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Viscous incompressible free-surface flow down an inclined perturbed plane

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Abstract The plane stationary free boundary value problem for the Navier-Stokes equations is studied. This problem models the viscous fluid free-surface flow down a perturbed inclined plane. For sufficiently small data the solvability and uniqueness results are proved in Hölder spaces. The asymptotic behavior of the solution is investigated.

Keywords Stationary Navier-Stokes equations · Unbounded domains · Noncompact free-surface flows

Mathematics Subject Classification (2010) 35Q30 · 76D05 · 35R35

1 Introduction

We consider a plane stationary flow of a viscous incompressible fluid moving under the gravity force along the fixed unbounded bottom $S = \{x \in \mathbb{R}^2 : x_2 = \varepsilon^2 \varphi_0(x_1)\}$, where $\text{supp}\varphi_0 \subset (-1, 1)$, and $\varepsilon > 0$ is a small positive parameter. So, S is a slightly perturbed

Dedicated to the memory of Professor M.Padula.

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plane $\{x \in \mathbb{R}^2 : x_2 = 0\}$ and the perturbation has a compact support. We assume that the vector $\mathbf{e}^\gamma = (-\cos\gamma, \sin\gamma)$ makes an angle $\gamma \in (0, \frac{\pi}{2})$ with x_1-axis and coincides with the direction of the gravitational force. The free boundary Γ of the fluid is a priori unknown and we look for Γ in the form $\Gamma = \{x \in \mathbb{R}^2 : x_2 = \psi(x_1) = 1 + \varepsilon\Psi(x_1)\}$[1], where $\psi(x_1) > 0$ for $x_1 \in (-\infty, +\infty)$ and

$$\lim_{|x_1|\to\infty} \Psi(x_1) = 0.$$

Thus, we have to find the velocity vector $\mathbf{u}(x) = \big(u_1(x), u_2(x)\big)$, the pressure function $p(x)$ and the function $\Psi(x_1)$ that solve in the unknown domain

$$\Omega = \{x \in \mathbb{R}^2 : \varepsilon^2\varphi_0(x_1) < x_2 < \psi(x_1) = 1 + \varepsilon\Psi(x_1),\ x_1 \in \mathbb{R}\} \tag{1.1}$$

the following boundary value problem for the Navier–Stokes system of equations

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = -g\mathbf{e}^\gamma & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } S, \\ \mathbf{u}\cdot\mathbf{n} = 0, \boldsymbol{\tau}\cdot\mathcal{S}(\mathbf{u})\mathbf{n} = 0 & \text{on } \Gamma, \\ \left(\frac{\psi'(x_1)}{\sqrt{1+\psi'(x_1)^2}}\right)' = \sigma^{-1}\big(-p(x) + \mathbf{n}\cdot\mathcal{S}(\mathbf{u})\mathbf{n}\big)\big|_\Gamma, \\ \lim_{|x_1|\to\infty}\psi(x_1) = 1, \\ \int_{\sigma_t} u_1(x)dx_2 = \frac{g\sin\alpha}{3\nu}, \end{cases} \tag{1.2}$$

where $\alpha = \frac{\pi}{2} - \gamma$, $\boldsymbol{\tau}$ and $\mathbf{n}$ are unit vectors directed respectively along the tangent and the outward normal to the free boundary Γ, $\nu > 0$ and $\sigma > 0$ are the constant coefficients of viscosity and surface tension, g is the acceleration of gravity, $\mathcal{S}(\mathbf{u})$ is the deformation tensor with the elements $\mathcal{S}_{ij}(\mathbf{u}) = \nu\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)$, $i, j = 1, 2$, σ_t is the cross-section of the domain Ω by the line $x_1 = t$, $\nabla = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\Big)$, div $\mathbf{u} = \nabla\cdot\mathbf{u}$, $\Delta\mathbf{u} = \nabla^2\mathbf{u}$, $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2$.

Note that the left hand side of Eq. $(1.2)_5$ is equal to the curvature $K(x)$ of the free boundary Γ and the condition $(1.2)_7$ prescribes the flow rate of the fluid over the cross-sections of the domain Ω. It shows that the fluid moves only due to the gravity force.

Mathematical problems for the stationary flows of a viscous incompressible fluid with a free boundary were a subject of many papers. Many references on this topic can be found—e.g.—in the bibliographies of [7,15,17,18], etc. Coating flows with the static or dynamic contact angles were investigated in [3,4,8,16,19–22]. In all these papers, that are concerned both with compact and semi-infinite free boundaries, the following iteration scheme proposed in [4,13], is used. It consists of solving the Stokes problem for the velocity and pressure in a given domain defined by the previous iteration and of finding the free boundary for the next iteration from the equation

[1] Without loss of generality we suppose that the height of the fluid at infinity is equal to 1.

$$K(x) = \sigma^{-1}\big(- p(x) + \mathbf{n} \cdot \mathcal{S}(\mathbf{u})\mathbf{n}\big)|_\Gamma \tag{1.3}$$

and some additional boundary conditions (in our case, $(1.2)_6$) depending on the concrete problem under consideration. The velocity and pressure in (1.3) are obtained from the above-mentioned Stokes problem. This scheme of the solution of the complete nonlinear problem can be illustrated by the diagram

$$\Gamma_0 \Rightarrow \Omega_0 \Rightarrow (\mathbf{u}_0, p_0) \Rightarrow \Gamma_1 \Rightarrow \Omega_1 \Rightarrow (\mathbf{u}_1, p_1) \Rightarrow \cdots \Gamma_k \Rightarrow \Omega_k \Rightarrow (\mathbf{u}_k, p_k) \Rightarrow \cdots$$

Note that in this scheme the construction of $(\mathbf{u}, p)$ is separated from the construction of the free boundary Γ at every step.

The method described above can not be applied directly to the free boundary problems with free boundaries that are graphs over the whole line $-\infty \leq x_1 \leq +\infty$ (like in the problem (1.2)). The pressure q_k in the k-th approximation obtained from the Stokes problem in the domain Ω_k has the gradient decaying exponentially as $x_1 \to \pm\infty$, but $q_k(x)$ itself may tend to different constants q_k^+ and q_k^-. We always can normalize $q_k(x)$ by setting $q_k^+ = 0$, however, it can happen that

$$q_{*k} = q_k^+ - q_k^- \neq 0. \tag{1.4}$$

The pressure drop q_{*k} is the functional of the data of the Stokes problem (see formula (2.17) below) and in general $q_{*k} \neq 0$. This makes it problematic to find the next approximation Γ_{k+1} of the free boundary from (1.3) or, more exactly, from the boundary value problem

$$\begin{cases} \Psi''_{k+1}(x_1) - g\sigma^{-1}\cos\alpha\Psi_{k+1}(x_1) \\ = \sigma^{-1}\Big(- q_k(x) + \mathbf{n} \cdot \mathcal{S}(\mathbf{v}_k)\mathbf{n}\Big)\Big|_{\Gamma_k} \equiv \Phi_{k+1}(x_1) \\ \lim_{|x_1|\to\infty} \Psi_{k+1}(x_1) = 0. \end{cases} \tag{1.5}$$

that has a unique solution $\Psi_{k+1}(x_1)$ if and only if the right-hand side Φ_{k+1} of Eq. $(1.5)_1$ decays sufficiently rapidly as $|x_1| \to \infty$, which is possible only if $q_{*k} = 0$.

In order to overcome this difficulty, in [7] and independently in [1] a different scheme was proposed which was based on linearization of the problem on an appropriate exact solution in the unperturbed "uniform" flow domain $\Omega_0 = \{x \in \mathbb{R}^2 : 0 < x_2 < 1\}$. The main difference of this scheme from the previous one is that on each step of iterations the determination of the velocity vector $\mathbf{u}$ and the pressure function p is not separated from the determination of the free boundary Γ (i.e., of the functions Ψ describing Γ) and all the auxiliary problems are solved in the same fixed domain, i.e. in the strip Ω_0. This scheme can be illustrated as

$$(\mathbf{u}_0, p_0, \Psi_0) \Rightarrow (\mathbf{u}_1, p_1, \Psi_1) \Rightarrow \cdots \Rightarrow (\mathbf{u}_k, p_k, \Psi_k) \Rightarrow \cdots$$

Note that the arising linearized problem contains more boundary conditions than it is allowed by general ADN-elliptic theory and contains additionally the unknown function Ψ_k in the boundary conditions prescribed on the "free surface" $\{x \in \mathbb{R}^2 : x_2 = 1\}$ of the "uniform" domain Ω_0. The solvability of the problem (1.2) and of the corresponding linear problem was proved in [7] and in [1]. In [7] the proofs are based on L^2-theory for the above-mentioned generalized elliptic problems (see [5] for general results of such type), and in [1]—on the detailed investigation of the pseudo-differential operator corresponding to the linearized problem.

Analogous results for two-fluid flows down a perturbed inclined infinite plane and in a perturbed inclined infinite channel with one moving wall were obtained in [11,12]. The three-dimensional stationary free-surface flow over an inclined plane was studied in [2], and the non-stationary flow in [23].

Both methods proposed in [7] and [1] are rather complicated, and it is difficult to apply them in different (from L^2) functional settings. Moreover, these methods do not work in more complicated geometries. For example, they cannot be applied in the case where we have two nonintersecting free boundaries tending to some constants as $x_1 \to +\infty$ and $x_1 \to -\infty$.

In this paper, we propose a new iteration scheme which allows to investigate such cases. Having in mind further applications, we realize this scheme for the problem (1.2) and prove its solvability in the Hölder spaces. The scheme consists of the following: as in [1,7], we map the unknown flow domain onto the strip Ω_0 and consider the problem in the fixed domain. However, now we separate finding the solutions $(\mathbf{v}_k, q_k)$ of the Stokes problem from determination of the functions Ψ_k describing the free boundary. In order to guarantee that on every step of iterations the pressure drop $q_{*k} = 0$, we introduce a smooth function $H_0(x_1)$ and look for Ψ_k is the form $\Psi_k(x_1) = \chi_k H_0(x_1) + \Upsilon_k(x_1)$. The constants χ_k are chosen so that the pressure $q_k(x)$ in the k-th iteration satisfies the condition $q_{*k} = 0$. This gives us the possibility to solve the problem (1.5) for the $(k+1)$-th iteration, hence all $\big(\mathbf{u}_k(x), p_k(x), \chi_k, \Upsilon_k(x_1)\big)$ are well defined. Finally, we prove that the sequence of iterations

$$\big(\mathbf{u}_k(x), p_k(x), \psi_k(x_1) = 1 + \varepsilon\chi_k H_0(x_1) + \varepsilon\Upsilon_k(x_1)\big)$$

converges to the solution $\big(\mathbf{u}(x), p(x), \psi(x_1)\big)$ of the problem (1.2).

2 Function spaces and auxiliary results

2.1 Definition of function spaces

Let $\Omega_0 = \{x : 0 < x_2 < 1\}$ be the strip in $\mathbb{R}^2$, $S_0 = \{x : x_2 = 0\}$ and $\Gamma_0 = \{x : x_2 = 1\}$.

Let us introduce function spaces which we use in the paper. Denote by $C^{l+\delta}(\Omega_0; \beta)$ ($l \geq 0$ is an integer, $\delta \in (0, 1)$, $\beta > 0$) the Banach space consisting of l-times differentiable in Ω_0 functions having finite norm

$$\|v\|_{C^{l+\delta}(\Omega_0;\beta)} = \|\exp\left(\beta\sqrt{1 + x_1^2}\right) v\|_{C^{l+\delta}(\Omega_0)}, \tag{2.1}$$

where $C^{l+\delta}(\Omega_0)$ is the usual Hölder space of functions.

Analogously, we define $C^{l+\delta}(\mathbb{R},\beta)$ as the space of functions on $\mathbb{R}$ with finite norm

$$\|v\|_{C^{l+\delta}(\mathbb{R};\beta)} = \|\exp\left(\beta\sqrt{1+x_1^2}\right)v\|_{C^{l+\delta}(\mathbb{R})}. \tag{2.2}$$

2.2 Transformation of the domain

Consider the function $\omega(y_1,y_2;\Psi)$ given by the formula

$$\omega(y_1,y_2;\Psi) = \zeta(y_2)\int_{-1}^{1} K(\tau)\,\varepsilon\varphi_0(y_1+\tau y_2)d\tau + (1-\zeta(y_2))\int_{-1}^{1} K(\tau)\Psi(y_1+\tau y_2)d\tau, \tag{2.3}$$

where $K(\tau)$ is an infinitely smooth function such that

$$\operatorname{supp} K \subset (-1,1), \quad \int_{-1}^{1} K(\tau)d\tau = 1, \quad \int_{-1}^{1} \tau K(\tau)d\tau = 0, \tag{2.4}$$

and ζ is an infinitely smooth cut-off function with $\zeta(y_2) = 1$ for $|y_2| \le \frac{1}{4}$ and $\zeta(y_2) = 0$ for $|y_2| \ge \frac{1}{2}$. Then ω satisfies the boundary conditions

$$\omega|_{y_2=0} = \varepsilon\varphi_0(y_1), \quad \partial_{y_2}\omega\Big|_{y_2=0} = 0, \quad \omega|_{y_2=1} = \Psi(y_1), \quad \partial_{y_2}\omega\Big|_{y_2=1} = 0,$$

where $\partial_{y_i}\omega = \frac{\partial\omega}{\partial y_i}$, $i = 1,2$.

Define the transformation $X(y)$:

$$x_1 = y_1, \quad x_2 = y_2 + \varepsilon\omega(y;\Psi) \tag{2.5}$$

which maps Ω_0 onto the domain Ω given by formula (1.1). Let

$$\mathcal{L} = \begin{pmatrix} 1 & 0 \\ \varepsilon\partial_{y_1}\omega & 1+\varepsilon\partial_{y_2}\omega \end{pmatrix}$$

be the Jacobi matrix of this transformation; then $L = det\mathcal{L}$ is the Jacobian and $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$ is the co-factors matrix of $\mathcal{L}$. It is clear that $L = 1+\varepsilon\partial_{y_2}\omega$,

$$\widehat{\mathcal{L}} = I + \begin{pmatrix} \varepsilon\partial_{y_2}\omega & 0 \\ -\varepsilon\partial_{y_1}\omega & 0 \end{pmatrix}, \qquad \left(\mathcal{L}^{-1}\right)^T = I + \begin{pmatrix} 0 & -\frac{\varepsilon\partial_{y_1}\omega}{1+\varepsilon\partial_{y_2}\omega} \\ 0 & -\frac{\varepsilon\partial_{y_2}\omega}{1+\varepsilon\partial_{y_2}\omega} \end{pmatrix}, \tag{2.6}$$

$$L^{-1}\mathcal{L} = I + \begin{pmatrix} -\frac{\varepsilon\partial_{y_2}\omega}{1+\varepsilon\partial_{y_2}\omega} & 0 \\ \frac{\varepsilon\partial_{y_1}\omega}{1+\varepsilon\partial_{y_2}\omega} & 0 \end{pmatrix}. \tag{2.7}$$

The lemma below is proved by direct calculations using the formulas (2.3)–(2.7).

Lemma 2.1 *Let* $\varphi_0 \in C^{l+3+\delta}(\mathbb{R})$, $\operatorname{supp}\varphi_0 \subset (-1,1)$, $\Psi \in C^{l+3+\delta}(\mathbb{R},\beta)$, $\beta > 0$. *Then* $\omega \in C^{l+3}(\Omega_0,\beta)$ *and the following estimates*

$$\|\omega\|_{C^{l+3+\delta}(\Omega_0;\beta)} \le c\Big(\varepsilon\|\varphi_0\|_{C^{l+3+\delta}([-1,1])} + \|\Psi\|_{C^{l+3+\delta}(\mathbb{R},\beta)}\Big), \tag{2.8}$$

$$\begin{aligned}&\|X(y)-y\|_{C^{l+3+\delta}(\Omega_0;\beta)} + \|a_{ij}(y)-\delta_{ij}\|_{C^{l+3+\delta}(\Omega_0;\beta)}\\ &\le c\varepsilon\Big(\varepsilon\|\varphi_0\|_{C^{l+3+\delta}([-1,1])} + \|\Psi\|_{C^{l+3+\delta}(\mathbb{R},\beta)}\Big)\end{aligned} \tag{2.9}$$

hold. Here $a_{ij}(y) = \big(\partial X_j^{-1}/\partial x_i\big)\big|_{x=X(y)}$, $i,j=1,2$, *are the elements of the matrix* $\mathcal{L}^{-T} = (\mathcal{L}^{-1})^T$.

2.3 Stokes problem

Consider in Ω_0 the Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{w} + \nabla s = \mathbf{f} & \text{in } \Omega_0,\\ \operatorname{div}\,\mathbf{w} = 0 & \text{in } \Omega_0,\\ \mathbf{w} = \mathbf{a} & \text{on } S_0,\\ \mathbf{w}\cdot\mathbf{n} = b,\ \boldsymbol{\tau}\cdot\mathcal{S}(\mathbf{w})\cdot\mathbf{n} = d & \text{on } \Gamma_0.\end{cases} \tag{2.10}$$

Theorem 2.1 *Let* $\mathbf{f} \in C^{\delta}(\Omega_0;\beta)$, $\mathbf{a} \in C^{2+\delta}(\mathbb{R};\beta)$, $b \in C^{2+\delta}(\mathbb{R};\beta)$, $d \in C^{1+\delta}(\mathbb{R};\beta)$, *where* $\beta \in (0,\beta_*)$ *with sufficiently small* β_*, *and let the following compatibility condition*

$$\int_{\mathbb{R}} b(y_1)dy_1 - \int_{\mathbb{R}} a_2(y_1)dy_1 = 0 \tag{2.11}$$

be satisfied.

(i) There exists a unique[2] *solution* $\mathbf{w} \in C^{2+\delta}(\Omega_0;\beta)$, $\nabla s \in C^{\delta}(\Omega_0;\beta)$ *of problem* (2.10) *satisfying the estimate*

$$\begin{aligned}\|\mathbf{w}\|_{C^{2+\delta}(\Omega_0;\beta)} + \|\nabla s\|_{C^{\delta}(\Omega_0;\beta)} \le c\Big(&\|\mathbf{f}\|_{C^{\delta}(\Omega_0;\beta)} + \|\mathbf{a}\|_{C^{2+\delta}(\mathbb{R};\beta)}\\ &+\|b\|_{C^{2+\delta}(\mathbb{R};\beta)} + \|d\|_{C^{1+\delta}(\mathbb{R};\beta)}\Big).\end{aligned} \tag{2.12}$$

Moreover, the pressure function s *exponentially tends to certain constant limits* s^+ *and* s^- *as* $y_1 \to +\infty$ *and* $y_1 \to -\infty$.

(ii) The difference $s_* = s^+ - s^-$ *is uniquely determined by the data of problem* (2.10). *There holds the following formula*

[2] The pressure s is unique up to an additive constant.

$$s_* = s^+ - s^- = \int\limits_{\Omega} \mathbf{f} \cdot \mathbf{W}^0 dy + \int\limits_{S_0} \Big(3\nu a_1 - a_2 Q^0\Big)\, dy_1 + \int\limits_{\Gamma_0} \Big(b Q^0 + \frac{3}{2} d\Big)\, dy_1, \tag{2.13}$$

where

$$W_1^0(y) = \frac{3y_2(2-y_2)}{2}, \quad W_2^0(y) \equiv 0, \quad Q^0(y) = -3\nu y_1$$

is the Poiseuille solution in Ω_0 *satisfying the boundary conditions*

$$\mathbf{W}^0(y)|_{y_2=0} = 0, \quad W_2^0(y)|_{y_2=1} = 0,$$

and having the unit flux.

(iii) *If* $s^+ = s^-$*, then the pressure* s *can be normalized by the condition* $\lim_{|y_1|\to\infty} s(y) = 0$*. In this case the estimate*

$$\|s\|_{C^{1+\delta}(\Omega_0;\beta)} \le c\Big(\|\mathbf{f}\|_{C^{\delta}(\Omega_0;\beta)} + \|\mathbf{a}\|_{C^{2+\delta}(\mathbb{R};\beta)} + \|b\|_{C^{2+\delta}(\mathbb{R};\beta)} + \|d\|_{C^{1+\delta}(\mathbb{R};\beta)}\Big) \tag{2.14}$$

holds.

Remark Inequality (4.4) with a small ε shows that also the exponent β should be chosen small. It can be shown that the constants in the inequalities (2.12) and (2.14) remain bounded when β is decreasing. The same is true for the estimate (2.19).

Proof (i) The first statement of the theorem is well known, for the proofs of analogous results in similar domains with noncompact boundaries see [7,8,18], etc.

(ii) Let us prove formula (2.13). Multiply equations $(2.10)_1$ by $\mathbf{W}^0$, integrate over the domain $\Omega_{0k} = \{y \in \Omega_0 : |y_1| < k\}$ and, taking into account the identity (note that $div\,\mathbf{w} = 0$)

$$\nu \Delta w_i W_i^0 = \nu \frac{\partial}{\partial y_k}\big(\mathcal{S}_{ik}(\mathbf{W}^0) w_i\big) - \nu \frac{\partial}{\partial y_k}\big(\mathcal{S}_{ik}(\mathbf{w}) W_i^0\big) - \nu \Delta W_i^0 w_i,$$

reduce the obtained expression to the form

$$\begin{aligned}\int\limits_{\Omega_{0k}} \mathbf{f} \cdot \mathbf{W}^0 dy &= \int\limits_{\Omega_{0k}} \big(-\nu \Delta \mathbf{w} + \nabla s\big) \cdot \mathbf{W}^0 dy \\ &= -\int\limits_{\Omega_{0k}} \nu \Delta \mathbf{W}^0 \cdot \mathbf{w} dy - 3\nu \int\limits_{S_{0k}} a_1 dy_1 - \frac{3}{2} \int\limits_{\Gamma_{0k}} d\, dS \\ &\quad + \nu \int\limits_{\sigma(k)} \Big(\mathcal{S}_{21}(\mathbf{W}^0) w_2 - \mathcal{S}_{11}(\mathbf{w}) W_1^0\Big)\Big|_{y_1=k} dy_2\end{aligned}$$

$$-\nu \int\limits_{\sigma(-k)} \left(\mathcal{S}_{21}(\mathbf{W}^0)w_2 - \mathcal{S}_{11}(\mathbf{u})W_1^0\right)\Big|_{y_1=-k} dy_2$$

$$+ \int\limits_{\sigma(k)} \left(sW_1^0\right)\Big|_{y_1=k} - \int\limits_{\sigma(-k)} \left(sW_1^0\right)\Big|_{y_1=-k}.$$

Here $S_{0k} = S_0 \cap \{y_1 : |y_1| < k\}$, $\Gamma_{0k} = \Gamma_0 \cap \{y_1 : |y_1| < k\}$, $\sigma(\pm k) = \{y \in \Omega_0 : y_1 = \pm k\}$. Passing in the last identity to a limit as $k \to \infty$ and using the flux condition we derive

$$\int\limits_{\Omega_0} \mathbf{f} \cdot \mathbf{W}^0 dy = -\nu \int\limits_{\Omega_0} \Delta\mathbf{W}^0 \cdot \mathbf{w} dy - 3\nu \int\limits_{S_0} a_1\, dy_1 - \frac{3}{2} \int\limits_{\Gamma_0} d\, dS + (s^+ - s^-). \tag{2.15}$$

In view of the equation $-\nu\Delta\mathbf{W} + \nabla Q = 0$ we have

$$-\nu \int\limits_{\Omega_0} \Delta\mathbf{W}^0 \cdot \mathbf{w} dy = -\int\limits_{\Omega_0} \nabla Q^0 \cdot \mathbf{w} dy = \int\limits_{S_0} a_2 Q^0\big|_{y_2=0} dy_1 - \int\limits_{\Gamma_0} bQ^0\big|_{\Gamma_0} dS. \tag{2.16}$$

From (2.15) and (2.16) it follows that

$$s_* = s^+ - s^- = \int\limits_{\Omega_0} \mathbf{f} \cdot \mathbf{W}^0 dy + \int\limits_{S_0} \left(3\nu a_1 - a_2 Q^0\right) dy_1 + \int\limits_{\Gamma_0} \left(bQ^0 + \frac{3}{2} d\right) dS. \tag{2.17}$$

(iii) If $s^+ = s^- = 0$, then estimate (2.14) follows from (2.12). □

2.4 Boundary value problem for the ordinary differential equation

Consider the problem

$$\begin{cases} \Upsilon''(y_1) - \gamma_0 \Upsilon(y_1) = G(y_1), & y_1 \in \mathbb{R}, \\ \lim_{|y_1| \to \infty} \Upsilon(y_1) = 0, \end{cases} \tag{2.18}$$

where γ_0 is a positive constant. The theorem below follows from the representation of the solution Υ of (2.18) in terms of the Green function.

Theorem 2.2 *Let $G \in C^{1+\delta}(\mathbb{R}; \beta)$, $\beta > 0$. Then problem (2.18) has a unique solution $\Upsilon \in C^{3+\delta}(\mathbb{R}; \beta)$ and the following estimate*

$$\|\Upsilon\|_{C^{3+\delta}(\mathbb{R};\beta)} \le c\|G\|_{C^{1+\delta}(\mathbb{R};\beta)} \tag{2.19}$$

holds.

3 Linearization of the free boundary problem

Let

$$v_1^0(x) = \frac{g\sin\alpha}{2\nu}x_2(2-x_2),\quad v_2^0(x)=0,\quad p^0(x) = g\cos\alpha(1-x_2),\quad \psi_0(x_1)=1$$

be the exact Poiseuille type solution of (1.2) for $\varepsilon = 0$, i.e.,

$$-\nu\Delta\mathbf{v}^0(x) + (\mathbf{v}^0\cdot\nabla)\mathbf{v}^0 + \nabla p^0(x) = -g\mathbf{e}^\gamma,\quad div_x\mathbf{v}^0 = 0, \tag{3.1}$$

and

$$\begin{aligned}
&\mathbf{v}^0|_{S_0} = 0,\quad v_2^0|_{\Gamma_0} = 0,\quad \frac{\partial v_1^0}{\partial x_2}\Big|_{\Gamma_0} = 0,\\
&\left(\frac{\psi_0'}{\sqrt{1+\psi_0'^2}}\right)' = \sigma^{-1}\left(-p^0 + \frac{\partial v_2^0}{\partial x_2}\right)|_{\Gamma_0}.\\
&\int_0^1 v_1^0(x)dx_2 = \frac{g\sin\alpha}{3\nu}.
\end{aligned}$$

Using the formulas

$$\boldsymbol{\tau} = \frac{(1,\varepsilon\Psi(x_1))}{\sqrt{1+\varepsilon^2\Psi'^2(x_1)}},\quad \mathbf{n} = \frac{(-\varepsilon\Psi(x_1),1)}{\sqrt{1+\varepsilon^2\Psi'^2(x_1)}},\quad x\in\Gamma, \tag{3.2}$$

it is easy to calculate that on the perturbed boundaries $S = \{x : x_2 = \varepsilon^2\varphi_0(x_1)\}$ and $\Gamma = \{x : x_2 = 1+\varepsilon\Psi(x_1)\}$ of the domain Ω the function $\mathbf{v}^0$ satisfies the boundary conditions

$$\begin{aligned}
\mathbf{v}^0|_S &= \frac{g\sin\alpha}{2\nu}\varepsilon^2\varphi_0(x_1)(2-\varepsilon^2\varphi_0(x_1),0),\\
\boldsymbol{\tau}\cdot\mathcal{S}(\mathbf{v}^0)\mathbf{n}|_\Gamma &= -(1+\varepsilon^2\Psi'^2(x_1))^{-1}\varepsilon\Big(\frac{g\sin\alpha}{\nu}\Psi(x_1) - \varepsilon g\sin\alpha\Psi'^2(x_1)\Psi(x_1)\Big),\\
\mathbf{v}^0(x)\cdot\mathbf{n}(x)|_\Gamma &= -\frac{g\sin\alpha}{2\nu}\frac{(1-\varepsilon^2\Psi^2)\varepsilon\Psi'(x_1)}{\sqrt{1+\varepsilon^2\Psi'^2(x_1)}}.
\end{aligned} \tag{3.3}$$

Substitute

$$\mathbf{u}(x) = \mathbf{v}^0(x) + \varepsilon\mathbf{V}(x),\quad p(x) = p^0(x)+\varepsilon q(x),\quad \psi(x_1) = 1+\varepsilon\Psi(x_1),$$

into (1.2), introduce a new vector-field $\mathbf{v} = \widehat{\mathcal{L}}\widehat{\mathbf{V}}(y))$, $\widehat{\mathbf{V}}(y) = \mathbf{V}(X(y))$, whose components are given by[3]

[3] We performe this change of the unknown function in order to keep the divergence equal to zero.

$$v_1(y) = V_1(X(y))\big(1 + \varepsilon\partial_{y_2}\omega(y)\big), \quad v_2(y) = V_2(X(y)) - \varepsilon V_1(X(y))\partial_{y_1}\omega(y),$$

and make the change of variables $x = X(y)$ in (1.2). Since

$$\begin{aligned} \operatorname{div}_y \mathbf{v}(y) &= \operatorname{div}_x \mathbf{V}(x)\big|_{x=X(y)}\big(1 + \varepsilon\partial_{y_2}\omega(y)\big) = 0, \\ v_2(y)|_{y_2=1} &= \big(\mathbf{V}(x)\cdot\mathbf{n}(x)\big)\big|_{x_2=1+\varepsilon\Psi(y_1)}\sqrt{1+\varepsilon^2\Psi'^2(y_1)}, \end{aligned}$$

we get the following problem in the strip Ω_0:

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla q = -(\mathbf{v}^0\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v}^0 + \mathbf{F}(\mathbf{v}, q, \Psi) & \text{in } \Omega_0, \\ \operatorname{div}\ \mathbf{v} = 0 & \text{in } \Omega_0, \\ v_1\big|_{y_2=0} = A_1, \quad v_2\big|_{y_2=0} = A_2 \\ v_2|_{y_2=1} = \frac{g\sin\alpha}{2\nu}\Psi' + B(\Psi), \\ \nu\left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1}\right)\Big|_{y_2=1} = g\sin\alpha\Psi + D(\mathbf{v}, \Psi), \\ \Psi'' - g\sigma^{-1}\cos\alpha\Psi = \sigma^{-1}\big(-q(y) + 2\nu\frac{\partial v_2}{\partial y_2}\big)\big|_{y_2=1} + \Phi(\mathbf{v}, \Psi), \\ \lim_{|y_1|\to\infty}\Psi(y_1) = 0. \end{cases} \tag{3.4}$$

Let us compute $\mathbf{F}$. Since $\mathbf{v}^0(x)$ satisfies (3.1) we have

$$\begin{aligned} \mathbf{F} = \nu(\widehat{\nabla}^2 - \nabla^2)\widehat{\mathbf{V}} - (\widehat{\nabla} - \nabla)q + \nu\nabla^2(\widehat{\mathbf{V}} - \mathbf{v}) - \varepsilon(\widehat{\mathbf{V}}\cdot\widehat{\nabla})\widehat{\mathbf{V}} \\ -(\widehat{\mathbf{v}}^0\cdot(\widehat{\nabla} - \nabla))\widehat{\mathbf{V}} - ((\widehat{\mathbf{v}}^0 - \mathbf{v}^0)\cdot\nabla)\widehat{\mathbf{V}} - (\mathbf{v}^0\cdot\nabla)(\widehat{\mathbf{V}} - \mathbf{v}) \\ -(\widehat{\mathbf{V}}\cdot(\widehat{\nabla} - \nabla))\widehat{\mathbf{v}}^0 - (\mathbf{v}\cdot\nabla)(\widehat{\mathbf{v}}^0 - \mathbf{v}^0) - ((\widehat{\mathbf{V}} - \mathbf{v})\cdot\nabla)\mathbf{v}^0, \end{aligned} \tag{3.5}$$

where $\nabla = \nabla_y$ and $\widehat{\nabla} = \mathcal{L}^{-T}\nabla$ is the transformed gradient ∇_x. Making use of $\widehat{\mathbf{V}} - \mathbf{v} = (L^{-1}\mathcal{L} - I)\mathbf{v}$, $\widehat{\nabla} - \nabla = (\mathcal{L}^{-T} - I)\nabla$, we obtain

$$\begin{aligned} \mathbf{F} = {} & \nu(\widehat{\nabla}^2 - \nabla^2)\widehat{\mathbf{V}} - (\widehat{\nabla} - \nabla)q + \nu\nabla^2\left(-\frac{\varepsilon\partial_{y_2}\omega}{L}v_1, \frac{\varepsilon\partial_{y_1}\omega}{L}v_1\right) - \varepsilon(\widehat{\mathbf{V}}\cdot\widehat{\nabla})\widehat{\mathbf{V}} \\ & + \left(\frac{\varepsilon\partial_{y_1}\omega}{L}\widehat{V}_1 + \frac{\varepsilon\partial_{y_2}\omega}{L}\widehat{V}_2\right)\partial_{y_2}\widehat{\mathbf{v}}^0 + \left(\frac{\varepsilon\partial_{y_2}\omega}{L}v_1^0\partial_{y_1} - \frac{\varepsilon\partial_{y_1}\omega}{L}v_1^0\partial_{y_2}\right)\widehat{\mathbf{V}} \\ & + (\mathbf{v}^0\cdot\nabla)\left(-\frac{\varepsilon\partial_{y_2}\omega}{L}v_1, \frac{\varepsilon\partial_{y_1}\omega}{L}v_1\right) - \left(\widehat{V}_1\frac{\varepsilon\partial_{y_1}\omega}{L} + \widehat{V}_2\frac{\varepsilon\partial_{y_2}\omega}{L}\right)\partial_{y_2}\widehat{\mathbf{v}}^0 \\ & - \left(-\frac{\varepsilon\partial_{y_2}\omega}{L}v_1\partial_{y_1} + \frac{\varepsilon\partial_{y_1}\omega}{L}v_1\partial_{y_1}\right)\widehat{\mathbf{v}}^0 + (\mathbf{v}\cdot\nabla)\left(\frac{\varepsilon\partial_{y_2}\omega}{L}v_1^0, -\frac{\varepsilon\partial_{y_1}\omega}{L}v_1^0\right), \end{aligned}$$

where $L = 1 + \varepsilon\partial_{y_2}\omega$.

As for the boundary conditions, it is straightforward to compute

$$\begin{aligned} A_1(y_1) &= -\varepsilon\frac{g\sin\alpha}{2\nu}\varphi_0(y_1)(2 - \varepsilon^2\varphi_0(y_1)), \\ A_2(y_1) &= \varepsilon^3\frac{g\sin\alpha}{2\nu}\varphi_0(y_1)(2 - \varepsilon^2\varphi_0(y_1))\varphi_0'(y_1) \end{aligned}$$

and

$$B(y_1) = -\varepsilon^2 \frac{g \sin\alpha}{2\nu} \Psi^2(y_1)\Psi'(y_1).$$

In addition, as

$$\begin{aligned}
\boldsymbol{\tau}(x) \cdot \mathcal{S}(\mathbf{V}(x))\mathbf{n}(x) &= (1 - \varepsilon^2\Psi'^2(x_1))\mathcal{S}_{12}(\mathbf{V}) + \varepsilon\Psi'\mathcal{S}_{11}(\mathbf{V}) - \varepsilon\Psi'\mathcal{S}_{22}(\mathbf{V}), \quad x \in \Gamma,\\
\widehat{\mathcal{S}}_{12}(\mathbf{V}) &= \frac{\partial \widehat{V}_2}{\partial y_1} + \frac{\partial \widehat{V}_1}{\partial y_2} - \frac{\varepsilon\partial_{y_1}\omega}{1+\varepsilon\partial_{y_2}\omega}\frac{\partial \widehat{V}_2}{\partial y_2} - \frac{\varepsilon\partial_{y_2}\omega}{1+\varepsilon\partial_{y_2}\omega}\frac{\partial \widehat{V}_1}{\partial y_2}\\
&= \frac{\partial v_2}{\partial y_1} + \frac{\partial v_1}{\partial y_2} - \frac{\partial}{\partial y_1}\left(\frac{\varepsilon\partial_{y_1}\omega}{1+\varepsilon\partial_{y_2}\omega} v_2\right) - \frac{\partial}{\partial y_2}\left(\frac{\varepsilon\partial_{y_2}\omega}{1+\varepsilon\partial_{y_2}\omega} v_1\right)\\
&\quad - \frac{\varepsilon\partial_{y_1}\omega}{L}\frac{\partial \widehat{V}_2}{\partial y_2} - \frac{\varepsilon\partial_{y_2}\omega}{L}\frac{\partial \widehat{V}_1}{\partial y_2},\\
\mathbf{n} \cdot \widehat{\mathcal{S}}(\widehat{\mathbf{V}})\mathbf{n} &= 2n_1n_2\widehat{\mathcal{S}}_{12}(\widehat{\mathbf{V}}) + n_1^2\widehat{\mathcal{S}}_{11}(\widehat{\mathbf{V}}) + n_2^2\widehat{\mathcal{S}}_{22}(\widehat{\mathbf{V}}),\\
\frac{\partial V_2(x)}{\partial x_2} &= \frac{\partial \widehat{V}_2}{\partial y_2} - \frac{\varepsilon\partial_{y_2}\omega}{L}\frac{\partial \widehat{V}_2}{\partial y_2}\\
&= \frac{\partial v_2}{\partial y_2} - \frac{\varepsilon\partial_{y_2}\omega}{L}\frac{\partial \widehat{V}_2}{\partial y_2} + \frac{\partial}{\partial y_2}\left(\frac{\varepsilon\omega_{y_1}}{L} v_1\right),
\end{aligned}$$

we obtain

$$\begin{aligned}
D(y_1) &= \nu\varepsilon\left(\partial_{y_1}\omega\frac{\partial \widehat{V}_1}{\partial y_2} - \partial_{y_2}\omega(y)\frac{\partial \widehat{V}_2}{\partial y_2}\right)\Big|_{y_2=1}\\
&\quad + \varepsilon\Big(\Psi'(y_1)\widehat{\mathcal{S}}_{11}(\widehat{\mathbf{V}}) - \Psi'(y_1)\widehat{\mathcal{S}}_{22}(\widehat{\mathbf{V}}) + \varepsilon\Psi'^2(y_1)\widehat{\mathcal{S}}_{21}(\widehat{\mathbf{V}})\Big)\Big|_{y_2=1}\\
&\quad + \nu\varepsilon\left(\frac{\partial}{\partial y_2}\left(\frac{\partial_{y_2}\omega}{1+\varepsilon\partial_{y_2}\omega}v_1\right) - \frac{\partial}{\partial y_1}\left(\frac{\partial_{y_1}\omega}{1+\varepsilon\partial_{y_2}\omega}v_1\right)\right)\Big|_{y_2=1}\\
&\quad - \varepsilon^2 g\sin\alpha\Psi'^2(y_1)\Psi(y_1),\\
\widehat{\mathcal{S}}_{ij}(\widehat{\mathbf{V}}) &= \nu\sum_{m=1}^{2}\left(a_{jm}(y)\frac{\partial \widehat{V}_j}{\partial y_m} + a_{im}(y)\frac{\partial \widehat{V}_i}{\partial y_m}\right),
\end{aligned}$$

$$\begin{aligned}
\Phi(y_1) &= \frac{1}{1+\varepsilon^2\Psi'^2(y_1)}\Bigg(\bigg(-2\varepsilon\Psi'(y_1)\widehat{\mathcal{S}}_{12}(\widehat{\mathbf{V}}) + \varepsilon^2\Psi'^2(y_1)\widehat{\mathcal{S}}_{11}(\widehat{\mathbf{V}})\\
&\quad + 2\varepsilon^2\nu\Psi'^2(y_1)\frac{\partial \widehat{V}_2}{\partial y_2} + 2\nu(a_{22}(y)-1)\frac{\partial \widehat{V}_2}{\partial y_2}\bigg)\Big|_{y_2=1} + \varepsilon g\sin\alpha\Psi(y_1)\Psi'(y_1)\Bigg)\\
&\quad + 2\nu\varepsilon\frac{\partial}{\partial y_2}\left(\frac{\partial_{y_1}\omega}{1+\varepsilon\partial_{y_2}\omega}v_1\right)\Big|_{y_2=1} + \frac{1}{2}\varepsilon^2\frac{d}{dy_1}\left(\Psi'^3(y_1)\int_0^1\frac{ds}{\left(1+s\varepsilon\Psi'^2(y_1)\right)^{3/2}}\right).
\end{aligned}$$

Assume that

$$\|\mathbf{v}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|\nabla q\|_{C^{l+\delta}(\Omega_0;\beta)} + \|\Psi\|_{C^{l+3+\delta}(\mathbb{R},\beta)} \le A_0.$$

For sufficiently small ε from (2.8) (2.9) follow the estimates

$$\|\mathbf{F}\|_{C^{l+\delta}(\Omega_0;\beta)} \le \varepsilon C\Big(\|\mathbf{v}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|\nabla q\|_{C^{l+\delta}(\Omega_0;\beta)} + \|\Psi\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big),$$
$$\|\mathbf{F}(\mathbf{v}_1, q_1, \Psi_1) - \mathbf{F}(\mathbf{v}_2, q_2, \Psi_2)\|_{C^{l+\delta}(\Omega_0;\beta)}$$
$$\le \varepsilon C\Big(\|\mathbf{v}_1-\mathbf{v}_2\|_{C^{l+2+\delta}(\Omega_0;\beta)}+\|\nabla q_1-\nabla q_2\|_{C^{l+\delta}(\Omega_0;\beta)} + \|\Psi_1 - \Psi_2\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big); \tag{3.6}$$

$$\|B\|_{C^{l+2+\delta}(\mathbb{R};\beta)} \le \varepsilon C\|\Psi\|_{C^{l+3+\delta}(\mathbb{R};\beta)},$$
$$\|B(\Psi_1) - B(\Psi_2)\|_{C^{l+2+\delta}(\mathbb{R};\beta)} \le \varepsilon C\|\Psi_1 - \Psi_2\|_{C^{l+3+\delta}(\mathbb{R};\beta)}; \tag{3.7}$$

$$\|D\|_{C^{l+1+\delta}(\mathbb{R};\beta)} \le \varepsilon C\Big(\|\mathbf{v}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|\Psi\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big),$$
$$\|D(\mathbf{v}_1, \Psi_1) - D(\mathbf{v}_2, \Psi_2)\|_{C^{l+1+\delta}(\mathbb{R};\beta)}$$
$$\le \varepsilon C\Big(\|\mathbf{v}_1 - \mathbf{v}_2\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|\Psi_1 - \Psi_2\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big); \tag{3.8}$$

$$\|\Phi\|_{C^{l+1+\delta}(\mathbb{R};\beta)} \le \varepsilon C\Big(\|\mathbf{v}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|\Psi\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big),$$
$$\|\Phi(\mathbf{v}_1, \Psi_1) - \Phi(\mathbf{v}_2, \Psi_2)\|_{C^{l+2+\delta}(\mathbb{R};\beta)}$$
$$\le \varepsilon C\Big(\|\mathbf{v}_1 - \mathbf{v}_2\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|\Psi_1 - \Psi_2\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big), \tag{3.9}$$

where

$$C = C(A_0, \|\varphi_0\|_{C^{l+3+\delta}(-1,1)}).$$

Moreover,

$$\|\mathbf{A}\|_{C^{l+2+\delta}(\mathbb{R};\beta)} \le c\varepsilon|\sin\alpha|\Big(\|\varphi_0\|_{C^{l+3+\delta}(-1,1)} + \varepsilon\|\varphi_0\|^2_{C^{l+3+\delta}(-1,1)}\Big), \tag{3.10}$$

where the constant c is independent of $|\sin\alpha|$ and ε.

4 Successive approximations

Let the function $H_0 \in C^{l+3+\delta}(\mathbb{R};\beta)$ be such that

$$\int_{-\infty}^{\infty} \Big(\frac{1}{2\nu}H_0'(y_1)Q^0(y_1) + \frac{3}{2}H_0(y_1)\Big)\,dy_1 = \kappa_0 \ne 0. \tag{4.1}$$

We choose H_0 as the solution of the problem

$$\begin{cases} H_0''(y_1) - \gamma_0 H_0(y_1) = h_0(y_1), \\ \lim_{|y_1|\to\infty} H_0(y_1) = 0, \end{cases} \tag{4.2}$$

where $\gamma_0 = g\sigma^{-1}\cos\alpha$, $h_0 \in C^{l+3+\delta}(\mathbb{R};\beta)$,

$$\|h_0\|_{C^{l+1+\delta}(\mathbb{R};\beta)} \le c\varepsilon \quad \text{and} \quad \int\limits_{\mathbb{R}} h_0(y_1)\,dy_1 = -1. \tag{4.3}$$

Relations (4.3) are possible if the condition

$$0 < \beta \le c_0\varepsilon \le \beta_*. \tag{4.4}$$

is satisfied. Then in virtue of (2.19),

$$\|H_0\|_{C^{l+3+\delta}(\mathbb{R};\beta)} \le c\|h_0\|_{C^{l+1+\delta}(\mathbb{R};\beta)} \le c\varepsilon. \tag{4.5}$$

Integrating (4.2) we get

$$\int\limits_{\mathbb{R}} \big(H_0''(y_1) - \gamma_0 H_0(y_1)\big)\,dy_1 = -g\sigma^{-1}\cos\alpha \int\limits_{\mathbb{R}} H_0(y_1)\,dy_1 = \int\limits_{\mathbb{R}} h_0(y_1)\,dy_1 = -1.$$

On the other hand, integrating by parts we obtain

$$\kappa_0 = \int\limits_{-\infty}^{\infty} \Big(\frac{1}{2\nu} H_0'(y_1) Q^0(y_1) + \frac{3}{2} H_0(y_1)\Big)\,dy_1 = 3\int\limits_{\mathbb{R}} H_0(y_1)\,dy_1.$$

From this formula and (4.3) we conclude that

$$\kappa_0 = \frac{3\sigma}{g\cos\alpha} \ne 0. \tag{4.6}$$

Setting $\Psi = \chi_* H_0 + \Upsilon$ in (3.4) with the constant χ_* that will be defined later leads to

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla q = -(\mathbf{v}^0\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{v}^0 + \\ \qquad + \mathbf{F}(\mathbf{u}, q, \chi_* H_0 + \Upsilon) & \text{in } \Omega_0, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega_0, \\ v_1\big|_{y_2=0} = A_1, \quad v_2\big|_{y_2=0} = A_2, \\ v_2|_{y_2=1} = \chi_* \frac{g\sin\alpha}{2\nu} H_0' + \frac{g\sin\alpha}{2\nu}\Upsilon' + B(\chi_* H_0 + \Upsilon), \\ \nu\Big(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1}\Big)\Big|_{x_2=1} = g\sin\alpha\,\chi_* H_0 + g\sin\alpha\,\Upsilon \\ \qquad + D(\mathbf{u}, \chi_* H_0 + \Upsilon), \\ \Upsilon'' - g\sigma^{-1}\cos\alpha\,\Upsilon = \chi_*\big(-H_0'' + g\sigma^{-1}\cos\alpha\, H_0\big) \\ + \sigma^{-1}\Big(-q(y) + 2\nu\frac{\partial v_2}{\partial y_2}\Big)\Big|_{y_2=1} + \Phi(\mathbf{u}, \chi_* H_0 + \Upsilon), \\ \lim_{|y_1|\to\infty} \Upsilon(y_1) = 0. \end{cases} \tag{4.7}$$

Take as zero approximation $(\mathbf{u}_0, p_0, \psi_0, \chi_0) = (\mathbf{v}^0, p^0, 1, 0)$, then define

$$\chi_1 = -\frac{1}{g\sin\alpha\, k_0}\int_{S_0}\left(3\nu A_1 - A_2 Q^0\right) dy_1, \tag{4.8}$$

and solve the following problems:

$$\begin{cases} -\nu\Delta\mathbf{v}_1 + \nabla q_1 = 0 & \text{in } \Omega_0, \\ \text{div } \mathbf{v}_1 = 0 & \text{in } \Omega_0, \\ \mathbf{v}_1\big|_{y_2=0} = \mathbf{A} = (A_1, A_2), & \\ v_{2\,1}|_{y_2=1} = \chi_1 \frac{g\sin\alpha}{2\nu} H_0', & \\ \nu\left(\frac{\partial v_{1\,1}}{\partial y_2} + \frac{\partial v_{2\,1}}{\partial y_1}\right)\big|_{y_2=1} = \chi_1 g\sin\alpha H_0; & \end{cases} \tag{4.9}$$

and

$$\begin{cases} \Upsilon_1'' - g\sigma^{-1}\cos\alpha\Upsilon_1 = \chi_1(-H_0'' + g\sigma^{-1}\cos\alpha H_0) \\ +\sigma^{-1}\left(-q_1(y) + 2\nu\frac{\partial v_{2\,1}}{\partial y_2}\right)\big|_{y_2=1} + \Phi(\mathbf{u}_1, \psi_0), \\ \lim_{|y_1|\to\infty}\Upsilon_1(y_1) = 0, \end{cases} \tag{4.10}$$

where $\mathbf{u}_1 = \mathbf{v}^0 + \varepsilon\mathbf{v}_1$, $p_1 = p^0 + \varepsilon q_1$. Define $\psi_1 = 1 + \varepsilon(\chi_1 H_0 + \Upsilon_1)$. From (4.8), (2.12), (2.19) and the definition of $\mathbf{A}$ it follows that

$$\begin{gathered} \chi_1 \sim \varepsilon, \quad \|\Upsilon_1\|_{C^{l+3+\delta}(\mathbb{R};\beta)} \sim \varepsilon, \\ \|\mathbf{v}_1\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_1\|_{C^{l+1+\delta}(\Omega_0;\beta)} \sim \varepsilon|\sin\alpha|. \end{gathered} \tag{4.11}$$

The following approximations are defined by

$$\mathbf{u}_{n+1} = \mathbf{v}^0 + \varepsilon\mathbf{v}_{n+1}, \quad p_{n+1} = p^0 + \varepsilon q_{n+1}, \quad \psi_{n+1} = 1 + \varepsilon(\chi_{n+1}H_0 + \Upsilon_{n+1}),$$

where

$$\begin{aligned} \chi_{n+1} = -(g\sin\alpha k_0)^{-1}\Bigg[&\int_{\Omega_0}\left(-(\mathbf{v}^0\cdot\nabla)\mathbf{v}_n - (\mathbf{v}_n\cdot\nabla)\mathbf{v}^0\right)\cdot\mathbf{W}^0\,dy \\ &+\int_{\Omega_0}\mathbf{F}(\mathbf{u}_n, p_n, \chi_n H_0 + \Upsilon_n)\cdot\mathbf{W}^0 dy + \int_{S_0}\left(3\nu A_1 - A_2 Q^0\right) dy_1 \\ &+\int_{\Gamma_0}\left(B(\chi_n H_0 + \Upsilon_n)Q^0 + \frac{3}{2}D(\mathbf{u}_n, \chi_n H_0 + \Upsilon_n)\right) dy_1 \\ &+g\sin\alpha\int_{\Gamma_0}\left(\frac{1}{2\nu}\Upsilon_n' Q^0 + \frac{3}{2}\Upsilon_n\right) dy_1\Bigg], \end{aligned} \tag{4.12}$$

$(\mathbf{v}_{n+1}, q_{n+1})$ are solutions of the problems:

$$\begin{cases} -\nu\Delta\mathbf{v}_{n+1} + \nabla q_{n+1} = -(\mathbf{v}^0\cdot\nabla)\mathbf{v}_n - (\mathbf{v}_n\cdot\nabla)\mathbf{v}^0 \\ +\mathbf{F}(\mathbf{u}_n, p_n, \chi_n H_0 + \Upsilon_n) & \text{in } \Omega_0, \\ \operatorname{div}\, \mathbf{v}_{n+1} = 0 & \text{in } \Omega_0, \\ \mathbf{v}_{n+1}\big|_{y_2=0} = \mathbf{A}, \\ v_{2\,n+1}|_{y_2=1} = \chi_{n+1}\frac{g\sin\alpha}{2\nu}H_0' + \frac{g\sin\alpha}{2\nu}\Upsilon_n' + B\big(\chi_n H_0 + \Upsilon_n\big), \\ \nu\Big(\frac{\partial v_{1\,n+1}}{\partial y_2} + \frac{\partial v_{2\,n+1}}{\partial y_1}\Big)\Big|_{y_2=1} = \chi_{n+1} g\sin\alpha\, H_0 + g\sin\alpha\Upsilon_n \\ +D\big(\mathbf{u}_n, \chi_n H_0 + \Upsilon_n\big). \end{cases} \tag{4.13}$$

Finally, Υ_{n+1} are solutions of

$$\begin{cases} \Upsilon_{n+1}'' - g\sigma^{-1}\cos\alpha\Upsilon_{n+1} = \chi_{n+1}(-H_0'' + g\sigma^{-1}\cos\alpha H_0) \\ +\sigma^{-1}\Big(-q_{n+1}(y) + 2\nu\frac{\partial v_{2\,n+1}}{\partial y_2}\Big)\Big|_{y_2=1} + \Phi(\mathbf{u}_{n+1}, \chi_n H_0 + \Upsilon_n), \\ \lim_{|y_1|\to\infty}\Upsilon_{n+1}(y_1) = 0. \end{cases} \tag{4.14}$$

Notice that χ_{n+1} are chosen so that

$$q_{*\,n+1} = \lim_{x_1\to\infty} q_{n+1}(x) - \lim_{x_1\to-\infty} q_{n+1}(x) = 0.$$

Indeed, applying (2.13) to the problem (4.13), we obtain

$$\begin{aligned} q_{n+1}^+ - q_{n+1}^- = &\int_{\Omega_0}\big(-(\mathbf{v}^0\cdot\nabla)\mathbf{v}_n - (\mathbf{v}_n\cdot\nabla)\mathbf{v}^0 + \mathbf{F}(\mathbf{u}_n, p_n, \chi_n H_0 + \Upsilon_n)\big)\cdot\mathbf{W}^0\,dy \\ &+\int_{S_0}\Big(3\nu A_1 - A_2 Q^0\Big)\,dy_1 + \int_{\Gamma_0}\Big(B(\chi_n H_0 + \Upsilon_n)Q^0 \\ &+\frac{3}{2}D(\mathbf{u}_n, \chi_n H_0 + \Upsilon_n)\Big)\,dy_1 \\ &+\frac{3}{2}\int_{\Gamma_0}(\chi_{n+1} g\sin\alpha H_0 + g\sin\alpha\Upsilon_n + D(\mathbf{u}_n, \chi_n H_0 + \Upsilon_n))dy_1, \end{aligned}$$

or, equivalently,

$$\begin{aligned} q_{n+1}^+ - q_{n+1}^- = &\ \chi_{n+1} g\sin\alpha\kappa_0 \\ &+\int_{\Omega_0}\big(-(\mathbf{v}^0\cdot\nabla)\mathbf{v}_n - (\mathbf{v}_n\cdot\nabla)\mathbf{v}^0 + \mathbf{F}(\mathbf{u}_n, p_n, \chi_n H_0 + \Upsilon_n)\big)\cdot\mathbf{W}^0\,dy \\ &+\int_{S_0}\Big(3\nu A_1 - A_2 Q^0\Big)\,dy_1 + \int_{\Gamma_0}\Big(B(\chi_n H_0 + \Upsilon_n)Q^0 \end{aligned}$$

$$+\frac{3}{2}D(\mathbf{u}_n,\chi_n H_0+\Upsilon_n)\Big)\,dy_1$$
$$+g\sin\alpha\int\limits_{\Gamma_0}\left(\frac{1}{2\nu}\Upsilon_n' Q^0+\frac{3}{2}\Upsilon_n\right)dy_1,$$

where

$$\kappa_0=\int\limits_{\Gamma_0}\left(\frac{1}{2\nu}H_0''+\frac{3}{2}H_0\right)dy_1\neq 0.$$

Substituting (4.12) into the last formula, we get

$$q_{n+1}^{+}-q_{n+1}^{-}=0.$$

5 Convergence of successive approximations; existence of the solution

Assume that

$$\varepsilon|\sin\alpha|^{-1}\ll 1\ \ as\ \ \varepsilon,\,\alpha\to 0. \tag{5.1}$$

From (2.12), (3.6), (3.7), (3.8), (3.10), (2.19) follow the inequalities

$$\begin{aligned}
&|\chi_1|\le c\varepsilon\|\varphi_0\|_{C^{l+3+\delta}(-1,1)},\\
&\|\mathbf{v}_1\|_{C^{l+2+\delta}(\Omega_0;\beta)}+\|q_1\|_{C^{l+1+\delta}(\Omega_0;\beta)}\\
&\quad\le c\Big(\|\mathbf{A}\|_{C^{l+2+\delta}(\mathbb{R};\beta)}+|\sin\alpha||\chi_1|\|H_0\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big)\\
&\quad\le c\varepsilon|\sin\alpha|\Big(\|\varphi_0\|_{C^{l+3+\delta}(-1,1)}+|\chi_1|\Big),\\
&\|\Upsilon_1\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\\
&\quad\le c\Big(\|\mathbf{v}_1\|_{C^{l+2+\delta}(\Omega_0;\beta)}+\|q_1\|_{C^{l+1+\delta}(\Omega_0;\beta)}+|\chi_1|\|H_0\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big)\\
&\quad\le c\varepsilon\Big(\|\varphi_0\|_{C^{l+3+\delta}(-1,1)}+|\chi_1|\Big);
\end{aligned}\tag{5.2}$$

$$\begin{aligned}
|\chi_{n+1}|\le{}& c\Big(\|\mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)}+\|\Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big)\\
&+c\varepsilon|\sin\alpha|^{-1}\Big(\|\mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)}+\|q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)}\\
&+\|\Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)}+\varepsilon|\chi_n|\Big)+c\varepsilon\|\varphi_0\|_{C^{l+3+\delta}(-1,1)},
\end{aligned}\tag{5.3}$$

$$\begin{aligned}
&\|\mathbf{v}_{n+1}\|_{C^{l+2+\delta}(\Omega_0;\beta)}+\|q_{n+1}\|_{C^{l+1+\delta}(\Omega_0;\beta)}\\
&\le c\Big(\|\mathbf{F}\|_{C^{l+\delta}(\Omega_0;\beta)}+\|\mathbf{A}\|_{C^{l+2+\delta}(\mathbb{R};\beta)}+\|B\|_{C^{l+2+\delta}(\mathbb{R};\beta)}+\|D\|_{C^{l+1+\delta}(\mathbb{R};\beta)}\\
&\quad+c|\sin\alpha|\big(\|\mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)}+\varepsilon|\chi_{n+1}|+\|\Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\big)\Big)
\end{aligned}\tag{5.4}$$

$$
\begin{aligned}
&\le c\varepsilon\Big(\|\mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)} + \|\Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} \\
&\quad + \varepsilon|\chi_n| + |\sin\alpha|\|\varphi_0\|_{C^{l+3+\delta}(-1,1)}\Big) \\
&\quad + c|\sin\alpha|\Big(\|\mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \varepsilon|\chi_{n+1}| + \|\Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big), \\
&\|\Upsilon_{n+1}\|_{C^{l+3+\delta}(\mathbb{R};\beta)} \\
&\le c\Big(\|\mathbf{v}_{n+1}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_{n+1}\|_{C^{l+1+\delta}(\Omega_0;\beta)} + \varepsilon|\chi_{n+1}| + \|\Phi\|_{C^{l+1+\delta}(\mathbb{R};\beta)}\Big) \\
&\le c\varepsilon\Big(\|\Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + \varepsilon|\chi_n|\Big) + c\Big(\|\mathbf{v}_{n+1}\|_{C^{l+2+\delta}(\Omega_0;\beta)} \\
&\quad + \|q_{n+1}\|_{C^{l+1+\delta}(\Omega_0;\beta)} + \varepsilon|\chi_{n+1}|\Big).
\end{aligned}
\tag{5.5}
$$

From (5.3)–(5.5) it is easy to deduce the following estimates:

$$
\begin{aligned}
&\|\Upsilon_{n+1}\|_{C^{l+3+\delta}(\mathbb{R};\beta)} \\
&\quad \le c\varepsilon\Big(\|\mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)} + \|\Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + \varepsilon|\chi_n|\Big) \\
&\qquad + c\varepsilon|\sin\alpha|\|\varphi_0\|_{C^{l+3+\delta}(-1,1)}; \\
&\|\mathbf{v}_{n+1}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_{n+1}\|_{C^{l+1+\delta}(\Omega_0;\beta)} \\
&\quad \le c\big(\varepsilon + |\sin\alpha|\big)\Big(\|\mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)} \\
&\qquad + \|\Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + \varepsilon|\chi_n|\Big) + c\varepsilon|\sin\alpha|\|\varphi_0\|_{C^{l+3+\delta}(-1,1)}; \\
&|\chi_{n+1}| \\
&\quad \le c\big(\varepsilon + |\sin\alpha|\big)\Big(\|\mathbf{v}_{n-1}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_{n-1}\|_{C^{l+1+\delta}(\Omega_0;\beta)} \\
&\qquad + \|\Upsilon_{n-1}\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + \varepsilon|\chi_{n-1}|\Big) \\
&\qquad + c\varepsilon|\sin\alpha|^{-1}\Big(\|\mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)} \\
&\qquad + \|\Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + \varepsilon|\chi_n|\Big) + c\varepsilon\|\varphi_0\|_{C^{l+3+\delta}(-1,1)}.
\end{aligned}
\tag{5.6}
$$

Denote

$$
Z_n = \|\mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)} + \|\Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + |\chi_n|.
$$

For sufficiently small ε and $|\sin\alpha|$ it follows from (5.6) that

$$
Z_{n+1} \le \varrho\big(Z_n + Z_{n-1}\big) + c\varepsilon\|\varphi_0\|_{C^{l+3+\delta}(-1,1)} \quad \text{with} \quad \varrho < \frac{1}{2}.
$$

Hence if Z_n and Z_{n-1} satisfy

$$
Z_m \le c\varepsilon(1-2\varrho)^{-1}\|\varphi_0\|_{C^{l+3+\delta}(-1,1)} \equiv A_*, \tag{5.7}
$$

then the same inequality holds for Z_{n+1}.

Since $Z_0 = 0$ and Z_1 satisfies (5.7) (in view of (5.2)), (5.7) holds for all $m \geq 1$. Let us estimate the differences

$$R_n = \|\mathbf{v}_{n+1} - \mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_{n+1} - q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)} + \|\Upsilon_{n+1} - \Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + |\chi_{n+1} - \chi_n|.$$

We have

$$\begin{aligned} &|\chi_{n+1} - \chi_n| \\ &\quad \leq c\Big(\|\mathbf{v}_n - \mathbf{v}_{n-1}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|\Upsilon_n - \Upsilon_{n-1}\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big) \\ &\quad + \frac{c}{|\sin\alpha|}\Big(\|\mathbf{F}_n - \mathbf{F}_{n-1}\|_{C^{l+1+\delta}(\Omega_0;\beta)} + \|B_n - B_{n-1}\|_{C^{l+2+\delta}(\mathbb{R};\beta)} \\ &\quad + \|D_n - D_{n-1}\|_{C^{l+1+\delta}(\mathbb{R};\beta)}\Big) \\ &\quad \leq c\Big(\|\mathbf{v}_n - \mathbf{v}_{n-1}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|\Upsilon_n - \Upsilon_{n-1}\|_{C^{l+3+\delta}(\mathbb{R};\beta)}\Big) \\ &\quad + \frac{c\varepsilon}{|\sin\alpha|}\Big(\|\mathbf{v}_n - \mathbf{v}_{n-1}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_n - q_{n-1}\|_{C^{l+1+\delta}(\Omega_0;\beta)} \\ &\quad + \|\Upsilon_n - \Upsilon_{n-1}\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + \varepsilon|\chi_n - \chi_{n-1}|\Big); \end{aligned} \tag{5.8}$$

$$\begin{aligned} &\|\mathbf{v}_{n+1} - \mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_{n+1} - q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)} \\ &\quad \leq c\big(\varepsilon + |\sin\alpha|\big)\Big(\|\mathbf{v}_n - \mathbf{v}_{n-1}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_n - q_{n-1}\|_{C^{l+1+\delta}(\Omega_0;\beta)} \\ &\quad + \|\Upsilon_n - \Upsilon_{n-1}\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + \varepsilon|\chi_n - \chi_{n-1}|\Big) + c\varepsilon|\sin\alpha||\chi_{n+1} - \chi_n|; \end{aligned} \tag{5.9}$$

$$\begin{aligned} &\|\Upsilon_{n+1} - \Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} \\ &\quad \leq \varepsilon c|\chi_{n+1} - \chi_n| + c\Big(\|\mathbf{v}_{n+1} - \mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_{n+1} - q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)}\Big) \\ &\quad + c\varepsilon\Big(\|\mathbf{v}_{n+1} - \mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|\Upsilon_n - \Upsilon_{n-1}\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + \varepsilon|\chi_n - \chi_{n-1}|\Big). \end{aligned} \tag{5.10}$$

Then for sufficiently small ε and $|\sin\alpha|$ we obtain from (5.8)–(5.10)

$$\begin{aligned} R_n &= \|\mathbf{v}_{n+1} - \mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_{n+1} - q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)} \\ &\quad + \|\Upsilon_{n+1} - \Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + |\chi_{n+1} - \chi_n| \\ &\leq \varrho\Big(\|\mathbf{v}_n - \mathbf{v}_{n-1}\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|\mathbf{v}_{n-1} - \mathbf{v}_{n-2}\|_{C^{l+2+\delta}(\Omega_0;\beta)} \\ &\quad + \|q_n - q_{n-1}\|_{C^{l+1+\delta}(\Omega_0;\beta)} + \|q_{n-1} - q_{n-2}\|_{C^{l+1+\delta}(\Omega_0;\beta)} \\ &\quad + \|\Upsilon_n - \Upsilon_{n-1}\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + \|\Upsilon_{n-1} - \Upsilon_{n-2}\|_{C^{l+3+\delta}(\mathbb{R};\beta)} \\ &\quad + |\chi_n - \chi_{n-1}| + |\chi_{n-1} - \chi_{n-2}|\Big) = \varrho\big(R_{n-1} + R_{n-2}\big) \end{aligned} \tag{5.11}$$

with $\varrho < \frac{1}{2}$.

Note that in view of (5.7)

$$R_n \le Z_{n+1} + Z_n \le 2A_*.$$

If $n = 2m$, then

$$\begin{aligned} R_{2n} &\le \varrho\big(R_{2n-1} + R_{2n-2}\big) \le \varrho^2\big(R_{2n-2} + 2R_{2n-3} + R_{2n-4}\big) \le \quad \cdots \\ &\cdots \le \varrho^n\big(C_n^0 R_n + C_n^1 R_{n-2} + \cdots + C_n^n R_0\big) \le 2A_*\varrho^n\big(C_n^0 + C_n^1 + \cdots + C_n^n\big) \\ &= 2A_*(2\varrho)^n = 2A_*\lambda^n \end{aligned}$$

with $\lambda < 1$. The case $n = 2m + 1$ is similar. Now it is standard to show that

$$\begin{aligned} &\|\mathbf{v}_m - \mathbf{v}_n\|_{C^{l+2+\delta}(\Omega_0;\beta)} + \|q_m - q_n\|_{C^{l+1+\delta}(\Omega_0;\beta)} \\ &\quad + \|\Upsilon_m - \Upsilon_n\|_{C^{l+3+\delta}(\mathbb{R};\beta)} + |\chi_m - \chi_n| \to 0 \quad \text{as} \quad n, m \to \infty, \end{aligned}$$

hence the sequence $\{\mathbf{v}_n, q_n, \Upsilon_n, \chi_n\}$ converges in $C^{l+2+\delta}(\Omega_0;\beta) \times C^{l+1+\delta}(\Omega_0;\beta) \times C^{l+3+\delta}(\mathbb{R};\beta) \times \mathbb{R}$ to $\{\mathbf{v}, q, \Upsilon, \chi_*\}$. Obviously, $\mathbf{u}(x) = \mathbf{v}^0(x) + \mathbf{v}(x)$, $p(x) = p^0(x) + q(x)$ and $\psi(x_1) = 1 + \varepsilon\chi_* H_0(x_1) + \varepsilon\Upsilon(x_1)$ solve problem (1.2). Thus, we have proved the main result of the paper:

Theorem 5.1 *Assume that* $\varphi_0 \in C^{l+3+\delta}(-1, 1)$, $\operatorname{supp}\varphi_0 \subset (-1, 1)$, *and the numbers* $\varepsilon, \alpha, \varepsilon|\sin\alpha|^{-1}$ *are sufficiently small. Then problem* (1.2) *has a unique solution* $(\mathbf{u}, p, \psi)$. *This solution admits the representation*

$$\mathbf{u}(x) = \mathbf{v}^0(x) + \varepsilon\mathbf{v}(x), \;\; p(x) = p^0(x) + \varepsilon q(x), \;\; \psi(x_1) = 1 + \varepsilon\chi_* H_0(x_1) + \varepsilon\Upsilon(x_1),$$

where

$$v_1^0(x) = \frac{g\sin\alpha}{2\nu}x_2(2 - x_2), \;\; v_2^0(x) = 0, \;\; p^0(x) = g\cos\alpha(1 - x_2),$$

χ_* *is a constant,*

$$\mathbf{v} \in C^{l+2+\delta}(\Omega_0;\beta), \quad q \in C^{l+1+\delta}(\Omega_0;\beta), \quad H_0, \; \Upsilon \in C^{l+3+\delta}(\mathbb{R};\beta)$$

with $0 < \beta \le c_0\varepsilon$.

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