

Height estimates and half-space type theorems in weighted product spaces with nonnegative Bakry–Émery–Ricci curvature

Henrique F. de Lima¹ · Márcio S. Santos²

Received: 30 March 2016 / Accepted: 7 November 2016 / Published online: 16 November 2016
© Università degli Studi di Ferrara 2016

Abstract We prove height estimates concerning compact hypersurfaces with nonzero constant weighted mean curvature and whose boundary is contained into a slice of a weighted product space of nonnegative Bakry–Émery–Ricci curvature. As applications of our estimates, we obtain half-space type results related to complete noncompact hypersurfaces properly immersed in such an ambient space.

Keywords Weighted product spaces · Bakry–Émery–Ricci tensor · Compact hypersurfaces · Height estimates · Half-space type theorems

Mathematics Subject Classification Primary 53C42; Secondary 53B30 · 53C50

1 Introduction

In 1969, Heinz [17] showed that a compact graph of positive constant mean curvature H in the 3-dimensional Euclidean space \mathbb{R}^3 with boundary on a plane can reach at most height $\frac{1}{H}$ from this plane. A hemisphere in \mathbb{R}^3 of radius $\frac{1}{H}$ shows that this estimate is optimal. In particular, Heinz's result motivated several authors to approach the problem of obtaining a priori estimates for the height function of a compact hypersurface whose

✉ Henrique F. de Lima
henrique@dme.ufcg.edu.br

Márcio S. Santos
marciomat28@gmail.com

¹ Departamento de Matemática, Universidade Federal de Campina Grande, Campina Grande, Paraíba 58429-970, Brazil

² Departamento de Matemática, Universidade Federal da Paraíba, João Pessoa, Paraíba 58051-900, Brazil

boundary is contained into a slice of a Riemannian product space (see, for instance, [2, 3, 8, 13, 18, 23]). More recently, extending the ideas of Alías and Dajczer [4], García-Martínez, Impera and Rigoli [15] proved sharp height estimates concerning compact hypersurfaces immersed in certain types of warped product spaces, with some constant higher order mean curvature and whose boundary is contained into a slice.

Concerning the Lorentzian setting, López [22] obtained a sharp estimate for the height of compact constant mean curvature spacelike surfaces with boundary contained in a spacelike plane of the 3-dimensional Lorentz–Minkowski space \mathbb{L}^3 . Later on, the first author [10] established a height estimate for compact spacelike hypersurfaces with some positive constant higher order mean curvature and whose boundary is contained in a spacelike hyperplane of the $(n + 1)$ -dimensional Lorentz–Minkowski space \mathbb{L}^{n+1} . As in [22], through the computation of the height of the hyperbolic caps of \mathbb{L}^{n+1} , he showed that his estimate is sharp. Afterwards, this same author jointly with Colares [9] generalized the results of [10] to the context of Lorentzian product spaces of the type $-\mathbb{R} \times \mathbb{P}^n$, where the fiber \mathbb{P}^n is supposed to have nonnegative sectional curvature. Proceeding with this picture, García-Martínez and Impera [14] extended the results of [9] to the so-called generalized Robertson–Walker spacetimes.

On the other hand, into the branch of the geometric analysis, many problems lead us to consider Riemannian manifolds endowed with a measure that has a smooth positive density with respect to the Riemannian one. This turns out to be compatible with the metric structure of the manifold and the resulting spaces are the *weighted manifolds*, which are also called manifolds with density or smooth metric measure spaces in the literature.

More precisely, given a complete n -dimensional Riemannian manifold (M^n, g) and a smooth function $f : M^n \rightarrow \mathbb{R}$, the weighted manifold M_f^n associated to M^n and f is the triple $(M^n, g, d\mu = e^{-f} dM)$, where dM denotes the standard volume element of M^n . Appearing naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows and many other objects, weighted manifolds are proved to be important nontrivial generalizations of Riemannian manifolds and, nowadays, there are several geometric investigations concerning them. For a brief overview of results in this scope, we refer to the articles of Morgan [24] and Wei–Wylie [27].

We point out that a theory of Ricci curvature for weighted manifolds goes back to Lichnerowicz [20, 21] and it was later developed by Bakry and Émery in the seminal work [6], where they introduced the *Bakry–Émery–Ricci tensor* Ric_f of a weighted manifold M_f^n as being the following extension of the standard Ricci tensor Ric of M^n :

$$\text{Ric}_f = \text{Ric} + \text{Hess} f. \quad (1.1)$$

So, it is natural to try to extend results stated in terms of the Ricci curvature to analogous results for the Bakry–Émery–Ricci tensor.

Here, our purpose is to revisit the problem of measure the vertical height of a compact hypersurface with nonzero constant weighted mean curvature and whose boundary is supposed, now, to be contained into a slice of a weighted product space. In this setting, under the assumption that the ambient space has nonnegative Bakry–Émery–Ricci curvature, we are able to establish optimal height estimates (see

Theorems 1 and 3). As application of these estimates, we extend the ideas of [10, 14] and [15] in order to prove half-space type theorems (see Theorems 2 and 4).

2 Preliminaries

In what follows, let us consider an $(n + 1)$ -dimensional product space \overline{M}^{n+1} of the form $I \times \mathbb{P}^n$, where $I \subset \mathbb{R}$ is an open interval, \mathbb{P}^n is an n -dimensional connected Riemannian manifold and \overline{M}^{n+1} is endowed with the standard product metric

$$\langle \cdot, \cdot \rangle = \epsilon \pi_I^*(dt^2) + \pi_{\mathbb{P}}^*(\langle \cdot, \cdot \rangle_{\mathbb{P}}),$$

where $\epsilon = \pm 1$, π_I and $\pi_{\mathbb{P}}$ denote the canonical projections from $I \times \mathbb{P}^n$ onto each factor, and $\langle \cdot, \cdot \rangle_{\mathbb{P}}$ is the Riemannian metric on \mathbb{P}^n . For simplicity, we will just write $\overline{M}^{n+1} = \epsilon I \times \mathbb{P}^n$ and $\langle \cdot, \cdot \rangle = \epsilon dt^2 + \langle \cdot, \cdot \rangle_{\mathbb{P}}$. In this setting, for a fixed $t_0 \in I$, we say that $\mathbb{P}_{t_0}^n = \{t_0\} \times \mathbb{P}^n$ is a *slice* of \overline{M}^{n+1} .

Throughout this paper, we will consider Σ^n a connected hypersurface immersed into \overline{M}^{n+1} . In the case where \overline{M}^{n+1} is Lorentzian (that is, when $\epsilon = -1$) we will assume that Σ^n is a *spacelike hypersurface*, that is, the metric induced on Σ^n via the immersion is a Riemannian metric. Since ∂_t is a globally defined timelike vector field on $-I \times \mathbb{P}^n$, it follows that there exists a unique unitary timelike normal field N globally defined on Σ^n which is in the same time-orientation as ∂_t , so that its corresponding angle function $\Theta = \langle N, \partial_t \rangle$ satisfies $\Theta \leq -1$. In that case, we will refer to N as the future-pointing Gauss map of Σ^n . When \overline{M}^{n+1} is a Riemannian product space (that is, when $\epsilon = 1$), Σ^n will be assumed to be a *two-sided* hypersurface in \overline{M}^{n+1} . This condition means that there is a globally defined unit normal vector field N .

Denoting by $\overline{\nabla}$, ∇ and $\widehat{\nabla}$ the gradients with respect to the metrics of $\epsilon I \times_{\rho} \mathbb{P}^n$, Σ^n and \mathbb{P} , respectively, a simple computation shows that the gradient of π_I on \overline{M}^{n+1} is given by

$$\overline{\nabla} \pi_I = \epsilon \langle \overline{\nabla} \pi_I, \partial_t \rangle \partial_t = \epsilon \partial_t. \tag{2.1}$$

So, from (2.1) we conclude that the gradient of the (vertical) height function $h = (\pi_I)|_{\Sigma}$ of Σ^n is given by

$$\nabla h = (\overline{\nabla} \pi_I)^{\top} = \epsilon \partial_t^{\top} = \epsilon \partial_t - \Theta N, \tag{2.2}$$

where $(\cdot)^{\top}$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M}^{n+1})$ along Σ^n . Thus, we get the following relation

$$|\nabla h|^2 = \epsilon(1 - \Theta^2), \tag{2.3}$$

where $|\cdot|$ denotes the norm of a vector field on Σ^n .

Now, let $\epsilon I \times \mathbb{P}^n$ be a semi-Riemannian product space endowed with a weighted function f . For a Riemannian hypersurface Σ^n immersed in $\epsilon I \times \mathbb{P}^n$, the *f-divergence*

operator on Σ^n is defined by

$$\operatorname{div}_f(X) = e^f \operatorname{div}(e^{-f} X),$$

for all tangent vector field X on Σ^n and, for a smooth function $u : \Sigma^n \rightarrow \mathbb{R}$, its drifting Laplacian is given by

$$\Delta_f u = \operatorname{div}_f(\nabla u) = \Delta u - \langle \nabla u, \nabla f \rangle. \tag{2.4}$$

According to Gromov [16], the weighted mean curvature, or simply f -mean curvature, H_f of Σ^n is given by

$$nH_f = nH + \epsilon \langle \bar{\nabla} f, N \rangle \tag{2.5}$$

where H denotes its standard mean curvature function of Σ^n with respect to its orientation N .

3 The Riemannian setting

It follows from a splitting theorem due to Fang, Li and Zhang (see [12, Theorem 1.1]) that if a weighted product space $I \times \mathbb{P}^n$ with bounded weighted function f is such that $\overline{\operatorname{Ric}}_f \geq 0$, then f must be constant along I . So, motivated by this result, along this section we will consider weighted product spaces $I \times \mathbb{P}^n$ whose weighted function f does not depend on the parameter $t \in I$, that is, $\langle \bar{\nabla} f, \partial_t \rangle = 0$ and, for the sake of simplicity, we will denote them by $I \times \mathbb{P}_f^n$.

Now, we present our first height estimate.

Theorem 1 *Let $I \times \mathbb{P}_f^n$ be a weighted Riemannian product space with $\overline{\operatorname{Ric}}_f \geq 0$ and let Σ^n be a compact hypersurface with boundary contained into the slice $\{s\} \times \mathbb{P}^n$, for some $s \in I$, and whose angle function Θ does not change sign. If Σ^n has nonzero constant f -mean curvature such that $nH_f^2 \leq |A|^2$, where A denotes the Weingarten operator of Σ^n with respect to its unit normal vector field N , then the height function h of Σ^n satisfies*

$$|h - s| \leq \frac{1}{|H_f|}. \tag{3.1}$$

Proof Initially, we will show that

$$\Delta_f \Theta = -\Theta(\widetilde{\operatorname{Ric}}_f(N^*, N^*) + |A|^2), \tag{3.2}$$

where $\widetilde{\operatorname{Ric}}_f$ stands for the Bakry–Émery–Ricci tensor of \mathbb{P}^n and $N^* = N - \Theta \partial_t$ is the projection of N onto \mathbb{P}^n .

Indeed, from Proposition 6 of [5] (see also Proposition 1 of [1]) we have that

$$\Delta \Theta = -n \partial_t^\top(H) - \Theta(\widetilde{\operatorname{Ric}}(N^*, N^*) + |A|^2), \tag{3.3}$$

where ∂_t^\top stands for the tangential component of ∂_t along Σ^n and $\widetilde{\text{Ric}}$ is the Ricci tensor of \mathbb{P}^n .

On the other hand, using our assumption that $\langle \bar{\nabla} f, \partial_t \rangle = 0$ and that H_f is constant, from (1.1) and (2.5) we get

$$\begin{aligned} \partial_t^\top(H) &= \partial_t^\top \left(H_f - \frac{1}{n} \langle \bar{\nabla} f, N \rangle \right) \\ &= -\frac{1}{n} \partial_t^\top \langle \bar{\nabla} f, N \rangle \\ &= \frac{1}{n} \Theta \overline{\text{Hess}}_f(N, N) + \frac{1}{n} \langle A(\partial_t^\top), \bar{\nabla} f \rangle. \end{aligned} \tag{3.4}$$

Moreover, taking into account once more that $\langle \bar{\nabla} f, \partial_t \rangle = 0$, it is not difficult to verify that

$$\overline{\text{Hess}}_f(N, N) = \widetilde{\text{Hess}}_f(N^*, N^*), \tag{3.5}$$

where $\widetilde{\text{Hess}}_f$ stands for the Hessian of f on \mathbb{P}^n .

Consequently, since

$$\nabla \Theta = -A(\partial_t^\top),$$

from (3.6) and (3.5) we get

$$n\partial_t^\top(H) = \Theta \widetilde{\text{Hess}}_f(N^*, N^*) + \langle \nabla \Theta, \bar{\nabla} f \rangle. \tag{3.6}$$

So, from the Eqs. (3.3) and (3.6) we deduce (3.3).

Moreover, since f is constant along \mathbb{R} , from (2.1) we get that

$$\langle \nabla f, \nabla h \rangle = -\langle \bar{\nabla} f, N \rangle \Theta. \tag{3.7}$$

But, from Proposition 7.35 of [25] we have that

$$\bar{\nabla}_X \partial_t = 0 \tag{3.8}$$

for every $X \in \mathfrak{X}(\Sigma)$. Thus, from (2.2) and (3.8), we get

$$\nabla_X(\nabla h) = \nabla_X(\partial_t^\top) = \Theta AX. \tag{3.9}$$

Hence, from (3.9) we have

$$\Delta h = nH\Theta. \tag{3.10}$$

Therefore, from (2.4), (2.5), (3.7) and (3.10) we obtain

$$\Delta_f h = nH_f\Theta. \tag{3.11}$$

Now, we define on Σ^n the function

$$\varphi = H_f h + \Theta. \tag{3.12}$$

From 3.11 and 3.2 we get that

$$\Delta_f \varphi = -\Theta(|A|^2 - nH_f^2 + \widetilde{\text{Ric}}_f(N^*, N^*)). \tag{3.13}$$

Consequently, since $\widetilde{\text{Ric}}_f(N^*, N^*) = \overline{\text{Ric}}_f(N, N) \geq 0$, $nH_f^2 \leq |A|^2$ and choosing N such that $-1 \leq \Theta \leq 0$, from (3.13) we get that $\Delta_f \varphi \geq 0$. Thus, we conclude from the maximum principle that $\varphi \leq \varphi|_{\partial\Sigma}$ and, hence, from (3.12) we have that

$$H_f h - 1 \leq H_f h + \Theta \leq H_f s. \tag{3.14}$$

We then consider the two possible cases. In the case that $H_f > 0$, from (3.11) we have $\Delta_f h \leq 0$ and, from the maximum principle, $h \geq s$ on Σ^n . Thus, from (3.14) we conclude that

$$h - s \leq \frac{1}{H_f}. \tag{3.15}$$

Finally, in the case that $H_f < 0$, from (3.11) we have $\Delta_f h \geq 0$ and, again from the maximum principle, $h \leq s$ on Σ^n . Thus, from (3.14) we must have

$$s - h \leq -\frac{1}{H_f}. \tag{3.16}$$

Therefore, estimate (3.1) follows from (3.15) and (3.16). □

Remark 1 Concerning Theorem 1, we note that if Σ^n is locally a graph over \mathbb{P}^n , then its angle function Θ is either $\Theta < 0$ or $\Theta > 0$ along Σ^n . Hence, the assumption that Θ does not change sign is generally weaker than that of Σ^n being a local graph. Moreover, as it was already observed by Espinar and Rosenberg [11] when they made allusion to immersions into the Euclidean space, the condition that Θ does not change sign can also be regarded as the image of the Gauss map of the hypersurface lying in a closed hemisphere of the Euclidean sphere.

On the other hand, we also observe that the hypothesis $nH_f^2 \leq |A|^2$ is automatically satisfied in the case that the weighted function f is constant. Furthermore, taking into account Heinz’s estimate [17] previously commented in the introduction, we see that our estimate (3.1) is optimal.

From Theorem 1, and according to the ideas of [10, 14, 15], we obtain the following half-space type result.

Theorem 2 *Let $\mathbb{R} \times \mathbb{P}_f^n$ be a weighted Riemannian product space with $\overline{\text{Ric}}_f \geq 0$ and \mathbb{P}^n compact. Let Σ^n be a complete noncompact two-sided hypersurface properly immersed in $\mathbb{R} \times \mathbb{P}_f^n$, whose angle function Θ does not change sign. If Σ^n has nonzero constant f -mean curvature such that $nH_f^2 \leq |A|^2$, then Σ^n cannot lie in a half-space of $\mathbb{R} \times \mathbb{P}$. In particular, Σ^n must have at least one top and one bottom end.*

Proof Suppose by contradiction that, for instance, $\Sigma^n \subset (-\infty, \tau] \times \mathbb{P}$, for some $\tau \in \mathbb{R}$. Thus, for each $s < \tau$ we define

$$\Sigma_s^+ = \{(t, x) \in \Sigma^n : t \geq s\}.$$

Since \mathbb{P} is compact and Σ^n is properly immersed in $\mathbb{R} \times \mathbb{P}_f^n$, we have that Σ_s^+ is a compact hypersurface contained in a slab of width $\tau - s$ and with boundary in $\{s\} \times \mathbb{P}$. Thus, we can apply Theorem 1 to get that Σ_s^+ is contained in a slab of width $\frac{1}{|H_f|}$, so that it must be $\tau - s \leq \frac{1}{|H_f|}$. Consequently, choosing s sufficiently small we violate this estimate, reaching a contradiction.

Analogously, if we suppose that $\Sigma^n \subset [\tau, +\infty) \times \mathbb{P}$ with $\tau \in \mathbb{R}$, for each $s > \tau$ we define Σ_s^- by

$$\Sigma_s^- = \{(t, x) \in \Sigma; t \leq s\}.$$

Hence, since Σ_s^- is a compact hypersurface with boundary in $\{s\} \times \mathbb{P}$, we can reason as in the previous case and obtain another contradiction. \square

4 The Lorentzian setting

As a consequence of a splitting theorem due to Case (see [7, Theorem 1.2]), if a weighted Lorentzian product space $-\mathbb{R} \times \mathbb{P}$ is endowed with a bounded weighted function f and if $\overline{\text{Ric}}_f(V, V) \geq 0$, for all timelike vector field V on $-I \times \mathbb{P}$, then f must be constant along \mathbb{R} . Motivated by this result, in what follows we will consider weighted Lorentzian product spaces $-I \times \mathbb{P}^n$ whose weighted function f does not depend on the parameter $t \in I$, that is, $\langle \overline{\nabla} f, \partial_t \rangle = 0$ and we will denote them by $-I \times \mathbb{P}_f^n$.

We proceeding with our second height estimate.

Theorem 3 *Let $-I \times \mathbb{P}_f^n$ be a weighted Lorentzian product space with $\overline{\text{Ric}}_f \geq 0$ and let Σ^n be a compact spacelike hypersurface with boundary contained into the slice $\{s\} \times \mathbb{P}^n$, for some $s \in I$. If Σ^n has nonzero constant f -mean curvature such that $nH_f^2 \leq |A|^2$, where A denotes the Weingarten operator of Σ^n with respect to its future-pointing unit normal vector field N , then the height function h of Σ^n satisfies*

$$|h - s| \leq \frac{\max_{\partial \Sigma} |\Theta| - 1}{|H_f|}. \tag{4.1}$$

Proof Since from Proposition 1 of [1] gives us

$$\Delta \Theta = n \partial_t^\top (H) + \Theta (\widetilde{\text{Ric}}(N^*, N^*) + |A|^2),$$

in a similar way of the proof of Theorem 1 we get that

$$\Delta_f \Theta = \Theta (\widetilde{\text{Ric}}_f(N^*, N^*) + |A|^2). \tag{4.2}$$

Moreover, also analogously as in the proof of Theorem 1, we have that

$$\Delta_f h = -nH_f \Theta. \tag{4.3}$$

So, we define on Σ^n the function

$$\varphi = -H_f h - \Theta. \tag{4.4}$$

From 4.3 and 4.2 we get that

$$\Delta_f \varphi = -\Theta(|A|^2 - nH_f^2 + \widetilde{\text{Ric}}_f(N^*, N^*)).$$

Consequently, since $\widetilde{\text{Ric}}_f(N^*, N^*) = \overline{\text{Ric}}_f(N, N) \geq 0, nH_f^2 \leq |A|^2$ and choosing N future-pointing (that is, $\Theta \leq -1$), we get that $\Delta_f \varphi \geq 0$. Thus, we conclude from the maximum principle that $\varphi \leq \varphi|_{\partial\Sigma}$ and, hence, from (4.4) we have that

$$-H_f h + 1 \leq -H_f h - \Theta \leq -H_f s + \max_{\partial\Sigma} |\Theta|. \tag{4.5}$$

We then consider the two possible cases. In the case that $H_f > 0$, from (4.3) we have $\Delta_f h \geq 0$ and, from the maximum principle, $h \leq s$ on Σ^n . Thus, from (4.5) we conclude that

$$s - h \leq \frac{\max_{\partial\Sigma} |\Theta| - 1}{H_f}. \tag{4.6}$$

In the case that $H_f < 0$, from (4.3) we have $\Delta_f h \leq 0$ and, again from the maximum principle, $h \geq s$ on Σ^n . Thus, from (4.5) we must have

$$h - s \leq \frac{1 - \max_{\partial\Sigma} |\Theta|}{H_f}. \tag{4.7}$$

Therefore, estimate (4.1) follows from (4.6) and (4.7). □

Remark 2 Taking into account the height estimate of [10] mentioned in the introduction, we see that our estimate (4.1) is also sharp.

Finally, reasoning as in the proof of Theorem 2, from Theorem 3 we get the following

Theorem 4 *Let $-\mathbb{R} \times \mathbb{P}_f^n$ be a weighted Lorentzian product space with $\overline{\text{Ric}}_f \geq 0$ and \mathbb{P}^n compact. Let Σ^n be a complete noncompact spacelike hypersurface properly immersed in $-\mathbb{R} \times \mathbb{P}_f^n$, with bounded angle function Θ . If Σ^n has nonzero constant f -mean curvature such that $nH_f^2 \leq |A|^2$, then Σ^n cannot lie in a half-space of $-\mathbb{R} \times \mathbb{P}$. In particular, Σ^n must have at least one top and one bottom end.*

Remark 3 We recall that an integral curve of the unit timelike vector field ∂_t is called a comoving observer and, for a fixed point $p \in \Sigma^n$, $\partial_t(p)$ is called an instantaneous comoving observer. In this setting, among the instantaneous observers at p , $\partial_t(p)$

and $N(p)$ appear naturally. From the orthogonal decomposition $N(p) = N^*(p) - \Theta(p)\partial_t(p)$, we have that $|\Theta(p)|$ corresponds to the energy $E(p)$ that $\partial_t(p)$ measures for the normal observer $N(p)$. Furthermore, the speed $|\nu(p)|$ of the Newtonian velocity $\nu(p) := E^{-1}(p)N^*(p)$ that $\partial_t(p)$ measures for $N(p)$ satisfies the equation $|\nu(p)|^2 = \tanh(\cosh^{-1}|\Theta(p)|)$. Hence, the boundedness of the angle function Θ of the spacelike hypersurface Σ^n means, physically, that the speed of the Newtonian velocity that the instantaneous comoving observer measures for the normal observer do not approach the speed of light 1 on Σ^n (cf. [26, Sections 2.1 and 3.1]). In this direction, as it was already observed by Latorre and Romero [19], the assumption of Θ be bounded on a complete spacelike hypersurface is a mild hypothesis to supply the noncompactness of it.

Acknowledgements The first author is partially supported by CNPq, Brazil, Grant 303977/2015-9. The authors would like to thank the referee for giving valuable suggestions and comments which improved the paper.

References

1. Albuje, A.L., Aledo, J.A., Alías, L.J.: On the scalar curvature of hypersurfaces in spaces with a Killing field. *Adv. Geom.* **10**, 487–503 (2010)
2. Aledo, J.A., Espinar, J.M., Gálvez, J.A.: Surfaces with constant curvature in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. Height estimates and representation. *Bull. Braz. Math. Soc.* **38**, 533–554 (2007)
3. Aledo, J.A., Espinar, J.M., Gálvez, J.A.: Height estimates for surfaces with positive constant mean curvature in $M^2 \times \mathbb{R}$. *Ill. J. Math.* **52**, 203–211 (2008)
4. Alías, L.J., Dajczer, M.: Constant mean curvature hypersurfaces in warped product spaces. *Proc. Edinb. Math. Soc.* **50**, 511–526 (2007)
5. Alías, L.J., Dajczer, M., Ripoll, J.: A Bernstein-type theorem for Riemannian manifolds with a Killing field. *Ann. Glob. Anal. Geom.* **31**, 363–373 (2007)
6. Bakry, D., Émery, M.: Diffusions hypercontractives. In: Azéma, J., Yor, M. (eds.) *Séminaire de Probabilités XIX, 1983/84. Lecture Notes in Mathematics*, vol. 1123, pp. 177–206. Springer, Berlin (1985)
7. Case, J.S.: Singularity theorems and the Lorentzian splitting theorem for the Bakry–Émery–Ricci tensor. *J. Geom. Phys.* **60**, 477–490 (2010)
8. Cheng, X., Rosenberg, H.: Embedded positive constant r -mean curvature hypersurfaces in $M^m \times \mathbb{R}$. *Ann. Braz. Acad. Sci.* **77**, 183–199 (2005)
9. Colares, A.G., de Lima, H.F.: Space-like hypersurfaces with positive constant r -mean curvature in Lorentzian product spaces. *Gen. Relativ. Gravity* **40**, 2131–2147 (2008)
10. de Lima, H.F.: A sharp height estimate for compact spacelike hypersurfaces with constant r -mean curvature in the Lorentz–Minkowski space and application. *Differ. Geom. Appl.* **26**, 445–455 (2008)
11. Espinar, J.M., Rosenberg, H.: Complete constant mean curvature surfaces and Bernstein type theorems in $M^2 \times \mathbb{R}$. *J. Differ. Geom.* **82**, 611–628 (2009)
12. Fang, F., Li, X.-D., Zhang, Z.: Two generalizations of Cheeger–Gromoll splitting theorem via Bakry–Émery–Ricci curvature. *Ann. Inst. Fourier* **59**, 563–573 (2009)
13. Gálvez, J.A., Martínez, A.: Estimates in surfaces with positive constant Gauss Curvature. *Proc. Am. Math. Soc.* **128**, 3655–3660 (2000)
14. García-Martínez, S.C., Impera, D.: Height estimates and half-space theorems for spacelike hypersurfaces in generalized Robertson–Walker spacetimes. *Differ. Geom. Appl.* **32**, 46–67 (2014)
15. García-Martínez, S.C., Impera, D., Rigoli, M.: A sharp height estimates for compact hypersurfaces with constant k -mean curvature in warped product spaces. *Proc. Edinb. Math. Soc.* **58**, 403–419 (2015)
16. Gromov, M.: Isoperimetry of waists and concentration of maps. *Geom. Funct. Anal.* **13**, 178–215 (2003)
17. Heinz, E.: On the nonexistence of a surface of constant mean curvature with finite area and prescribed rectifiable boundary. *Arch. Ration. Mech. Anal.* **35**, 249–252 (1969)

18. Hoffman, D., de Lira, J.H., Rosenberg, H.: Constant mean curvature surfaces in $M^2 \times \mathbb{R}$. *Trans. Am. Math. Soc.* **358**, 491–507 (2006)
19. Latorre, J.M., Romero, A.: Uniqueness of noncompact spacelike hypersurfaces of constant mean curvature in generalized Robertson–Walker spacetimes. *Geom. Dedic.* **93**, 1–10 (2002)
20. Lichnerowicz, A.: Variétés Riemanniennes à tenseur C non négatif. *C. R. Acad. Sci. Paris Sér. A-B* **271**, A650–A653 (1970)
21. Lichnerowicz, A.: Variétés Kähleriennes à première classe de Chern non négative et variétés Riemanniennes à courbure de Ricci généralisée non négative. *J. Differ. Geom.* **6**, 47–94 (1971)
22. López, R.: Area monotonicity for spacelike surfaces with constant mean curvature. *J. Geom. Phys.* **52**, 353–363 (2004)
23. Manzano, J.M.: Estimates for constant mean curvature graphs in $M \times \mathbb{R}$. *Rev. Mat. Iberoam.* **29**, 1263–1281 (2013)
24. Morgan, F.: Manifolds with density. *Not. Am. Math. Soc.* **52**, 853–858 (2005)
25. O’Neill, B.: *Semi-Riemannian Geometry with Applications to Relativity*. Academic Press, London (1983)
26. Sachs, R.K., Wu, H.: *General Relativity for Mathematicians*, Graduate Texts in Mathematics. Springer, New York (1977)
27. Wei, G., Willie, W.: Comparison Geometry for the Bakry–Emery–Ricci Tensor. *J. Differ. Geom.* **83**, 377–405 (2009)