Ann Univ Ferrara (2017) 63:33–50
DOI 10.1007/s11565-017-0279-x

Algebraic methods for vector bundles on non-Kähler elliptic fibrations

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Received: 4 March 2017 / Accepted: 13 March 2017 / Published online: 21 March 2017

Abstract We survey some parts of the vast literature on holomorphic vector bundles on compact complex manifolds, focusing on the rank-two case vector bundles on non-Kähler elliptic fibrations. It is by no means intended to be a complete overview of this wide topic, but we rather focus on results obtained by the author and his collaborators.

Keywords Holomorphic vector bundles · Complex compact manifolds · Non-Kähler manifolds · Elliptic fibrations · Moduli spaces

Mathematics Subject Classification 32L10 · 32J18 · 32G13 · 14J60 · 14F05 · 13D09

1 Introduction

The study of vector bundles over elliptic fibrations has been a very active area of research in both mathematics and physics over the past 35 years; in fact, there is by now a well understood theory for projective elliptic fibrations (see Donagi [35,36], Donagi–Pantev [37], Friedman [41], Friedman et al. [42], Bridgeland [12,13], Bridgeland–

Dedicated to the memory of Alexandru Lascu.

The author has been partly supported by a research grant of Max Planck Institute for Mathematics in Bonn, March 2016 and by the Fields Institute *Focus Program on Topology, Stratified Spaces and Particle Physics*, Toronto, August 8–26, 2016.

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Maciocia [14], Bartocci et al. [6,7], Hernandez Ruiperez and Muñoz Porras [45], Căldăraru et al. [30,31]).

Not very much is known for non-Kähler elliptic fibrations of complex dimension greater than two. One of the motivations for the study of vector bundles on non-Kähler elliptic threefolds comes from recent developments in superstring theory, where six-(real)dimensional non-Kähler manifolds occur in string compactifications with non-vanishing background H-field (see, for example [8,33,44]).

The "space"of our universe is considered to be of the form $\mathbb{R}^4 \times X$, where $\mathbb{R}^4$ is the Minkowski space and X is a complex 3-dimensional manifold of Calabi–Yau type (i.e. $\omega_X \cong \mathcal{O}_X$), not neccessarily Kähler. Thus, the study of moduli spaces of vector bundles over a non-Kähler Calabi–Yau type threefold is interesting also for physicists.

The purpose of this paper is to survey some parts of the theory of holomorphic vector bundles on non-Kähler elliptic fibrations, with emphasis on personal contributions.

The outline of the paper is as follows. In Sect. 2, we recall first examples and some general results about holomorphic vector bundles over non-algebraic surfaces. Section 3 is devoted to results concerning moduli spaces of rank-two vector bundles on non-Kähler elliptic surfaces. In Sect. 4, we discuss the moduli spaces of vector bundles on non-Kähler elliptic Calabi–Yau type threefolds and more generally, on elliptic fibrations which are principal elliptic bundles.

2 First examples and first results

Let X be a compact complex manifold. Unless otherwise stated, all manifolds are assumed to be compact. For definitions and the proofs of some of the results see [15]. A holomorphic vector bundle V of rank r over the complex manifold X is called *filtrable* if there exists a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_r = V,$$

where $\mathcal{F}_k$ is a coherent subsheaf of rank k. A holomorphic vector bundle V of rank r over a complex manifold X is called *reducible* if it admits a coherent analytic subsheaf $\mathcal{F}$ such that

$$0 < rank(\mathcal{F}) < r,$$

and *irreducible* otherwise. Clearly, for rank 2 non-filtrable is equivalent to irreducible.

Remark 1 Of course, every holomorphic (algebraic) vector bundle V over a projective manifold X is filtrable (since $H^0(X, V \otimes H^n)$ is non-zero for big n, we obtain a rank one subsheaf of V, and so on; here H denotes an ample line bundle on X), but on non-algebraic manifolds there exist holomorphic vector bundles which are non-filtrable.

The first paper on holomorphic vector bundles in the non-algebraic case was by Elencwajg–Forster (see [40]). In 1982, they constructed in this paper irreducible vector bundles of rank 2 on a complex 2-torus X without divisors (curves); i.e. with Neron-Severi group $NS(X) = 0$. This was done by comparing the versal deformation of a filtrable rank 2 vector bundle with the space parametrising extensions producing filtrable rank 2 vector bundles. They proved that, in general, the versal deformation has a bigger dimension, hence it contains also non-filtrable vector bundles.

The next example is due to Schuster, see [58].

Example 1 Let X be a K3-surface with Picard group $Pic(X) = 0$. Then the tangent bundle T_X is irreducible.

Since $H^0(X, T_X) = 0$ it follows easily that the rank 2 vector bundle T_X has no coherent analytic subsheaves (see, for example [40] or [15], p. 92).

The next example is due to Coandă, see [15], p. 104.

Example 2 Let X be a K3-surface with Picard group $Pic(X) = 0$. We have an exact sequence of holomorphic vector bundles over X

$$0 \to \mathcal{O}_X \to T_X \otimes T_X \to S^2 T_X \to 0.$$

Then, $S^2 T_X$ is an irreducible holomorphic vector bundle of rank 3.

It is worth to mention here the following results, where holomorphic vector bundles play a key role.

In 1982, Schuster proved in [58] that for any compact complex surface X, every coherent sheaf $\mathcal{F}$ on X has global resolutions with locally free sheaves (vector bundles). One of the main steps in the proof was to use rank 2 holomorphic vector bundles from the versal deformation of T_X.

A long standing problem for compact complex manifolds was to decide if every coherent sheaf has global resolutions with locally free sheaves (i.e. with vector bundles). A negative answer came only after 20 years. In 2002, C. Voisin proved that this is false for some Kähler compact complex manifolds of dimension ≥ 3, see [62]. In 2012, Vuletescu gave some new examples of non-Kähler compact complex manifolds of dimension 3 and coherent sheaves $\mathcal{F}$ on X having no global resolution by vector bundles, see [66]. The proof that these sheaves do not admit a locally free resolution is very different from Voisin's argument. The manifold X is a Calabi–Eckmann manifold i.e. a principal elliptic bundle over the base $P^1(\mathbb{C}) \times P^1(\mathbb{C})$, which is diffeomorphic to the product of two real spheres of dimension 3.

Another paper on holomorphic vector bundles over non-algebraic complex surfaces was by Brînzănescu–Flondor in 1985; see [20]. Let X be a non-algebraic surface, let V be a rank 2 holomorphic vector bundle on X and, let $c_1(V),\ c_2(V)$ be the Chern classes of the vector bundle V. We have the following result:

Proposition 1 *Let X be a non-algebraic surface and let $a \in NS(X)$ be fixed. Then, for every holomorphic rank 2 vector bundle V over X with $c_1(V) = a$, we have*

$$c_2(V) \geq min\{a^2/4,\ 2\chi(\mathcal{O}_X) + (c_1(X).a + a^2)/2\}.$$

In the same paper, [20], one defined a bound for a holomorphic rank 2 vector bundle on a non-algebraic surface with fixed Chern class $a \in NS(X)$ to be filtrable. In another paper, see [21], one gives the range of Chern classes $c_1,\ c_2$ of simple filtrable rank two holomorphic vector bundles over complex surfaces X without divisors. These results were extended later by Toma [60] to the case of complex surfaces of algebraic dimension zero.

The results of the paper [20] were extended by Bănică– Le Potier [4], to the case of holomorphic vector bundles of any rank over a non-algebraic surface.

We need some notation. The Chern classes and the rank can be defined for any analytic coherent sheaf $\mathcal{F}$ over X (see, for example [15], p. 12 and p. 17). If $\mathcal{F}$ is locally free, then we have $c_1(\mathcal{F}) = c_1(\det(\mathcal{F})) \in NS(X)$. Generally, by the quoted result of Schuster, see [58], any analytic coherent sheaf $\mathcal{F}$ over a complex surface has a resolution

$$0 \to V_2 \to V_1 \to \mathcal{F} \to 0,$$

with V_i locally free sheaves. Then

$$c_1(\mathcal{F}) = c_1(V_1) - c_1(V_2) \in NS(X).$$

Now, let $\mathcal{F}$ be an analytic coherent sheaf over a surface X of rank $r > 0$, with Chern classes $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$. The *discriminant* $\Delta(\mathcal{F})$ is defined by

$$\Delta(\mathcal{F}) := \frac{1}{r}\left(c_2(\mathcal{F}) - \frac{r-1}{2r}c_1^2(\mathcal{F})\right).$$

The extension of Proposition 1 is given by the following result in [4]:

Theorem 1 *Let X be a non-algebraic surface and $\mathcal{F}$ a torsion-free coherent sheaf over X of rank r, with Chern classes $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$. Then $\Delta(\mathcal{F}) \geq 0$.*

Other simpler proofs were given later by Brînzănescu [19] and by Vuletescu [65] (or [15], p. 95).

Remark 2 In the case X is a primary Kodaira surface there is a kind of converse of this theorem, namely one shows that a topological complex vector bundle V of any rank has a holomorphic complex structure if and only if it satisfies the conditions

$$c_1(V) \in NS(X) \; and \; \Delta(V) \geq 0;$$

see the paper by Aprodu et al. [1].

For a non-algebraic surface X, $a \in NS(X)$ and r a positive integer we can define the following rational positive number

$$m(r,a) := -\frac{1}{2r} max\{\Sigma_1^r (a/r - \mu_i)^2, \;\; \mu_i \in NS(X) \;\; with \;\; \Sigma_1^r \mu_i = a\}.$$

Remark 3 When X is a 2-torus and $r = 2$ an explicit description of the invariants $m(2,a)$ is given in [20].

The extension of the results for filtrable bundles in [20] is the following result of Bănică and Le Potier [4]:

Theorem 2 *A rank r, $r \geq 2$ topological complex vector bundle V over a non-algebraic surface X admits a filtrable holomorphic structure if and only if*

$$c_1(V) \in NS(X) \text{ and } \Delta(V) \geq m(r, c_1(V)),$$

except when X is a K3-surface with $a(X) = 0, c_1(V) \in rNS(X)$ and $\Delta(V) = \frac{1}{r}$. In this excepted case V admits no holomorphic structures.

Other relevant papers on the topic of holomorphic vector bundles over non-algebraic surfaces are authored by Braam and Hurtubise [11], Teleman [59], Vuletescu [64,65, 67].

3 Vector bundles on non-Kähler elliptic surfaces

Let $X \stackrel{\pi}{\to} B$ be a minimal non-Kähler elliptic surface with B a smooth curve of genus g. It is well-known that $X \stackrel{\pi}{\to} B$ is a quasi-bundle over the base B, that is, all the smooth fibres are isomorphic to a fixed elliptic curve E and the singular fibres (in a finite number) are multiples of elliptic curves.

Remark 4 For $g = 0$, X is a Hopf surface, for $g = 1$, X is a Kodaira surface and, for $g \geq 2$, X is called a properly elliptic surface.

Let E^* denote the dual of E (we fix a non-canonical identification $E^* = Pic^0(E)$ by fixing an origin on E). The Jacobian surface associated to $X \stackrel{\pi}{\to} B$ is

$$J(X) = B \times E^* \stackrel{p_1}{\to} B,$$

and X is obtained from the relative Jacobian $J(X)$ by a finite number of logarithmic transformations [51]. We have the following result (see [16–18]):

Theorem 3 *For any minimal non-Kähler elliptic surface we have the isomorphism:*

$$NS(X)/Tors(NS(X)) \cong Hom(J_B, Pic^0(E)),$$

where $NS(X)$ is the Neron–Severi group of the surface and J_B denotes the Jacobian variety of the curve B.

This result was extended by Brînzănescu–Ueno for torus quasi-bundles over curves, see [25].

Remark 5 In the case of elliptic surfaces, from the above theorem we get:

For any Chern class $c = c_1(L)$, with $L \in Pic(X)$ a line bundle, the class $\overline{c} \in NS(X)/Tors(NS(X))$, if it is non-zero, defines a covering map $\overline{c} : B \to Pic^0(E)$, which gives us a section of the Jacobian $J(X)$. This is exactly the *spectral curve* associated to the line bundle L, defined by Hitchin (see [46]).

Let V be a holomorphic rank-2 vector bundle on X, with fixed $c_1(V) = c_1 \in NS(X)$ and $c_2(V) = c_2 \in \mathbb{Z}$. Now, we fix also the determinant line bundle of V, denoted by $\delta = det(V)$. It defines an involution on the relative Jacobian $J(X) = B \times E^*$ of X:

$$i_\delta : J(X) \to J(X),\ (b, \lambda) \to (b, \delta_b \otimes \lambda^{-1}),$$

where δ_b denotes the restriction of δ to the fibre $E_b = \pi^{-1}(b)$, which has degree zero (see Lemma 2.2 in [22]). Taking the quotient of $J(X)$ by this involution, each fibre of p_1 becomes $E^*/i_\delta \cong \mathbb{P}^1$ and the quotient $J(X)/i_\delta$ is isomorphic to a ruled surface $\mathbb{F}_\delta$ over B. Let $\eta : J(X) \to \mathbb{F}_\delta$ be the canonical map.

The main existence result of holomorphic rank-2 vector bundles over non-Kähler elliptic surfaces is the following (see [22]):

Theorem 4 *Let X be a minimal non-Kähler elliptic surface over a smooth curve B of genus g and fix a pair (c_1, c_2) in $NS(X) \times \mathbb{Z}$. Set $m_{c_1} := m(2, c_1)$ and denote $\overline{c}_1$ the class of c_1 in $NS(X)$ modulo $2NS(X)$; moreover, let $e_{\overline{c}_1}$ be the invariant of the ruled surface $\mathbb{F}_{\overline{c}_1}$ determined by $\overline{c}_1$. Then, there exists a holomorphic rank-2 vector bundle on X with Chern classes c_1 and c_2 if and only if*

$$\Delta(2, c_1, c_2) \geq (m_{c_1} - d_{\overline{c}_1}/2),$$

where $d_{\overline{c}_1} := (e_{\overline{c}_1} + 4m_{c_1})/2$. Note that both $d_{\overline{c}_1}$ and $(m_{c_1} - d_{\overline{c}_1}/2)$ are non-negative numbers. Furthermore, if

$$(m_{c_1} - d_{\overline{c}_1}/2) \leq \Delta(2, c_1, c_2) < m_{c_1},$$

then the corresponding vector bundles are non-filtrable.

Let us suppose for the moment that the minimal non-Kähler elliptic surface $X \xrightarrow{\pi} B$ (which is a quasi-bundle) has no multiple fibres, i.e. it is a principal elliptic bundle. The set of all holomorphic line bundles on X with trivial Chern class is given by the zero component $Pic^0(X)$ of the Picard group $Pic(X)$. By Proposition 1.6 in [59], one has

$$Pic^0(X) \cong Pic^0(B) \times \mathbb{C}^*,$$

and any fibre of $X \xrightarrow{\pi} B$ is $E \cong \mathbb{C}^*/ < \tau >$, where $< \tau >$ is the multiplicative cyclic group generated by a fixed complex number τ, with $|\tau| > 1$. In particular, there exists a universal Poincaré line bundle $\mathcal{U}$ on $X \times Pic^0(X)$, whose restriction to

$$X \times \mathbb{C}^* := X \times \{0\} \times \mathbb{C}^* \subset X \times Pic^0(X)$$

is constructed in terms of constant automorphy factors (for details, see [22,53]).

The main tool to study vector bundles on any elliptic surface X is by taking restrictions to the smooth fibres. Note that if X is non-Kähler, then the restriction of any line bundle on X to a smooth fibre of π always has degree zero; see [22]. For a rank two vector bundle V over X, its restriction to a generic fibre of π is semistable; more precisely, its restriction to a fibre $\pi^{-1}(b)$ is unstable on at most an isolated set of points $b \in B$ and, these isolated points are called the *jumps* of the bundle. Furthermore, there exists a divisor S_V in the relative Jacobian $J(X) = B \times E^*$ of X, called the *spectral curve* or *cover* of the bundle, that encodes the isomorphism class of the bundle over each fibre of π. The spectral curve can be constructed as follows. If the surface X does not have multiple fibres, then there exists a universal bundle $\mathcal{U}$ on $X \times Pic^0(X)$, whose restriction to $X \times \mathbb{C}^*$ is also denoted $\mathcal{U}$; we associate to the rank-2 vector bundle V the sheaf on $B \times \mathbb{C}^*$ defined by

$$\tilde{\mathcal{L}} := R^1\pi_*(s^*V \otimes \mathcal{U}),$$

where $s : X \times \mathbb{C}^* \to X$ is the projection onto the first factor, *id* is the identity map, and π also denotes the projection $\pi := \pi \times id : X \times \mathbb{C}^* \to B \times \mathbb{C}^*$. This sheaf is supported on a divisor $\tilde{S}_V$, defined with multiplicity, that descends to a divisor S_V in $J(X)$ of the form

$$S_V := \Sigma_1^k(\{x_i\} \times E^*) + \overline{C},$$

where $\overline{C}$ is a bisection of $J(X)$ and $x_1, x_2, \ldots, x_k$ are points in B that correspond to the jumps of V. The spectral curve of V is defined to be the divisor S_V. The line bundle $\tilde{\mathcal{L}}$ also descends to a line bundle $\mathcal{L}$ on $J(X)$ (see [22,23]).

If the fibration π has multiple fibres, then one can associate to X a principal E-bundle $\pi' : X' \to B'$ over a m-cyclic covering $\epsilon : B' \to B$, where the integer m depends on the multiplicities of the singular fibres. The map ϵ induces natural m-cyclic coverings $J(X') \to J(X)$ and $\psi : X' \to X$. By replacing X with X' (which does not have multiple fibres) in the above construction, we obtain the spectral cover S_{ψ^*V} of the vector bundle π^*V as a divisor in $J(X')$. Then, we define the spectral cover S_V of V to be the projection of S_{ψ^*V} in $J(X)$. This construction led to a natural definition of a twisted Fourier–Mukai transform Φ for locally free sheaves on X, in particular, $\Phi(V) = \mathcal{L}$. For more details, see [23], Sect. 3 and Theorem 3.1.

Recall that the determinant line bundle $\delta = det(V)$ defines the following involution on $J(X)$:

$$i_\delta : B \times E^* \to B \times E^*, \ i_\delta(b, \lambda) = (b, \delta_b \otimes \lambda^{-1}),$$

where δ_b denotes the restriction of δ to the fibre $E_b = \pi^{-1}(b)$. The spectral curve S_V of V is invariant with respect to this involution. The quotient of $J(X) = B \times E^*$ by the involution is a ruled surface $\mathbb{F}_\delta := J(X)/i_\delta$ over B. Let $\eta : J(X) \to \mathbb{F}_\delta$ be the canonical map. By construction, the spectral curve S_V of the bundle V descends to the quotient $\mathbb{F}_\delta$; in fact, it is a pullback via η of a divisor on $\mathbb{F}_\delta$ of the form

$$\mathcal{G}_V := \Sigma_1^k f_i + A,$$

where f_i is the fibre of the ruled surface over the point x_i and A is a section of the ruling such that $\eta^*A = \overline{C}$. The divisor $\mathcal{G}_V$ is called *the graph of V*.

The degree of a vector bundle can be defined on any compact complex manifold M of dimension d. A theorem of Gauduchon's [43] states that any hermitian metric on M is conformally equivalent to a metric (called now a *Gauduchon metric*), whose associated $(1,1)$-form ω satisfies $\partial\overline{\partial}\omega^{d-1} = 0$. Suppose that M is endowed with such a metric and let L be a holomorphic line bundle on M. The *degree of* L *with respect to* ω is defined (see [28]), up to a constant factor, by

$$\deg(L) := \int_M F \wedge \omega^{d-1},$$

where F is the curvature of a hermitian connection on L, compatible with $\overline{\partial}_L$. Any two such forms differ by an exact $\partial\overline{\partial}$- exact form. Since $\partial\overline{\partial}\omega^{d-1} = 0$, the degree is independent of the choice of connection and is therefore well-defined. This degree is an extension of that in the Kähler case, where we get the usual topological degree. In general, this degree is not a topological invariant, for it can take values in a continuum.

Having defined the degree of holomorphic line bundles, we define the *degree* of a torsion-free coherent sheaf $\mathcal{V}$ by $\deg(\mathcal{V}) := \deg(\det \mathcal{V})$, where $\det \mathcal{V}$ is the determinant line bundle of $\mathcal{V}$, and the *slope of* $\mathcal{V}$ by

$$\mu(\mathcal{V}) := \deg(\mathcal{V})/rank(\mathcal{V}).$$

Now, we define the notion of stability: A torsion-free coherent sheaf $\mathcal{V}$ on M is *stable* if and only if for every coherent subsheaf $\mathcal{S} \subset \mathcal{V}$ with $0 < rk(\mathcal{S}) < rk(\mathcal{V})$, we have $\mu(\mathcal{S}) < \mu(\mathcal{V})$.

Fix a rank-2 vector bundle V on a minimal non-Kähler elliptic surface X and let δ be its determinant line bundle; there exists a sufficient condition on the spectral cover of V that ensures its stability (see [24]):

Proposition 2 *Suppose that the spectral cover of V includes an irreducible bisection $\overline{C}$ of $J(X)$. Then V is irreducible, and hence it is also stable with respect to any Gauduchon metric.*

Let X be a minimal non-Kähler elliptic surface and consider a pair (c_1, c_2) in $NS(X) \times \mathbb{Z}$. We fix a Gauduchon metric on X. For a fixed line bundle δ on X with $c_1(\delta) = c_1$, let $\mathcal{M}_{\delta,c_2}$ be the moduli space of stable (with respect to the fixed Gauduchon metric) holomorphic rank-2 vector bundles with invariants $\det(V) = \delta$ and $c_2(V) = c_2$. Note that, for any $c_1 \in NS(X)$, one can choose a line bundle δ on X such that

$$c_1(\delta) \in c_1 + 2NS(X) \quad \text{and} \quad m(2, c_1) = -\frac{1}{2}(c_1(\delta)/2)^2;$$

moreover, if there exist line bundles a and δ' such that $\delta = a^2\delta'$, then there is a natural isomorphism between the moduli spaces $\mathcal{M}_{\delta,c_2}$ and $\mathcal{M}_{\delta',c_2}$, defined by $V \to a \otimes V$.

This moduli space can be identified, via the Kobayashi–Hitchin correspondence, with the moduli space of gauge-equivalence classes of Hermitian–Einstein connections in the fixed differentiable rank-2 vector bundle determined by δ and c_2 (see, for example

[28,52]). In particular, if the determinant δ is the trivial line bundle $\mathcal{O}_X$, then there is a one-to-one correspondence between $\mathcal{M}_{\mathcal{O}_X,c_2}$ and the moduli space of $SU(2)$-instantons, that is, anti-selfdual connections. We can define the map

$$G : \mathcal{M}_{\delta,c_2} \to Div(\mathbb{F}_\delta)$$

that associates to each stable vector bundle its graph in $Div(\mathbb{F}_\delta)$, called the *graph map*. In [11,53], the stability properties of vector bundles on Hopf surfaces were studied by analysing the image and the fibres of this map. In particular, it was shown [53] that the moduli space admits a natural Poisson structure with respect to which the graph map is a Lagrangian fibration whose generic fibre is an abelian variety, i.e. the map G admits an algebraically completely integrable system structure. For the general case, the moduli spaces $\mathcal{M}_{\delta,c_2}$ are studied by Brînzănescu and Moraru [24].

We have the following results (see [24]):

Theorem 5 *Let $X \xrightarrow{\pi} B$ be a non-Kähler elliptic surface and let $\mathcal{M}_{\delta,c_2}$ be defined as above. Then:*

(i) There are necessary and sufficient conditions such that $\mathcal{M}_{\delta,c_2}$ is nonempty (see Theorem 4).
(ii) If $c_2 - c_1^2/2 > g - 1$ (g is the genus of B), the moduli space $\mathcal{M}_{\delta,c_2}$ is smooth on the open dense subset of regular bundles (a regular bundle is a vector bundle for which its restriction to any fibre has its automorphism group of the smallest dimension).
(iii) If $g \le 1$, the moduli space $\mathcal{M}_{\delta,c_2}$ is smooth of dimension $8\Delta(2, c_1, c_2)$ and $G : \mathcal{M}_{\delta,c_2} \to Div(\mathbb{F}_\delta)$ is an algebraically completely integrable Hamiltonian system.
(iv) The generic fibre of the graph map $G : \mathcal{M}_{\delta,c_2} \to Div(\mathbb{F}_\delta)$ is a Prym variety (for Prym varieties, see [55]).
(v) Let $\mathbb{P}_{\delta,c_2}$ be the set of divisors in $\mathbb{F}_\delta$ of the form $\Sigma_1^k f_i + A$, where A is a section of the ruling and the f_i's are fibres of the ruled surface, that are numerically equivalent to $\eta_*(B_0) + c_2 f$. For $c_2 \ge 2$, the graph map is surjective on $\mathbb{P}_{\delta,c_2}$. For $c_2 < 2$, see [24].
(vi) Explicit descriptions of the the singular fibres of G are given, see [24].

Special results on the moduli space $\mathcal{M}_{\delta,c_2}$ in the case of primary Kodaira surfaces are given in [2].

4 Vector bundles on higher-dimensional non-Kähler elliptic fibrations

Let M be an n-dimensional compact complex manifold, $T = V/\Lambda$ an m-dimensional complex torus and $X \xrightarrow{\pi} M$ a principal bundle with fiber T. The theory of principal torus bundles is developed in great detail in [47]; see also [25]. It is well known that such bundles are described by elements of $H^1(M, \mathcal{O}_M(T))$, where $\mathcal{O}_M(T)$ denotes the sheaf of local holomorphic maps from M to T. Considering the exact sequence of groups

$$0 \to \Lambda \to V \to T \to 0$$

and taking local sections we obtain the following exact sequence

$$0 \to \Lambda \to \mathcal{O}_M \otimes V \to \mathcal{O}_M(T) \to 0.$$

Passing to the cohomology we have the long exact sequence

$$\cdots \to H^1(M, \Lambda) \to H^{0,1}_M \otimes V \to H^1(M, \mathcal{O}_M(T)) \xrightarrow{c^{\mathbb{Z}}}$$
$$\xrightarrow{c^{\mathbb{Z}}} H^2(M, \Lambda) \to H^{0,2}_M \otimes V \to \cdots$$

By taking the image of the co-cycle defining the bundle via the map $c^{\mathbb{Z}}$ we obtain a characteristic class $c^{\mathbb{Z}}(X) \in H^2(M, \Lambda) = H^2(M, \mathbb{Z}) \otimes \Lambda$ and also a characteristic class $c(X) \in H^2(M, \mathbb{C}) \otimes V$.

Concerning some important sheaves on X we have the identifications (see [47]):

$$\mathcal{K}_X = \pi^* \mathcal{K}_M, \quad R^i \pi_* \mathcal{O}_X = \mathcal{O}_M \otimes_{\mathbb{C}} H^{0,i}(T) \tag{1}$$

and the exact sequence

$$0 \to \Omega^1_M \to \pi_* \Omega^1_X \to \mathcal{O}_M \otimes_{\mathbb{C}} H^{1,0}(T) \to 0. \tag{2}$$

All the information concerning the topology of the bundle $X \to M$ is given by the following invariants

(a) The exact sequence (2) gives rise to an element $\gamma \in \mathrm{Ext}^1(\mathcal{O}_M \otimes H^{1,0}(T), \Omega^1_M) = H^1(\Omega^1_M) \otimes H^{1,0}(T)^*$. Thus γ is a map $H^{1,0}(T) \to H^{1,1}(M)$.

(b) The first non-trivial d_2- differential in the Leray spectral sequence ($d_2 : E_2^{0,1} \to E_2^{2,0}$) of the sheaf $\mathbb{C}_X$. We obtain in this way a map $\delta : H^1(T, \mathbb{C}) \to H^2(M, \mathbb{C})$. In the same way we may define the maps $\delta^{\mathbb{Z}} : H^1(T, \mathbb{Z}) \to H^2(M, \mathbb{Z})$.

(c) The first non-trivial d_2-differential in the Leray spectral sequence of $\mathcal{O}_X$, where $d_2 : H^0(R^1 \pi_* \mathcal{O}_X) \to H^2(\pi_* \mathcal{O}_X)$. Via the identifications (1) we get a map $\epsilon : H^{0,1}(T) \to H^{0,2}(M)$.

(d) The characteristic classes $c^{\mathbb{Z}}(X)$ and $c(X)$ defined above.

These invariants are related by the following theorem of Höfer (see [47]):

Theorem 6 *Let $X \xrightarrow{\pi} M$ be a holomorphic principal T-bundle. Then:*

1. *The Borel spectral sequence* ([10]) ${}^{p,q}E_2^{s,t} = \sum H^{i,s-i}(M) \otimes H^{p-i,t-p+i}(T)$ *degenerates on E_3- level and the d_2-differential is given by ϵ and γ.*
2. *The Leray spectral sequence $E_2^{s,t} = H^s(M, \mathbb{C}) \otimes H^t(T, \mathbb{C})$ degenerates on E_3- level and the d_2- differential is given by δ.*
3. *Via the identification $H^1(T, \mathbb{Z}) = Hom(\Lambda, \mathbb{Z})$ the characteristic class $c^{\mathbb{Z}}$ and the map $\delta^{\mathbb{Z}}$ coincide.*
4. *δ is determined by $\delta^{\mathbb{Z}}$ via scalar extension.*

5. *If $H^2(M)$ has Hodge decomposition then δ determines ϵ and γ and conversely.*

Firstly, in this section, we shall be concerned with the study of the (coarse) relative moduli space of fibrewise degree-zero line bundles over a principal elliptic bundle $X \to S$, where S is a compact complex manifold, with fiber $E := E_\tau := \mathbb{C}/\Lambda$ ($\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$). Also we make the assumption that $\delta \neq 0$. In particular, $X \to S$ does not have the topology of a product. We should note here that if S is Kähler, then X is non-Kähler if and only if $\delta \neq 0$, see [47].

We shall need in the sequel the following result of Deligne, [34], in the formulation of [47], Proposition 5.2.

Theorem 7 *Let $X \to S$ be a principal elliptic bundle. Then the following statements are equivalent:*

(a) The Leray spectral sequence for $\mathbb{C}_X$ degenerates at the E_2-level;
(b) $\delta : H^1(E, \mathbb{C}) \to H^2(X, \mathbb{C})$ is the zero map;
(c) The restriction map $H^2(X, \mathbb{C}) \to H^2(E, \mathbb{C})$ takes a non-zero value in $H_E^{1,1}$.

In our case the preceding theorem has a very important consequence

Corollary 1 *Let $X \to S$ be a principal elliptic bundle with S a compact complex manifold and $\delta \neq 0$. Then for any vector bundle $\mathcal{F}$ over X and any $s \in S$ the bundle $\mathcal{F}|_{X_s}$ has degree 0.*

Let $X \xrightarrow{\pi} S$ be an elliptic principal bundle with typical fibre an elliptic curve E_τ and base S a smooth manifold. Let $F : (An/S)^{op} \to (Sets)$ be the functor from the category of analytic spaces over S to the category of sets, given, for any commutative diagram

$$\begin{array}{ccc} X_T & \longrightarrow & X \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ T & \longrightarrow & S, \end{array} \tag{3}$$

where $X_T := X \times_S T$, by

$$F(T) := \{\mathcal{L} \text{ invertible on } X_T \mid \deg(\mathcal{L}|_{X_{T,t}}) = 0, \quad \text{for all} \quad t \in T\}/\sim\},$$

where $\mathcal{L}_1 \sim \mathcal{L}_2$ if there is a line bundle L on T such that $\mathcal{L}_1 \simeq \mathcal{L}_2 \otimes \pi^* L$.

A variety J over S will be called the *relative Jacobian of* X if

(i) it corepresents the functor F, see [49] Definition 2.2.1, i.e. there is a natural transformation $F \xrightarrow{\sigma} \mathrm{Hom}_S(-, J)$ and for any other variety N/S with a natural transformation $F \xrightarrow{\sigma'} \mathrm{Hom}_S(-, N)$ there is a unique S-morphism $J \xrightarrow{\nu} N$ such that $\nu_* \circ \sigma = \sigma'$.
(ii) for any point $s \in S$ the map $F(\{s\}) \to \mathrm{Hom}_S(\{s\}, J) \simeq J_s$ is bijective. Then each fibre J_s is the Jacobian of the fibre $X_s \simeq E$.

If X is projective, the existence of the relative Jacobian is well known, because it can be identified with the coarse relative moduli space of stable locally free sheaves of rank 1 and degree 0 on the fibres of X, see [30,49]. The relative Jacobian exists also in our non-Kähler case. It is just the product $S \times E^*$ and has the following special properties (see [26]):

Theorem 8 *(i) The functor F is corepresented by $J := S \times E^*$.*
(ii) For any point $s \in S$ the map $F(\{s\}) \to \mathrm{Hom}_S(\{s\}, J) \simeq J_s \simeq E^*$ is bijective.
(iii) The map $\sigma(T)$ is injective for any complex space T.
(iv) The functor F is locally representable by $J = S \times E^*$, i.e. if $U \subset S$ is a trivializing open subset, $\sigma(U)$ is bijective.

It will follow from Theorem 9 that the relative Jacobian $J = S \times E^*$ is only a coarse moduli space under our assumption on X. However, by property (iv) of the theorem one can find a system of local universal sheaves which will form a twisted sheaf as in [30], Chapter 4.

In the following we replace the relative Jacobian J by $S \times E$ via the canonical isomorphism between E and E^*. Then the local trivializations $X_i \xrightarrow{\theta_i} S_i \times E$ are at the same time isomorphisms between X_i and $J_i := S_i \times E$. The local universal sheaves $\mathcal{U}_i$ on $X_{iJ} = J \times_S X_i = J_i \times_{S_i} X_i$ are then given as pull backs of the universal sheaf $\mathcal{O}_{E\times E}(\Delta) \otimes p_2^* \mathcal{O}_E(-p_0)$ for the classical Jacobian of the elliptic curve E, after fixing an origin $p_0 \in E$ and where Δ is the diagonal.

Denoting by ρ_i the composition of maps

$$X_{iJ} \xrightarrow{id\times\theta_i} J \times_S (S_i \times E) \simeq S_i \times E \times E \to E \times E,$$

and by p_X the projection from X_{iJ} to X_i, the local universal sheaf becomes

$$\mathcal{U}_i = \rho_i^*(\mathcal{O}_{E\times E}(\Delta) \otimes p_2^* \mathcal{O}_E(-p_0)) \simeq \mathcal{O}_{X_{iJ}}(\Gamma_i) \otimes p_X^* \mathcal{O}_{X_i}(-s_i),$$

where Γ_i is the inverse of the diagonal (or the graph of the map θ_i) and s_i is the section of X_i corresponding to the reference point p_0 under the isomorphism θ_i, see [30], Proposition 4.2.3.

To measure the failure of these bundles to glue to a global universal one let us consider the line bundles $\mathcal{M}_{ij} := \mathcal{U}_i \otimes \mathcal{U}_j^{-1}$ over $J \times_S X_{ij}$. Then the restriction of $\mathcal{M}_{ij}$ to a fibre X_s of the projection $J \times_s X_i \xrightarrow{q_i} J$ is trivial because both $\mathcal{U}_j$ and $\mathcal{U}_i$ restrict to isomorphic sheaves. It follows that there are invertible sheaves $\mathcal{F}_{ij}$ on $J_{ij} = S_{ij} \times E$ such that $\mathcal{M}_{ij} = q_i^* \mathcal{F}_{ij}$.

This collection of line bundles satisfies the following properties:

1. $\mathcal{F}_{ii} = \mathcal{O}_{J_i}$;
2. $\mathcal{F}_{ji} = \mathcal{F}_{ij}^{-1}$;
3. $\mathcal{F}_{ij} \otimes \mathcal{F}_{jk} \otimes \mathcal{F}_{ki} =: \mathcal{F}_{ijk}$ is trivial, with trivialization induced by the canonical one of $\mathcal{M}_{ij} \otimes \mathcal{M}_{jk} \otimes \mathcal{M}_{ki}$;
4. $\mathcal{F}_{ijk} \otimes \mathcal{F}_{jkl}^{-1} \otimes \mathcal{F}_{kli} \otimes \mathcal{F}_{lij}^{-1}$ is canonically trivial.

These conditions tell us that the collection $\{\mathcal{F}_{ij}\}$ represents a gerb (see [37]) and gives rise to an element $\alpha \in H^2(J, \mathcal{O}_J^*)$. More explicitly, α is defined as follows. We may assume that the sheaves $\mathcal{F}_{ij}$ are already trivial with trivializations $a_{ij} : \mathcal{O}_J \simeq \mathcal{F}_{ij}$ over J_{ij}.

If $c_{ijk} : \mathcal{O}_J \simeq \mathcal{F}_{ijk}$ is the isomorphism which is induced by the canonical trivialization of $\mathcal{M}_{ij} \otimes \mathcal{M}_{jk} \otimes \mathcal{M}_{ki}$, then

$$a_{ij} \otimes a_{jk} \otimes a_{ki} = \alpha_{ijk} c_{ijk} \tag{4}$$

with scalar functions α_{ijk} which then define a cocycle for the sheaf $\mathcal{O}_J^*$, thus defining the class $\alpha \in H^2(J, \mathcal{O}_J^*)$, see [30], Section 4.3. It is straightforward to prove:

Lemma 1 *The sheaves $\mathcal{U}_i$ can be glued to a global universal sheaf if and only if the class $\alpha = 0$.*

The element α is related to the element $\xi \in H^1(S, \mathcal{O}_S(E))$ which is defined by the cocycle of the elliptic bundle $X \to S$, using the Ogg–Shafarevich group $\text{Ш}_S(J)$ of J, see [30], Section 4.4. There is an exact sequence

$$0 \to \text{Br}(S) \to \text{Br}(J) \xrightarrow{\pi} \text{Ш}_S(J) \to 0,$$

where $\text{Br}(S) \simeq H^2(J, \mathcal{O}_J^*)$ is the analytic Brauer group of S and $\text{Ш}_S(J)$ is isomorphic to $H^1(S, \mathcal{O}_S(E))$ in our setting. We have the following result (see [30], Theorem 4.4.1):

Theorem 9 $\xi = \pi(\alpha)$.

Because $\xi \neq 0$ in our case, $\alpha \neq 0$, and thus the local universal sheaves cannot be glued to a global universal sheaf by preserving the bundle structure on the elliptic fibres.

The collection of local universal sheaves above can be considered as an α-twisted sheaf with which one can define a Fourier–Mukai transform. Recall the definition of an α-twisted sheaf on a complex space or on an appropriate scheme X. Let $\alpha \in C^2(\mathfrak{U}, \mathcal{O}_X^*)$ be a Čech 2-cocycle, given by an open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ and sections $\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)$. An α-twisted sheaf on X will be a pair of families $(\{\mathcal{F}_i\}_{i \in I}, \{\varphi_{ij}\}_{i,j,\in I})$ with $\mathcal{F}_i$ a sheaf of $\mathcal{O}_X$-modules on U_i and $\varphi_{ij} : \mathcal{F}_j|_{U_i \cap U_j} \to \mathcal{F}_i|_{U_i \cap U_j}$ isomorphisms such that

- φ_{ii} is the identity for all $i \in I$.
- $\varphi_{ij} = \varphi_{ji}^{-1}$, for all $i, j \in I$.
- $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{kl}$ is multiplication by α_{ijk} on $\mathcal{F}_i|_{U_i \cap U_j \cap U_k}$ for all $i, j, k \in I$.

It is easy to see that the coherent α-twisted sheaves on X make up an abelian category and thus give rise to a derived category $\mathcal{D}^\flat(X, \alpha)$. For further properties of α-twisted sheaves, see [30].

With the notation above, the family $(\mathcal{U}_i)$ becomes a twisted sheaf $\mathcal{U}$ w.r.t. the cocycle $p_J^* \alpha$ of the sheaf $\mathcal{O}_{J \times_S X}^*$ as follows. The trivializations a_{ij} of the $\mathcal{F}_{ij}$ induce isomorphisms $\phi_{ij} : \mathcal{U}_j \simeq \mathcal{U}_i$ which satisfy the definition of a twisted sheaf because of identity 4. We also need the dual $\mathcal{V}$ of $\mathcal{U}$ on $J \times_S X$ which locally over S_i is given by

$$\mathcal{V}_i = \rho_i^*(\mathcal{O}_{E\times E}(-\Delta) \otimes p_2^*\mathcal{O}_E(p_0)) \simeq \mathcal{O}_{X_{iJ}}(-\Gamma_i) \otimes p_X^*\mathcal{O}_{X_i}(s_i).$$

It follows that $\mathcal{V}_i$ is α^{-1}-twisted. We let $\mathcal{V}^0$ and $\mathcal{U}^0$ denote the extensions of $\mathcal{V}$ and $\mathcal{U}$ to $J \times X$ by zero.

The following theorem (see [26]) supplies us with the main tool for the treatment of the moduli spaces $M_X(n, 0)$ of relatively semistable vector bundles on X of rank n and degree 0 on the fibres X_s (for vector bundles on elliptic curves see [3,61]). It is an analogue of the Theorem 6.5.4 [30] (see also [31]):

Theorem 10 *Let $X \xrightarrow{\pi} S$ be an elliptic principal fiber bundle, where S has trivial canonical bundle. Let $\alpha \in Br(J)$ be the obstruction to the existence of the universal sheaf on $J \times_S X$ and let $\mathcal{U}$ be the associated $p_J^*(\alpha)$-twisted universal sheaf on $J \times_S X$ with its dual $\mathcal{V}$ as above.*

Then the twisted Fourier–Mukai transform $\Psi : \mathcal{D}^\flat(J, \alpha) \to \mathcal{D}^\flat(X)$, given by $\Psi(\mathcal{F}) := Rp_{X*}(\mathcal{V}^0 \otimes^L Lp_J{}^*\mathcal{F})$ is an equivalence of categories, where p_J and p_X are the product projections

$$J \xleftarrow{p_J} J \times X \xrightarrow{p_X} X \tag{5}$$

Note here that $\mathcal{V}^0 \otimes^L Lp_J{}^*\mathcal{F})$ is a complex in the category of untwisted sheaves on $J \times X$.

Remark 6 Similar results were obtained in different settings by Ben-Bassat [9] and Burban–Kreussler [29]. Related results were obtained in [32].

In the sequel we shall work with the adjoint transform $\Phi(-) = Rp_{J*}(\mathcal{U}^0 \otimes^L Lp_X^*(-))$ of Ψ with kernel $\mathcal{U}^0$. It is the reverse equivalence, see [14], 8.4, [6,48,56] for the untwisted situation.

Now, we shall apply the twisted Fourier–Mukai transform to the moduli problem for rank-n relatively semi-stable vector bundles on the principal elliptic bundle X. By Deligne's theorem (Theorem 7), the degree of the restriction $\mathcal{F}_s$ of any vector bundle $\mathcal{F}$ on X is 0 for any $s \in S$. Therefore we consider the set $MS_X(n, 0)$ of rank-n vector bundles on X which are fibrewise semistable (see [3,42]) and of degree zero, together with its quotient

$$M_X(n, 0) := MS_X(n, 0)/\sim$$

of equivalence classes, where two bundles are defined to be equivalent if they are fibrewise S-equivalent (for S-equivalence see [49]).

We denote by $\Phi^i(\mathcal{F})$ the i-th term of the complex $\Phi(\mathcal{F})$. We say that the sheaf $\mathcal{F}$ is Φ-WIT$_i$ (the weak index theorem holds) if $\Phi^i(\mathcal{F}) \neq 0$ and $\Phi^j(\mathcal{F}) = 0$ for any $j \neq i$. Moreover if $\mathcal{F}$ is WIT$_i$ and $\Phi^i(\mathcal{F})$ is locally free we say that $\mathcal{F}$ is IT$_i$, see [54].

Let $\mathcal{F}$ be a WIT_1 sheaf on X. The spectral cover $C(\mathcal{F})$ of $\mathcal{F}$ is the 0-th Fitting subscheme (see [39,54]) of J given by the Fitting ideal sheaf $Fitt_0(\Phi^1(\mathcal{F}))$ of $\Phi^1(\mathcal{F})$. For details see [26].

In this way we obtain a map from $M_X(n,0)$ to $S \times \text{Sym}^n E$, where $\text{Sym}^n E := E^n/\mathfrak{S}_n$ is the n-th symmetric power of E as the quotient of E^n by the symmetric group $\mathfrak{S}_n$. Then $S \times \text{Sym}^n E$ is a complex manifold of dimension $n + \dim(S)$ and can be thought of as the relative space of cycles of degree n in E. We will show that this map is part of a transformation of functors with target $\text{Hom}_S(-, S \times \text{Sym}^n E)$ and that $S \times \text{Sym}^n E$ corepresents the moduli functor $\mathcal{M}_X(n,0)$ for $M_X(n,0)$ defined as follows.

For any complex space T over S let the set $\mathcal{M}_X(n,0)(T)$ be defined by

$$\mathcal{M}_X(n,0)(T) := \mathcal{MS}_X(n,0)(T)/\sim,$$

where $\mathcal{MS}_X(n,0)(T)$ is the set of vector bundles on X_T of rank n and fibre degree 0, and where the equivalence relation $\mathcal{F} \sim \mathcal{G}$ is defined by S-equivalence of the restricted sheaves $\mathcal{F}_t$ and $\mathcal{G}_t$ on the fibres X_{T_t}. The functor property is then defined via pull backs.

We are going to describe the spectral cover as a functor below. For that let $T \to S$ be a complex space over S and let Φ_T be the Fourier–Mukai transform for the product $J_T \times X_T$ with the pull back $\mathcal{U}_T$ of $\mathcal{U}$ as kernel. By [7], Proposion 2.7 and Corollary 2.12, any bundle $\mathcal{F}_T$ in $\mathcal{MS}_X(n,0)(T)$ is also $\Phi_T - WIT_1$ and admits a spectral cover $C(\mathcal{F}_T) \subset T \times E$ defined by the Fitting ideal $Fitt_0\Phi^1_T(\mathcal{F}_T)$ (see also [45]).

Lemma 2 *If T is reduced, then $C(\mathcal{F}_T)$ is flat over T.*

For the proof, one uses the Douady's flatness criterion [38]; see [26].

Lemma 3 *The spectral cover is compatible with base change: For any morphism $h : T' \to T$ over S and any bundle $\mathcal{F}_T$ in $\mathcal{MS}_X(n,0)(T)$,*

$$h^*C(\mathcal{F}_T) \simeq C(h^*\mathcal{F}_T)$$

For the proof, see [26].

The spectral covers $C(\mathcal{F}_T)$ lead us to consider the relative Douady functors

$$\mathcal{D}^n : (An/S)^{op} \to (Sets),$$

where (An/S) denotes the category of complex analytic spaces over S and where a set $\mathcal{D}^n(T)$ for a morphism $T \to S$ is defined as the set of analytic subspaces $Z \subset T \times E$ which are flat over T and have 0-dimensional fibres of constant length n. The Douady functor $\mathcal{D}^n$ is represented by a complex space $D^n(S \times E/S)$ over S, see [57]. For a point $s \in S$, $\mathcal{D}^n(\{s\})$ is the set of 0-dimensional subspaces of length n and can be identified with the symmetric product $\text{Sym}^n(E)$ because it is well known that the Hilbert–Chow morphism, in our case the Douady–Barlet morphism, $\mathcal{D}^n(\{s\}) \to \{s\} \times \text{Sym}^n(E)$ is an isomorphism for the smooth curve E, see [5] Ch.V. It is then easy to show that also the relative Douady–Barlet morphism $D^n(S \times E/S) \to S \times \mathrm{Sym}^n(E)$ is an isomorphism. This implies that for any complex space T over S there is a bijection

$$\mathcal{D}^n(T) \xrightarrow{\sim} \mathrm{Hom}_S(T, S \times \mathrm{Sym}^n(E)). \tag{6}$$

One should note here that the behavior of families of cycles is more difficult to describe than of those for the Douady space.

Let now $\mathcal{D}^n_r$ resp. $\mathcal{M}_X(n,0)_r$ be the restriction of the functors $\mathcal{D}^n$ and $\mathcal{M}_X(n,0)$ to the category (Anr/S) of reduced complex analytic spaces. By the Lemmas 2 and 3 the spectral covers give rise to a transformation of functors

$$\mathcal{M}_X(n,0)_r \xrightarrow{\gamma} \mathcal{D}^n_r \simeq \mathrm{Hom}_S(-, S \times \mathrm{Sym}^n(E)), \tag{7}$$

where for a reduced space T over S and for a class $[\mathcal{F}_T]$ in $\mathcal{M}_X(n,0)(T)$ we have $\gamma(T)(\mathcal{F}_T) = C(\mathcal{F}_T)$. Note that by flatness and compatibility with restriction to fibres, $C(\mathcal{F}_T)$ depends only on the equivalence class of $\mathcal{F}_T$. We are now able to present the theorem which generalises Theorem 8, see [26].

Theorem 11 *The spectral cover induces a transformation of functors* $\gamma : \mathcal{M}_X(n,0)_r \to \mathrm{Hom}_S(-, S \times \mathrm{Sym}^n(E))$ *with the following properties.*

(i) The functor $\mathcal{M}_X(n,0)_r$ *is corepresented by* $S \times \mathrm{Sym}^n(E)$ *via the transformation* γ,
(ii) For any point $s \in S$ *the induced map* $M_{X_s}(n,0) \to \mathrm{Sym}^n(E)$ *is bijective.*
(iii) The map $\gamma(T)$ *is injective for any reduced complex space* T *over* S.
(iv) $\mathcal{M}_X(n,0)_r$ *is locally representable by* $S \times \mathrm{Sym}^n(E)$, *i.e. if* $U \subset S$ *is a trivializing open subset for* X *and* T *is a complex space over* U, *then* $\gamma(T)$ *is bijective.*

The proof of the next result is based also on the spectral cover; see [27].

Theorem 12 *Consider an elliptic principal bundle* $X \xrightarrow{\pi} S$ *over a surface* S, *with at least one non-zero characteristic (Chern) class and with invariant* $\epsilon = 0$. *If* S *has no curves, then, up to a twist by a line bundle, every rank-2 irreducible vector bundle* V *on* X *is a pull-back from* S.

When S is a projective manifold, a similar result was obtained by Verbitsky; see [63].

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