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On the local regularity of suitable weak solutions to the generalized Navier–Stokes equations

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Abstract We consider the Navier–Stokes equations in three spatial dimensions and present a new proof of the Caffarelli–Kohn–Nirenberg theorem, based on a generalized notion of a local suitable weak solution, involving the local pressure. By estimating the integrals involving the pressure in terms of velocity, the pressure term is cancelled in the local decay estimates. In particular, our proof shows that the Caffarelli–Kohn–Nirenberg theorem holds for any open set Ω without any restriction on the size and the regularity of the boundary. In addition, the method forms a basis for proving partial regularity results to other fluid models such as non-Newtonian models or models with heat conduction.

Keywords Navier–Stokes equations · Non-Newtonian fluids · Suitable weak solutions · Existence · Regularity

Mathematics Subject Classification (2000) 35B65 · 35Q30 · 76D05 · 76N10

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be any domain and let $0 < T < +\infty$. Set $Q = \Omega \times (0, T)$. We consider the following generalized Navier–Stokes equations (g-NSE)

$$\text{(g-NSE)} \qquad \begin{cases} \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad Q, \\ \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \operatorname{div} \boldsymbol{S} + \nabla p = -\operatorname{div} \boldsymbol{f} & \text{in} \quad Q, \end{cases}$$

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where

$$\begin{aligned} \boldsymbol{u} = (u^1, u^2, u^3) &= \text{unknown velocity of the fluid,} \\ p &= \text{unknown pressure,} \\ -\text{div} \boldsymbol{f} &= \text{external force,} \\ \boldsymbol{S} &= \text{deviatoric stress tensor.} \end{aligned}$$

The system (g-NSE) will be completed with the following boundary and initial conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \partial\Omega \times (0, T), \tag{1.1}$$

$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{on} \quad \Omega \times \{0\}, \tag{1.2}$$

where $\boldsymbol{u}_0$ is a given initial velocity distribution.

1.1 Models for the constitutive law $\boldsymbol{S}$

Due to friction the deviatoric stress $\boldsymbol{S}$ depends on $\boldsymbol{D}(\boldsymbol{u})$, where

$$D_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u^j + \partial_j u^i) \quad (i, j = 1, 2, 3).$$

In addition, in case of heat conducting fluids $\boldsymbol{S}$ can depend also on the temperature θ of the fluid which is due to heat transfer. We present various models which are well-known models of fluid motions.

(i) *Newtonian fluid with constant viscosity*: Here $\boldsymbol{S}$ is proportional to $\boldsymbol{D}(\boldsymbol{u})$, i.e. there exists a constant $\nu > 0$ which is called the *viscosity* of the fluid such that

$$\boldsymbol{S} = 2\nu \boldsymbol{D}(\boldsymbol{u}). \tag{1.3}$$

Owing to $\text{div}\boldsymbol{u} = 0$ we have $\text{div}\boldsymbol{S} = 2\nu\text{div}\boldsymbol{D}(\boldsymbol{u}) = \nu\Delta\boldsymbol{u}$. Thus, (g-NSE) turns into the usual NSE

$$\text{(NSE)} \qquad \begin{cases} \text{div}\boldsymbol{u} = 0 \quad \text{in} \quad Q, \\ \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = -\text{div}\boldsymbol{f} \quad \text{in} \quad Q. \end{cases}$$

(ii) *Newtonian fluid with non-constant viscosity*: There exists a bounded measurable function $\nu : Q \to \mathbb{R}$, such that

$$\boldsymbol{S} = \nu \boldsymbol{D}(\boldsymbol{u}), \qquad 0 < \nu_0 \le \nu(x, t) \le \nu_1 < +\infty \quad \text{for a. e.} (x, t) \in Q. \tag{1.4}$$

where $\nu_0, \nu_1 = \text{const} > 0$.

(iii) *Non-Newtonian fluids with shear dependent viscosity*: There exists a positive function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$S = \mu(|D(u)|)D(u). \tag{1.5}$$

In the engineering practice one often makes use of the so called power law model, where

$$\mu(s) = (1+s^2)^{\frac{q-2}{2}} \quad \text{or} \quad \mu(s) = s^{q-2}, \quad 1 < q < \infty. \tag{1.6}$$

Here we distinguish between the following three cases

First case :	$1 < q < 2$	*shear thinning*,
Second case :	$q = 2$	*Newtonian*,
Third case :	$2 < q < +\infty$	*shear thickening*.

(iv) *Heat conducting fluids*: Due to heat conduction, the viscosity may depend on the temperature θ, such that

$$S = \nu(\theta)D(u). \tag{1.7}$$

and the generalized NSE being coupled by the equation of heat transport,

$$\partial_t\theta + u\cdot\nabla\theta - \operatorname{div}(\kappa\nabla\theta) = \nu(\theta)|D(u)|^2, \tag{1.8}$$

where $\kappa > 0$ denotes the heat capacity due to Fourier's law. The Eq. (1.8) will be completed by appropriate initial and boundary conditions.

For further details on fluid mechanical background see [2,13].

1.2 Notion of a weak solution

First, let us introduce the function spaces which will be used in what follows. By $W^{k,q}(\Omega)$, $W_0^{k,q}(\Omega)$ ($k \in \mathbb{N}$; $1 \le q \le +\infty$) we denote the usual Sobolev spaces (see, e. g. [1]). Spaces of vector valued functions will be denoted by bold letters, i.e. instead of $W^{k,q}(\Omega;\mathbb{R}^m)$, $L^q(\Omega;\mathbb{R}^m)$, etc. we write shorter $\boldsymbol{W}^{k,q}(\Omega)$, $\boldsymbol{L}^q(\Omega)$, etc.

Let $\boldsymbol{C}^\infty_{0,\sigma}(\Omega)$ denote the space of solenoidal smooth functions having compact support in Ω. We define

$$\boldsymbol{L}^q_\sigma(\Omega) := \text{closure of } \boldsymbol{C}^\infty_{0,\sigma(\Omega)} \text{ w.r.t. the norm in } \boldsymbol{L}^q(\Omega),$$
$$\boldsymbol{W}^{1,q}_{0,\sigma}(\Omega) := \text{ closure of } \boldsymbol{C}^\infty_{0,\sigma}(\Omega) \text{ w.r.t. the norm in } \boldsymbol{W}^{1,q}_0(\Omega).$$

In particular, we set

$$\boldsymbol{V} := \boldsymbol{W}^{1,2}_{0,\sigma}(\Omega), \quad \boldsymbol{H} := \boldsymbol{L}^2_\sigma(\Omega).$$

Let X be a Banach space with norm $\|\cdot\|_X$. Then, by $L^q(a,b;X)$ we denote the space of Bochner measurable functions, such that

$$\|f\|^q_{L^q(a,b;X)} = \int_a^b \|f(t)\|^q_X dt < +\infty \quad \text{if} \quad 1 \le q < +\infty,$$

$$\|f\|_{L^\infty(a,b;X)} = \operatorname{ess\,sup}_{t\in(a,b)} \|f(t)\|_X < +\infty$$

(for details see [18]).

Now, we introduce the notion of a weak solution. For the sake of simplicity we only consider the case of NSE with $\nu = 1$ and $\boldsymbol{f} = \boldsymbol{0}$.

Definition 1.1 Let $\boldsymbol{u}_0 \in \boldsymbol{H}$. A function $\boldsymbol{u} : Q \to \mathbb{R}^3$ is called a *weak solution* to the NSE if

(i) $\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}) \cap L^2(0, T; \boldsymbol{V})$.

(ii) For every $\boldsymbol{\varphi} \in C^1([0, T); \boldsymbol{C}^\infty_{0,\sigma}(\Omega))$:

$$\int_Q -\boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} - \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{\varphi} + 2\boldsymbol{D}\boldsymbol{u} : \boldsymbol{D}\boldsymbol{\varphi} dxdt = \int_\Omega \boldsymbol{u}_0 \boldsymbol{\varphi}(\cdot, 0) dx. \tag{1.9}$$

(iii) In addition, $\boldsymbol{u}$ is called a *Leray–Hopf solution* if the following energy inequality is fulfilled:

$$\|\boldsymbol{u}(t)\|^2_{\boldsymbol{H}} + 2 \int_0^t \int_\Omega |\nabla \boldsymbol{u}|^2 dxds \le \|\boldsymbol{u}_0\|^2_{\boldsymbol{H}} \quad \text{for a.e.} \quad t \in (0, T). \tag{1.10}$$

The existence of a Leray–Hopf solution is well-known and can be found in Leray [14] for the case $\Omega = \mathbb{R}^3$ and in Hopf [11] for general bounded domains. However, the notion of a Leray–Hopf solution is not sufficient for the study of local regularity properties, since in general there is no control of the local energy. For this reason, first Scheffer [17] introduced the notion of a suitable weak solution and then he proves the partial regularity for such solutions. Later, this notion has been also used in the celebrated paper by Caffarelli–Kohn–Nirenberg [4] to obtain the partial regularity, proving that the 1-dimensional parabolic Hausdorff measure of the singular set is zero. For more simplified proofs of this result see [12,15]. Recently a new proof of the Caffarelli–Kohn–Nirenberg theorem has been given by Vasseur in [21].

Let us now recall the notion of a suitable weak solution due to Scheffer. A pair $(\boldsymbol{u}, p)$ is called *suitable weak solution* to (NSE) if $\boldsymbol{u}$ is a weak solution to (NSE), $p \in L^{\frac{3}{2}}(Q)$ and the following local energy inequality is fulfilled for all nonnegative $\phi \in C^\infty_0(Q)$ and for almost all $t \in (0, T)$:

$$\int_\Omega |\boldsymbol{u}(t)|^2 \phi(t) dx + 2 \int_0^t \int_\Omega |\nabla \boldsymbol{u}|^2 \phi dxds$$
$$\le \int_0^t \int_\Omega |\boldsymbol{u}|^2 (\partial_t \phi + \Delta \phi) + (|\boldsymbol{u}|^2 + 2p) \boldsymbol{u} \cdot \nabla \phi dxds. \tag{1.11}$$

Clearly, in order to establish such suitable weak solutions to (NSE) one has to define a pressure function p, which is only available if Ω is a uniform C^2 domain (cf. [5]). Therefore, we cannot expect to obtain such solutions for general domains. The same difficulties occur if one considers other models like heat conducting fluids or non-Newtonian fluids. The purpose of this paper is to provide a new method in constructing suitable weak solutions for general domains and general models. As we will see below, we may introduce a notion of a *local suitable weak solution* to the NSE instead defining a global pressure, we introduce a local pressure decomposition $p = \partial_t p_h + p_0$. However, this notion requires some study on the steady Stokes equation, which will be the subject of Sect. 2.

2 Local estimate of the pressure

2.1 The definition of the spatial pressure

The aim of this section is the construction of a special operator which maps every $\boldsymbol{F} \in W^{-1,q}(\Omega)$ into a class $[p] \in L^q_{\mathrm{loc}}(\Omega)/\mathbb{R}$. To begin with, we recall a well-known result due to Galdi et al. [7] (for a similar result on Lipschitz domains see [3]), which is the following

Lemma 2.1 (Galdi–Simader–Sohr) *Let $G \subset \mathbb{R}^n$ be a bounded domain with $\partial G \in C^1$. Then, for every $\boldsymbol{F} \in \boldsymbol{W}^{-1,q}(G)$ $(1 < q < \infty)$ there exists exactly one pair $(\boldsymbol{v}, p) \in \boldsymbol{W}^{1,q}_{0,\sigma}(G) \times L^q_0(G)$*[1] *such that*

$$\operatorname{div} \boldsymbol{v} = 0 \quad in \quad G, \quad -\Delta \boldsymbol{v} + \nabla p = \boldsymbol{F} \quad in \quad \boldsymbol{W}^{-1,q}(G), \tag{2.1}$$

$$\boldsymbol{v} = \boldsymbol{0} \quad on \quad \partial G. \tag{2.2}$$

In addition, there holds

$$\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^q(G)} + \|p\|_{L^q(G)} \le C_q \|\boldsymbol{F}\|_{\boldsymbol{W}^{-1,q}(G)}. \tag{2.3}$$

As an immediate consequence of Lemma 2.1 we have the

Lemma 2.2 *For every $1 < q < \infty$ there holds*

$$\boldsymbol{W}^{-1,q}(G) = \boldsymbol{W}^{-1,q}_{\mathrm{div}}(G) \oplus \boldsymbol{W}^{-1,q}_{\mathrm{grad}}(G), \tag{2.4}$$

where

$$\boldsymbol{W}^{-1,q}_{\mathrm{div}}(G) = \{-\Delta \boldsymbol{v} \mid \boldsymbol{v} \in \boldsymbol{W}^{1,q}_{0,\sigma}(G)\}, \quad \boldsymbol{W}^{-1,q}_{\mathrm{grad}}(G) = \{\nabla p \mid p \in L^q_0(G)\}.$$

Lemma 2.2 enables us to define the surjective bounded operator $\mathscr{P}_{q,G} : \boldsymbol{W}^{-1,q}(G) \to L^q_0(G)$ by setting $\mathscr{P}_{q,G}\boldsymbol{F} = p$, where $\boldsymbol{F} - \nabla p \in \boldsymbol{W}^{-1,q}_{\mathrm{div}}(G)$. Clearly,

[1] Here $L^q_0(G)$ means the space of all $f \in L^q(G)$ with $\int_G f dx = 0$.

$\mathscr{P}_{q,G}\nabla p = p$ for all $p \in L^q_0(G)$. Consequently, $\nabla \mathscr{P}_{q,G}$ defines a projection of $\boldsymbol{W}^{-1,q}(G)$ onto $\boldsymbol{W}^{-1,q}_{\text{grad}}(G)$.

Notice, due to uniqueness of the weak solution we have $\mathscr{P}_{q,G}\boldsymbol{F} = \mathscr{P}_{s,G}\boldsymbol{F}$ for all $\boldsymbol{F} \in \boldsymbol{W}^{-1,q}(G) \cap \boldsymbol{W}^{-1,s}(G)$ $(1 < s, q < +\infty)$. Thus, in what follows $\mathscr{P}_G$ stands for $\mathscr{P}_{q,G}$ for some $1 < q < +\infty$.

By means of elliptic regularity we have the following regularity property of $\mathscr{P}_G$:

Lemma 2.3 *Let $G \subset \mathbb{R}^n$ be a bounded C^2 domain. Let $\boldsymbol{f} \in \boldsymbol{L}^q(G) \hookrightarrow \boldsymbol{W}^{-1,q}(G)$. Then $\mathscr{P}_G\boldsymbol{f} \in W^{1,q}(G)$ and there holds the estimate*

$$\|\nabla \mathscr{P}_G \boldsymbol{f}\|_{\boldsymbol{L}^q(G)} \le c\|\boldsymbol{f}\|_{\boldsymbol{L}^q(G)}, \tag{2.5}$$

where $c = \text{const} > 0$ depending on q, n and the geometry of G only.

2.2 The time dependent case

Next, let $\boldsymbol{F} \in L^s(a, b, \boldsymbol{W}^{-1,q}(G))$. With help of Pettis' theorem (e.g. see [24]; Chap. V.4.) we may define $\mathscr{P}_G : L^s(a, b, \boldsymbol{W}^{-1,q}(G)) \to L^s(a, b, L^q_0(G))$ according to

$$(\mathscr{P}_G\boldsymbol{F})(t) = \mathscr{P}_G(\boldsymbol{F}(t)) \quad \text{for a. e. } t \in (a, b).$$

Furthermore, for every $\boldsymbol{F} \in L^s(a, b, \boldsymbol{W}^{-1,q}(G))$ we define

$$\mathscr{P}_G\partial_t\boldsymbol{F} = \partial_t\mathscr{P}_G\boldsymbol{F} \quad \text{in the sense of distributions.}$$

Using the above definition of the projections $\mathscr{P}_{q,G}$ we have the following

Lemma 2.4 *Let $\boldsymbol{u} \in L^2(0, T; \boldsymbol{V}) \cap L^\infty(0, T; \boldsymbol{H})$ be a weak solution to the (NSE) with $\boldsymbol{f} = \boldsymbol{0}$. Then for every bounded subdomain $G \subset \Omega$ with $\partial G \in C^2$ there holds*

$$-\int_0^T\int_G (\boldsymbol{u} + \nabla p_{h,G}) \cdot \partial_t\boldsymbol{\varphi}\, dxdt - \int_0^T\int_G (\boldsymbol{u}\otimes\boldsymbol{u} + p_{1,G}\boldsymbol{I}) : \nabla\boldsymbol{\varphi}\, dxdt$$
$$+\int_0^T\int_G (\nabla\boldsymbol{u} - p_{2,G}\boldsymbol{I}) : \nabla\boldsymbol{\varphi}\, dxdt = 0 \tag{2.6}$$

for all $\boldsymbol{\varphi} \in \boldsymbol{C}^\infty_0(G \times (0, T))$, where[2]

$$p_{h,G} = -\mathscr{P}_{2,G}\boldsymbol{u},$$
$$p_{1,G} = -\mathscr{P}_{\frac{3}{2},G}\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}),$$
$$p_{2,G} = \mathscr{P}_{2,G}\Delta\boldsymbol{u}.$$

[2] Here $\boldsymbol{I}$ stands for the identity matrix in $\mathbb{R}^{3\times 3}$.

In addition, we have the estimates,

$$\|\nabla p_{h,G}(t)\|_{L^m(G)} \leq c\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^m(G)} \quad (1 < m \leq 6), \tag{2.7}$$

$$\|p_{1,G}(t)\|_{L^{3/2}(G)} \leq c\|\boldsymbol{u}(t) \otimes \boldsymbol{u}(t)\|_{\boldsymbol{L}^{3/2}(G)}, \tag{2.8}$$

$$\|p_{2,G}(t)\|_{L^2(G)} \leq c\|\nabla \boldsymbol{u}(t)\|_{\boldsymbol{L}^2(G)}, \tag{2.9}$$

for almost all $t \in (0, T)$. *Here* $c = \text{const} > 0$ *depends on the geometry of* G *and in* (2.7) *on* m *only. In particular, if* G *is the ball* $B_R(x_0)$ *then* c *in* (2.7) *depends only on* m*, while in* (2.8) *and* (2.9) c *is an absolute constant.*

Proof According to Lemma 2.1 there exists $\boldsymbol{v}_h, \boldsymbol{v}_2 \in L^2(0, T; \boldsymbol{W}^{1,2}_{0,\sigma}(G)), \boldsymbol{v}_1 \in L^{\frac{3}{2}}(0, T; \boldsymbol{W}^{1,\frac{3}{2}}_{0,\sigma}(G))$ such that

$$-\Delta \boldsymbol{v}_h + \nabla p_h = -\boldsymbol{u}, \tag{2.10}$$

$$-\Delta \boldsymbol{v}_2 + \nabla p_2 = \Delta \boldsymbol{u}, \tag{2.11}$$

$$-\Delta \boldsymbol{v}_1 + \nabla p_1 = -\operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}). \tag{2.12}$$

Since $\boldsymbol{u}$ is a weak solution to (NSE) we see that

$$\int_0^T \int_G -\nabla \boldsymbol{v}_h : \nabla \boldsymbol{\varphi}_t + (\nabla \boldsymbol{v}_1 + \nabla \boldsymbol{v}_2) : \nabla \boldsymbol{\varphi} dx dt = 0$$

for all $\boldsymbol{\varphi} \in C^1_0(0, T; \boldsymbol{C}^\infty_{0,\sigma}(G))$. Introducing the Steklov mean of $f \in L^1(Q)$ by

$$f_\lambda(x, t) := \frac{1}{\lambda} \int_t^{t+\lambda} f(x, \tau) d\tau, \quad (x, t) \in G \times (0, T - \lambda), \quad 0 < \lambda < T,$$

the above identity leads to

$$\int_0^{T-\lambda} \int_G \Big(\nabla \partial_t (\boldsymbol{v}_h)_\lambda + \nabla (\boldsymbol{v}_1)_\lambda + \nabla (\boldsymbol{v}_2)_\lambda\Big) : \nabla \boldsymbol{\varphi} dx dt = 0 \quad \forall \boldsymbol{\varphi} \in C^1_0(0, T-\lambda; \boldsymbol{C}^\infty_{0,\sigma}(G)) \tag{2.13}$$

for all $0 < \lambda < T$.

Let $0 < \lambda < T$ be arbitrarily chosen. Fix $\eta \in C^1_0(0, T - \lambda)$ and define

$$\boldsymbol{w}(x) := \int_0^{T-\lambda} \Big(\partial_t (\boldsymbol{v}_h)_\lambda(x, t) + (\boldsymbol{v}_1)_\lambda(x, t) + (\boldsymbol{v}_2)_\lambda(x, t)\Big) \eta(t) dt, \quad x \in G.$$

Inserting $\boldsymbol{\varphi}(x,t) = \boldsymbol{\psi}(x)\eta(t)$ for $\boldsymbol{\psi} \in \boldsymbol{C}^\infty_{0,\sigma}(G)$ into (2.13) and using Fubini's theorem, it follows that

$$\int_G \nabla \boldsymbol{w} : \nabla \boldsymbol{\psi} dx = 0 \quad \forall \boldsymbol{\psi} \in \boldsymbol{C}^\infty_{0,\sigma}(G).$$

By the definition of $\boldsymbol{v}_h$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ we see that $\boldsymbol{w} \in \boldsymbol{W}^{1,\frac{3}{2}}_{0,\sigma}(G)$. Clearly, $\boldsymbol{w}$ solves the Stokes system (2.1), (2.2) with $\boldsymbol{F} = 0$. Thus, thanks to Lemma 2.1 we have $\boldsymbol{w} = 0$. Recalling the definition of $\boldsymbol{w}$ this shows that

$$\int_{G\times(0,T-\lambda)} \Big(\partial_t(\boldsymbol{v}_h)_\lambda + (\boldsymbol{v}_1)_\lambda + (\boldsymbol{v}_2)_\lambda\Big) \cdot \boldsymbol{\varphi} dxdt = 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{C}^\infty_0(G \times (0, T-\lambda)).$$

Whence, $\partial_t(\boldsymbol{v}_h)_\lambda + (\boldsymbol{v}_1)_\lambda + (\boldsymbol{v}_2)_\lambda = 0$ a. e. in $G \times (0, T-\lambda)$. Now, observing

$$\int_0^{T-\lambda}\int_G \Big(\nabla\partial_t(\boldsymbol{v}_h)_\lambda + \nabla(\boldsymbol{v}_1)_\lambda + \nabla(\boldsymbol{v}_2)_\lambda\Big) : \nabla\boldsymbol{\varphi} dxdt = 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{C}^\infty_0(G \times (0, T-\lambda))$$

applying integration by parts and passing to the limit $\lambda \to 0$ we arrive at

$$\int_0^T\int_G -\nabla \boldsymbol{v}_h : \nabla \boldsymbol{\varphi}_t + \nabla \boldsymbol{v}_1 : \nabla\boldsymbol{\varphi} + \nabla \boldsymbol{v}_2 : \nabla\boldsymbol{\varphi} dxdt = 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{C}^\infty_0(G \times (0, T)). \tag{2.14}$$

Whence, the assertion follows from (2.10)–(2.12) using (2.14).

In order to verify (2.7) we may chose $N \subset (0, T)$ with Lebesgue measure zero such that $\boldsymbol{u}(t) \in \boldsymbol{L}^6(G)$ for all $t \in (0, T) \setminus N$. Then (2.7) is an immediate consequence of (2.5) (cf. Lemma 2.3). Similarly, (2.8) and (2.9) follows by using (2.3) (cf. Lemma 2.1). □

Remark 2.5 1. Owing to div$\boldsymbol{u} = 0$ we see that both $p_{h,G}(t)$ and $p_{2,G}(t)$ defined above are harmonic for a. e. $t \in (0, T)$.

2. As it is readily seen the statement of Lemma 2.4 remains true if one replaces the convective term div$\boldsymbol{u} \otimes \boldsymbol{u}$ by div$\boldsymbol{H}$ for a general matrix $\boldsymbol{H} \in \boldsymbol{L}^{\frac{3}{2}}(G \times (0, T))$ and replacing $p_{1,G}$ by $-\mathscr{P}_G$div$\boldsymbol{H}$.

3 Local suitable weak solutions to the Navier–Stokes equations

Based on the local projections introduced in Sect. 2 we are in a position to introduce a notion of local suitable weak solutions, which reads as follows

Definition 3.1 A weak solution $\boldsymbol{u}$ to (NSE) is called a *local suitable weak solution*, if for every ball $B \subset\subset \Omega$ there holds

$$\int_B |\boldsymbol{v}_B(t)|^2\phi(t)dx + 2\int_0^t\int_B |\nabla \boldsymbol{v}_B|^2\phi dxds$$
$$\leq \int_0^t\int_B |\boldsymbol{v}_B|^2(\partial_t\phi + \Delta\phi) + (|\boldsymbol{u}|^2 + 2p_{0,B})\boldsymbol{v}_B\cdot\nabla\phi dxds$$
$$+\int_0^t\int_B 2u^iu^j\partial_i(\partial_j p_h\phi) - |\boldsymbol{u}|^2\nabla p_{h,B}\cdot\nabla\phi dxds, \tag{3.1}$$

for all nonnegative $\phi \in C_0^\infty(B\times(0,T))$ and for almost all $t \in (0,T)$, where $\boldsymbol{v}_B = \boldsymbol{u} + \nabla p_{h,B}$ and

$$p_{h,B} = -\mathscr{P}_{2,B}\boldsymbol{u},$$
$$p_{0,B} = \mathscr{P}_{2,B}\Delta\boldsymbol{u} - \mathscr{P}_{\frac{3}{2},B}\mathrm{div}(\boldsymbol{u}\otimes\boldsymbol{u}).$$

For the definition of the operator $\mathscr{P}_{q,B}$ see Sect. 2.

Now, we obtain the following result on the existence of a local suitable weak solution

Theorem 3.2 *For every $\boldsymbol{u}_0 \in \boldsymbol{H}$ there exists a local suitable weak solution to (NSE).*

Proof Let $\eta \in C^\infty(\mathbb{R})$ with $\eta \equiv 1$ on $(-\infty, 1)$ and $\eta \equiv 0$ in $(2, +\infty)$. Set $\eta_\varepsilon(\tau) = \eta(\varepsilon\tau)$ $(\tau \in \mathbb{R};\ \varepsilon > 0)$. Clearly, there exists a unique weak solution $\boldsymbol{u}_\varepsilon \in L^2(0,T;\boldsymbol{V})\cap L^\infty(0,T;\boldsymbol{H})$ to the approximate system

$$(\mathrm{NSE})_\varepsilon \qquad \begin{cases} \mathrm{div}\boldsymbol{u}_\varepsilon = 0 & \text{in } Q,\\ \partial_t\boldsymbol{u}_\varepsilon + \mathrm{div}(\eta_\varepsilon(|\boldsymbol{u}_\varepsilon|^2)\boldsymbol{u}_\varepsilon\otimes\boldsymbol{u}_\varepsilon) - \Delta\boldsymbol{u}_\varepsilon + \nabla p_\varepsilon = 0 & \text{in } Q,\\ \boldsymbol{u}_\varepsilon(0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases}$$

Integration by parts gives

$$\|\boldsymbol{u}_\varepsilon\|^2_{L^\infty(0,T\boldsymbol{H})} + 2\int_Q \|\nabla\boldsymbol{u}_\varepsilon\|^2 dxdt \leq 2\|\boldsymbol{u}_0\|^2_{\boldsymbol{H}}. \tag{3.2}$$

Furthermore, by using Hölder's inequality and Sobolev's embedding theorem we infer that

$$\|\boldsymbol{u}_\varepsilon\|_{L^\alpha(0,T\boldsymbol{L}^\beta(B))} \leq c\|\boldsymbol{u}_0\|_{\boldsymbol{H}} \quad \forall\alpha,\beta\in[2,+\infty) \quad \text{with} \quad \frac{2}{\alpha}+\frac{3}{\beta}=\frac{3}{2}. \tag{3.3}$$

Thus, by means of reflexivity and Banach–Alaoglu's theorem[3] one finds a sequence (ε_k) with $\varepsilon_k \to 0$ and a function $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})\cap L^\infty(0,T;\boldsymbol{H})$ such that

[3] Note, $L^\infty(0,T;\boldsymbol{H})$ can be identified with $(L^1(0,T;\boldsymbol{H}))^*$.

$$\boldsymbol{u}_{\varepsilon_k} \rightharpoonup \boldsymbol{u} \quad \text{in} \quad L^2(0, T; \boldsymbol{V}), \tag{3.4}$$

$$\boldsymbol{u}_{\varepsilon_k} \overset{*}{\rightharpoonup} \boldsymbol{u} \quad \text{in} \quad L^\infty(0, T; \boldsymbol{H}) \quad \text{as} \quad k \to +\infty. \tag{3.5}$$

In addition, we have for a. e. $t \in (0, T)$

$$\boldsymbol{u}_{\varepsilon_k}(t) \rightharpoonup \boldsymbol{u}(t) \quad \text{in} \quad \boldsymbol{H} \quad \text{as} \quad k \to +\infty. \tag{3.6}$$

Next, fix a ball $B \subset\subset \Omega$. Define

$$\begin{aligned} p^k_{h,B} &= -\mathscr{P}_{2,B}\boldsymbol{u}_{\varepsilon_k}, \\ p^k_{1,B} &= \mathscr{P}_{2,B}\Delta\boldsymbol{u}_{\varepsilon_k}, \\ p^k_{2,B} &= -\mathscr{P}_{\frac{3}{2},B}\mathrm{div}(\eta_\varepsilon(|\boldsymbol{u}_{\varepsilon_k}|^2)\boldsymbol{u}_{\varepsilon_k} \otimes \boldsymbol{u}_{\varepsilon_k}). \end{aligned}$$

Owing to (2.5) and (3.2) we have

$$\begin{aligned} \|p^k_{h,B}\|_{L^2(0,T;L^2(B))} &\le c\|\boldsymbol{u}_{\varepsilon_k}\|_{L^2(0,T;\boldsymbol{H})} \le cT^{\frac{1}{2}}\|\boldsymbol{u}_0\|_{\boldsymbol{H}}, \\ \|\nabla p^k_{h,B}\|_{L^2(0,T;\boldsymbol{L}^2(B))} &\le c\|\nabla\boldsymbol{u}_{\varepsilon_k}\|_{L^2(0,T;\boldsymbol{H})} \le c\|\boldsymbol{u}_0\|_{\boldsymbol{H}}. \end{aligned}$$

Thus, eventually passing to a subsequence there exists $p_h \in L^2(0, T; W^{1,2}(B))$ such that

$$p^k_{h,B} \rightharpoonup p_h \quad \text{in} \quad L^2(0, T; W^{1,2}(B)) \quad \text{as} \quad k \to +\infty.$$

Observing (3.4) by virtue of the boundedness of $\mathscr{P}_{2,B}$ there holds $p_h = p_{h,B} = \mathscr{P}_{2,B}\boldsymbol{u}$. In addition, thanks to (3.6) we find

$$p^k_{h,B}(t) \rightharpoonup p_{h,B}(t) \quad \text{in} \quad L^2(B) \quad \text{as} \quad k \to +\infty \tag{3.7}$$

for almost all $t \in (0, T)$. Let $t \in (0, T)$ such that (3.7) is true. Let $x \in B$ and $B_r(x) \subset B$. Since $p^k_{h,B}(t)$ and $p_{h,B}(t)$ are harmonic functions by using the mean value property we have

$$p^k_{h,B}(x, t) = \frac{1}{B_r(x)} \int\limits_{B_r(x)} p^k_{h,B}(y, t)dy \to \frac{1}{B_r(x)} \int\limits_{B_r(x)} p_{h,B}(y, t)dy = p_{h,B}(x, t) \quad \text{as} \quad k \to +\infty.$$

This shows that

$$p^k_{h,B} \to p_{h,B} \quad \text{a. e. in } B \times (0, T) \quad \text{as} \quad k \to +\infty. \tag{3.8}$$

On the other hand, by (3.2) for every $0 < \tau < 1$ and every multi-index $\alpha \in \mathbb{N}_0^3$ we have

$$\|D^\alpha p^k_{h,B}\|_{L^\infty(\tau B\times(0,T))} \le c(\alpha,\tau)\|\boldsymbol{u}_0\|_{\boldsymbol{H}} \quad \forall k \in \mathbb{N}.$$

Then, by virtue of (3.8) taking into account that $p^k_{h,B}$ are harmonic, we deduce that

$$p^k_{h,B} \to p_{h,B} \quad \text{in} \quad L^3(0,T;W^{2,3}(\tau B)) \quad \text{as} \quad k \to +\infty. \tag{3.9}$$

Define, $\boldsymbol{v}^k_B := \boldsymbol{u}_{\varepsilon_k} + \nabla p^k_{h,B}$ a.e. in $B \times (0,T)$. By Lemma 2.4 we see that $\boldsymbol{v}^k_B \in L^\infty(0,T;\boldsymbol{L}^2(B)) \cap L^2(0,T;\boldsymbol{W}^{1,2}(B))$ solving the equation

$$\partial_t \boldsymbol{v}^k_B = \Delta \boldsymbol{v}^k_B - \operatorname{div}(\eta_{\varepsilon_k}(|\boldsymbol{u}_{\varepsilon_k}|^2)\boldsymbol{u}_{\varepsilon_k} \otimes \boldsymbol{u}_{\varepsilon_k}) - \nabla p^k_{1,B} - \nabla p^k_{2,B} \quad \text{in} \quad B \times (0,T)$$

in the sense of distributions. According to (3.2) and the boundedness of $\mathscr{P}_{\frac{3}{2},B}$ we infer that

$$\begin{aligned}&\left\|\Delta \boldsymbol{v}^k_B - \operatorname{div}(\eta_{\varepsilon_k}(|\boldsymbol{u}_{\varepsilon_k}|^2)\boldsymbol{u}_{\varepsilon_k} \otimes \boldsymbol{u}_{\varepsilon_k}) - \nabla p^k_{1,B} - \nabla p^k_{2,B}\right\|_{L^{3/2}(0,T;W^{-1,3/2}(B))}\\ &\le c(1+\|\boldsymbol{u}_0\|^2_{\boldsymbol{H}}).\end{aligned}$$

Hence, the sequence $(\boldsymbol{v}^k_B)$ is relatively compact in $L^1(B\times(0,T))$. Taking into account (3.2), (3.3), (3.5) and (3.9) we deduce that

$$\boldsymbol{v}^k_B \to \boldsymbol{v}_B = \boldsymbol{u} + \nabla p_{h,B} \quad \text{in} \quad \boldsymbol{L}^3(B\times(0,T)) \quad \text{as} \quad k \to +\infty.$$

Once more appealing to (3.9) we get

$$\boldsymbol{u}_{\varepsilon_k} \to \boldsymbol{u} \quad \text{in} \quad \boldsymbol{L}^3(B\times(0,T)) \quad \text{as} \quad k \to +\infty. \tag{3.10}$$

Consequently, there holds

$$\boldsymbol{u}_{\varepsilon_k} \otimes \eta_{\varepsilon_k}(|\boldsymbol{u}_{\varepsilon_k}|^2)\boldsymbol{u}_{\varepsilon_k} \to \boldsymbol{u}\otimes\boldsymbol{u} \quad \text{in} \quad \boldsymbol{L}^{\frac{3}{2}}(B\times(0,T)) \quad \text{as} \quad k \to +\infty. \tag{3.11}$$

This shows that $\boldsymbol{u}$ is a weak solution to (NSE). In addition, recalling the definition of $p^k_{1,B}$ and $p^k_{2,B}$ from (3.5) and (3.11) we deduce that

$$p^k_{1,B} \rightharpoonup p_{1,B} \quad \text{in} \quad L^2(B\times(0,T)), \tag{3.12}$$

$$p^k_{2,B} \to p_{2,B} \quad \text{in} \quad L^{3/2}(B\times(0,T)) \quad \text{as} \quad k \to +\infty. \tag{3.13}$$

Now, it only remains to verify the validity of the local energy inequality (3.1). To this aim we fix $t \in (0,T)$ such that (3.6) and (3.7) are fulfilled. Then, let $\phi \in$

$C_0^\infty(B \times (0,T))$ be a nonnegative function. In $(\mathrm{NSE})_{\varepsilon_k}$ testing with $\boldsymbol{v}_B^k\phi$ and using integration by parts we are led to[4]

$$\int_B |\boldsymbol{v}_B^k(t)|^2\phi(t)dx + 2\int_0^t\int_B |\nabla \boldsymbol{v}_B^k|^2\phi dxds$$
$$= \int_0^t\int_B |\boldsymbol{v}_B^k|^2(\partial_t\phi + \Delta\phi) - H_k(|\boldsymbol{u}_{\varepsilon_k}|^2)\boldsymbol{u}_{\varepsilon_k}\cdot\nabla\phi + (2p_{1,B}^k + 2p_{2,B}^k)\boldsymbol{v}_B^k\cdot\nabla\phi dxds$$
$$+ \int_0^t\int_B 2\eta_{\varepsilon_k}(|\boldsymbol{u}_{\varepsilon_k}|^2)\boldsymbol{u}_{\varepsilon_k}\otimes\boldsymbol{u}_{\varepsilon_k} : \nabla(\nabla p_{h,B}^k\phi) + 2\eta_{\varepsilon_k}(|\boldsymbol{u}_{\varepsilon_k}|^2)|\boldsymbol{u}_{\varepsilon_k}|^2\boldsymbol{u}_{\varepsilon_k}\cdot\nabla\phi dxds, \tag{3.14}$$

where

$$H_k(\xi) = \int_0^\xi \eta_{\varepsilon_k}(\tau)d\tau, \quad \xi \geq 0.$$

By means of (3.4), (3.5), (3.7) and (3.9) using Banach–Steinhaus' theorem we see that

$$\int_B |\boldsymbol{v}_B(t)|^2\phi(t)dx + 2\int_0^t\int_B |\nabla\boldsymbol{v}_B|^2\phi dxds$$
$$\leq \liminf_{k\to\infty}\left(\int_B |\boldsymbol{v}_B^k(t)|^2\phi(t)dx + 2\int_0^t\int_B |\nabla\boldsymbol{v}_B^k|^2\phi dxds\right). \tag{3.15}$$

On the other hand, with the help of (3.9), (3.10), (3.11), (3.12) and (3.13) we verify

$$\lim_{k\to\infty}\int_0^t\int_B |\boldsymbol{v}_B^k|^2(\partial_t\phi + \Delta\phi) - H_k(|\boldsymbol{u}_{\varepsilon_k}|^2)\boldsymbol{u}_{\varepsilon_k}\cdot\nabla\phi + (2p_{1,B}^k + 2p_{1,B}^k)\boldsymbol{v}_B^k\cdot\nabla\phi dxds$$
$$+ \lim_{k\to\infty}\int_0^t\int_B 2\eta_{\varepsilon_k}(|\boldsymbol{u}_{\varepsilon_k}|^2)\boldsymbol{u}_{\varepsilon_k}\otimes\boldsymbol{u}_{\varepsilon_k} : \nabla(\nabla p_{h,B}^k\phi) + 2\eta_{\varepsilon_k}(|\boldsymbol{u}_{\varepsilon_k}|^2)|\boldsymbol{u}_{\varepsilon_k}|^2\boldsymbol{u}_{\varepsilon_k}\cdot\nabla\phi dxds$$
$$= \int_0^t\int_B |\boldsymbol{v}_B|^2(\partial_t\phi + \Delta\phi) - |\boldsymbol{u}|^2\boldsymbol{u}\cdot\nabla\phi + (2p_{1,B} + 2p_{2,B})\boldsymbol{v}_B\cdot\nabla\phi dxds$$
$$+ \int_0^t\int_B 2\boldsymbol{u}\otimes\boldsymbol{u} : \nabla(\nabla p_{h,B}\phi) + 2|\boldsymbol{u}|^2\boldsymbol{u}\cdot\nabla\phi dxds. \tag{3.16}$$

4 Here we argue as in the proof of Lemma 2.4 replacing $\boldsymbol{u}\otimes\boldsymbol{u}$ by $\eta_{\varepsilon_k}(|\boldsymbol{u}_{\varepsilon_k}|^2)\boldsymbol{u}_{\varepsilon_k}\otimes\boldsymbol{u}_{\varepsilon_k}$ (cf. also Remark 2.5, 2.).

Thus, from (3.14), (3.15) and (3.16) it follows that

$$\int_B |\boldsymbol{v}_B(t)|^2\phi(t)dx + 2\int_0^t\int_B |\nabla \boldsymbol{v}_B|^2\phi dxds$$
$$\leq \int_0^t\int_B |\boldsymbol{v}_B|^2(\partial_t\phi + \Delta\phi) + (|\boldsymbol{u}|^2 + 2p_{1,B} + 2p_{2,B})\boldsymbol{v}_B\cdot\nabla\phi dxds$$
$$+ \int_0^t\int_B 2\boldsymbol{u}\otimes\boldsymbol{u} : \nabla(\nabla p_{h,B}\phi) - |\boldsymbol{u}|^2\nabla p_{h,B}\cdot\nabla\phi dxds.$$

This completes the proof of (3.1). □

Remark 3.3 1. Besides the local energy inequality (3.1) we may write an alternative one. In fact, the last integral on the right of (3.1) can be rewritten as follows. Using integration by parts together with the identity $2(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = 2\,\mathrm{curl}\,\boldsymbol{u}\times\boldsymbol{u} + \nabla|\boldsymbol{u}|^2$ we get

$$\int_0^t\int_B 2\boldsymbol{u}\otimes\boldsymbol{u} : \nabla(\nabla p_{h,B}\phi) - |\boldsymbol{u}|^2\nabla p_{h,B}\cdot\nabla\phi dxds$$
$$= -2\int_0^t\int_B (\mathrm{curl}\,\boldsymbol{u}\times\boldsymbol{u})\cdot\nabla p_{h,B}\phi dxdt. \tag{3.17}$$

2. Observing that

$$2\mathscr{P}_{q,B}(\mathrm{curl}\,\boldsymbol{u}(t)\times\boldsymbol{u}(t)) = 2\mathscr{P}_{q,B}\mathrm{div}(\boldsymbol{u}(t)\otimes\boldsymbol{u}(t)) - \mathscr{P}_{q,B}\nabla|\boldsymbol{u}(t)|^2$$
$$= -2p_{1,B}(t) - |\boldsymbol{u}(t)|^2 + (|\boldsymbol{u}(t)|^2)_B \quad \forall\, 1 < q \leq 3,$$

using Lemma 2.3 and Sobolev's embedding theorem we obtain the estimate

$$\begin{cases} \left\|2p_{1,B}(t) + |\boldsymbol{u}(t)|^2 - (|\boldsymbol{u}(t)|^2)_B\right\|_{L^q(B)} \leq c\|\,\mathrm{curl}\,\boldsymbol{u}(t)\times\boldsymbol{u}(t)\|_{\boldsymbol{L}^{\frac{3q}{3+q}}(B)} \\ \text{for a.e. } t\in(0,T), \quad \forall\,\frac{3}{2} < q \leq 3. \end{cases} \tag{3.18}$$

4 Caccioppoli-type inequalities in terms of *u* only

In this section we will derive two Caccioppoli-type inequalities, which play the main role in the proof of the partial regularity. First let us introduce the notations which will be used below. For $x_0 \in \mathbb{R}^3$ and $0 < R < +\infty$ we denote by $B_R = B_R(x_0)$ the usual ball with center x_0 and radius R. In addition, for $X_0 = (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$ we introduce the parabolic cylinder

$$Q_R = Q_R(X_0) := B_R(x_0) \times (t_0 - R^2, t_0).$$

For the proof of partial regularity of suitable weak solutions it will be important to obtain decay estimates for quantities like $R^{-\alpha}\|\boldsymbol{u}\|_{L^q(Q_R)}$ which are invariant under the natural scaling of the Navier–Stokes equations. The main quantities we are going to work with are the following,

$$\begin{aligned}
A(R) = A(R, X_0) &:= R^{-1}\|\boldsymbol{u}\|^2_{L^3(t_0-R^2,t_0;\boldsymbol{L}^{18/5}(B_R))},\\
B(R) = B(R, X_0) &:= R^{-1}\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_R)},\\
\widetilde{B}(R) = \widetilde{B}(R, X_0) &:= R^{-1}\|\operatorname{curl} \boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_R)},\\
D(R) = D(R, X_0) &:= R^{-2}\|\boldsymbol{u}\|^2_{L^3(t_0-R^2,t_0;\boldsymbol{L}^{9/4}(B_R))},\\
E(R) = E(R, X_0) &:= R^{-\frac{4}{3}}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^3(Q_R)}.
\end{aligned}$$

Lemma 4.1 *Let $\boldsymbol{u} \in L^2(0, T; \boldsymbol{V}) \cap L^\infty(0, T; \boldsymbol{H})$ be a local suitable weak solution to (NSE). Then for every $Q_{2R} = Q_{2R}(X_0) \subset Q$ we have the following two Caccioppoli-type inequalities*

$$A(R) + B(R) + E(R) \le c\Big(D(2R) + [E(2R)]^{3/2}\Big), \tag{4.1}$$

$$A(R) + B(R) + E(R) \le c\Big(D(2R) + \widetilde{B}(2R)E(2R)\Big), \tag{4.2}$$

where $c > 0$ denotes an absolute constant.

Proof 1° *Proof of* (4.1). Let $R \le r < \rho \le 2R$ be arbitrarily chosen. Set $\sigma := \frac{r+\rho}{2}$. Let $\phi \in C_0^\infty(Q)$ such that $0 \le \phi \le 1$ in Q, $\phi \equiv 0$ in $Q \setminus B_\sigma \times (t_0 - \sigma^2, t_0 + \sigma^2)$, $\phi \equiv 1$ on Q_r and $|\partial_t\phi| + |\Delta\phi| + |\nabla\phi|^2 \le c(\rho - r)^{-2}$ in Q. Then, from (3.1) with $B = B_\rho$ replacing ϕ by ϕ^2 therein we get

$$\begin{aligned}
&\int\limits_{B_\rho} |\boldsymbol{v}_B(t)|^2\phi^2(t)dx + 2\int\limits_{t_0-\rho^2}^{t}\int\limits_{B_\rho} |\nabla \boldsymbol{v}_B|^2\phi^2 dxds\\
&\quad\le \int\limits_{t_0-\rho^2}^{t}\int\limits_{B_\rho} |\boldsymbol{v}_B|^2(\partial_t\phi^2 + \Delta\phi^2)dxds + 2\int\limits_{t_0-\rho^2}^{t}\int\limits_{B_\rho} (|\boldsymbol{u}|^2 + 2p_{0,B})\boldsymbol{v}_B \cdot \phi\nabla\phi dxds\\
&\qquad + \int\limits_{t_0-\rho^2}^{t}\int\limits_{B_\rho} 2u^iu^j\partial_i(\partial_j p_h\phi^2) + 2|\boldsymbol{u}|^2\nabla p_{h,B} \cdot \phi\nabla\phi dxds, \qquad (4.3)\\
&\quad= I_1 + I_2 + I_3
\end{aligned}$$

for almost all $t \in (t_0 - \rho^2, t_0)$.

(i) Since $p_{h,B} = -\mathscr{P}_{2,B}\boldsymbol{u}$, according to Lemma 2.3 we have

$$I_1 \le c(\rho - r)^{-2} \int\limits_{Q_\rho} |\boldsymbol{u}|^2 + |\nabla p_{h,B}|^2 dxds \le c(\rho - r)^{-2} \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(B_\rho)}.$$

(ii) Recalling that $p_{0,B} = p_{1,B} + p_{2,B} = -\mathscr{P}_{2,B}\mathrm{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \mathscr{P}_{\frac{3}{2},B}\Delta\boldsymbol{u}$ by virtue of Lemma 2.1 we find

$$\|p_{1,B}\|_{\boldsymbol{L}^{3/2}(Q_\rho)} \le c\|\boldsymbol{u}\|^2_{\boldsymbol{L}^3(Q_\rho)},$$
$$\|p_{2,B}\|_{\boldsymbol{L}^2(Q_\rho)} \le c\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(Q_\rho)}.$$

Thus, with help of Hölder's inequality together with the estimates above we easily deduce

$$I_2 \le c(\rho - r)^{-1}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^3(Q_\rho)}\|\boldsymbol{v}_B\|_{\boldsymbol{L}^3(Q_\rho)} + c(\rho - r)^{-1}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(Q_\rho)}\|\boldsymbol{v}_B\|_{\boldsymbol{L}^2(Q_\rho)}.$$

On the other hand, by using Lemma 2.3 having $\|\boldsymbol{v}_B\|_{\boldsymbol{L}^3(Q_\rho)} \le c\|\boldsymbol{u}\|_{\boldsymbol{L}^3(Q_\rho)}$ and $\|\boldsymbol{v}_B\|_{\boldsymbol{L}^2(Q_\rho)} \le c\|\boldsymbol{u}\|_{\boldsymbol{L}^2(Q_\rho)}$, we apply Young's inequality to arrive at

$$I_2 \le c(\rho - r)^{-1}\|\boldsymbol{u}\|^3_{\boldsymbol{L}^3(Q_\rho)} + c(\rho - r)^{-2}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)} + \frac{1}{4}\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)}.$$

(iii) Again using Hölder's inequality recalling that $\Delta p_{h,B} = 0$ in $B \times (0, T)$ and arguing as above we infer

$$\begin{aligned} I_3 &\le c(\rho - r)^{-1}\|\boldsymbol{u}\|^3_{\boldsymbol{L}^3(Q_\rho)} + c\|\boldsymbol{u}\|^2_{\boldsymbol{L}^3(Q_\rho)} \left(\int\limits_{Q_\sigma} |\nabla^2 p_{h,B}|^3 dxds \right)^{1/3} \\ &\le c(\rho - r)^{-1}\|\boldsymbol{u}\|^3_{\boldsymbol{L}^3(Q_\rho)}. \end{aligned}$$

Inserting estimates of $I_1 - I_3$ into (4.3) and applying Hölder's inequality we are let to

$$\begin{aligned} &\|\phi\boldsymbol{v}_B\|^2_{L^\infty(t_0-\rho^2;\boldsymbol{L}^2(B_\rho))} + \|\phi\nabla\boldsymbol{v}_B\|^2_{\boldsymbol{L}^2(Q_\rho)} \\ &\quad \le c\rho(\rho - r)^{-2}\|\boldsymbol{u}\|^2_{L^3(t_0-\rho^2;\boldsymbol{L}^{9/4}(B_\rho))} + c(\rho - r)^{-1}\|\boldsymbol{u}\|^3_{\boldsymbol{L}^3(Q_\rho)} + \frac{1}{4}\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)}. \end{aligned} \tag{4.4}$$

By means of Sobolev's embedding theorem using Hölder's inequality we get

$$
\begin{aligned}
&\|\phi \boldsymbol{v}_B\|^2_{L^3(t_0-\rho^2;\boldsymbol{L}^{18/5}(B_\rho))}\\
&\quad \le c\rho(\rho-r)^{-2}\|\boldsymbol{u}\|^2_{L^3(t_0-\rho^2;\boldsymbol{L}^{9/4}(B_\rho))} + c(\rho-r)^{-1}\|\boldsymbol{u}\|^3_{\boldsymbol{L}^3(Q_\rho)} + \frac{1}{4}\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)}.
\end{aligned}
$$

Recalling $\Delta p_{h,B} = 0$ and applying Lemma 2.3 we see that

$$
\begin{aligned}
\|\phi \boldsymbol{u}\|^2_{L^3(t_0-\rho^2;\boldsymbol{L}^{18/5}(B_\rho))} &\le 2\|\phi \boldsymbol{v}_B\|^2_{L^3(t_0-\rho^2;\boldsymbol{L}^{18/5}(B_\rho))} + 2\|\nabla p_{h,B}\|^2_{L^3(t_0-\rho^2;\boldsymbol{L}^{18/5}(B_\sigma))}\\
&\le 2\|\phi \boldsymbol{v}_B\|^2_{L^3(t_0-\rho^2;\boldsymbol{L}^{18/5}(B_\rho))} + c(\rho-r)^{-1}\|\boldsymbol{u}\|^2_{L^3(t_0-\rho^2;\boldsymbol{L}^{9/4}(B_\rho))}.
\end{aligned}
$$

Combining the last two inequalities we deduce that

$$
\begin{aligned}
&\|\boldsymbol{u}\|^2_{L^3(t_0-r^2;\boldsymbol{L}^{18/5}(B_r))}\\
&\quad \le cR(\rho-r)^{-2}\Big(\|\boldsymbol{u}\|^2_{L^3(t_0-4R^2;\boldsymbol{L}^{9/4}(B_{2R}))} + \|\boldsymbol{u}\|^3_{\boldsymbol{L}^3(Q_{2R})}\Big) + \frac{1}{4}\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)}.
\end{aligned}
\tag{4.5}
$$

Similarly, one proves

$$
\begin{aligned}
\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_r)} &\le 2\|\phi\nabla \boldsymbol{v}_B\|^2_{\boldsymbol{L}^2(Q_r)} + \|\nabla^2 p_{h,B}\|_{\boldsymbol{L}^2(Q_\sigma)}\\
&\le cR(\rho-r)^{-2}\Big(\|\boldsymbol{u}\|^2_{L^3(t_0-4R^2;\boldsymbol{L}^{9/4}(B_{2R}))} + \|\boldsymbol{u}\|^3_{\boldsymbol{L}^3(Q_{2R})}\Big)\\
&\quad + \frac{1}{4}\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)}.
\end{aligned}
\tag{4.6}
$$

By means of (4.5) and (4.6) we obtain

$$
\begin{aligned}
&\|\boldsymbol{u}\|^2_{L^3(t_0-r^2;\boldsymbol{L}^{18/5}(B_r))} + \|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_r)}\\
&\quad \le\ cR(\rho-r)^{-2}\Big(\|\boldsymbol{u}\|^2_{L^3(t_0-4R^2;\boldsymbol{L}^{9/4}(B_{2R}))} + \|\boldsymbol{u}\|^3_{\boldsymbol{L}^3(Q_{2R})}\Big) + \frac{1}{2}\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)}
\end{aligned}
$$

with an absolute constant $c > 0$. By a well-known algebraic iteration argument (see, e. g. [8]) we get

$$
\begin{aligned}
&\|\boldsymbol{u}\|^2_{L^3(t_0-R^2;\boldsymbol{L}^{18/5}(B_R))} + \|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_R)}\\
&\qquad \le\ cR^{-1}\Big(\|\boldsymbol{u}\|^2_{L^3(t_0-4R^2;\boldsymbol{L}^{9/4}(B_{2R}))} + \|\boldsymbol{u}\|^3_{\boldsymbol{L}^3(Q_{2R})}\Big).
\end{aligned}
$$

Multiplying both sides by R^{-1} we complete the proof of the first inequality (4.1).

$2°$ *Proof of* (4.2). For given $R \le r < \rho \le 2R$ let $\phi \in C_0^\infty(Q)$ denote the same cut-off function as in 1°. Appealing to Remark 3.3/1. besides (4.3) we have the inequality[5]

[5] Here we have used the fact that $(\operatorname{curl} \boldsymbol{u} \times \nabla p_{h,B}) \cdot \nabla p_{h,B} = 0$ a.e. in Q_ρ.

$$\int\limits_{B_\rho} |\boldsymbol{v}_B(t)|^2\phi^2(t)dx + 2\int\limits_{t_0-\rho^2}^{t}\int\limits_{B_\rho}|\nabla \boldsymbol{v}_B|^2\phi^2 dxds$$
$$\leq \int\limits_{t_0-\rho^2}^{t}\int\limits_{B_\rho}|\boldsymbol{v}_B|^2(\partial_t\phi^2+\Delta\phi^2)dxds + \int\limits_{t_0-\rho^2}^{t}\int\limits_{B_\rho}(|\boldsymbol{u}|^2+2p_{0,B})\boldsymbol{v}_B\cdot\phi\nabla\phi dxds$$
$$-2\int\limits_{t_0-\rho^2}^{t}\int\limits_{B_\rho}(\operatorname{curl}\boldsymbol{u}\times\boldsymbol{v}_B)\cdot\nabla p_{h,B}\phi^2 dxds, \tag{4.7}$$
$$= I_1 + I_2 + I_3'.$$

(i) Clearly, as in 1° we see that

$$I_1 \leq \rho(\rho-r)^{-2}\|\boldsymbol{u}\|^2_{L^3(t_0-\rho^2,t_0;\boldsymbol{L}^{9/4}(B_\rho))}.$$

(ii) Making use of (3.18) with $q=2$ with help of Hölder's inequality arguing similar as in 1° we find

$$I_2 \leq c(\rho-r)^{-1}\|\operatorname{curl}\boldsymbol{u}\times\boldsymbol{u}\|_{\boldsymbol{L}^{6/5}(Q_\rho)}\|\phi\boldsymbol{v}_B\|_{L^6(t_0-\rho^2,t_0;\boldsymbol{L}^2(B_\rho))}$$
$$+c(\rho-r)^{-1}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(Q_\rho)}\|\boldsymbol{v}_B\|_{\boldsymbol{L}^2(Q_\rho)}$$
$$\leq c\rho^{\frac{2}{3}}(\rho-r)^{-2}\|\operatorname{curl}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^3(Q_\rho)}+c\rho(\rho-r)^{-2}\|\boldsymbol{u}\|^2_{L^3(t_0-\rho^2,t_0;\boldsymbol{L}^{9/4}(B_\rho))}$$
$$+\frac{1}{4}\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)}+\frac{1}{4}\|\phi\boldsymbol{v}_B\|^2_{L^\infty(t_0-\rho^2,t_0;\boldsymbol{L}^2(B_\rho))}.$$

(iii) Similarly as above using Lemma 2.2 we estimate

$$I_3' \leq c\rho^{\frac{2}{3}}(\rho-r)^{-2}\|\operatorname{curl}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^3(Q_\rho)}+\frac{1}{4}\|\phi\boldsymbol{v}_B\|^2_{L^\infty(t_0-\rho^2,t_0;\boldsymbol{L}^2(B_\rho))}.$$

Inserting the estimates of I_1, I_2 and I_3' into (4.7) we are led to

$$\|\phi\boldsymbol{v}_B\|^2_{L^\infty(t_0-\rho^2;\boldsymbol{L}^2(B_\rho))}+\|\phi\nabla\boldsymbol{v}_B\|^2_{\boldsymbol{L}^2(Q_\rho)}$$
$$\leq c\rho(\rho-r)^{-2}\|\boldsymbol{u}\|^2_{L^3(t_0-\rho^2;\boldsymbol{L}^{9/4}(B_\rho))}+c\rho^{\frac{2}{3}}(\rho-r)^{-2}\|\operatorname{curl}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^3(Q_\rho)}$$
$$+\frac{1}{4}\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)}. \tag{4.8}$$

Using the same reasoning as in 1° from (4.8) we deduce that

$$\|\boldsymbol{u}\|^2_{L^3(t_0-r^2;\boldsymbol{L}^{18/5}(B_r))} + \|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_r)}$$
$$\leq cR(\rho-r)^{-2}\Big(\|\boldsymbol{u}\|^2_{L^3(t_0-4R^2;\boldsymbol{L}^{9/4}(B_{2R}))} + cR^{-\frac{1}{3}}\|\operatorname{curl}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_{2R})}\|\boldsymbol{u}\|^2_{\boldsymbol{L}^3(Q_{2R})}\Big)$$
$$+\frac{1}{2}\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_\rho)},$$

where $c > 0$ denotes an absolute constant. As in 1° from the last estimate we obtain the second inequality (4.2). □

5 Partial regularity

On the basis of the two Caccioppoli-type inequalities (4.1) and (4.2), we are now in a position to prove the following local regularity result

Theorem 5.1 *Let* $\boldsymbol{u}$ *be a local suitable weak solution to (NSE).*

1. *There exists* $\varepsilon_1 > 0$ *such that for any* $Q_R(X_0) \subset Q$ *there holds*

$$R^{-\frac{4}{3}}\left(\int\limits_{Q_R(X_0)} |\boldsymbol{u}|^3 dxdt\right)^{\frac{2}{3}} \leq \varepsilon_1 \quad\Longrightarrow\quad \boldsymbol{u}|_{Q_{R/2}} \in \boldsymbol{C}^0(\overline{Q_{R/2}(X_0)}). \tag{5.1}$$

2. *There exists* $\varepsilon_2 > 0$*, such that for every* $X_0 \subset Q$ *there holds*

$$\limsup_{R\to 0} R^{-1}\int\limits_{Q_R(X_0)} |\operatorname{curl}\boldsymbol{u}|^2 dxdt \leq \varepsilon_2$$
$$\Longrightarrow\quad \exists\rho > 0: \quad \boldsymbol{u}|_{Q_\rho(X_0)} \in \boldsymbol{C}^0(\overline{Q_\rho(X_0)}). \tag{5.2}$$

In particular, if $S(\boldsymbol{u})$ *is the set of possible singularities then*

$$\mathcal{P}_1(S(\boldsymbol{u})) = 0, \tag{5.3}$$

where $\mathcal{P}_1$ *stands for the one-dimensional parabolic Hausdorff measure.*

Proof 1° Let $Q_R = Q_R(X_0) \subset Q$ be fixed. Let $\zeta \in C_0^\infty(B_{R/2})$ be a cut-off function, with $\zeta \equiv 1$ on $B_{R/4}$. Noticing, that

$$\boldsymbol{u}\otimes\boldsymbol{u} \in L^{3/2}(0,T;L^{9/5}(\Omega;\mathbb{R}^{3\times 3})) \cap L^3(0,T;L^{9/7}(\Omega;\mathbb{R}^{3\times 3})),$$
$$p_{1,B_{R/2}} \in L^{3/2}(0,T;L^{9/5}(B_{R/2})) \cap L^3(0,T;L^{9/7}(B_{R/2}))$$

by the $L^p - L^q$ theory of the heat equation (cf. [10]) there exists a unique weak solution

$$\boldsymbol{W} \in L^{\frac{3}{2}}(t_0-R^2/4,t_0;\boldsymbol{W}^{2,\frac{9}{5}}) \cap L^3(t_0-R^2/4,t_0;\boldsymbol{W}^{2,\frac{9}{7}})$$

to the system of the heat equations

$$\begin{cases} \partial_t \boldsymbol{W} - \Delta \boldsymbol{W} & = -\zeta \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{I} \zeta p_{1,B_{R/2}} \quad \text{in} \quad \mathbb{R}^3 \times (t_0 - R^2/4, t_0), \\ \boldsymbol{W}(t_0 - R^2/4) & = 0 \quad \text{in} \quad \mathbb{R}^3. \end{cases}$$

Then, applying divergence to both sides of the above system we se that the vector function $\boldsymbol{w} = \operatorname{div} \boldsymbol{W}$ is a weak solution to the system

$$\text{(S)} \qquad \begin{cases} \partial_t \boldsymbol{w} - \Delta \boldsymbol{w} = -\operatorname{div}(\zeta \boldsymbol{u} \otimes \boldsymbol{u}) - \nabla(\zeta p_{1,B_{R/2}}) \quad \text{in} \quad \mathbb{R}^3 \times (t_0 - R^2/4, t_0), \\ \boldsymbol{w}(t_0 - R^2/4) = 0 \quad \text{in} \quad \mathbb{R}^3. \end{cases}$$

By the $L^p - L^q$ theory of the heat equation (cf. [10]) making use of Gagliardo–Nirenberg's inequality we infer

$$\begin{aligned} & \|\boldsymbol{w}\|_{L^3(t_0 - R^2/4, t_0; \boldsymbol{L}^{9/4})} \\ & \quad \le \|\nabla \boldsymbol{w}\|_{L^3(t_0 - R^2/4, t_0; \boldsymbol{L}^{9/7})} \le \|\nabla^2 \boldsymbol{W}\|_{L^3(t_0 - R^2/4, t_0; \boldsymbol{L}^{9/7})} \\ & \quad \le c\|\boldsymbol{u}\|^2_{L^3(t_0 - R^2/4, t_0; \boldsymbol{L}^{18/5}(B_{R/2}))} + c\|p_{1,B_{R/2}}\|_{L^{3/2}(t_0 - R^2/4, t_0; \boldsymbol{L}^{9/5}(B_{R/2}))}. \end{aligned}$$

In addition, verifying that

$$-\operatorname{div}(\zeta \boldsymbol{u} \otimes \boldsymbol{u}) - \nabla(\zeta p_{1,B_{R/2}}) \in L^1(0, T; \boldsymbol{L}^{3/2})$$

it follows that $\boldsymbol{w} \in L^\infty(t_0 - R^2/4, t_0; \boldsymbol{L}^{3/2})$.

Next, applying Hölder's inequality together with Sobolev's inequality, making use of Lemma 2.1 and (4.1), from the inequality above we infer

$$R^{-2}\|\boldsymbol{w}\|^2_{L^3(t_0 - R^2/4, t_0; \boldsymbol{L}^{9/4}(B_R))} \le c[A(R/2)]^2 \le c([E(R)]^2 + [E(R)]^3). \tag{5.4}$$

Next, set $\boldsymbol{U} := \boldsymbol{u} - \boldsymbol{w}$ a.e. in $Q_{R/2}$. Clearly, according to Lemma 2.2 $\boldsymbol{U}$ satisfies the following equation

$$\partial_t \boldsymbol{U} - \Delta \boldsymbol{U} = -\partial_t \nabla p_{h,B_{R/2}} - \nabla p_{2,B_{R/2}} \quad \text{in} \quad Q_{R/4} \tag{5.5}$$

in the sense of distributions. Define,

$$\boldsymbol{V} := \boldsymbol{U} + \nabla p_{h,B_{R/2}} + \nabla P \quad \text{a.e. in} \quad Q_{R/4},$$

where

$$P(x,t) = \int_{t_0 - R^2/4}^{t} p_{2,B_{R/2}}(x,s)ds, \quad (x,t) \in Q_{R/4}.$$

Recalling that both $p_{h,B_{R/2}}$ and $p_{2,B_{R/2}}$ are harmonic we see that $\boldsymbol{V}$ is a caloric function in $Q_{R/4}$, i.e.

$$\partial_t \boldsymbol{V} - \Delta \boldsymbol{V} = 0 \quad \text{in} \quad Q_{R/4}.$$

Thus, we get for $0 < \tau < \frac{1}{8}$

$$\|\boldsymbol{V}\|^2_{L^3(t_0-4\tau^2R^2,t_0;\boldsymbol{L}^{9/4}(B_{2\tau R}))} \leq c\tau^4 \|\boldsymbol{V}\|^2_{L^3(t_0-R^2/16,t_0;\boldsymbol{L}^{9/4}(B_{R/4}))}.$$

On the other hand, using the mean value property of harmonic functions, from the definition of $\boldsymbol{V}$ we get for $0 < \tau < \frac{1}{8}$

$$\begin{aligned}\|\boldsymbol{U}\|^2_{L^3(t_0-4\tau^2R^2,t_0;\boldsymbol{L}^{9/4}(B_{2\tau R}))} &\leq c\tau^{\frac{8}{3}} \|\boldsymbol{U}\|^2_{L^3(t_0-R^2/16,t_0;\boldsymbol{L}^{9/4}(B_{R/4}))} \\ &\quad + c\tau^{\frac{8}{3}} \|\nabla p_{h,B_{R/2}}\|^2_{L^3(t_0-R^2/4,t_0;\boldsymbol{L}^{9/4}(B_{R/2}))} \\ &\quad + c\tau^{\frac{8}{3}} R \|p_{2,B_{R/2}}\|^2_{L^2(Q_{R/2})}.\end{aligned}$$

By means of Lemmas 2.1 and 2.3, using (4.1) we infer

$$\begin{aligned}D_0(2\tau R) &\leq c\tau^{\frac{2}{3}}(D_0(R/2) + D(R) + B(R/2)) \\ &\leq c\tau^{\frac{2}{3}}(D_0(R/2) + E(R)) + c[E(R)]^{\frac{3}{2}},\end{aligned} \tag{5.6}$$

with an absolute constant $c > 0$, where

$$D_0(\rho) = \rho^{-2}\|\boldsymbol{U}\|^2_{L^3(t_0-\rho^2,t_0;\boldsymbol{L}^{9/4}(B_\rho))}, \quad 0 < \rho \leq \frac{R}{2}.$$

Alternatively, using (4.2) we have

$$D_0(2\tau R) \leq c\tau^{\frac{2}{3}}(D_0(R/2) + E(R)) + c\widetilde{B}(R)E(R). \tag{5.7}$$

Applying (4.1) and then using (5.6) and (5.4) we estimate

$$\begin{aligned}E(\tau R) &\leq cD(2\tau R)) + c\tau^{-2}[E(R)]^{\frac{3}{2}} \\ &\leq cD_0(2\tau R)) + c\tau^{-2}([E(R)]^{\frac{3}{2}} + [E(R)]^3) \\ &\leq c\tau^{\frac{2}{3}}(D_0(R/2) + E(R)) + c\tau^{-2}([E(R)]^{\frac{3}{2}} + [E(R)]^3) \\ &\leq c\tau^{\frac{2}{3}}E(R) + c\tau^{-2}([E(R)]^{\frac{3}{2}} + [E(R)]^3).\end{aligned}$$

Thus, there exists an absolute constant c_1, such that for every $0 < \tau < 1$ and for every $Q_R = Q_R(X_0) \subset Q$,

$$E(\tau R) \leq c_1\tau^{\frac{2}{3}}E(R) + c_1\tau^{-2}([E(R)]^{\frac{3}{2}} + [E(R)]^3). \tag{5.8}$$

If $E(R) \le \frac{1}{4(2c_1)^{16}}$ then with $\tau := \frac{1}{(2c_1)^3}$ we get

$$E(\tau R) \le \tau^{\frac{1}{3}} E(R).$$

In fact, from (5.8) and the definition of τ it follows that

$$\begin{aligned} E(\tau R) &\le \frac{1}{2}\tau^{\frac{1}{3}} E(R) + 2c_1 \tau^{-2} E(R)^{\frac{1}{2}} E(R) \\ &\le \frac{1}{2}\tau^{\frac{1}{3}} E(R) + (2c_1)^7 \frac{1}{2(2c_1)^8} E(R) = \tau^{\frac{1}{3}} E(R). \end{aligned}$$

Define,

$$\varepsilon_1 := \frac{1}{16 \cdot (2c_1)^{16}}.$$

Let $Q_R(X_0) \subset Q$, such that $E(R, X_0) \le \varepsilon_1$. Then, observing $Q_{R/2}(Y) \subset Q_R(X_0)$ for every $Y \in Q_{R/2}(X_0)$ we have

$$E(R/2, Y) \ \le \ 4E(R, X_0) \le \frac{1}{4 \cdot (2c_1)^{16}}.$$

Hence, (5.8) gives

$$E(\tau R/2, Y) \le \tau^{\frac{1}{3}} E(R/2, Y) \quad \forall Y \in Q_{R/2}(X_0).$$

Thus, by using a standard iteration argument the above inequality yields

$$E(\sigma, Y) \le C\left(\frac{\sigma}{R}\right)^{\frac{1}{3}} E(R/2, Y) \le C\varepsilon_1 \left(\frac{\sigma}{R}\right)^{\frac{1}{3}} \quad \forall Y \in Q_{R/2}(X_0), \ \forall 0 < \sigma \le \frac{R}{2} \tag{5.9}$$

with an absolute constant $C > 0$. By a similar reasoning as in [22] and [23] from (5.9) we get $\boldsymbol{u} + \nabla p_{h,B_R} \in \boldsymbol{C}^{0,\alpha}(\overline{Q_{R/2}(X_0)})$ for some $0 < \alpha < 1$ and $\nabla p_{h,B_R} \in \boldsymbol{C}^0(Q_R(X_0))$. Whence, the claim.

2° Given $Q_R = Q_R(X_0) \subset Q$ let $\boldsymbol{w} \in L^{\frac{3}{2}}(t_0 - R^2/4, t_0; \boldsymbol{W}^{1,\frac{9}{5}}) \cap L^\infty(t_0 - R^2/4, t_0; \boldsymbol{L}^{\frac{3}{2}})$ denote the unique solution to the heat equation

$$\begin{cases} \partial_t \boldsymbol{w} - \Delta \boldsymbol{w} = -\zeta \operatorname{curl} \boldsymbol{u} \times \boldsymbol{u} - \zeta \nabla\Big(p_{1,B_{R/2}} + \frac{|\boldsymbol{u}|^2}{2}\Big) \ \text{ in } \ \mathbb{R}^3 \times (t_0 - R^2/4, t_0), \\ \boldsymbol{w}(t_0 - R^2/4) = 0 \ \text{ in } \ \mathbb{R}^3. \end{cases} \tag{S'}$$

Once more using the $L^p - L^q$ theory of the heat equation (cf. [10]) we see that

$$\begin{aligned}\|\boldsymbol{w}\|_{L^\infty(t_0-R^2/4,t_0;\boldsymbol{L}^{3/2})} + \|\boldsymbol{w}\|_{L^3(t_0-R^2/4,t_0;\boldsymbol{L}^{9/4})}\\ \le c\|\operatorname{curl}\boldsymbol{u}\times\boldsymbol{u}\|_{L^{6/5}(t_0-R^2/4,t_0;\boldsymbol{L}^{7/5}(B_{R/2}))}\\ + c\Big\|\nabla\Big(p_{1,B_{R/2}} + \frac{|\boldsymbol{u}|^2}{2}\Big)\Big\|_{L^{6/5}(t_0-R^2/4,t_0;\boldsymbol{L}^{7/5}(B_{R/2}))}.\end{aligned}$$

By the aid of Lemma 2.3 and Hölder's inequality taking into account (4.2) we are led to

$$\begin{aligned}R^{-2}\|\boldsymbol{w}\|^2_{L^3(t_0-R^2/4,t_0;\boldsymbol{L}^{9/4})} &\le cR^{-2}\|\operatorname{curl}\boldsymbol{u}\|^2_{\boldsymbol{L}^2(Q_R)}\|\boldsymbol{u}\|^2_{L^3(t_0-R^2/4,t_0;\boldsymbol{L}^{18/5}(B_{R/2}))}\\ &= c\widetilde{B}(R)A(R/2)\\ &\le cE(R)(\widetilde{B}(R) + [\widetilde{B}(R)]^2). \qquad (5.10)\end{aligned}$$

Next, setting $\boldsymbol{U} := \boldsymbol{u} - \boldsymbol{w}$ we see that $\boldsymbol{U}$ verifies (5.5). Accordingly, as in 1° with help of (5.7), (5.10) and (4.2) we estimate

$$\begin{aligned}E(\tau R) &\le cD(2\tau R)) + c\tau^{-2}\widetilde{B}(R)E(R)\\ &\le cD_0(2\tau R)) + c\tau^{-2}(\widetilde{B}(R) + [\widetilde{B}(R)]^2)E(R)\\ &\le c\tau^{\frac{2}{3}}(D_0(R/2) + E(R)) + c\tau^{-2}(\widetilde{B}(R) + [\widetilde{B}(R)]^2)E(R)\\ &\le c\tau^{\frac{2}{3}}E(R) + c\tau^{-2}(\widetilde{B}(R) + [\widetilde{B}(R)]^2)E(R).\end{aligned}$$

Thus, there exists an absolute constant $c_2 > 0$, such that for every $0 < \tau < 1$ there holds

$$E(\tau R) \le c_2\tau^{\frac{2}{3}}E(R) + c_2\tau^{-2}(\widetilde{B}(R) + [\widetilde{B}(R)]^2)E(R). \qquad (5.11)$$

Define $\tau := \frac{1}{(2c_2)^3}$ and $\varepsilon_2 := \frac{1}{4(2c_2)^{16}}$. Let $X_0 \in Q$, such that $\limsup_{R\to 0}\widetilde{B}(R, X_0) \le \varepsilon_2$. Arguing as above we see that there exists $R_0 > 0$ such that

$$E(\tau R, X_0) \le \tau^{\frac{1}{3}}E(R, X_0) \quad \forall\, 0 < R \le R_0,$$

which shows that $\lim_{R\to 0}E(R, X_0) = 0$. According to the first statement of the theorem $\boldsymbol{u}$ is continuous in a neighbourhood of X_0.

Finally, the set of singular points $S(\boldsymbol{u})$ is containted in the set of all $X_0 \in Q_T$ for which $\limsup_{R\to 0^+} R^{-1}\int_{Q_R(X_0)}|\nabla\boldsymbol{u}|^2dxdt > 0$. Thus, as in [4] one proves $\mathcal{P}_1(S(\boldsymbol{u})) = 0$. This completes the proof of the theorem. □

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