psi_2|, \quad \psi_1,\psi_2\in\mathbb{R}, \tag{2.3}$$

where $k_0>0$, $\varrho>0$.

Note that the hypothesis (H2) was first introduced by Lasiecka and Tataru [12] in 1993.

Consider the following inequalities, that will help us in some estimations; we omit their proof.

Lemma 1 *The following inequalities hold,*

$$\int_0^1\left(\int_0^\infty g(s)\left(\phi(t)-\phi(t-s)\right)ds\right)^2 dx \le d_1\left(g\circ\phi_x\right)(t), \tag{2.4}$$

$$\int_0^1\left(\int_0^\infty g'(s)\left(\phi_x(t)-\phi_x(t-s)\right)ds\right)^2 dx \le -g(0)\left(g'\circ\phi_x\right)(t), \tag{2.5}$$

$$\int_0^1\left(\int_0^\infty g(s)\left(\phi_x(t)-\phi_x(t-s)\right)ds\right)^2 dx \le g_0\left(g\circ\phi_x\right)(t), \tag{2.6}$$

$$\int_0^1\left(\int_0^\infty g'(s)\left(\phi(t)-\phi(t-s)\right)ds\right)^2 dx \le -d_2(g'\circ\phi_x)(t), \tag{2.7}$$

where d_1, d_2 are positive constants and

$$(g\circ\nu)(t)=\int_0^1\int_0^\infty g(s)(\nu(x,t)-\nu(x,t-s))^2\,ds\,dx.$$

Here are some notations that will help us for the computation of energy

$$\eta^t(x,s)=\phi(x,t)-\phi(x,t-s),\ \ (x,t,s)\in(0,1)\times\mathbb{R}_+\times\mathbb{R}_+,$$

which was adopted in articles [6,15]. Here η^t is the relative history of ϕ and verifies

$$\begin{aligned}
&\eta_t^t+\eta_s^t-\phi_t=0,\ \ (x,t,s)\in(0,1)\times\mathbb{R}_+\times\mathbb{R}_+.\\
&\eta^t\,(0,s)=\eta^t\,(1,s)=0, t, s>0\\
&\eta^t(x,0)=0,\ \ \eta^0(x,s)=\eta_0(x,s), x\in(0,1)\,, t>0,
\end{aligned} \tag{2.8}$$

then, the system (1.1) is equivalent to

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, x \in (0,1), t > 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^\infty g(s)\,\phi_{xx}(t-s)\,ds \\ +\chi(t) f(\phi_t) = 0, x \in (0,1), t > 0, \\ \eta_t^t + \eta_s^t = \phi_t,\ s, t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), x \in (0,1), \\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x), x \in (0,1), \\ \eta^t(x,0) = 0,\ \eta^0(x,s) = \eta_0(x,s), x \in (0,1), s > 0 \\ u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = \eta^t(0,s) = \eta^t(1,s) = 0, s, t > 0. \end{cases} \tag{2.9}$$

In order to be able to use Poincaré's inequality for u , we introduce

$$\bar{u}(x,t) = u(x,t) - t\int_0^1 u_1(x)\,dx - \int_0^1 u_0(x)\,dx.$$

Using $(2.9)_1$, we have

$$\int_0^1 \bar{u}(x,t)\,dx = 0, \forall t \geq 0.$$

In what follows, we will work with $\bar{u}$ but, for convenience, we write u instead of $\bar{u}$.

3 Well-posedness

In this section, we give the existence and uniqueness result for problem (2.9) using the semigroup theory. First, we introduce the vector function

$$\Phi = \left(u, u_t, \phi, \phi_t, \eta^t\right)^T,$$

and the two new dependent variables

$$v = u_t, \psi = \phi_t.$$

Note that, the second equation of (2.9) can be rewritten as follows

$$J\phi_{tt} - l\phi_{xx} + bu_x + \xi\phi - \int_0^\infty g(s)\,\eta_{xx}^t(x,s)\,ds + \chi(t) f(\phi_t) = 0,$$

then the system (2.9) is equivalent to

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi + \mathcal{A}\Phi = \Gamma(\Phi), \\ \Phi(x,0) = \Phi_0(x) = (u_0, u_1, \phi_0, \phi_1, \eta_0)^T, \end{cases} \tag{3.1}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the linear operator defined by

$$\mathcal{A}\Phi = \begin{pmatrix} -v \\ -\frac{\mu}{\rho} u_{xx} - \frac{b}{\rho}\phi_x \\ -\psi \\ -\frac{l}{J}\phi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\phi - \frac{1}{J}\int_0^\infty g(s)\,\eta^t_{xx}(x,s)\,ds \\ \eta^t_s - \psi \end{pmatrix},$$

$$\Gamma(\Phi) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\chi(t)}{J} f(\psi) \\ 0 \end{pmatrix},$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H} = H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times L_g,$$

such that

$$\begin{aligned} H^1_*(0,1) &= H^1(0,1) \cap L^2_*(0,1), \\ L^2_*(0,1) &= \left\{ \varphi \in L^2(0,1) : \int_0^1 \varphi(x)\,dx = 0 \right\}, \\ L_g &= \left\{ \varphi : \mathbb{R}_+ \longrightarrow H^1_0(0,1), \int_0^1 \int_0^\infty g(s)\,\varphi_x^2 ds dx < \infty \right\}, \end{aligned}$$

the space L_g is endowed with the following inner product

$$\langle \varphi_1, \varphi_2 \rangle_{L_g} = \int_0^1 \int_0^\infty g(s)\,\varphi_{1x}(s)\,\varphi_{2x}(s)\,dsdx.$$

For any $\Phi = \left(u, v, \phi, \psi, \eta^t\right)^T \in \mathcal{H}$, $\tilde{\Phi} = \left(\tilde{u}, \tilde{v}, \tilde{\phi}, \tilde{\psi}, \tilde{\eta}^t\right)^T \in \mathcal{H}$, we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned} \left\langle \Phi, \tilde{\Phi} \right\rangle_{\mathcal{H}} &= \rho \int_0^1 v\tilde{v}dx + \mu \int_0^1 u_x \tilde{u}_x dx + J \int_0^1 \psi\tilde{\psi}dx + b \int_0^1 \left(u_x\tilde{\phi} + \tilde{u}_x\phi\right) dx \\ &\quad + \xi \int_0^1 \phi\tilde{\phi}dx + l \int_0^1 \phi_x\tilde{\phi}_x dx + \left\langle \eta^t, \tilde{\eta}^t \right\rangle_{L_g}. \end{aligned}$$

The domain of $\mathcal{A}$ is given by:

$$\begin{aligned} D(\mathcal{A}) = \Big\{ &\Phi \in \mathcal{H} \mid u \in H^2_*(0,1) \cap H^1_*(0,1);\ \phi \in H^2(0,1) \cap H^1_0(0,1); \\ &v \in H^1_*(0,1);\ \psi \in H^1_0(0,1);\ \eta^t \in L_g \Big\}, \end{aligned}$$

where

$$H_*^2(0,1)=\left\{\varphi\in H^2(0,1):\varphi_x(0)=\varphi_x(1)=0\right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$. Now, we can give the following existence result.

Remark 1 Note that, the inner product $\langle\Phi,\Phi\rangle_{\mathcal{H}}$ is positive. Indeed

$$\begin{aligned}\langle\Phi,\Phi\rangle_{\mathcal{H}}=\rho\int_0^1 v^2dx+\mu\int_0^1 u_x^2dx+J\int_0^1\psi^2dx+2b\int_0^1 u_x\phi dx\\+\xi\int_0^1\phi^2dx+l\int_0^1\phi_x^2dx+\int_0^1\int_0^\infty g(s)\left(\eta_x^t(x,s)\right)^2dsdx,\end{aligned}$$

and it can easily be verified that

$$\begin{aligned}\mu u_x^2+2bu_x\phi+\xi\phi^2=\frac{1}{2}\Bigg[\mu\left(u_x+\frac{b}{\mu}\phi\right)^2+\xi\left(\phi+\frac{b}{\xi}u_x\right)^2\\+\left(\mu-\frac{b^2}{\xi}\right)u_x^2+\left(\xi-\frac{b^2}{\mu}\right)\phi^2\Bigg],\end{aligned}$$

thanks to (1.3), we conclude that

$$\begin{aligned}\langle\Phi,\Phi\rangle_{\mathcal{H}}\geq\rho\int_0^1 v^2dx+J\int_0^1\psi^2dx+\xi_1\int_0^1\phi^2dx+\mu_1\int_0^1 u_x^2dx\\+l\int_0^1\phi_x^2dx+\int_0^1\int_0^\infty g(s)\left(\eta_x^t(x,s)\right)^2dsdx,\end{aligned}$$

where $\xi_1=\frac{1}{2}\left(\xi-\frac{b^2}{\mu}\right)>0$ and $\mu_1=\frac{1}{2}\left(\mu-\frac{b^2}{\xi}\right)>0$.

Theorem 1 *Let $\Phi_0\in\mathcal{H}$ and assume that $(H_1)-(H_3)$ hold. Then, there exists a unique solution $\Phi\in C(\mathbb{R}_+,\mathcal{H})$ of problem (3.1). Moreover, if $\Phi_0\in D(\mathcal{A})$ then*

$$\Phi\in C(\mathbb{R}_+,D(\mathcal{A}))\cap C^1(\mathbb{R}_+,\mathcal{H}).$$

Proof We use the semigroup approach. It is sufficient to show that $\mathcal{A}$ is a maximal monotone operator. First, we give the expression of $\langle\mathcal{A}\Phi,\Theta\rangle_{\mathcal{H}}$ for any $\Phi=\left(u,v,\phi,\psi,\eta^t\right)^T\in\mathcal{H}$, $\Theta=(\Theta_1,\Theta_2,\Theta_3,\Theta_4,\Theta_5)^T\in\mathcal{H}$. Then, by a simple calculation using the integration by parts, we have

$$
\begin{aligned}
\langle \mathcal{A}\Phi, \Theta\rangle_{\mathcal{H}} = \mu \int_0^1 (u_x \Theta_{2x} - v_x \Theta_{1x})\, dx + b \int_0^1 (\phi \Theta_{2x} - v_x \Theta_3)\, dx \\
+ l \int_0^1 (\psi \Theta_{3xx} - \phi_{xx} \Theta_4)\, dx + \xi \int_0^1 (\phi \Theta_4 - \psi \Theta_3)\, dx \\
+ b \int_0^1 (u_x \Theta_4 - \psi \Theta_{1x})\, dx - \int_0^1 \Theta_4 \int_0^\infty g(s)\, \eta^t_{xx}(x,s)\, ds dx \\
+ \int_0^1 \int_0^\infty g(s) \left(\eta^t_s(x,s) - \psi\right)_x \Theta_{5x} ds dx.
\end{aligned}
$$

Therefore, using the integration by parts and the boundary conditions, we can conclude that

$$
\begin{aligned}
\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} &= -\int_0^1 \psi \int_0^\infty g(s)\, \eta^t_{xx}(x,s)\, ds dx \\
&\quad + \int_0^1 \int_0^\infty g(s) \left(\eta^t_s(x,s) - \psi\right)_x \eta^t_x(x,s)\, ds dx \\
&= \int_0^1 \int_0^\infty g(s)\, \eta^t_{sx}(x,s)\, \eta^t_x(x,s)\, ds dx.
\end{aligned}
$$

Again, integrating by parts with respect to s and using the fact that $\eta^t_x(x,0) = 0$, we obtain (see also Lemma 1)

$$
\begin{aligned}
\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} &= \int_0^\infty g(s) \int_0^1 \eta^t_{sx}(x,s)\, \eta^t_x(x,s)\, dx ds \\
&= -\frac{1}{2} \int_0^1 \int_0^\infty g'(s) \left(\eta^t_x(x,s)\right)^2 ds dx \\
&= -\frac{1}{2} \left(g' \circ \phi_x\right)(t) \geq 0.
\end{aligned}
$$

Thus, $\mathcal{A}$ is monotone. Next, we prove that the operator $(I + \mathcal{A})$ is surjective. Given $K = (k_1, k_2, k_3, k_4, k_5)^T \in \mathcal{H}$, we prove that there exists a unique $\Phi \in D(\mathcal{A})$ such that

$$
(I + \mathcal{A})\, \Phi = K. \tag{3.2}
$$

That is,

$$\begin{cases} u - v = k_1 \in H^1_* (0, 1), \\ \rho v - \mu u_{xx} - b\phi_x = \rho k_2 \in L^2_* (0, 1), \\ \phi - \psi = k_3 \in H^1_0 (0, 1), \\ J\psi - l\phi_{xx} + bu_x + \xi\phi - \int_0^\infty g(s)\, \eta^t_{xx}(x, s)\, ds = Jk_4 \in L^2 (0, 1), \\ \eta^t + \eta^t_s - \psi = k_5 \in L_g. \end{cases} \tag{3.3}$$

Using (3.3)$_5$, we obtain

$$\eta^t = e^{-s} \int_0^s e^{\varsigma} (\psi + k_5 (\varsigma))\, d\varsigma. \tag{3.4}$$

Inserting $u - v = k_1$, $\phi - \psi = k_3$ and (3.4) in (3.3)$_2$ and (3.3)$_4$, we obtain

$$\begin{cases} \rho u - \mu u_{xx} - b\phi_x = h_1 \in L^2_* (0, 1), \\ (J + \xi)\phi + bu_x - \left(l + \int_0^\infty g(s) \left(1 - e^{-s}\right) ds\right) \phi_{xx} = h_2 \in L^2 (0, 1), \end{cases} \tag{3.5}$$

where

$$\begin{cases} h_1 = \rho k_2 + \rho k_1, \\ h_2 = Jk_4 + Jk_3 + \int_0^\infty g(s)\, e^{-s} \int_0^s e^{\varsigma} (k_5 - k_3)_{xx}\, d\varsigma ds. \end{cases}$$

To solve (3.5), we consider the following variational formulation

$$B((u, \phi), (u_1, \phi_1)) = \mathcal{G}(u_1, \phi_1), \tag{3.6}$$

where $B : \left[H^1_* (0, 1) \times H^1_0 (0, 1)\right]^2 \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} B((u, \phi), (u_1, \phi_1)) = {} & \rho \int_0^1 u u_1 dx + \mu \int_0^1 u_x u_{1x} dx + (J + \xi) \int_0^1 \phi\phi_1 dx \\ & + b \int_0^1 (u_x \phi_1 + \phi u_{1x})\, dx \\ & + \left(l + \int_0^\infty g(s) \left(1 - e^{-s}\right) ds\right) \int_0^1 \phi_x \phi_{1x} dx, \end{aligned} \tag{3.7}$$

and $\mathcal{G} : \left[H^1_* (0, 1) \times H^1_0 (0, 1)\right] \longrightarrow \mathbb{R}$ is the linear functional given by

$$\mathcal{G}(u_1, \phi_1) = \int_0^1 h_1 u_1 dx + \int_0^1 h_2 \phi_1 dx.$$

Now, for $V = H^1_* (0, 1) \times H^1_0 (0, 1)$ equipped with the norm

$$\|(u, \phi)\|^2_V = \|u\|^2_2 + \|\phi\|^2_2 + \|u_x\|^2_2 + \|\phi_x\|^2_2,$$

we have

$$B\left((u,\phi),(u,\phi)\right)=\rho\int_0^1 u^2dx+\mu\int_0^1 u_x^2dx+(J+\xi)\int_0^1\phi^2dx+2b\int_0^1\phi u_xdx$$
$$+\left(l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right)\int_0^1\phi_x^2dx.$$

On the other hand, we can write

$$\mu u_x^2+2bu_x\phi+(J+\xi)\phi^2=\frac{1}{2}\left[\mu\left(u_x+\frac{b}{\mu}\phi\right)^2+(J+\xi)\left(\phi+\frac{b}{J+\xi}u_x\right)^2\right.$$
$$\left.+\left(\mu-\frac{b^2}{J+\xi}\right)u_x^2+\left(J+\xi-\frac{b^2}{\mu}\right)\phi^2\right],$$

by using (1.3), we deduce that

$$B\left((u,\phi),(u,\phi)\right)\geq\rho\int_0^1 u^2dx+\kappa_1\int_0^1 u_x^2dx+\kappa_2\int_0^1\phi^2dx$$
$$+\left(l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right)\int_0^1\phi_x^2dx,$$

where

$$\kappa_1=\frac{1}{2}\left(\mu-\frac{b^2}{J+\xi}\right)>0,\kappa_2=\frac{1}{2}\left(J+\xi-\frac{b^2}{\mu}\right)>0,$$

then, for some $M_0>0$

$$\begin{aligned}\left|B\left((u,\phi),(u,\phi)\right)\right|&\geq\rho\|u\|_2^2+\kappa_1\|u_x\|_2^2+\kappa_2\|\phi\|_2^2\\&\quad+\left(l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right)\|\phi_x\|_2^2\\&\geq M_0\left(\|u\|_2^2+\|\phi\|_2^2+\|u_x\|_2^2+\|\phi_x\|_2^2\right)\\&=M_0\|(u,\phi)\|_V^2.\end{aligned}$$

Thus, B is coercive. On the other hand, by using Cauchy-Schwarz and Poincaré's inequalities, we obtain

$$\begin{aligned}&\left|B\left((u,\phi),(u_1,\phi_1)\right)\right|\\&\quad\leq\rho\|u\|_2\|u_1\|_2+\mu\|u_x\|_2\|u_{1x}\|_2+(J+\xi)\|\phi\|_2\|\phi_1\|_2\\&\qquad+b\left(\|u_x\|_2\|\phi_1\|_2+\|\phi\|_2\|u_{1x}\|_2\right)\\&\qquad+\left(l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right)\|\phi_x\|_2\|\phi_{1x}\|_2\end{aligned}$$

$$\leq \zeta_1 \left(\|u\|_2 + \|\phi\|_2 + \|u_x\|_2 + \|\phi_x\|_2\right) \times \left(\|u_1\|_2 + \|\phi_1\|_2 + \|u_{1x}\|_2 + \|\phi_{1x}\|_2\right)$$
$$\leq 16\zeta_1 \left\|(u,\phi)\right\|_V \left\|(u_1,\phi_1)\right\|_V .$$

Similarly, we can show that

$$\left|\mathcal{G}\left(u_1,\phi_1\right)\right| \leq \zeta_2 \left\|(u_1,\phi_1)\right\|_V .$$

Consequently, by the Lax-Milgram Lemma, the system (3.5) has a unique solution

$$(u,\phi) \in H_*^1(0,1) \times H_0^1(0,1),$$

satisfying

$$B\left((u,\phi),(u_1,\phi_1)\right) = \mathcal{G}\left(u_1,\phi_1\right), \forall \left(u_1,\phi_1\right) \in V.$$

The substitution of u and ϕ into $(3.3)_1$ and $(3.3)_3$ yields

$$(v,\psi) \in H_*^1(0,1) \times H_0^1(0,1).$$

Similarly, inserting ψ in (3.4) and bearing in mind $(3.3)_5$, we obtain $\eta^t \in L_g$. Moreover, if we take $u_1 = 0 \in H_*^1(0,1)$ in (3.6), we get

$$(J+\xi)\int_0^1 \phi\phi_1 dx + b\int_0^1 u_x\phi_1 dx$$
$$+\left(l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right)\int_0^1 \phi_x\phi_{1x}dx = \int_0^1 h_2\phi_1 dx.$$

Hence, we obtain

$$\left(l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right)\int_0^1 \phi_x\phi_{1x}dx$$
$$= \int_0^1 \left(h_2-(J+\xi)\phi-bu_x\right)\phi_1 dx, \forall\phi_1 \in H_0^1(0,1). \tag{3.8}$$

By noting that $h_2-(J+\xi)\phi-bu_x \in L^2(0,1)$, we obtain $\phi \in H^2(0,1) \cap H_0^1(0,1)$ and, consequently, (3.8) takes the form

$$\int_0^1 \left(-\left(l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right)\phi_{xx} - h_2 + (J+\xi)\phi + bu_x\right)\phi_1 dx$$
$$= 0, \forall\phi_1 \in H_0^1(0,1).$$

Therefore, we obtain

$$-\left(l+\int_0^\infty g(s)\left(1-e^{-s}\right)ds\right)\phi_{xx} + (J+\xi)\phi + bu_x = h_2.$$

This gives (3.5)$_2$. Similarly, if we take $\phi_1 = 0 \in H_0^1(0,1)$ in (3.6), we get

$$\mu u_{xx} = \rho u - b\phi_x - h_1 \text{ in } L_*^2(0,1),$$

using the fact that $u_x(0) = u_x(1) = 0$, then we conclude

$$u \in H_*^2(0,1) \cap H_*^1(0,1).$$

Hence, there exists a unique $\Phi \in D(\mathcal{A})$ such that (3.2) is satisfied. Therefore, $\mathcal{A}$ is a maximal monotone operator. Now, we prove that the operator Γ defined in (3.1) is locally Lipschitz in $\mathcal{H}$. Let $\Phi = \left(u, v, \phi, \psi, \eta^t\right)^T \in \mathcal{H}$ and $\Phi_1 = \left(u_1, v_1, \phi_1, \psi_1, \eta_1^t\right)^T \in \mathcal{H}$, then we have

$$\|\Gamma(\Phi) - \Gamma(\Phi_1)\|_{\mathcal{H}} \leq \gamma_1 \|f(\psi) - f(\psi_1)\|_{L^2}.$$

By using (2.3), Hölder and Poincaré inequalities, we can get

$$\|f(\psi) - f(\psi_1)\|_{L^2} \leq \gamma_1 k_0(\|\psi\|_{2\varrho}^{\varrho} + \|\psi_1\|_{2\varrho}^{\varrho}) \|\psi - \psi_1\| \leq \gamma_1 \|\psi_x - \psi_{1x}\|_{L^2},$$

which gives us

$$\|\Gamma(\Phi) - \Gamma(\Phi_1)\|_{\mathcal{H}} \leq \gamma_2 \|\Phi - \Phi_1\|_{\mathcal{H}}.$$

Then the operator Γ is locally Lipschitz in $\mathcal{H}$. Consequently, the well-posedness result follows from the Hille–Yosida theorem. □

4 Technical Lemmas

In this section, we use the multipliers method to construct the Lyapunov functional that must be equivalent to the energy of system (2.9). To achieve our goal we state and prove the following lemmas.

Lemma 2 *The energy functional E, defined by*

$$E(t) = \frac{1}{2}\int_0^1 \left\{\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + 2bu_x\phi + \xi\phi^2 + (\delta - g_0)\phi_x^2\right\} dx + \frac{1}{2}(g \circ \phi_x)(t), \tag{4.1}$$

satisfies

$$E'(t) = -\chi(t)\int_0^1 \phi_t f(\phi_t)\, dx + \frac{1}{2}\left(g' \circ \phi_x\right)(t) \leq 0. \tag{4.2}$$

Proof Multiplying the first equation of (2.9) by u_t, the second equation of (2.9) by ϕ_t, using (2.8), integrating over (0, 1) and summing them up, we obtain

$$\frac{d}{2dt}\int_0^1 \left\{\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + 2bu_x\phi + +\xi\phi^2 + (\delta - g_0)\,\phi_x^2\right\} dx$$
$$+\chi\,(t)\int_0^1 \phi_t f\,(\phi_t)\,dx - \int_0^1 \phi_t \int_0^\infty g(s)\eta_{xx}^t(x,s)dsdx = 0. \tag{4.3}$$

We estimate the last term of (4.3) as follows

$$\begin{aligned}
&-\int_0^1 \phi_t \int_0^\infty g(s)\eta_{xx}^t(x,s)dsdx\\
&= -\int_0^1 \left(\eta_t^t + \eta_s^t\right)\int_0^\infty g(s)\eta_{xx}^t(x,s)dsdx\\
&= -\int_0^\infty g(s)\int_0^1 \eta_t^t\eta_{xx}^t(x,s)dxds - \int_0^\infty g(s)\int_0^1 \eta_s^t\eta_{xx}^t(x,s)dxds,
\end{aligned}$$

integrating by parts, we have

$$-\int_0^1 \phi_t \int_0^\infty g(s)\eta_{xx}^t(x,s)dsdx = \frac{d}{2dt}\,(g\circ\phi_x)\,(t) - \frac{1}{2}\left(g^{'}\circ\phi_x\right)(t). \tag{4.4}$$

By substituting (4.4) in (4.3), bearing in mind (4.1), yields (4.2). □

Remark 2 The energy $E(t)$ defined by (4.1) is non-negative. In fact, by the same technique as in Remark (1), we can write

$$\mu u_x^2 + 2bu_x\phi + \xi\phi^2 > \mu_1 u_x^2 + \xi_1\phi^2.$$

Consequently,

$$E(t) > \frac{1}{2}\int_0^1 \left\{\rho u_t^2 + \mu_1 u_x^2 + J\phi_t^2 + \xi_1\phi^2 + (\delta - g_0)\,\phi_x^2\right\} dx + \frac{1}{2}\,(g\circ\phi_x)\,(t).$$

Lemma 3 *Let* (u, ϕ) *be the solution of* (2.9). *Then for any positive constant* ε_1 *the functional*

$$F_1\,(t) = J\int_0^1 \phi_t\phi dx + \frac{b\rho}{\mu}\int_0^1 \phi\int_0^x u_t\,(y)\,dydx,$$

satisfies

$$F_1'\,(t) \le -\frac{l}{2}\int_0^1 \phi_x^2 dx - \xi_1\int_0^1 \phi^2 dx + \left(J + \frac{b^2\rho^2}{4\mu^2\varepsilon_1}\right)\int_0^1 \phi_t^2 dx$$
$$+ k_1\int_0^1 f^2\,(\phi_t)\,dx + \varepsilon_1\int_0^1 u_t^2 dx + \frac{g_0}{2l}\,(g\circ\phi_x)\,(t), \tag{4.5}$$

where c, k_1 *are positive constants.*

Proof By differentiating $F_1(t)$ with respect to t, using (2.9) and integration by parts

$$\begin{aligned} F_1'(t) &= -\delta \int_0^1 \phi_x^2 dx - b\int_0^1 u_x \phi dx - \xi \int_0^1 \phi^2 dx \\ &\quad + \int_0^1 \phi_x \left(\int_0^\infty g(s)\phi_x(t-s)\,ds \right) dx - \chi(t)\int_0^1 \phi f(\phi_t)\,dx + J\int_0^1 \phi_t^2 dx \\ &\quad + \frac{b\rho}{\mu}\int_0^1 \phi_t \left(\int_0^x u_t(y)\,dy \right) dx + \frac{b\rho}{\mu}\int_0^1 \phi \frac{d}{dt}\left(\int_0^x u_t(y)\,dy \right) dx. \end{aligned} \tag{4.6}$$

Using Cauchy-Schwarz inequality, we have

$$\int_0^1 \left(\int_0^x u_t(y)\,dy \right)^2 dx \le \int_0^1 \left(\int_0^1 u_t dx \right)^2 dx \le \int_0^1 u_t^2 dx.$$

By a simple computation it's easy to prove that

$$\frac{d}{dt}\left(\int_0^x u_t(y)\,dy \right) = \frac{\mu}{\rho} u_x + \frac{b}{\rho}\phi.$$

Using Young's inequality and (2.6), for any $\varepsilon_1, \delta_1, \delta_2 > 0$, we obtain

$$\begin{aligned} &\int_0^1 \phi_x \left(\int_0^\infty g(s)\phi_x(t-s)\,ds \right) dx \\ &= -\int_0^1 \phi_x \left(\int_0^\infty g(s)(\phi_x(t) - \phi_x(t-s))\,ds \right) dx + g_0 \int_0^1 \phi_x^2 dx \\ &\le (\delta_1 + g_0)\int_0^1 \phi_x^2 dx + \frac{1}{4\delta_1}\int_0^1 \left(\int_0^\infty g(s)(\phi_x(t) - \phi_x(t-s))\,ds \right)^2 dx \\ &\le (\delta_1 + g_0)\int_0^1 \phi_x^2 dx + \frac{1}{4\delta_1} g_0 (g \circ \phi_x)(t), \end{aligned} \tag{4.7}$$

$$\frac{b\rho}{\mu}\int_0^1 \phi_t \left(\int_0^x u_t(y)\,dy \right) dx \le \frac{b^2\rho^2}{4\mu^2\varepsilon_1}\int_0^1 \phi_t^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx, \tag{4.8}$$

and

$$\begin{aligned} -\chi(t)\int_0^1 \phi f(\phi_t)\,dx &\le \chi(t)\delta_2 \int_0^1 \phi^2 dx + \frac{\chi(t)}{4\delta_2}\int_0^1 f^2(\phi_t)\,dx \\ &\le \chi(0)\delta_2 \int_0^1 \phi^2 dx + \frac{\chi(0)}{4\delta_2}\int_0^1 f^2(\phi_t)\,dx. \end{aligned} \tag{4.9}$$

By substituting (4.7)–(4.9) into (4.6) and letting $\delta_1 = \frac{l}{2}$, $\delta_2 = \frac{1}{\chi(0)}\xi_1$, we obtain (4.5). □

Lemma 4 *Let* (u, ϕ) *be the solution of* (2.9). *Then for any positive constants* $\varepsilon_2, \varepsilon_3$ *the functional*

$$F_2(t) = -J \int_0^1 \phi_t \int_0^\infty g(s)(\phi(t) - \phi(t-s))dsdx,$$

satisfies

$$\begin{aligned} F_2'(t) \le & -J\frac{g_0}{2}\int_0^1 \phi_t^2 dx + 3\varepsilon_2 \int_0^1 \phi_x^2 dx + \varepsilon_3 \int_0^1 u_x^2 dx \\ & + k_2 \int_0^1 f^2(\phi_t)dx - \frac{Jd_2}{2g_0}(g' \circ \phi_x)(t) + c\left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right)(g \circ \phi_x)(t), \end{aligned} \tag{4.10}$$

where k_2, c, d_2 *are positive constants.*

Proof First, we note that

$$\begin{aligned} & \frac{\partial}{\partial t}\left(\int_0^\infty g(s)(\phi(t) - \phi(t-s))ds\right) \\ & = \frac{\partial}{\partial t}\left(\int_{-\infty}^t g(t-s)(\phi(t) - \phi(s))ds\right) \\ & = \int_{-\infty}^t g'(t-s)(\phi(t) - \phi(s))ds + \int_{-\infty}^t g(t-s)\phi_t(t)ds \\ & = g_0\phi_t(t) + \int_0^\infty g'(s)(\phi(t) - \phi(t-s))ds. \end{aligned}$$

Then, by a simple differentiation of $F_2(t)$ and using (2.9), we have

$$\begin{aligned} F_2'(t) = & \ \delta \int_0^1 \phi_x \int_0^\infty g(s)(\phi_x(t) - \phi_x(t-s))dsdx - Jg_0 \int_0^1 \phi_t^2 dx \\ & -J \int_0^1 \phi_t \int_0^\infty g'(s)(\phi(t) - \phi(t-s))dsdx \\ & + b \int_0^1 u_x \int_0^\infty g(s)(\phi(t) - \phi(t-s))dsdx \\ & + \xi \int_0^1 \phi \int_0^\infty g(s)(\phi(t) - \phi(t-s))dsdx \\ & - \int_0^1 \int_0^\infty g(s)\,\phi_x(x, t-s)\,ds \int_0^\infty g(s)(\phi_x(t) - \phi_x(t-s))dsdx \\ & + \chi(t) \int_0^1 f(\phi_t) \int_0^\infty g(s)(\phi(t) - \phi(t-s))dsdx, \end{aligned} \tag{4.11}$$

and together with (2.9), by using Young's inequality, (2.4), (2.6) and (2.7) we have the following estimations:

$$\begin{aligned}
&\delta \int_0^1 \phi_x \int_0^\infty g(s)(\phi_x(t)-\phi_x(t-s))dsdx \\
&\quad \le \varepsilon_2 \int_0^1 \phi_x^2 dx + \frac{\delta^2}{4\varepsilon_2} \int_0^1 \left(\int_0^\infty g(s)(\phi_x(t)-\phi_x(t-s))ds \right)^2 dx \\
&\quad \le \varepsilon_2 \int_0^1 \phi_x^2 dx + \frac{\delta^2}{4\varepsilon_2} g_0 (g \circ \phi_x)(t), \qquad (4.12)\\
&\quad - J \int_0^1 \phi_t \int_0^\infty g'(s)(\phi(t)-\phi(t-s))dsdx \\
&\quad \le J\delta_3 \int_0^1 \phi_t^2 dx - \frac{J}{4\delta_3} \int_0^1 \left(\int_0^\infty g'(s)(\phi(t)-\phi(t-s))ds \right)^2 dx \\
&\quad \le J\delta_3 \int_0^1 \phi_t^2 dx - \frac{J d_2}{4\delta_3} (g' \circ \phi_x)(t), \qquad (4.13)\\
&b \int_0^1 u_x \int_0^\infty g(s)(\phi(t)-\phi(t-s))dsdx \\
&\quad \le \varepsilon_3 \int_0^1 u_x^2 dx + \frac{b^2}{4\varepsilon_3} \int_0^1 \left(\int_0^\infty g(s)(\phi(t)-\phi(t-s))ds \right)^2 dx \\
&\quad \le \varepsilon_3 \int_0^1 u_x^2 dx + \frac{b^2 d_1}{4\varepsilon_3} (g \circ \phi_x)(t), \qquad (4.14)\\
&\quad - \int_0^1 \int_0^\infty g(s)\phi_x(t-s)ds \int_0^\infty g(s)(\phi_x(t)-\phi_x(t-s))dsdx \\
&\quad = \int_0^1 \left(\int_0^\infty g(s)(\phi_x(t)-\phi_x(t-s))ds \right)^2 dx \\
&\quad - g_0 \int_0^1 \phi_x(t) \int_0^\infty g(s)(\phi_x(t)-\phi_x(t-s))dsdx \\
&\quad \le c\left(1+\frac{1}{\varepsilon_2}\right)(g \circ \phi_x)(t) + \varepsilon_2 \int_0^1 \phi_x^2 dx, \qquad (4.15)\\
&\chi(t) \int_0^1 f(\phi_t) \int_0^\infty g(s)(\phi(t)-\phi(t-s))dsdx \\
&\quad \le \chi(t)\varepsilon_2 \int_0^1 f^2(\phi_t)dx + \frac{\chi(t)d_1}{4\varepsilon_2}(g \circ \phi_x)(t) \\
&\quad \le \chi(0)\varepsilon_2 \int_0^1 f^2(\phi_t)dx + \frac{\chi(0)d_1}{4\varepsilon_2}(g \circ \phi_x)(t). \qquad (4.16)
\end{aligned}$$

By using Young's and Poincaré inequalities and (2.4),

$$\xi \int_0^1 \phi \int_0^\infty g(s)(\phi(t)-\phi(t-s))dsdx \le \varepsilon_2 \int_0^1 \phi_x^2 dx + \frac{\xi^2 d_1}{4\varepsilon_2}(g \circ \phi_x)(t). \qquad (4.17)$$

By substituting (4.12)–(4.17) into (4.11), we have

$$F_2'(t) \leq -J(g_0-\delta_3)\int_0^1 \phi_t^2 dx + 3\varepsilon_2 \int_0^1 \phi_x^2 dx + \varepsilon_3 \int_0^1 u_x^2 dx$$
$$+ k_2 \int_0^1 f^2(\phi_t)dx - \frac{Jd_2}{4\delta_3}(g' \circ \phi_x)(t) + c\left(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\right)(g\circ\phi_x)(t).$$

Finally, letting $\delta_3 = \dfrac{g_0}{2}$, we obtain (4.10). □

Lemma 5 *Let (u, ϕ) be the solution of (2.9). Then for any positive constant ε_4 the functional*

$$F_3(t) = J\int_0^1 \phi_x u_t dx + J\int_0^1 \phi_t u_x dx - \frac{\rho}{\mu}\int_0^1 u_t \left(\int_0^\infty g(s)\,\phi_x(t-s)\,ds\right)dx,$$

satisfies

$$F_3'(t) \leq -\frac{b}{2}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx + k_3 \int_0^1 f^2(\phi_t)\,dx$$
$$+ c\,(g\circ\phi_x)(t) + c\varepsilon_4 \int_0^1 u_t^2 dx - \frac{c}{\varepsilon_4}\left(g'\circ\phi_x\right)(t). \tag{4.18}$$

Proof First, we note that

$$\frac{d}{dt}\left(\int_0^\infty g(s)\phi_x(t-s)ds\right)$$
$$= \frac{d}{dt}\left(\int_{-\infty}^t g(t-s)\phi_x(s)ds\right)$$
$$= g(0)\phi_x(t) + \int_{-\infty}^t g'(t-s)\phi_x(s)ds$$
$$= \int_0^\infty g'(s)\left(\phi_x(t-s)-\phi_x(t)\right)ds + g(0)\phi_x(t) + \phi_x(t)\int_0^\infty g'(s)ds$$
$$= \int_0^\infty g'(s)\left(\phi_x(t-s)-\phi_x(t)\right)ds.$$

By differentiating $F_3(t)$, using (2.9) and then integrating by parts, we obtain

$$F_3'(t) = \frac{bJ}{\rho}\int_0^1 \phi_x^2 dx - b\int_0^1 u_x^2 dx - \xi\int_0^1 \phi u_x dx - \chi(t)\int_0^1 u_x f(\phi_t)\,dx$$
$$- \frac{b}{\mu}\int_0^1 \phi_x\left(\int_0^\infty g(s)\,\phi_x(t-s)\,ds\right)dx$$
$$- \frac{\rho}{\mu}\int_0^1 u_t\left(\int_0^\infty g'(s)\left(\phi_x(t-s)-\phi_x(t)\right)ds\right)dx \tag{4.19}$$

Using Young's inequality, (2.5) and (2.6)

$$
\begin{aligned}
&-\frac{\rho}{\mu}\int_0^1 u_t\left(\int_0^\infty g'(s)\left(\phi_x(t-s)-\phi_x(t)\right)ds\right)dx\\
&\quad\le \frac{\rho}{\mu}\varepsilon_4\int_0^1 u_t^2dx-\frac{\rho}{4\mu\varepsilon_4}g(0)\left(g'\circ\phi_x\right)(t),\\
&-\frac{b}{\mu}\int_0^1 \phi_x\left(\int_0^\infty g(s)\phi_x(t-s)\,ds\right)dx\\
&\quad=\frac{b}{\mu}\int_0^1 \phi_x\left(\int_0^\infty g(s)\left(\phi_x(t)-\phi_x(t-s)\right)ds\right)dx-\frac{b}{\mu}g_0\int_0^1\phi_x^2dx\\
&\quad\le\left(\delta_4-\frac{b}{\mu}g_0\right)\int_0^1\phi_x^2dx+\frac{b^2}{4\delta_4\mu^2}\left(g\circ\phi_x\right)(t). \qquad (4.20)
\end{aligned}
$$

By letting $\delta_4=\frac{b}{\mu}g_0$, we get

$$
-\frac{b}{\mu}\int_0^1 \phi_x\left(\int_0^\infty g(s)\phi_x(t-s)\,ds\right)dx\le c\left(g\circ\phi_x\right)(t), \qquad (4.21)
$$

$$
\begin{aligned}
-\chi(t)\int_0^1 u_x f(\phi_t)\,dx&\le\chi(t)\delta_5\int_0^1 u_x^2dx+\frac{1}{4\delta_5}\chi(t)\int_0^1 f^2(\phi_t)\,dx\\
&\le\chi(0)\delta_5\int_0^1 u_x^2dx+\frac{1}{4\delta_5}\chi(0)\int_0^1 f^2(\phi_t)\,dx. \qquad (4.22)
\end{aligned}
$$

By using Young's and Poincaré inequalities, we have

$$
-\xi\int_0^1\phi u_x dx\le\frac{\xi^2}{b}\int_0^1\phi_x^2dx+\frac{b}{4}\int_0^1 u_x^2dx. \qquad (4.23)
$$

Insert (4.20)–(4.23) in (4.19). So, the estimate (4.19) becomes

$$
\begin{aligned}
F_3'(t)\le&-\left(b-\chi(0)\delta_5-\frac{b}{4}\right)\int_0^1 u_x^2dx+c\int_0^1\phi_x^2dx+c\varepsilon_4\int_0^1 u_t^2dx\\
&+\frac{1}{4\delta_5}\chi(0)\int_0^1 f^2(\phi_t)\,dx+c\left(g\circ\phi_x\right)(t)-\frac{c}{\varepsilon_4}\left(g'\circ\phi_x\right)(t),
\end{aligned}
$$

and letting $\delta_5=\frac{b}{4\chi(0)}$, the proof is, hence, complete. □

Lemma 6 *Let* (u,ϕ) *be the solution of* (2.9). *Then the functional*

$$
F_4(t)=-\rho\int_0^1 u_t u\,dx,
$$

satisfies

$$F_4'(t) \leq \frac{3\mu}{2}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx - \rho\int_0^1 u_t^2 dx. \tag{4.24}$$

Proof

$$\begin{aligned} F_4'(t) &= -\rho\int_0^1 u_t^2 dx + \rho\int_0^1 u_{tt} u dx \\ &= -\rho\int_0^1 u_t^2 dx - \mu\int_0^1 u_{xx} u dx - b\int_0^1 \phi_x u dx \\ &= \mu\int_0^1 u_x^2 dx + b\int_0^1 \phi u_x dx - \rho\int_0^1 u_t^2 dx, \end{aligned}$$

by using Young's and Poincaré inequalities, we obtain easily the estimation (4.24). □

Now, we define the Lyapunov functional $L(t)$ by

$$L(t) = NE(t) + N_1 F_1(t) + N_2 F_2(t) + N_3 F_3(t) + F_4(t), \tag{4.25}$$

where $N,\ N_1,\ N_2,\ N_3$ are positive constants.

Lemma 7 *Let (u, ϕ) be the solution of (2.9). Then, there exist two positive constants b_1 and b_2 such that the Lyapunov functional (4.25) satisfies*

$$b_1 E(t) \leq L(t) \leq b_2 E(t), \forall t \geq 0, \tag{4.26}$$

and

$$L'(t) \leq -c_1 E(t) + c_2\left(g \circ \phi_x\right)(t) + c_3\int_0^1 \left(\phi_t^2 + f^2(\phi_t)\right) dx. \tag{4.27}$$

Proof From (4.25), we get

$$\begin{aligned} &|L(t) - NE(t)| \\ &\leq JN_1\int_0^1 |\phi_t \phi|\, dx + \frac{b\rho}{\mu} N_1 \int_0^1 \left|\phi\int_0^x u_t(y)\, dy\right| dx \\ &\quad + JN_2\int_0^1 \left|\phi_t\int_0^\infty g(s)(\phi(t) - \phi(t-s)) ds\right| dx + JN_3\int_0^1 |\phi_x u_t|\, dx \\ &\quad + JN_3\int_0^1 |\phi_t u_x|\, dx + \frac{\rho}{\mu} N_3\int_0^1 \left|u_t\left(\int_0^\infty g(s)\,\phi_x(t-s)\, ds\right)\right| dx \\ &\quad + \rho\int_0^1 |u_t u|\, dx. \end{aligned}$$

By using Young's, Cauchy-Schwarz, and Poincaré inequalities, we have

$$|L(t) - NE(t)| \leq c\int_0^1 \left(\phi_t^2+\phi_x^2 + u_x^2 + u_t^2 + \phi^2\right) dx + c\,(g \circ \phi_x)\,(t) \leq cE(t),$$

that is

$$(N - c)\,E(t) \leq L(t) \leq (N + c)\,E(t).$$

Now, by choosing N (depending on N_1, N_2 and N_3) sufficiently large we obtain (4.26).

By differentiating $L(t)$, we obtain

$$\begin{aligned} L'(t) \leq &-\left(\frac{Jg_0}{2}N_2 - N_1\left(J + \frac{b^2\rho}{\mu^2}N_1\right)\right)\int_0^1 \phi_t^2 dx \\ &-\left(N_1\frac{l}{4} - N_3c - c\right)\int_0^1 \phi_x^2 dx \\ &-\left(N_3\frac{b}{4} - \frac{3\mu}{2}\right)\int_0^1 u_x^2 dx - N_1\xi_1\int_0^1 \phi^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx \\ &+\left(\frac{N}{2} - \frac{4c^2}{\rho}N_3^2 - N_2\frac{Jd_2}{2g_0}\right)\left(g' \circ \phi_x\right)(t)+k\int_0^1 f^2\left(\phi_t\right) dx \\ &+\left(\frac{g_0}{2l}N_1 + c\left(N_2 + \frac{12N_2^2}{lN_1} + \frac{4N_2^2}{bN_3}\right) + cN_3\right)(g \circ \phi_x)(t). \end{aligned} \tag{4.28}$$

By setting

$$\varepsilon_1 = \frac{\rho}{4N_1}, \varepsilon_2 = \frac{lN_1}{12N_2}, \varepsilon_3 = \frac{bN_3}{4N_2}, \varepsilon_4 = \frac{\rho}{4cN_3},$$

and by using (4.28), we get

$$\begin{aligned} L'(t) \leq &-\left(\frac{Jg_0}{2}N_2 - N_1\left(J + \frac{b^2\rho}{\mu^2}N_1\right)\right)\int_0^1 \phi_t^2 dx \\ &-\left(N_1\frac{l}{4} - N_3c - c\right)\int_0^1 \phi_x^2 dx \\ &-\left(N_3\frac{b}{4} - \frac{3\mu}{2}\right)\int_0^1 u_x^2 dx - N_1\xi_1\int_0^1 \phi^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx \\ &+\left(\frac{N}{2} - \frac{4c^2}{\rho}N_3^2 - N_2\frac{Jd_2}{2g_0}\right)\left(g' \circ \phi_x\right)(t)+k\int_0^1 f^2\left(\phi_t\right) dx \\ &+\left(\frac{g_0}{2l}N_1 + c\left(N_2 + \frac{12N_2^2}{lN_1} + \frac{4N_2^2}{bN_3}\right) + cN_3\right)(g \circ \phi_x)(t). \end{aligned}$$

First, we choose N_3 large enough such that

$$N_3\frac{b}{4}-\frac{3\mu}{2}>0.$$

For fixed $N_3>0$, we take N_1 large enough such that

$$N_1\frac{l}{4}-N_3c-c>0.$$

Then we select $N_2>0$ large so that

$$\frac{Jg_0}{2}N_2-N_1\left(J+\frac{b^2\rho}{\mu^2}N_1\right)>0.$$

Finally, we choose N large enough such that

$$\frac{N}{2}-\frac{4c^2}{\rho}N_3^2-N_2\frac{Jd_2}{2g_0}>0.$$

Consequently, we obtain the estimation (4.27) of $L'(t)$. □

5 Stability result

In this section, we state and prove our stability result.

Theorem 2 *Assume that (H1)–(H3) hold. Let $h(t)=\alpha(t)\chi(t)$ be a positive non-increasing function, then, for any $\Phi_0\in D(\mathcal{A})$satisfying, for some $c_0\geq 0$,*

$$\int_0^1\phi_{0x}^2(x,s)dx\leq c_0,\ \forall s>0, \tag{5.1}$$

there exist the positive constants a_1, a_2, a_3, such that

$$E(t)\leq a_1G_0^{-1}\left(\frac{a_2+a_3\int_0^t h(s)\int_s^\infty g(\tau)\,d\tau ds}{\int_0^t h(s)ds}\right), \tag{5.2}$$

where

$$G_0(t)=tG'(\varepsilon_0 t),\forall\varepsilon_0\geq 0.$$

Proof Multiplying (4.27) by $h(t)$, we get

$$h(t)L'(t)\leq -c_1h(t)E(t)+c_2h(t)(g\circ\phi_x)(t)+c_3h(t)\int_0^1\left(\phi_t^2+f^2(\phi_t)\right)dx. \tag{5.3}$$

We distinguish two cases

1) *G* is linear on $[0, \epsilon]$. By using (5.3) and the hypothesis (H_3) , we have

$$\begin{aligned} h(t)L'(t) &\leq -c_1 h(t)E(t) + c_2 h(t) \left(g \circ \phi_x \right)(t) + c_3 h(t) \int_0^1 \phi_t f(\phi_t) dx \\ &\leq -c_1 h(t)E(t) + c_2 h(t) \left(g \circ \phi_x \right)(t) - c_3 \alpha(t) E'(t). \end{aligned} \tag{5.4}$$

To estimate $h(t) \left(g \circ \phi_x \right)(t)$ we use the following technique

$$\begin{aligned} & h(t) \int_0^1 \int_0^t g(s)(\phi_x(x,t) - \phi_x(x,t-s))^2 \, ds \, dx \\ & \leq \chi(t) \int_0^1 \int_0^t \alpha(s) g(s)(\phi_x(x,t) - \phi_x(x,t-s))^2 \, ds \, dx \\ & \leq -\chi(t) \int_0^1 \int_0^t g'(s)(\phi_x(x,t) - \phi_x(x,t-s))^2 \, ds \, dx \\ & \leq -\chi(t) \int_0^1 \int_0^\infty g'(s)(\phi_x(x,t) - \phi_x(x,t-s))^2 \, ds \, dx \\ & \leq -2\chi(t) \, E'(t). \end{aligned}$$

On the other hand, by using (5.1) and the fact that $E(t)$ is non-increasing, for $t, s \in R_+$,

$$\begin{aligned} \int_0^1 (\phi_x(x,t) - \phi_x(x,t-s))^2 dx &\leq 2 \int_0^1 \phi_x^2(x,t) dx + 2 \int_0^1 \phi_x^2(x,t-s) dx \\ &\leq 4 \sup_{s>0} \int_0^1 \phi_x^2(x,s) dx + 2 \sup_{\tau>0} \int_0^1 \phi_{0x}^2(x,\tau) dx \\ &\leq \frac{8E(0)}{(\delta - g_0)} + 2c_0, \end{aligned}$$

then, we obtain

$$\begin{aligned} & h(t) \int_0^1 \int_t^\infty g(s)(\phi_x(x,t) - \phi_x(x,t-s))^2 \, ds \, dx \\ & \leq \left(\frac{8E(0)}{(\delta - g_0)} + 2c_0 \right) h(t) \int_t^\infty g(s) ds. \end{aligned}$$

Therefore, we deduce that, for all $t \in R_+$,

$$h(t) \left(g \circ \phi_x \right)(t) \leq -2\chi(t) \, E'(t) + \left(\frac{8E(0)}{(\delta - g_0)} + 2c_0 \right) h(t) \int_t^\infty g(s) ds. \tag{5.5}$$

Inserting (5.5) in (5.4) and using the fact that $E'(t) \le 0$, we get

$$\begin{aligned} &h(t)L'(t) + \left(c_3\alpha(t) + 2c_2\chi\,(t) + \tau_1\right) E'(t) \\ &\quad \le h(t)L'(t) + \left(c_3\alpha(t) + 2c_2\chi\,(t) + \tau_1\right) E'(t) - \tau_1 E'(t) \\ &\quad \le -c_1 h(t)E(t) + \beta h(t)v\,(t)\,, \end{aligned} \tag{5.6}$$

where $\beta = c_2\left(\dfrac{8E(0)}{(\delta - g_0)} + 2c_0\right)$, $v\,(t) = \int_t^\infty g(s)ds$, $\tau_1 > 0$.
Since $\alpha'(t) \le 0$, $\chi'(t) \le 0$, $h'(t) \le 0$, then (5.6) is equivalent to

$$L_1'(t) \le -c_1 h(t)E(t) + \beta h(t)v\,(t)\,, \tag{5.7}$$

where

$$L_1(t) = h(t)L(t) + \left(c_3\alpha(t) + 2c_2\chi\,(t) + \tau_1\right) E(t) \sim E(t). \tag{5.8}$$

It's easy to verify that this last relation holds. Indeed, we have from (4.26)

$$b_1 E(t) \le L(t) \le b_2 E(t), \forall t \ge 0,$$

and because $h(t)$, $\alpha(t)$ and $\chi\,(t)$ are positive non-increasing functions, then for every $t \ge 0$, we deduce that exists $m_1, m_2 > 0$, satisfying

$$m_1 E(t) \le L_1(t) \le m_2 E(t),$$

with $m_1 = \tau_1$, $m_2 = b_2 h(0) + c_3\alpha(0) + 2c_2\chi\,(0) + \tau_1$. This proves that (5.8) is checked.
Because $E(t)$ is a non-increasing function, for all $T \in R_+$, by using (5.7), we have

$$E(T)\int_0^T h(t)dt \le \left(\frac{L_1(0)}{c_1} + \frac{\beta}{c_1}\int_0^T h(t)v\,(t)\,dt\right). \tag{5.9}$$

Using the fact that $G_0^{-1}\,(t)$ is linear, then (5.9) can be rewritten as follows

$$E(T) \le \lambda G_0^{-1}\left(\frac{\dfrac{L_1(0)}{c_1} + \dfrac{\beta}{c_1}\int_0^T h(t)v\,(t)\,dt}{\int_0^T h(t)dt}\right), \lambda > 0,$$

which gives (5.2) with $a_1 = \lambda$, $a_2 = \dfrac{L_1(0)}{c_1}$ and $a_3 = \dfrac{\beta}{c_1}$. The proof is complete.
2) *G* is nonlinear on $[0, \epsilon]$. In this case, we use the same estimation in the above case of $h(t)\,(g \circ \phi_x)\,(t)$ for the second term of (5.3). It's left to estimate the last term

of (5.3). For that we first choose $0 \leq \epsilon_1 \leq \epsilon$, such that $sf(s) \leq \min(\epsilon, G(\epsilon))$, $\forall |s| \leq \epsilon_1$ and by using (H_3), for $s \neq 0$, it follows that

$$\begin{cases} s^2 + f^2(s) \leq G^{-1}(sf(s)), \forall |s| \leq \epsilon_1, \\ \nu_1 |s| \leq |f(s)| \leq \nu_2 |s|, \forall |s| \geq \epsilon_1, \end{cases}$$

and we consider the following two sets

$$I_1 = \{x \in (0,1) : |\phi_t| \leq \epsilon_1\}, I_2 = \{x \in (0,1) : |\phi_t| > \epsilon_1\}.$$

Now, we define $I(t)$ by

$$I(t) = \int_{I_1} \phi_t f(\phi_t) dx,$$

using Jensen's inequality and the hypothesis (H_3), we have

$$c_3 h(t) \int_0^1 \left(\phi_t^2 + f^2(\phi_t)\right) dx \leq c_3' h(t) G^{-1}(I(t)) - c_3' \alpha(t) E'(t). \tag{5.10}$$

Inserting (5.10) in (5.3), we obtain

$$L_1'(t) \leq -c_1 h(t) E(t) + \beta h(t) v(t) + c_3' h(t) G^{-1}(I(t)), \tag{5.11}$$

where

$$L_1(t) = h(t) L(t) + \left(c_3 \alpha(t) + 2c_2 \chi(t) + \tau_1\right) E(t) \sim E(t), \tau_1 > 0,$$

we use the same technique as in the precedent case to show that $L_1(t)$ is equivalent to $E(t)$.

Now, for $\varepsilon_0 < \epsilon_1$ and using the fact that $E'(t) \leq 0$, $G' > 0$, $G'' > 0$ on $(0, \epsilon]$, we find that the functional $L_2(t)$, defined by

$$L_2(t) = G'(\varepsilon_0 E(t)) L_1(t) + \tau_2 E(t) \sim E(t), \tau_2 > 0,$$

satisfies

$$\begin{aligned} L_2'(t) &= E'(t) \left(\varepsilon_0 G''(\varepsilon_0 E(t)) L_1(t) + \tau_2\right) + L_1'(t) G'(\varepsilon_0 E(t)) \\ &\leq -c_1 h(t) G_0(E(t)) + +\beta G'(\varepsilon_0 E(t)) h(t) v(t) \\ &\quad + c_3' h(t) G'(\varepsilon_0 E(t)) G^{-1}(I(t)). \end{aligned} \tag{5.12}$$

Note that, the equivalence between $L_2(t)$ and $E(t)$ is due to the fact that $G'(\varepsilon_0 E(t))$ is positive non-increasing function and $L_1(t) \sim E(t)$. Indeed, we have for all $t \geq 0$,

$$m_1 E(t) \leq L_1(t) \leq m_2 E(t),$$

and

$$0 < G'\left(\varepsilon_0 E(t)\right) \leq G'\left(\varepsilon_0 E(0)\right),$$

then

$$\tau_2 E\left(t\right) \leq L_2\left(t\right) \leq \left(G'\left(\varepsilon_0 E(0)\right) m_2 + \tau_2\right) E(t).$$

Therefore, there exists $\sigma_1, \sigma_2 > 0$, satisfying

$$\sigma_1 E\left(t\right) \leq L_2\left(t\right) \leq \sigma_2 E(t),$$

with $\sigma_1 = \tau_2$, $\sigma_2 = G'\left(\varepsilon_0 E(0)\right) m_2 + \tau_2$.
To estimate the last term of (5.12), we apply the following general Young's inequality

$$AB \leq G^*\left(A\right) + G\left(B\right), \text{ if } A \in \left(0, G'\left(\epsilon\right)\right], B \in \left(0, \epsilon\right],$$

where

$$G^*\left(s\right) = s\left(G'\right)^{-1}\left(s\right) - G\left(\left(G'\right)^{-1}\left(s\right)\right), \text{ if } s \in \left(0, G'\left(\epsilon\right)\right],$$

we deduce that

$$c_3' h(t) G'\left(\varepsilon_0 E(t)\right) G^{-1}\left(I\left(t\right)\right) \leq c_3' \varepsilon_0 h(t) G_0\left(E(t)\right) - c_3' \alpha(t) E'(t). \tag{5.13}$$

Substituting (5.13) in (5.12) and letting $\varepsilon_0 = \dfrac{c_1}{2c_3'}$, we have

$$L_2'\left(t\right) + c_3' \alpha(t) E'(t) \leq -kh(t) G_0\left(E(t)\right) + \beta G'\left(\varepsilon_0 E(t)\right) h(t) v\left(t\right), \tag{5.14}$$

which can be rewritten as

$$\begin{aligned} &\left(L_2\left(t\right) + c_3' \alpha(t) E(t)\right)' - c_3' \alpha'(t) E(t) \\ &\quad \leq -kh(t) G_0\left(E(t)\right) + \beta G'\left(\varepsilon_0 E(t)\right) h(t) v\left(t\right), \end{aligned} \tag{5.15}$$

since $\alpha'(t) \leq 0$, then (5.15) is equivalent to

$$L_3'\left(t\right) \leq -kh(t) G_0\left(E(t)\right) + \beta G'\left(\varepsilon_0 E(t)\right) h(t) v\left(t\right). \tag{5.16}$$

where

$$L_3\left(t\right) = L_2\left(t\right) + c_3' \alpha(t) E(t) \sim E(t),$$

this last relation is checked from the fact that $\alpha(t)$ is a positive non-increasing function and $L_2\left(t\right) \sim E(t)$. Indeed, for every $t \geq 0$, we have already

$$\sigma_1 E\left(t\right) \leq L_2\left(t\right) \leq \sigma_2 E(t),$$

then

$$\sigma_1 E(t) \leq L_3(t) \leq \sigma_3 E(t),$$

with $\sigma_3 = \sigma_2 + c'_3 \alpha(0)$.
By using (5.16), because $G_0(E(t))$ and $G'(\varepsilon_0 E(t))$ are non-increasing functions, then for all $T \in R_+$, we have

$$kG_0(E(T)) \int_0^T h(t)dt \leq \left(L_3(0) + \beta G'(\varepsilon_0 E(0)) \int_0^T h(t)v(t)\,dt \right),$$

that can be rewritten as follows

$$E(T) \leq G_0^{-1} \left(\frac{\frac{L_3(0)}{k} + \frac{\beta G'(\varepsilon_0 E(0))}{k} \int_0^T h(t)v(t)\,dt}{\int_0^T h(t)dt} \right),$$

which gives (5.2) with $a_1 = 1$, $a_2 = \dfrac{L_3(0)}{k}$ and $a_3 = \dfrac{\beta G'(\varepsilon_0 E(0))}{k}$. The proof is complete. □

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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