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Effective adjunction theory

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Abstract Here we investigate the property of effectivity for adjoint divisors. Among others, we prove the following results: A projective variety X with at most canonical singularities is uniruled if and only if for each very ample Cartier divisor H on X we have $H^0(X, m_0K_X + H) = 0$ for some $m_0 = m_0(H) > 0$. Let X be a projective 4-fold, L an ample divisor and t an integer with $t \geq 3$. If $K_X + tL$ is pseudo-effective, then $H^0(X, K_X + tL) \neq 0$.

Keywords Termination of adjunction · Uniruledness · Quasi polarized pair · Minimal model program · Canonical singularities

Mathematics Subject Classification 14E30 · 14J40 · 14J35 · 14N30

1 Introduction

Let X be a normal projective variety over the complex field $\mathbb{C}$; let K_X be its canonical divisor. We assume that X has at most canonical singularities.

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In the paper we fix a suitable Cartier divisor H on X and we discuss when the effectivity or non-effectivity of some adjoint divisors $aK_X + bH$ determines the geometry of X.

In the first part we consider the notion of *Termination of Adjunction*. This turns out to be rather delicate, since in the literature there are different meanings for such a property. The following are four possibilities, where m_0 and m are natural numbers.

(A) For every (for some) big Cartier divisor H there exists $m_0 = m_0(H) > 0$ such that $mK_X + H \notin \overline{Eff(X)}$ (i.e. it is not pseudo-effective) for $m \geq m_0$.
(B) For every big Cartier divisor H we have $H^0(X, m_0K_X + H) = 0$ for some $m_0 = m_0(H) > 0$.
(C) For every very ample Cartier divisor H we have $H^0(X, m_0K_X + H) = 0$ for some $m_0 = m_0(H) > 0$.
(D) For some (for every) big Cartier divisor H_0 we have $H^0(X, m_0K_X + kH_0) = 0$ for every $k > 0$ and some $m_0 = m_0(k) > 0$.

It is clear that (A) $\Longrightarrow$ (B) $\Longrightarrow$ (C) $\Longrightarrow$ (D).

We prove that these four definitions are equivalent and moreover that *Adjunction Terminates in the above sense if and only if* X *is uniruled* (see Theorem 3, Corollaries 1 and 2).

The results follow by some characterizations of pseudo-effective Cartier divisor (see Theorem 2), which are direct consequences of a fundamental result of Siu ([21]). The connection with uniruledness follows in turn from the fact that a projective variety X with canonical singularities is uniruled if and only if K_X is not pseudo-effective (see [3], Corollary 0.3, or [5], Corollary 1.3.3).

A characterization of rationally connected manifolds along the same lines has been given in [6].

The examples described in [14], Theorem 39, show that, for varieties with singularities worst then canonical, uniruledness is not connected to Termination of Adjunction.

We consider also the following more general definition.

(C') Let H be an effective Cartier divisor on X. We say that *Adjunction Terminates in the classical sense* for H if there exists an integer $m_0 \geq 1$ such that

$$H^0(X, H + mK_X) = 0$$

for every integer $m \geq m_0$.

We conjecture that such a definition is actually equivalent to the previous ones; a partial result in this direction is provided by Proposition 2. In dimension two, Castelnuovo and Enriques indeed proved that Condition (C') implies that X is uniruled (see [7] and also [19]).

In the second part of the paper we assume that X is a projective variety of dimension n with at most terminal $\mathbb{Q}$-factorial singularities. We take a nef and and big Cartier divisor L on X and we call (X, L) a quasi polarized pair.

The following is a straightforward consequence of Theorem D in [5], see Remark 5 at the beginning of Sect. 5.

Proposition 1 *Let* (X, L) *be a quasi polarized pair and* $t > 0$*. If* $K_X + tL \in \overline{Eff(X)}$*, then there exists* $N \in \mathbb{N}$ *such that* $H^0(X, N(K_X + tL)) \neq 0$.

Note that for $t = 0$ the statement of the Proposition would amount to Abundance Conjecture, together with MMP.

The next Conjecture is an effective version of the above Proposition.

Conjecture 1 *Let* (X, L) *be a quasi polarized pair and* $t > 0$*. If* $K_X + tL \in \overline{Eff(X)}$*, then* $H^0(X, K_X + tL) \neq 0$.

The case $t = 1$ is a version of the so-called Ambro–Ionescu–Kawamata conjecture, which is true for $n \leq 3$ (see Theorem 1.5 in [13]), while for $t = n - 1$ we recover a conjecture by Beltrametti and Sommese (see [4], Conjecture 7.2.7). Note that if Conjecture 1 holds for $t = 1$ then it holds also for every $t > 0$.

In the paper we consider the following conjecture.

Conjecture 2 *Let* (X, L) *be a quasi polarized pair and* $s > 0$*. Then* $H^0(X, K_X + tL) = 0$ *for every integer* t *with* $1 \leq t \leq s$ *if and only if* $K_X + sL$ *is not pseudo-effective.*

Since L is big, in particular pseudo-effective, then the *if* part is obvious. Note that Conjecture 2 for $s = 1$ implies Conjecture 1.

We prove that **Conjecture 2 is true for** $s = n$ (see Proposition 4); we actually show that this case happens if and only if the pair (X, L) is birationally equivalent (via a 0-reduction, see the definition in the next section) to the pair $(\mathbb{P}^n, \mathcal{O}(1))$.

For $s = n - 1$ **the conjecture was essentially proved by Höring**, see [13], Theorem 1.2. We prove a slightly more explicit version of his result (see Proposition 7), namely, we show that this case happens if and only if the pair (X, L) is birationally equivalent to a finite list of pairs.

Finally, we focus on the case $n = 4$ (see Theorem 8 and Proposition 6) and we generalize previous work by Fukuma ([11], Theorem 3.1).

2 Notation and preliminaries

Let X be a normal complex projective variety of dimension n. We adopt [15] and [16] as the standard references for our set-up. In particular, we denote by $Div(X)$ the group of all Cartier divisors on X and by $Num(X)$ the subgroup of numerically trivial divisors. The quotient group $N^1(X) = Div(X)/Num(X)$ is the Neron-Severi group of X.

In the vector space $N^1(X)_{\mathbb{R}} := N^1(X) \otimes \mathbb{R}$, whose dimension is $\rho(X) := rkN^1(X)$, we consider some convex cones.

(a) $Amp(X) \subset N^1(X)_{\mathbb{R}}$ the convex cone of all *ample* $\mathbb{R}$-divisor classes; it is an open convex cone.
(b) $Big(X) \subset N^1(X)_{\mathbb{R}}$ the convex cone of all *big* $\mathbb{R}$-divisor classes; it is an open convex cone.
(e) $Eff(X) \subset N^1(X)_{\mathbb{R}}$ the convex cone spanned by the classes of all effective $\mathbb{R}$-divisors.
(n) $Nef(X) = \overline{Amp(X)} \subset N^1(X)_{\mathbb{R}}$ the closed convex cone of all *nef* $\mathbb{R}$-divisor classes.

(p) $\overline{Eff(X)} = \overline{Big(X)} \subset N^1(X)_{\mathbb{R}}$ the closed convex cone of all *pseudo-effective* $\mathbb{R}$-divisor classes.

The above definitions actually lean on some fundamental results like the openess of the ample and big cones, the facts that $int\{\overline{Eff(X)}\} = Big(X)$ and $Nef(X) = \overline{Amp(X)}$; for more details see [16].

Note that $Amp(X) \subset Nef(X) \cap Big(X)$ and that there are no inclusions between $Nef(X)$ and $Big(X)$.

Note also that if $\pi : X' \to X$ is a birational morphism and D is a Cartier divisor on X then D is big (resp. pseudo-effective) if and only if π^*D is big (resp. pseudo-effective).

We consider projective varieties with singularities of special type, as in the Minimal Model Program. For reader convenience we recall their definition (see [15], Definition 2.11 and Definition 2.12).

Definition 1 Let X be a normal projective variety. We say that X has *canonical* (respectively *terminal*) singularities if

(i) K_X is $\mathbb{Q}$-Cartier, and
(ii) $\nu_*\mathcal{O}_{\tilde{X}}(mK_{\tilde{X}}) = \mathcal{O}_X(mK_X)$ for one (or for any) resolution of the singularities $\nu : \tilde{X} \to X$

(respectively

ii) $\nu_*\mathcal{O}_{\tilde{X}}(mK_{\tilde{X}} - E) = \mathcal{O}_X(mK_X)$ for one (or for any) resolution of the singularities $\nu : \tilde{X} \to X$, where $E \subset \tilde{X}$ is the reduced exceptional divisor).

In the category of projective varieties with canonical singularities the pseudo-effectivity of the canonical bundle is a birational invariant, as noticed by Mori in [18], (11.4.1). He actually conjectured the following beautiful result ([18], (11.4.2) and (11.5)), which was proved in [3], Corollary 0.3 and in [5], Corollary 1.3.3.

Theorem 1 *Let X be a projective variety with at most canonical singularities. Then X is uniruled if and only if K_X is not pseudo-effective.*

As for the invariance of the global sections of adjoint bundles (or of pluri-canonical bundles if L is trivial) we have the following.

Lemma 1 *Let $\pi : Y \to X$ be a birational morphism between projective varieties with at most canonical singularities, let L be a Cartier divisor on X and let $a, b \in \mathbb{N}$. Then*

$$H^0(X, aK_X + bL) = H^0(Y, aK_Y + b\pi^*(L)).$$

Proof Since Y and X have canonical singularities we have $\pi_* aK_Y = aK_X$. This is straightforward from the definition of canonical singularities and by taking a resolution of Y, $\nu : Y' \to Y$, and $\pi \circ \nu : Y' \to X$ as a resolution of X.

Since L is Cartier, by projection formula it follows

$$\pi_*(aK_Y + b\pi^*(L)) = \pi_*(aK_Y + \pi^*(bL)) = \pi_*(aK_Y) + bL = aK_X + bL;$$

by taking global sections we obtain our statement. □

3 Termination of adjunction

Much of this section is based on the following Lemma, which was proved in the analytic setting by Siu (see [21], Proposition 1). For reader convenience we provide an algebraic proof relying on [17] (see also [20], Chapter V, Corollary 1.4).

Lemma 2 *Let X be a smooth projective variety of dimension n and let H be a very ample divisor on X. If $G := (n+1)H + K_X$, then for every pseudo-effective divisor F on X we have $H^0(X, F+G) \neq 0$.*

Proof Since F is pseudo-effective we have that $F + H$ is big, hence there exists a positive integer $m > 0$ such that $m(F+H) \sim A + E$ with A ample and E effective (see for instance [16], Corollary 2.2.7). Let $D := \frac{1}{m}E$ and $L := F + H$, so that $L - D = \frac{1}{m}A$ is big and nef; apply [17], Proposition 9.4.23, to get $H^0(X, K_X + L + kH + \mathcal{I}(D)) \neq 0$. Since the multiplier ideal $\mathcal{I}(D)$ is an ideal of $\mathcal{O}_X$, it follows that $H^0(X, K_X + L + kH) \neq 0$ for every $k \geq n$, i.e. $H^0(X, K_X + F + (k+1)H) \neq 0$ as soon as $k + 1 \geq n + 1$. □

The following characterization of pseudo-effective divisors is probably well-known to the specialists; however, we did not find it explicitly in the literature.

Theorem 2 *Let X be a smooth projective variety and let F be a divisor on X. The following statements, where m and N denote natural numbers, are equivalent:*

(i) *$F \in \overline{Eff(X)}$ (i.e it is pseudo-effective).*
(ii) *There is a big divisor G such that $H^0(X, N(mF+G)) \neq 0$ for every $m > 0$ and for some $N > 0$.*
(iii) *There is a big divisor G such that $H^0(X, mF+G) \neq 0$ for all $m > 0$.*
(iv) *There is a very ample divisor G such that $H^0(X, mF+G) \neq 0$ for all $m > 0$.*
(v) *For every big divisor H we have $H^0(X, mF+kH) \neq 0$ for all $m > 0$ and all $k \geq k_0(H)$.*

Proof First of all note that the implications (v) $\Longrightarrow$ (iv), (iv) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (ii) are obvious. Moreover (ii) $\Longrightarrow$ (i) follows from $F \equiv \lim_{m\to+\infty} \frac{mF+G}{m}$.

The difficult part is to prove (i) $\Longrightarrow$ (v); for this we use Lemma 2 together with Kodaira's Lemma (see for instance [16], Proposition 2.2.6). Namely, let G be the divisor of Lemma 2; then $H^0(X, G) \neq 0$ (just take $F = \mathcal{O}_X$). If H is a big divisor on X, then by Kodaira's Lemma $H^0(X, kH - G) \neq 0$ for every $k \geq k_0(H)$. Hence

$$\begin{aligned}\dim H^0(X, mF + kH) &= \dim H^0(X, mF + k_0H - G + G + (k-k_0)H) \geq \\ &\geq \dim H^0(X, mF + (k-k_0)H + G) > 0,\end{aligned}$$

where the last inequality follows from Lemma 2 by taking as a pseudo-effective divisor $mF + (k-k_0)H$. □

Remark 1 Note that (i) $\Longrightarrow$ (iii) is just Lemma 2, while (i) $\Longrightarrow$ (ii) follows easily from $int\{\overline{Eff(X)}\} = Big(X)$; this last fact was first noticed by Mori in [18], (11.3) on p. 318. Indeed, let $G \in Big(X)$ and $F \in \overline{Eff(X)}$; then the set $[G, F) := \{G + mF : m \in \mathbb{R}^+\}$ is contained in $int\{\overline{Eff(X)}\} = Big(X)$.

The next Theorem proves the equivalence of the different definitions of *Termination of Adjunction* stated in the Introduction.

Theorem 3 *Let X be a projective variety with at most canonical singularities. The following statements, where m and m_0 denote natural numbers, are equivalent:*

(i) *X is uniruled (i.e. K_X is not pseudo-effective).*
(ii) *For every big Cartier divisor H there exists $m_0 = m_0(H) > 0$ such that $mK_X + H \notin \overline{Eff(X)}$ for $m \geq m_0$.*
(iii) *For every big Cartier divisor H we have $H^0(X, m_0K_X + H) = 0$ for some $m_0 = m_0(H) > 0$.*
(iv) *For every very ample Cartier divisor H we have $H^0(X, m_0K_X + H) = 0$ for some $m_0 = m_0(H) > 0$.*
(v) *For some big Cartier divisor H_0 we have $H^0(X, m_0K_X + kH_0) = 0$ for every $k > 0$ and some $m_0 = m_0(k) > 0$.*

Proof (i) $\Longrightarrow$ (ii) is implied by the properties of the cone described in Sect. 2; indeed, it follows by contradiction from $K_X \equiv \lim_{m\to+\infty} \frac{mK_X+H}{m}$. (ii) $\Longrightarrow$ (iii), (iii) $\Longrightarrow$ (iv) and (iv) $\Longrightarrow$ (v) are straightforward. (v) $\Longrightarrow$ (i) requires a resolution of the singularities $\nu : \tilde{X} \to X$. Assume by contradiction that X is not uniruled. Therefore also $\tilde{X}$ is not uniruled and $K_{\tilde{X}}$ is pseudo-effective. If H is any big Cartier divisor on X, then $\tilde{H} = \nu^*(H)$ is big and by [16], Corollary 2.2.7, we have $l\tilde{H} = A + N$ with A ample and N effective for some $l > 0$. It follows that $hl\tilde{H} = hA + hN$ with hA very ample for some $h > 0$. Hence, by Lemma 1, for every $m_0 > 0$ we have $\dim H^0(X, m_0K_X + (n+1)hlH) = \dim H^0(\tilde{X}, m_0K_{\tilde{X}} + (n+1)hl\tilde{H}) = \dim H^0(\tilde{X}, (m_0-1)K_{\tilde{X}} + (K_{\tilde{X}} + (n+1)hA) + (n+1)hN) \geq \dim H^0(\tilde{X}, (m_0 - 1)K_{\tilde{X}} + (K_{\tilde{X}} + (n+1)hA))$. Lemma 2 says that this last term is positive, thus contradicting our assumption. □

Remark 2 Note that Mori in [18], (11.4) on p. 318, suggests that in principle (i) could have been stronger then (iv): *We say that X is κ-uniruled if K_X is not pseudo-effective. We note that κ-uniruledness is slightly stronger than saying that adjunction terminates, i.e. $H^0(X, mK_X + H) = 0$ for each very ample divisor H and some $m = m(H) > 0$.*

The following two corollaries show that the two formulations, respectively *for some* and *for every*, of (A) and (D) in the Introduction are equivalent.

Corollary 1 *Let X be a projective variety with at most canonical singularities. The following statements, where m and m_0 denote natural numbers, are equivalent:*

(i) *For every big Cartier divisor H there exists $m_0 = m_0(H) > 0$ such that $mK_X + H \notin \overline{Eff(X)}$ for $m \geq m_0$.*
(ii) *For some big Cartier divisor H_0 there exists $m_0 = m_0(H_0) > 0$ such that $mK_X + H_0 \notin \overline{Eff(X)}$ for $m \geq m_0$.*

Proof It is obvious that (i) implies (ii). Conversely, if (ii) holds then K_X is not pseudo-effective, hence X is uniruled. It follows from Theorem 3 that (i) holds. □

Corollary 2 *Let X be a projective variety with at most canonical singularities. The following statements, where m and m_0 denote natural numbers, are equivalent:*

(i) *For some big Cartier divisor* H_0 *we have* $H^0(X, m_0K_X + kH_0) = 0$ *for every* $k > 0$ *and some* $m_0 = m_0(k) > 0$.
(ii) *For every big Cartier divisor* H *we have* $H^0(X, m_0K_X + kH) = 0$ *for every* $k > 0$ *and some* $m_0 = m_0(k, H) > 0$.

Proof It is obvious that (ii) implies (i). Conversely, if (i) holds then by Theorem 3 X is uniruled, i.e. K_X is not pseudo-effective. Assume by contradiction that there exist a big divisor H and some $k_0 > 0$ such that $H^0(X, mK_X + k_0H) \neq 0$ for every $m > 0$. Then $K_X = \lim_{m\to+\infty} \frac{mK_X+k_0H}{m}$ is pseudo-effective, a contradiction. □

As pointed out by the referee, since every divisor is a difference of very ample ones, (C) is actually equivalent to the following stronger condition.

(C*) For every Cartier divisor D we have $H^0(X, m_0K_X + D) = 0$ for some $m_0 = m_0(D) > 0$.

The following is a more general definition of *Termination of Adjunction.*

Definition 2 (Condition (C')) Let X be a normal projective variety; let H be an effective Cartier divisor on X. We say that *Adjunction Terminates in the classical sense* for H if there exists an integer $m_0 \geq 1$ such that

$$H^0(X, H + mK_X) = 0$$

for every integer $m \geq m_0$.

We conjecture that such a definition is actually equivalent to the previous ones. The following partial result in this direction is straightforward.

Proposition 2 *Let* X *be a projective variety with canonical singularities. Let* H *be any effective divisor and assume that Adjunction Terminates in the classical sense for* H*. Then* X *has negative Kodaira dimension.*

Proof Recall that the Kodaira dimension of a singular variety is defined to be the Kodaira dimension of any smooth model (see for instance [16], Example 2.1.5). Assume by contradiction that X has non-negative Kodaira dimension, i.e. $H^0(\tilde{X}, n_0K_{\tilde{X}}) \neq 0$ for some integer $n_0 \geq 1$, where $\nu : \tilde{X} \to X$ is any resolution of the singularities. Since X has canonical singularities, from Lemma 1 it follows that $H^0(X, n_0K_X) = H^0(\tilde{X}, n_0K_{\tilde{X}}) \neq 0$. Hence $H^0(X, H + nn_0K_X) \neq 0$ for every integer $n \geq 1$, contradicting the assumption that $H^0(X, H + mK_X) = 0$ for $m >> 0$. □

Together with the standard conjecture that negative Kodaira dimension implies uniruledness (see for instance [18], (11.5) on p. 319, and [3], Conjecture 0.1), from Proposition 2 it would follow that Termination of Adjunction in the classical sense implies uniruledness. In dimension two such an implication holds unconditionally, as it was proved by Castelnuovo and Enriques in [7] (for a modern proof we refer to [19]).

We conclude this section with a characterization of uniruled varieties which may suggest a different way to consider (effective) termination of adjunction. It follows as a straightforward consequence of Lemma 2 and the main result in [3].

Proposition 3 *Let X be a smooth projective variety of dimension n and let H be a very ample divisor on X. If $H^0(X, mK_X + (n+1)H) = 0$ for some natural number $m \geq 1$, then X is uniruled.*

Proof Assume by contradiction that X is not uniruled, so that K_X is pseudo-effective by [3]. Lemma 2 with $F = (m-1)K_X$ gives the sought-for contradiction. □

Theorem 3.1 in [8] gives a statement similar to the last proposition; there the variety is singular and H is just nef and big. However $m > 1$ and H has to be multiplied by a higher number, for instance n^2.

4 Quasi polarized pairs

A *quasi polarized pair* is a pair (X, L) where X is a projective variety with at most $\mathbb{Q}$-factorial terminal singularities and L is a nef and big Cartier divisor on X. If L is ample we call the pair (X, L) a *polarized pair*.

In [1], Sect. 4, following T. Fujita's ideas as revisited by A. Höring in [13] and using the MMP developed in [5], we described a MMP with scaling related to divisors of type $K_X + rL$, for r a positive rational number.

In particular we introduced the 0-**reduction** of a quasi polarized pair (X, L) (see [1], Definition 4.4) as quasi polarized pair (X', L') birational to (X, L) obtained from (X, L) via a Minimal Model Program with scaling:

$$(X, L) \sim (X, \Delta) := (X_0, \Delta_0) \rightarrow - - -- \rightarrow (X_s, \Delta_s) \sim (X', L'),$$

which contracts or flips all extremal rays $\mathbb{R}^+[C]$ on X such that $L \cdot C = 0$.

At every step of the MMP given above, we have a quasi polarized variety (X_i, L_i) with at most terminal $\mathbb{Q}$-factorial singularities.

If $\pi_i : (X_i, \Delta_i) \rightarrow (X_{i+1}, \Delta_{i+1})$ is birational then $L_i = \pi_i^*(L_{i+1})$, while if $\pi_i : (X_i, \Delta_i) \rightarrow (X_{i+1}, \Delta_{i+1})$ is a flip then L_i and $\pi_i^*(L_{i+1})$ are isomorphic in codimension one.

Remark 3 By using Lemma 1 and Hartogs theorem we deduce

$$H^0(X, aK_X + bL) = H^0(X', aK_{X'} + bL')$$

for $a, b \in \mathbb{N}$.

The following has been proved in [1], Theorem 5.1 and in [12], Proposition 1.3.

Theorem 4 *Let (X, L) be a quasi polarized pair. Then $K_X + tL$ is pseudo-effective for all $t \geq n$ unless the 0-reduction (X', L') is $(\mathbb{P}^n, \mathcal{O}(1))$. Actually, $K_X + (n-1)L$ is pseudo-effective unless (X', L') is one of the following pairs:*

- $(\mathbb{P}^n, \mathcal{O}(1))$,
- $(Q, \mathcal{O}(1)_{|Q})$, *where $Q \subset \mathbb{P}^{n+1}$ is a quadric,*
- $C_n(\mathbb{P}^2, \mathcal{O}(2))$, *a generalized cone over $(\mathbb{P}^2, \mathcal{O}(2))$,*
- *X has the structure of a $\mathbb{P}^{n-1}$-bundle over a smooth curve C and L restricted to any fiber is $\mathcal{O}(1)$.*

Moreover, except in the above cases, $K_{X'} + (n-1)L'$ is nef.

The **first-reduction** of a quasi polarized pair (X, L) (see [1], Definition 5.5) is a quasi polarized pair (X'', L'') birational to (X, L) obtained from a 0-reduction (X', L') via a morphism $\rho : X' \to X''$ consisting of a series of divisorial contractions to smooth points, which are weighted blow-ups of weights $(1, 1, b, \dots, b)$ with $b \geq 1$ (see [2], Theorem 1.1).

Remark 4 According to [1], Proposition 5.4, we have

$$H^0(K_X + tL) = H^0(K_{X''} + tL'')$$

for any $0 \leq t \leq n-2$.

The following has been proved in [1], Theorem 5.7.

Theorem 5 *Let (X, L) be a quasi polarized pair.*

$K_X + (n-2)L$ is not pseudo-effective if and only if any first-reduction (X'', L'') is either one of the pairs listed in the statement of Theorem 4 or one of the following pairs:

- *a del Pezzo variety, that is $-K_{X''} \sim_{\mathbb{Q}} (n-1)L$ with L ample,*
- $(\mathbb{P}^4, \mathcal{O}(2))$,
- $(\mathbb{P}^3, \mathcal{O}(3))$,
- *$(Q, \mathcal{O}(2)_{|Q})$, where $Q \subset \mathbb{P}^4$ is a quadric,*
- *X has the structure of a quadric fibration over a smooth curve C and L restricted to any fiber is $\mathcal{O}(1)_{|Q}$,*
- *X has the structure of a $\mathbb{P}^{n-2}$-bundle over a normal surface S and L restricted to any fiber is $\mathcal{O}(1)$,*
- *$n = 3$, X is fibered over a smooth curve Z with general fiber $\mathbb{P}^2$ and L restricted to it is $\mathcal{O}(2)$.*

If $K_X + (n-2)L$ is pseudo-effective then on any first-reduction (X'', L'') the divisor $K_{X''} + (n-2)L''$ is nef.

The following definition was given by Höring (see ([13], Definition 1.2).

Definition 3 A quasi polarized pair (X, L) is a (generalized) scroll if X is smooth and there is a fibration $X \to Y$ onto a projective manifold Y such that the general fiber F admits a birational morphism $\tau : F \to \mathbb{P}^m$ and that $\mathcal{O}_F(L) = \tau^*\mathcal{O}_{\mathbb{P}^m}(1)$. A quasi polarized pair (X, L) is birationally a scroll if there is a birational morphism $\nu : X' \to X$ such that $(X', \nu^* L)$ is a (generalized) scroll.

The next is Theorem 1.4 in [13].

Theorem 6 *Let (X, L) be a quasi polarized pair. If (X, L) is not birationally a scroll then $\Omega_X \otimes L$ is generically nef.*

A key step in the proofs of Theorem 7 and of Theorem 8 is the following lemma due to Höring (see [13], p. 741, Step 2 in the proof of Theorem 1.2).

Lemma 3 *Let (X, L) be a quasi polarized pair. Assume that $K_X+(n-2)L$ is pseudo-effective and that $K_X+(n-1)L$ is nef and big. Then*

$$L^{n-2}[(2(K_X^2+c_2(X))+6nLK_X+(n+1)(3n-2)L^2]>0.$$

We consider now a quasi polarized pair (X, L) and we assume moreover that X is smooth. We borrow from Y. Fukuma the following set-up for the computation of the Hilbert polynomial of K_X+tL.

Let

$$F_0(t):=\dim H^0(X, K_X+tL),$$
$$F_i(t):=F_{i-1}(t+1)-F_{i-1}(t) \text{ for every integer } i \text{ with } 1\leq i\leq n.$$

The following statement can be easily checked by reverse induction on $b\leq a$.

Lemma 4 *Fix an integer $a\geq 1$. If $F_0(t)=0$ for every integer t with $1\leq t\leq a$, then $F_{a-b}(c)=0$ for all integers b, c with $1\leq c\leq b\leq a$.*

If one defines

$$A_i(X, L):=F_{n-i}(1)$$

then it follows easily that

$$\dim H^0(X, K_X+tL)=\sum_{j=0}^{n}\binom{t-1}{n-j}A_j(X, L). \quad (1)$$

Moreover, by taking $a=n-i+1$ and $b=c=1$ in Lemma 4, we obtain the following implication.

Corollary 3 *If $H^0(X, K_X+tL)=0$ for every integer t with $1\leq t\leq n-i+1$, then $A_i(X, L)=0$.*

On the other hand, by Kawamata–Viehweg vanishing theorem and Serre duality, we have $\dim H^0(X, K_X+tL)=\chi(X, -tL)$; therefore from the Riemann–Roch theorem we obtain the following explicit computations (for further details, see [9], (2.2), and [10], Proposition 3.2).

Lemma 5 *Let (X, L) be a polarized manifold of dimension n and let $g(X, L)$ denote the sectional genus of (X, L). Then we have*

$$\begin{aligned}
A_0(X, L)&=L^n\\
A_1(X, L)&=g(X, L)+L^n-1\\
24\cdot A_2(X, L)&=L^{n-2}[(2(K_X^2+c_2(X))+6nLK_X+(n+1)(3n-2)L^2]\\
48\cdot A_3(X, L)&=(n-2)(n^2-1)L^n+n(3n-5)K_XL^{n-1}+\\
&\quad +2(n-1)K_X^2L^{n-2}+2c_2(X)(K_X+(n-1)L)L^{n-3}.
\end{aligned}$$

5 Polarized abundance

The aim of this section is to argue around the Conjectures stated in the introduction.

We start showing that Proposition 1 is a direct consequence of (the more general) Theorem D in [5].

Remark 5 Let (X, L) be a quasi-polarized variety and let t be a positive rational number. Then there exists an effective $\mathbb{Q}$-divisor Δ^t on X such that $\Delta^t \sim_{\mathbb{Q}} tL$ and (X, Δ^t) is Kawamata log terminal. This is well-known to the specialists, a proof can be found in [1]. If $K_X + tL \in \overline{Eff(X)}$, then $K_X + \Delta^t \in \overline{Eff(X)}$ and by [5], Theorem D, there exists an $\mathbb{R}$-divisor $D \geq 0$ such that $K_X + \Delta^t \sim_{\mathbb{R}} D$. That is, there exists $N \in \mathbb{N}$ such that $H^0(X, N(K_X + tL)) > 0$.

We consider Conjecture 2; for $s = n$ we recover the following easy fact.

Proposition 4 *Let (X, L) be a quasi polarized pair of dimension n. We have $H^0(X, K_X + tL) = 0$ for every integer t with $1 \leq t \leq n$ if and only if $K_X + nL$ is not pseudo-effective. Moreover this is the case if and only if the 0-reduction (X', L') of the pair (X, L) is $(\mathbb{P}^n, \mathcal{O}(1))$.*

Proof By Remark 3 we have $H^0(X, K_X + tL) = H^0(X', K_{X'} + tL')$ for any $t \geq 0$. Hence if $H^0(X, K_X + tL) = 0$ for every integer t with $1 \leq t \leq n$ then from Corollary 3 it follows that $A_1(X', L') = g(X', L') + L'^n - 1 = 0$. Since we have $g(X', L') = 0$ and $L'^n = 1$ if and only if $(X', L') = (\mathbb{P}^n, \mathcal{O}(1))$, the claim follows from [1], Theorem 5.1 (2). □

Next, for $s = n - 1$, the following is a slightly more explicit version of [13], Theorem 1.2; the proof is essentially the one of [13].

Theorem 7 *Let (X, L) be a quasi polarized pair of dimension n. We have $H^0(X, K_X + tL) = 0$ for every integer t with $1 \leq t \leq n - 1$ if and only if $K_X + (n-1)L$ is not pseudo-effective.*

That is, by Theorem 4, if and only if the 0-reduction (X', L') of the pair (X, L) is one of the following:

- (i) $(\mathbb{P}^n, \mathcal{O}(1))$,
- (ii) $(Q, \mathcal{O}(1)_{|Q})$*, where* $Q \subset \mathbb{P}^{n+1}$ *is a quadric,*
- (iii) $C_n(\mathbb{P}^2, \mathcal{O}(2))$*, a generalized cone over* $(\mathbb{P}^2, \mathcal{O}(2))$,
- (iv) *X has the structure of a $\mathbb{P}^{n-1}$-bundle over a smooth curve C and L restricted to any fiber F is $\mathcal{O}(1)$.*

Proof Let (X', L') be the 0-reduction of the pair (X, L) and let $(\tilde{X}', \tilde{L}')$ be its desingularization (namely, $\nu : \tilde{X}' \to X'$ and $\tilde{L}' = \nu^*(L')$).

By Remark 3 and Lemma 1 we have

$$H^0(X, K_X + tL) = H^0(X', K_{X'} + tL') = H^0(\tilde{X}', K_{\tilde{X}'} + t\tilde{L}')$$

for any $t \geq 0$.

The *if* part is obvious. In order to prove the *only if* part, assume that $H^0(X, K_X + tL) = H^0(X', K_{X'} + tL') = H^0(\tilde{X}', K_{\tilde{X}'} + t\tilde{L}') = 0$ for every integer t with $1 \leq t \leq n-1$. Corollary 3 implies that

$$A_2(\tilde{X}', \tilde{L}') = 0. \tag{2}$$

Assume by contradiction that (X', L') is not one of the pairs in (i), (ii), (iii), (iv); then, by Theorem 4, $K_{X'} + (n-1)L'$ is nef. The required contradiction is provided by [13], Theorem 1.2. □

The next step $s = n - 2$ should work as follows.

Conjecture 3 *Let (X, L) be a quasi polarized manifold of dimension n. We have $H^0(X, K_X + tL) = 0$ for every integer t with $1 \leq t \leq n-2$ if and only if $K_X + (n-2)L$ is not pseudo-effective, that is if and only if the first-reduction (X'', L'') is one of the pairs (X, L) listed in Theorems 4 and 5.*

Once again, the *if* part is obvious. Conversely, from Corollary 3 it follows that $A_3(X, L) = 0$, but the proof of the *only if* part seems to be elusive.

From now on, we focus on the case $n = 4$; here formula (1) reads simply as:

$$H^0(X, K_X + tL) = \binom{t-1}{4} A_0(X, L) + \binom{t-1}{3} A_1(X, L) + \binom{t-1}{2} A_2(X, L) + \binom{t-1}{1} A_3(X, L) + \binom{t-1}{0} A_4(X, L) \tag{3}$$

where

$$\begin{aligned} A_1(X, L) &= g(X, L) + L^4 - 1, \\ A_2(X, L) &= \dim H^0(X, K_X + 3L) - 2\dim H^0(X, K_X + 2L) \\ &\quad + \dim H^0(X, K_X + L), \\ A_3(X, L) &= \dim H^0(X, K_X + 2L) - \dim H^0(X, K_X + L), \\ A_4(X, L) &= \dim H^0(X, K_X + L). \end{aligned}$$

We prove the following generalization of [11], Theorem 3.1.

Theorem 8 *Let (X, L) be a polarized manifold of dimension 4 and let t be an integer with $t \geq 3$. If $K_X + tL$ is pseudo-effective, then $H^0(X, K_X + tL) \neq 0$. In particular,*

- $H^0(X, K_X + tL) \neq 0$ *for* $t \geq 5$
- $H^0(X, K_X + 4L) = 0$ *if and only if* (X, L) *is* $(\mathbb{P}^4, \mathcal{O}(1))$
- $H^0(X, K_X + 3L) = 0$ *if and only if* (X, L) *is either* $(Q, \mathcal{O}(1)_{|Q})$, *where* $Q \subset \mathbb{P}^5$ *is a quadric, or* X *has the structure of a* $\mathbb{P}^3$*-bundle over a smooth curve* C *and* L *restricted to any fiber is* $\mathcal{O}(1)$.

Proof Since L is ample (X, L) is a 0-reduction, in particular by Theorem 4 we can assume that $K_X + tL$ is nef for $t \geq 4$. We can also assume that $K_X + 3L$ is nef. Indeed, if not then (X, L) is one of the exceptions listed in the statement of Theorem 4. If (X, L) is $(\mathbb{P}^4, \mathcal{O}(1))$ or $(Q, \mathcal{O}(1))$, where $Q \subset \mathbb{P}^5$ is a quadric hypersurface, then Theorem 8 is obvious. The case of a generalized cone over $(\mathbb{P}^2, \mathcal{O}(2))$ does not occur since X is smooth, while the case of a $\mathbb{P}^3$-bundle over a smooth curve will be considered in Proposition 5.

Now, assume that $\Omega_X \otimes L$ is generically nef. By using the formulas in Lemma 5 and Miyaoka inequality as stated in [13], Corollary 2.11, with $D := 4L$, we compute:

$$A_2(X, L) \geq \frac{1}{24}\left(2(K_X + 3L)^2 L^2 + 6(K_X + 3L)L^3 + 2L^4\right)$$
$$A_3(X, L) \geq -\frac{1}{24}(K_X + 3L)L^3.$$

Hence from (3) and the nefness of $K_X + 3L$ it follows that

$$\dim H^0(X, K_X + tL) \geq (t-1)A_3(X, L) + \frac{(t-1)(t-2)}{2} A_2(X, L) > 0$$

for every $t \geq 3$.

Finally, assume that $\Omega_X \otimes L$ is not generically nef. By Theorem 6 and Lemma 1 we may assume that X is a (generalized) scroll and the claim is a consequence of the following proposition. □

Proposition 5 *Let (X, L) be a generalized scroll of dimension 4 and let t be an integer such that $t \geq 3$. If $K_X + tL$ is nef, then $H^0(X, K_X + tL) \neq 0$.*

Proof Let $X \to Y$ be the scroll fibration and let F be the generic fiber with a birational morphism $\tau : F \to \mathbb{P}^m$ as in Definition 3.

If $X = \mathbb{P}^4$ the claim is obvious; therefore we can assume that $m \leq 3$ and that $A_1(X, L) = g(X, L) + L^4 - 1 > 0$ (since we have $g(X, L) = 0$ and $L^4 = 1$ if and only if $(X, L) = (\mathbb{P}^4, \mathcal{O}(1))$). We also have that $A_0(X, L) = L^4 \geq 1$ and $A_4(X, L) = \dim H^0(X, K_X + L) \geq 0$.

If $m = 3$, then $K_X + sL|_F = \tau^*\mathcal{O}_{\mathbb{P}^3}(-4 + s)$, hence $H^0(X, K_X + sL) = 0$ for $s \leq 3$. Thus we have $A_2(X, L) = A_3(X, L) = 0$ and from (3) it follows that for $t \geq 4$ we have

$$\dim H^0(X, K_X + tL) \geq A_1(X, L) > 0$$

If $m = 2$, then $K_X + sL|_F = \tau^*\mathcal{O}_{\mathbb{P}^2}(-3 + s)$, hence $H^0(X, K_X + sL) = 0$ for $s \leq 2$. In particular, we have $A_3(X, L) = 0$ and $A_2(X, L) = \dim H^0(X, K_X + 3L)$.

For $t = 3$, i.e. if we assume $K_X + 3L$ is nef, by Theorem 1.2 in [13] we must have $H^0(X, K_X + 3L) \neq 0$ since $H^0(X, K_X + sL) = 0$ for $s \leq 2$.

For $t \geq 4$ we deduce from (3) that

$$\dim H^0(X, K_X + tL) \geq A_1(X, L) > 0.$$

If $m = 1$, then $K_X + sL|_F = \tau^*\mathcal{O}_{\mathbb{P}^1}(-2+s)$, hence $H^0(X, K_X + L) = 0$. In particular, we have $A_3(X, L) \geq 0$.

If $H^0(X, K_X+2L) = 0$, then $A_2(X, L) = \dim H^0(X, K_X+3L)$ and we conclude exactly as in the previous case $m = 2$.

If $H^0(X, K_X + 2L) \neq 0$, then $K_X + 2L$ is pseudo-effective and $K_X + 3L$ is pseudo-effective and big.

Passing to the 0-reduction we may assume that $K_X + 3L$ is nef and big. Therefore Lemma 3 applies and by Lemma 5 we get $A_2(X, L) > 0$.

Hence from (3) it follows that for $t \geq 3$ we have

$$\dim H^0(X, K_X + tL) \geq A_2(X, L) > 0.$$

□

The statement of Theorem 8 should hold also for $t = 2$, but we have only the following partial result.

Proposition 6 *Let (X, L) be a polarized manifold of dimension 4. If $K_X + 2L$ is pseudo-effective, then $H^0(X, K_X + 2L) \neq 0$ unless $\Omega_X < \frac{1}{2}L >$ is not generically nef.*

Proof By Theorem 5 and Remark 4 we may assume that $K_X + 2L$ is nef.

Assume that $\Omega_X < \frac{1}{2}L >$ is generically nef. By using the formula for $A_3(X, L)$ in Lemma 5 and Miyaoka inequality, as stated in [13], Corollary 2.11, with $D := 2L$, we compute:

$$A_3(X, L) \geq \frac{1}{16}(K_X + 2L)^2L^2 + \frac{1}{12}(K_X + 2L)L^3 + \frac{1}{48}L^4.$$

Hence from (3) it follows that

$$\dim H^0(X, K_X + tL) \geq A_3(X, L) > 0.$$

□

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