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Well-posedness in Sobolev spaces for semi-linear 3-evolution equations

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Abstract We prove local in time well-posedness of the Cauchy problem in Sobolev spaces for semi-linear 3-evolution equations of the first order. We require real principal part, but complex valued coefficients for the lower order terms. Therefore decay conditions on the imaginary parts are needed, as $x \to \infty$.

Keywords Non-linear evolution equations · Well-posedness in Sobolev spaces · Pseudo-differential operators

Mathematics Subject Classification (2000) Primary 35G25; Secondary 35A27

1 Introduction and main result

Let us consider the Cauchy problem

$$\begin{cases} P(t,x,u(t,x),D_t,D_x)u(t,x) = f(t,x), \quad (t,x) \in [0,T] \times \mathbb{R} \\ u(0,x) = u_0(x), \quad x \in \mathbb{R} \end{cases} \tag{1.1}$$

Dedicated to the memory of our friend and colleague Mariarosaria Padula.

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for the semi-linear operator

$$P := D_t + a_3(t)D_x^3 + a_2(t,x,u)D_x^2 + a_1(t,x,u)D_x + a_0(t,x,u), \tag{1.2}$$

where $D := \frac{1}{i}\partial$, $a_3 \in C([0,T];\mathbb{R})$, $a_j \in C([0,T];C^\infty(\mathbb{R}\times\mathbb{C}))$ and $x \mapsto a_j(t,x,w) \in \mathcal{B}^\infty(\mathbb{R})$ (here $\mathcal{B}^\infty(\mathbb{R})$ is the space of complex valued functions which are bounded on $\mathbb{R}$ together with all their derivatives), for $j = 0, 1, 2$.

We are so dealing with a semi-linear non-kowalewskian 3-evolution equation $Pu = f$ with the real characteristic (in the sense of Petrowski) $\tau = -a_3(t)\xi^3$. In the case $a_3(t) \equiv -1$, $a_2 \equiv a_0 \equiv 0$, $a_1(t,x,u) = -6u$, we recover the Korteweg-de Vries equation.

The aim of this paper is to give suitable decay conditions on the coefficients in order that the Cauchy problem (1.1) is locally in time well-posed in H^s with s great enough, and in H^∞.

The well-posedness result will be achieved by developing the linear technique of [5] (coming from the examples in [7,8] and used also in [3,4]), and applying then a fixed point argument, following the ideas of [1,2,9].

We consider here $x \in \mathbb{R}$ only for simplicity's sake; $x \in \mathbb{R}^n$, $n \geq 2$ could be considered with only technical changes in our proofs, see [10,13].

The assumption $a_3(t) \in \mathbb{R}$ is due to the necessary condition of the Lax-Mizohata Theorem (cf. [15]), while the assumptions $a_j(t,x,w) \in \mathbb{C}$ for $0 \leq j \leq 2$ imply some decay conditions on the coefficients because of the necessary condition of Ichinose (cf. [12]). We shall thus assume, in the following, that there exists a constant $C_3 > 0$ such that

$$a_3(t) \geq C_3 \quad \forall t \in [0,T], \tag{1.3}$$

and that there exist constants $C, \varepsilon > 0$ and a function $h : \mathbb{C} \to \mathbb{R}^+$ bounded on compact sets (for instance, h continuous) such that for all $(t,x,w) \in [0,T]\times\mathbb{R}\times\mathbb{C}$:

$$|\mathrm{Im}\, a_2(t,x,w)| \leq \frac{C}{\langle x\rangle^{1+\varepsilon}} h(w) \tag{1.4}$$

$$|\mathrm{Im}\, a_1(t,x,w)| \leq \frac{C}{\langle x\rangle^{1/2}} h(w) \tag{1.5}$$

$$|\mathrm{Re}\, a_2(t,x,w)| \leq C h(w) \tag{1.6}$$

$$|\partial_x \mathrm{Re}\, a_2(t,x,w)| \leq \frac{C}{\langle x\rangle^{1/2}} h(w) \tag{1.7}$$

$$|\partial_w a_2(t,x,w)| \leq \frac{C}{\langle x\rangle^{1/2}} h(w), \tag{1.8}$$

with the notation $\langle x\rangle := \sqrt{1+x^2}$.

Under the assumptions above we prove the following result:

Theorem 1.1 *Let P be as in (1.2) satisfying (1.3)–(1.8). Then the Cauchy problem (1.1) is locally in time well-posed in H^∞. More precisely, for every given $s > 5/2$*

and for all $f \in C([0,T]; H^s(\mathbb{R}))$ *and* $u_0 \in H^s(\mathbb{R})$*, there exists* $0 < T^* \leq T$ *and a unique solution* $u \in C([0,T^*]; H^s(\mathbb{R}))$ *of* (1.1) *satisfying the following inequality:*

$$\|u(t,\cdot)\|_s^2 \leq e^{\sigma t}\left(\|u_0\|_s^2 + \int_0^t \|f(\tau,\cdot)\|_s^2 d\tau\right) \qquad \forall t \in [0,T^*], \tag{1.9}$$

for some positive constant σ *depending on* s.

Remark 1.2 Estimate (1.9) gives local in time well-posedness of the Cauchy problem (1.1) in H^s, $s > 5/2$. By the same estimate we gain also H^∞ well-posedness: if the Cauchy data are $f \in C([0,T]; H^\infty(\mathbb{R}))$ and $u_0 \in H^\infty(\mathbb{R})$, then the solution $u \in C([0,T]; H^s)$ for every $s > 5/2$, and then by Sobolev's embeddings we immediately get $u \in C([0,T]; H^\infty)$.

Example 1.3 Let us consider the non-linear equation

$$P(t,x,u,D_t,D_x) = D_t u + a_3(t)D_x^3 + a_2(x,u)D_x^2 u = f(t,x)$$

with

$$\begin{aligned} f &\in C([0,T]; H^s(\mathbb{R})), \quad s \geq 5/2 \\ a_3(t) &\in C([0,T]; \mathbb{R}), \quad a_3(t) \geq C_3 > 0 \ \forall t \in [0,T] \\ a_2(x,w) &= i\frac{\sin x^\alpha}{(1+x^2)^{\frac{1+\varepsilon}{2}}}\frac{1}{1+w^2}, \qquad \alpha, \varepsilon > 0. \end{aligned}$$

Then

$$\begin{aligned} |\mathrm{Im}\, a_2| &\leq \frac{1}{\langle x\rangle^{1+\varepsilon}} \\ |\partial_w a_2| &= \left| -i\frac{\sin x^\alpha}{\langle x\rangle^{1+\varepsilon}}\frac{2w}{(1+w^2)^2}\right| \leq \frac{2}{\langle x\rangle^{1+\varepsilon}} \leq \frac{2}{\langle x\rangle^{1/2}}. \end{aligned}$$

Therefore Theorem 1.1 can be applied to get, for some $0 < T^* \leq T$, a unique solution $u \in C([0,T^*]; H^s(\mathbb{R}))$ of the Cauchy problem

$$\begin{cases} P(t,x,u,D_t,D_x)u(t,x) = f(t,x) & (t,x) \in [0,T^*] \times \mathbb{R} \\ u(0,x) = u_0(x) \in H^s(\mathbb{R}), & x \in \mathbb{R}. \end{cases}$$

The same result holds if, more in general, we take

$$a_2(t,x,w) = i a_2'(t,x) a_2''(w)$$

for some real valued functions $a_2' \in C([0,T]; \mathcal{B}^\infty(\mathbb{R}))$ satisfying (1.4) and $a_2'' \in C([0,T]; C^\infty(\mathbb{R}))$ with bounded derivative $\partial_w a_2''$.

Example 1.4 By simple computations it is easy to check that Example 1.3 works also considering, for example, $a_2(t,x,w) = \frac{ia_2'(t,x)}{\langle x+w\rangle^{1+\varepsilon}}$, or $a_2(t,x,w) = \frac{ia_2'(t,x)}{\langle x\rangle^{1+\varepsilon}+w^2}$, with a real valued function $a_2' \in C([0,T];\mathcal{B}^\infty(\mathbb{R}))$ satisfying (1.4).

2 Notation and main tools

The proof of Theorem 1.1 is based on the pseudo-differential calculus. In this paper we denote by $S^m := S^m(\mathbb{R}^2)$ the space of symbols $a(x,\xi)$ such that for every $\alpha,\beta\in\mathbb{N}$

$$\sup_{x,\xi\in\mathbb{R}} |\partial_\xi^\alpha D_x^\beta a(x,\xi)|\langle\xi\rangle_h^{-m+|\alpha|} < \infty,$$

where $\langle\xi\rangle_h := \sqrt{h^2+\xi^2}$, $h\geq 1$ fixed. Our symbols will be of the form $a(x,w,\xi)$, depending smoothly on a parameter $w\in\mathbb{C}$.

The idea of the proof is to fix $u\in B_r$,

$$B_r := \{u\in C([0,T];H^s) : \sup_{t\in[0,T]} \|u(t,\cdot)\|_s \leq r\},$$

with $r>0$ to be determined later on, to solve the linear Cauchy problem

$$\begin{cases} P(t,x,u,D_t,D_x)v = f \\ v(0,x) = u_0(x) \end{cases} \tag{2.1}$$

in the unknown $v(t,x)$ following [5], and then use a fixed point argument to find the solution of the non-linear Cauchy problem (1.1).

For this reason we recall now some definitions and results from [5]. According to [5, formula (2.4) and Remark 3.1], we define

$$\lambda_2(x,\xi) := M_2\int_0^x \langle y\rangle^{-1-\varepsilon}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^2}\right)dy \tag{2.2}$$

$$\lambda_1(x,\xi) := M_1\int_0^x \langle y\rangle^{-\frac{1}{2}}\psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^2}\right)dy\cdot\langle\xi\rangle_h^{-1} \tag{2.3}$$

where the constants $M_1, M_2 > 0$ have to be chosen in the sequel, $\psi\in C_0^\infty(\mathbb{R})$ satisfies $0\leq\psi\leq 1$ and

$$\psi(y) = \begin{cases} 1 & |y|\leq\frac{1}{2} \\ 0 & |y|\geq 1. \end{cases}$$

Then

$$|\lambda_2(x,\xi)| \le M_2 \int_0^{\langle x\rangle} \langle y\rangle^{-1-\varepsilon} dy \le C_2$$

$$|\lambda_1(x,\xi)| \le CM_1 \langle x\rangle^{\frac{1}{2}} \langle \xi\rangle_h^{-1} \chi_{\mathrm{supp}\,\psi}(x) \le C_1 M_1,$$

for some $C_2, C_1 > 0$, where $\chi_{\mathrm{supp}\,\psi}$ is the characteristic function of the support of $\psi(\langle x\rangle/\langle\xi\rangle_h^2)$.

Therefore, for $\Lambda(x,\xi) := \lambda_1(x,\xi) + \lambda_2(x,\xi)$, we have that

$$|\Lambda(x,\xi)| \le C_2' \tag{2.4}$$

for some $C_2' > 0$; moreover, from [5, Lemma 2.1] (with $\delta = 0$):

$$|\partial_\xi^\alpha D_x^\beta \Lambda(x,\xi)| \le \delta_{\alpha,\beta} \langle\xi\rangle_h^{-\alpha} \qquad \forall \alpha,\beta \in \mathbb{N}, \tag{2.5}$$

for some $\delta_{\alpha,\beta} > 0$.

This proves that the pseudo-differential operator $e^{\Lambda(x,D_x)}$ has symbol $e^{\Lambda(x,\xi)} \in S^0$, and then we can apply the following:

Lemma 2.1 *(*see Lemma 2.3, [5]*) Let* $\Lambda(x,\xi)$ *satisfy* (2.5). *There exists a constant* $h_0 \ge 1$ *such that for* $h \ge h_0$ *the operator* e^Λ *is invertible and*

$$(e^\Lambda)^{-1} = e^{-\Lambda}(I + R), \tag{2.6}$$

where I *is the identity operator and* R *is an operator of the form* $R = \sum_{n=1}^{+\infty} r^n$ *with principal symbol*

$$\tilde r(x,\xi) = \partial_\xi \Lambda(x,\xi) D_x \Lambda(x,\xi). \tag{2.7}$$

We conclude this section by recalling two results that will be crucial in determining the minimal assumptions needed on the coefficients a_j in (1.2) to get the well-posedness result here presented:

Theorem 2.2 (Sharp-Gårding inequality, *[14]) Let* $a(x,D_x)$ *be a pseudo-differential operator with symbol* $a(x,\xi) \in S^m$ *suche that* $\mathrm{Re}\, a(x,\xi) \ge 0$. *Then there exists* $c > 0$ *such that*

$$\mathrm{Re}\,\langle a(x,D_x)u,u\rangle \ge -c\|u\|^2_{(m-1)/2}. \tag{2.8}$$

Theorem 2.3 (Fefferman–Phong inequality, *[11]) Let* $a(x,\xi) \in S^m$ *with* $a(x,\xi) \ge 0$. *Then there exists* $c > 0$ *such that*

$$\mathrm{Re}\,\langle a(x,D_x)u,u\rangle \ge -c\|u\|^2_{(m-2)/2}. \tag{2.9}$$

3 Proof of Theorem 1.1

To start with the proof we fix $s > 5/2$, $f, u \in C([0,T]; H^s)$ and $u_0 \in H^s(\mathbb{R})$, and consider the linear Cauchy problem (2.1). A direct application of [5, Theorem 1.1 and Remark 1.5] immediately gives the existence of a unique solution $v \in C([0,T]; H^s)$ of problem (2.1) such that

$$\|v(t,\cdot)\|_s^2 \le C_s(u)\left(\|u_0\|_s^2 + \int_0^t \|f(\tau,\cdot)\|_s^2 d\tau\right) \qquad \forall t \in [0,T] \tag{3.1}$$

for some $C_s(u) > 0$, since assumption (1.4) gives no loss of derivatives ($\sigma = 2\delta = 0$ in [5, Theorem 1.1]). This is not enough for our purposes, since to proceed with the proof and apply a fixed point scheme we need to know precisely the constant $C_s(u)$. We thus quickly retrace in what follows the proof of Theorem 1.1 in [5], taking care of the dependence of the constants on the fixed function u, and taking advantage of the choice of $p = 3$.

We write

$$iP(t,x,u,D_t,D_x) = \partial_t + A(t,x,u,D_x)$$

with

$$A(t,x,u,D_x) := ia_3(t)D_x^3 + ia_2(t,x,u)D_x^2 + ia_1(t,x,u)D_x + ia_0(t,x,u)$$

and compute the symbol of the pseudo-differential operator $(e^\Lambda)^{-1}Ae^\Lambda$.

We have:

$$\begin{aligned}\sigma(Ae^\Lambda) = (ia_3\xi^3 + ia_2\xi^2 + ia_1\xi)e^\Lambda + (3ia_3\xi^2 + 2ia_2\xi)D_x e^\Lambda \\ + \frac{1}{2}(6ia_3\xi)D_x^2 e^\Lambda + \tilde{A}e^\Lambda\end{aligned} \tag{3.2}$$

for some $\tilde{A} \in S^0$.

To compute then $\sigma((e^\Lambda)^{-1}Ae^\Lambda)$ we need to write down the symbol of $(e^\Lambda)^{-1}$ by means of (2.6) and (2.7).

In the sequel it will be useful to estimate, from (2.2) and (2.3):

$$|\partial_\xi \lambda_2(x,\xi)| \le M_2 \left| \int_0^x \langle y\rangle^{-1-\varepsilon}\langle y\rangle \left(\partial_\xi \frac{1}{\langle\xi\rangle_h^2}\right)\psi'\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^2}\right)dy\right| \le C_2' M_2 \frac{\langle x\rangle^{1-\varepsilon}}{\langle\xi\rangle_h^3}\chi_{\operatorname{supp}\psi'} \tag{3.3}$$

$$|\partial_x\lambda_2(x,\xi)|| \le M_2\langle x\rangle^{-1-\varepsilon}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right) \le C_2' M_2\langle x\rangle^{-1-\varepsilon} \tag{3.4}$$

$$
\begin{aligned}
|\partial_\xi \lambda_1(x,\xi)| &\le M_1 \left| \int_0^x \langle y\rangle^{-\frac{1}{2}} \langle y\rangle \left(\partial_\xi \frac{1}{\langle\xi\rangle_h^2}\right) \psi'\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^2}\right) dy \right| \langle\xi\rangle_h^{-1} \\
&\quad + M_1 \left| \int_0^x \langle y\rangle^{-\frac{1}{2}} \psi\left(\frac{\langle y\rangle}{\langle\xi\rangle_h^2}\right) dy \cdot \left(\partial_\xi \frac{1}{\langle\xi\rangle_h}\right) \right| \\
&\le C_1' M_1 \left(\frac{\langle x\rangle^{3/2}}{\langle\xi\rangle_h^4} \chi_{\operatorname{supp}\psi'} + \frac{\langle x\rangle^{1/2}}{\langle\xi\rangle_h^2} \right) \qquad (3.5) \\
|\partial_x \lambda_1(x,\xi)| &\le M_1 \langle x\rangle^{-\frac{1}{2}} \psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right) \langle\xi\rangle_h^{-1} \le C_1' M_1 \frac{\langle x\rangle^{-\frac{1}{2}}}{\langle\xi\rangle_h} \qquad (3.6)
\end{aligned}
$$

for some $C_2', C_1' > 0$, where $\chi_{\operatorname{supp}\psi'}$ is the characteristic function of

$$
\operatorname{supp}\psi'\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right) \subseteq \left\{ x \in \mathbb{R} : \frac{1}{2}\langle\xi\rangle_h^2 \le \langle x\rangle \le \langle\xi\rangle_h^2 \right\}.
$$

Therefore

$$
|\tilde{r}(x,\xi)| = |\partial_\xi \Lambda(x,\xi) \cdot D_x \Lambda(x,\xi)| \le C_{M_1,M_2} \langle x\rangle^{-\frac{1}{2}-\varepsilon} \langle\xi\rangle_h^{-2};
$$

by simple computations we get that $\tilde{r}(x,\xi) \in S^{-2}$ and, by (2.6) and (2.7):

$$
(e^\Lambda)^{-1} = e^{-\Lambda}(I + \tilde{r} + R_{-3})
$$

with $\tilde{r}(x,D)$ a pseudo-differential operator with symbol $\tilde{r}(x,\xi)$ and R_{-3} an operator of order -3.

Then, from (3.2):

$$
\begin{aligned}
\sigma((e^\Lambda)^{-1} A e^\Lambda) &= (e^{-\Lambda} + e^{-\Lambda}\tilde{r}) \left(i a_3 \xi^3 + i a_2 \xi^2 + i a_1 \xi\right) e^\Lambda \\
&\quad + (e^{-\Lambda} + e^{-\Lambda}\tilde{r}) \left(3 i a_3 \xi^2 + 2 i a_2 \xi\right) (D_x \Lambda) e^\Lambda \\
&\quad + (e^{-\Lambda} + e^{-\Lambda}\tilde{r})(3 i a_3 \xi) \left(D_x^2 \Lambda + (D_x \Lambda)^2\right) e^\Lambda \\
&\quad - (\partial_\xi \Lambda) \left(i D_x a_2 \xi^2\right) - (\partial_\xi \Lambda) \left(i a_3 \xi^3 + i a_2 \xi^2\right) (D_x \Lambda) \\
&\quad - (\partial_\xi \Lambda) \left(3 i a_3 \xi^2\right) \left(D_x^2 \Lambda + (D_x \Lambda)^2\right) \\
&\quad + \frac{1}{2} \left(\partial_\xi^2 \Lambda + (\partial_\xi \Lambda)^2\right) \left(i a_3 \xi^3\right) \left(D_x^2 \Lambda + (D_x \Lambda)^2\right) + A_0' \\
&= i a_3 \xi^3 + i a_2 \xi^2 + i a_1 \xi + \tilde{r}(x,\xi) \left(i a_3 \xi^3\right) + \left(3 i a_3 \xi^2\right) (D_x \Lambda)
\end{aligned}
$$

$$
\begin{aligned}
&+ (2ia_2\xi)(D_x\Lambda) + (3ia_3\xi)\left(D_x^2\Lambda + (D_x\Lambda)^2\right) \\
&- (\partial_\xi\Lambda)\left(iD_x a_2\xi^2\right) - \tilde{r}(x,\xi)\left(ia_3\xi^3\right) + A_0'' \\
= {} & ia_3\xi^3 + \left[ia_2\xi^2 + \left(3ia_3\xi^2\right)(D_x\lambda_2)\right] \\
&+ \left[ia_1\xi + \left(3ia_3\xi^2\right)(D_x\lambda_1) + (2ia_2\xi)(D_x\lambda_2)\right. \\
&\left. + (3ia_3\xi)\left(D_x^2\lambda_2 + (D_x\lambda_2)^2\right) - (\partial_\xi\lambda_2)\left(iD_x a_2\xi^2\right)\right] + A_0
\end{aligned}
$$

for some $A_0', A_0'', A_0 \in S^0$, since $a_3 = a_3(t)$ and because of (3.3) and (3.5).

Therefore

$$
\sigma((e^\Lambda)^{-1}Ae^\Lambda) = A_3 + A_2 + A_1 + A_0,
$$

with $A_j \in S^j$ defined by:

$$
\begin{aligned}
A_3(t,\xi) &:= ia_3\xi^3 \\
A_2(t,x,u,\xi) &:= ia_2\xi^2 + (3ia_3\xi^2)(D_x\lambda_2) \\
A_1(t,x,u,\xi) &:= ia_1\xi + (3ia_3\xi^2)(D_x\lambda_1) + (2ia_2\xi)(D_x\lambda_2) \\
&\quad + (3ia_3\xi)(D_x^2\lambda_2 + (D_x\lambda_2)^2) - (\partial_\xi\lambda_2)(iD_x a_2\xi^2).
\end{aligned}
$$

Note that assumption (1.3) implies

$$
\operatorname{Re} A_3(t,\xi) = 0. \tag{3.7}
$$

As in the proof of Theorem 1.1 of [5], we look first for $M_2 > 0$ great enough to apply the Fefferman–Phong inequality (2.9) to

$$
\operatorname{Re} A_2 = -\operatorname{Im} a_2\xi^2 + 3a_3\xi^2\partial_x\lambda_2. \tag{3.8}
$$

By (1.4)

$$
|\operatorname{Im} a_2(t,x,u)\xi^2| \le \frac{C}{\langle x\rangle^{1+\varepsilon}} h(u)\langle\xi\rangle_h^2, \tag{3.9}
$$

while, by (1.3) and (3.4), for $|\xi| \ge h$ we have

$$
\begin{aligned}
3a_3(t)\xi^2\partial_x\lambda_2(x,\xi) &= 3M_2 a_3(t)\xi^2\langle x\rangle^{-1-\varepsilon}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right) \\
&\ge 3M_2 C_3\psi\langle x\rangle^{-1-\varepsilon}|\xi|^2 \\
&\ge \frac{3}{\sqrt{2}} M_2 C_3\psi\langle x\rangle^{-1-\varepsilon}\langle\xi\rangle_h^2.
\end{aligned} \tag{3.10}
$$

Substituting (3.9) and (3.10) in (3.8):

$$\begin{aligned}\text{Re } A_2 &\geq \frac{3}{\sqrt{2}} M_2 C_3 \psi \frac{\langle \xi \rangle_h^2}{\langle x \rangle^{1+\varepsilon}} - \frac{C}{\langle x \rangle^{1+\varepsilon}} h(u) \langle \xi \rangle_h^2 \\ &= \psi \left(\frac{3}{\sqrt{2}} C_3 M_2 - C h(u) \right) \frac{\langle \xi \rangle_h^2}{\langle x \rangle^{1+\varepsilon}} - C h(u) \frac{\langle \xi \rangle_h^2}{\langle x \rangle^{1+\varepsilon}} (1 - \psi) \\ &\geq \psi \left(\frac{3}{\sqrt{2}} C_3 M_2 - C h(u) \right) \frac{\langle \xi \rangle_h^2}{\langle x \rangle^{1+\varepsilon}} - 2 C h(u)\end{aligned}$$

since $\langle \xi \rangle_h^2 \leq 2 \langle x \rangle$ on supp $\left(1 - \psi\left(\frac{\langle x \rangle}{\langle \xi \rangle_h^2}\right)\right)$.

We thus choose $M_2 > \sqrt{2} C c_r / 3 C_3$, where

$$c_r := \sup_{\substack{(t,x) \in [0,T] \times \mathbb{R} \\ u \in B_r}} h(u)$$

is a positive constant because h maps compact sets into bounded sets by assumption and $\sup_{(t,x) \in [0,T] \times \mathbb{R}} |u(t,x)| \leq C_s \sup_{t \in [0,T]} \|u(t,\cdot)\|_s$ since $s > \frac{5}{2} > \frac{1}{2}$ by Sobolev embedding Theorem.

Then

$$\text{Re } A_2(t, x, u, \xi) \geq -2 C c_r$$

and, applying the Fefferman–Phong inequality (2.9) to the operator Re $A_2(t, x, u, \xi) + 2 C c_r$, we have that

$$\text{Re } \langle \text{Re } A_2 z, z \rangle \geq -c(1 + c_r) \|z\|_0^2 \tag{3.11}$$

for some fixed constant $c > 0$.

On the other hand, we can write the operator Im $A_2(t, x, u, D_x) = i \text{Re } a_2(t, x, u) D_x^2$ as

$$\text{Im } A_2 = \frac{\text{Im } A_2 + (\text{Im } A_2)^*}{2} + \frac{\text{Im } A_2 - (\text{Im } A_2)^*}{2} \tag{3.12}$$

with

$$\begin{aligned}\text{Re} \left\langle \frac{\text{Im } A_2 - (\text{Im } A_2)^*}{2} z, z \right\rangle &= \frac{1}{2} \text{Re } \langle \text{Im } A_2 z, z \rangle - \frac{1}{2} \text{Re } \langle z, \text{Im } A_2 z \rangle \\ &= \frac{1}{2} \text{Re } \langle \text{Im } A_2 z, z \rangle - \frac{1}{2} \text{Re } \overline{\langle \text{Im } A_2 z, z \rangle} = 0 \end{aligned} \tag{3.13}$$

and $\frac{\operatorname{Im} A_2+(\operatorname{Im} A_2)^*}{2}$ of order 1 since

$$\begin{aligned}\sigma(\operatorname{Im} A_2)+\sigma((\operatorname{Im} A_2)^*) &\sim \sigma(\operatorname{Im} A_2)+\sum_{\alpha\geq 0}\frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha\overline{\sigma(\operatorname{Im} A_2)}\\ &= i\operatorname{Re} a_2\xi^2 - i\operatorname{Re} a_2\xi^2 + \partial_\xi D_x(-i\operatorname{Re} a_2\xi^2)+B_0\\ &= -2\partial_x\operatorname{Re} a_2\xi + B_0\end{aligned}$$

for some $B_0 \in S^0$.

Let us now choose $M_1 > 0$ in order to apply the sharp-Gårding inequality (2.8) to

$$\tilde{A}_1(t,x,u,D_x) := A_1(t,x,u,D_x) - 2(\partial_x\operatorname{Re} a_2)D_x$$

with symbol

$$\begin{aligned}\tilde{A}_1(t,x,u,\xi) = {} & ia_1\xi + (3ia_3\xi^2)(D_x\lambda_1) + (2ia_2\xi)(D_x\lambda_2)\\ & + (3ia_3\xi)(D_x^2\lambda_2 + (D_x\lambda_2)^2) - (\partial_\xi\lambda_2)(iD_xa_2\xi^2) - 2(\partial_x\operatorname{Re} a_2)\xi.\end{aligned} \tag{3.14}$$

By (1.3) and (2.3):

$$\begin{aligned}\operatorname{Re}(3ia_3\xi^2D_x\lambda_1) &= 3a_3\xi^2\partial_x\lambda_1\\ &\geq 3c|\xi|^2M_1\langle x\rangle^{-1/2}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\langle\xi\rangle_h^{-1}\\ &\geq \frac{3}{\sqrt{2}}C_3M_1\psi\frac{\langle\xi\rangle_h}{\langle x\rangle^{1/2}}\end{aligned} \tag{3.15}$$

if $|\xi| \geq h$.

On the other hand, by (1.5):

$$|\operatorname{Re}(ia_1\xi)| = |\operatorname{Im} a_1|\cdot|\xi| \leq \frac{C}{\langle x\rangle^{1/2}}h(u)\langle\xi\rangle_h. \tag{3.16}$$

By (1.6) and (3.4):

$$\begin{aligned}|\operatorname{Re}[(2ia_2\xi)(D_x\lambda_2)] &= |2\operatorname{Re} a_2\xi\partial_x\lambda_2|\\ &\leq 2|\operatorname{Re} a_2|\langle\xi\rangle_hM_2\langle x\rangle^{-1-\varepsilon}\psi\left(\frac{\langle x\rangle}{\langle\xi\rangle_h^2}\right)\\ &\leq 2CM_2h(u)\psi\frac{\langle\xi\rangle_h}{\langle x\rangle^{1/2}}.\end{aligned} \tag{3.17}$$

By (1.3):

$$\operatorname{Re}[(3ia_3\xi)(D_x^2\lambda_2 + (D_x\lambda_2)^2) = 0. \tag{3.18}$$

By (1.7), (1.8) and (3.3):

$$\begin{aligned}|\mathrm{Re}\,[(\partial_\xi\lambda_2)(iD_x a_2\xi^2)]| &= |\partial_\xi\lambda_2|\cdot|\mathrm{Re}\,\partial_x\big(a_2(t,x,u)\big)|\cdot|\xi|^2\\ &\le cM_2\frac{\langle x\rangle^{1-\varepsilon}}{\langle\xi\rangle_h^3}\chi_{\mathrm{supp}\,\psi'}\cdot|\mathrm{Re}\,(\partial_x a_2)+\mathrm{Re}\,(\partial_w a_2)(\partial_x u)|\langle\xi\rangle_h^2\\ &\le cM_2\frac{1}{\langle\xi\rangle_h^2}\chi_{\mathrm{supp}\,\psi'}\cdot\frac{C}{\langle x\rangle^{1/2}}h(u)(1+|\partial_x u|)\langle\xi\rangle_h^2\\ &\le cCM_2h(u)(1+|\partial_x u|)\frac{\langle\xi\rangle_h}{\langle x\rangle^{1/2}} \end{aligned}\tag{3.19}$$

for some $c>0$.

By (1.7) and (1.8):

$$\begin{aligned}\left|\partial_x\mathrm{Re}\,\big(a_2(t,x,u(t,x))\big)\xi\right| &= |\partial_x(\mathrm{Re}\,a_2)+\mathrm{Re}\,(\partial_w a_2)(\partial_x u)|\cdot|\xi|\\ &\le \frac{C}{\langle x\rangle^{1/2}}h(u)(1+|\partial_x u|)\langle\xi\rangle_h.\end{aligned}\tag{3.20}$$

Substituting (3.15)–(3.20) in (3.14) and taking into account that $\langle x\rangle^{-1/2}\langle\xi\rangle_h\le 2$ on $\mathrm{supp}\,(1-\psi)$, we finally find a constant $c>0$, which depends also on the already chosen M_2, such that

$$\begin{aligned}\mathrm{Re}\,\tilde{A}_1 \ge &\left(\frac{3C_3}{\sqrt{2}}M_1\psi - Ch(u) - 2CM_2\psi h(u)\right.\\ &\left.- cCM_2h(u)(1+|\partial_x u|) - Ch(u)(1+|\partial_x u|)\right)\frac{\langle\xi\rangle_h}{\langle x\rangle^{1/2}}\\ = &\,\psi\left(\frac{3C_3}{\sqrt{2}}M_1 - C(M_2)h(u)(1+|\partial_x u|)\right)\frac{\langle\xi\rangle_h}{\langle x\rangle^{1/2}}\\ &-(1-\psi)C(M_2)h(u)(1+|\partial_x u|)\frac{\langle\xi\rangle_h}{\langle x\rangle^{1/2}}\\ \ge &-2C(M_2)C_r\end{aligned}$$

for some constant $C(M_2)>0$ which depends on the already chosen M_2, and for $M_1\ge\frac{\sqrt{2}C(M_2)}{3C_3}C_r$ with

$$C_r := \sup_{\substack{(t,x)\in[0,T]\times\mathbb{R}\\ u\in B_r}} h(u)(1+|\partial_x u|)\ge c_r.$$

Applying the sharp-Gårding inequality (2.8) to $\tilde{A}_1+2C(M_2)C_r$ we obtain that

$$\mathrm{Re}\,\langle\tilde{A}_1(t,x,u,D_x)z,z\rangle\ge -c(1+2C(M_2)C_r)\|z\|_0^2\tag{3.21}$$

for some fixed constant $c>0$.

Summing up, we have chosen $M_1, M_2 > 0$ sufficiently large so that $A_\Lambda := (e^\Lambda)^{-1} A e^\Lambda$ satisfies:

$$\mathrm{Re}\,\langle (e^\Lambda)^{-1} A e^\Lambda z, z\rangle \geq -\tilde{C}(1 + C_r)\|z\|_0^2 \tag{3.22}$$

for some fixed constant $\tilde{C} > 0$, because of (3.7), (3.11), (3.13) and (3.21).

Now, for every $z \in C([0, T]; H^3) \cap C^1([0, T]; L^2)$, from the identity $i P_\Lambda = \partial_t + A_\Lambda$, where $P_\Lambda := (e^\Lambda)^{-1} P e^\Lambda$, $A_\Lambda := (e^\Lambda)^{-1} A e^\Lambda$, we have:

$$\begin{aligned} \frac{d}{dt}\|z\|_0^2 &= 2\mathrm{Re}\,\langle \partial_t z, z\rangle = 2\mathrm{Re}\,\langle i P_\Lambda z, z\rangle - 2\mathrm{Re}\,\langle A_\Lambda z, z\rangle \\ &\leq 2(\|P_\Lambda z\|_0^2 + \|z\|_0^2) + \tilde{C}(1 + C_r)\|z\|_0^2 \\ &= 2\|P_\Lambda z\|_0^2 + (2 + \tilde{C}(1 + C_r))\|z\|_0^2. \end{aligned}$$

By Gronwall's Lemma:

$$\|z\|_0^2 \leq e^{(2+\tilde{C}(1+C_r))t}\left(\|z(0,\cdot)\|_0^2 + \int_0^t 2\|P_\Lambda z(\tau,\cdot)\|_0^2 d\tau\right).$$

By usual arguments we get also, for $s \geq 5/2$:

$$\|z\|_s^2 \leq e^{(3+\tilde{C}(1+C_r))t}\left(\|z(0,\cdot)\|_s^2 + \int_0^t \|P_\Lambda z(\tau,\cdot)\|_s^2 d\tau\right). \tag{3.23}$$

The a-priori estimate (3.23) gives existence and uniqueness of a solution $z \in C([0, T]; H^s)$ of the Cauchy problem

$$\begin{cases} P_\Lambda(t, x, u, D_t, D_x)z(t, x) = f_\Lambda(t, x) \\ z(0, x) = (u_0)_\Lambda(x) \end{cases} \tag{3.24}$$

equivalent to (1.1) for $f_\Lambda := (e^\Lambda)^{-1} f$, $(u_0)_\Lambda := (e^\Lambda)^{-1} u_0$; moreover the solution satisfies the following energy estimate:

$$\|z\|_s^2 \leq e^{(3+\tilde{C}(1+C_r))t}\left(\|(u_0)_\Lambda\|_s^2 + \int_0^t \|f_\Lambda(\tau,\cdot)\|_s^2 d\tau\right). \tag{3.25}$$

Remark now that z is a solution of (3.24) if and only if $v = e^\Lambda z$ is a solution of (2.1). Since $e^\Lambda \in S^0$, from (3.25) we thus have that the solution v of the Cauchy problem (2.1) satisfies:

$$\|v\|_s^2 \le c_1\|z\|_s^2 \le c_1 e^{(3+\tilde{C}(1+C_r))t}\left(\|(u_0)_\Lambda\|_s^2 + \int_0^t \|f_\Lambda(\tau,\cdot)\|_s^2 d\tau\right)$$
$$\le c_2 e^{(3+\tilde{C}(1+C_r))t}\left(\|u_0\|_s^2 + \int_0^t \|f(\tau,\cdot)\|_s^2 d\tau\right), \tag{3.26}$$

for some fixed constants $c_1, c_2 > 0$. Note that (3.26) implies (3.1) for $C_s(u) := c_2 e^{(3+\tilde{C}(1+C_r))T}$.

It is then defined a map

$$\begin{aligned} S:\ B_r &\to C([0,T];H^s) \\ u &\mapsto v \end{aligned}$$

which associates, to every fixed $u \in B_r$, the unique solution $v \in C([0,T];H^s)$ of the Cauchy problem (2.1), satisfying

$$\|v(t,\cdot)\|_s \le \sqrt{c_2}e^{\frac{1}{2}(3+\tilde{C}(1+C_r))t}(\|u_0\|_s + \sqrt{t}\|f(t,\cdot)\|_s) \qquad \forall t \in [0,T]. \tag{3.27}$$

We now choose $r > 2e\sqrt{c_2}\max\{\|u_0\|_s, \sup_{t\in[0,T]}\|f(t,\cdot)\|_s\}$. Then

$$\|v(t,\cdot)\|_s \le \frac{r}{2}(1+\sqrt{t})e^{\frac{1}{2}(3+\tilde{C}(1+C_r))t-1} < r$$

if $t \in [0, T_0]$ for T_0 sufficiently small.

For such a choice of T_0 we thus have that, for

$$u \in B_r^0 := \{u \in C([0,T_0];H^s) : \sup_{t\in[0,T_0]}\|u(t,\cdot)\|_s \le r\},$$

the Cauchy problem (2.1) admits a unique solution $v \in B_r^0$, i.e.

$$S:\ B_r^0 \to B_r^0.$$

We are now ready to use a fixed point argument. Fix $u, \tilde{u} \in B_r^0$, let $v = S(u)$ and $\tilde{v} = S(\tilde{u})$ the corresponding solutions of (2.1) and set $w = v - \tilde{v}$.

From

$$\begin{aligned} &D_t v + a_3(t)D_x^3 v + a_2(t,x,u)D_x^2 v + a_1(t,x,u)D_x v + a_0(t,x,u) = f(t,x) \\ &D_t \tilde{v} + a_3(t)D_x^3 \tilde{v} + a_2(t,x,\tilde{u})D_x^2 \tilde{v} + a_1(t,x,\tilde{u})D_x \tilde{v} + a_0(t,x,\tilde{u}) = f(t,x) \end{aligned}$$

we have that

$$\begin{aligned} &D_t w + a_3(t)D_x^3 w + a_2(t,x,u)D_x^2 v - a_2(t,x,\tilde{u})D_x^2\tilde{v} \\ &\quad + a_1(t,x,u)D_x v - a_1(t,x,\tilde{u})D_x\tilde{v} + a_0(t,x,u) - a_0(t,x,\tilde{u}) = 0, \end{aligned}$$

i.e.

$$
\begin{aligned}
&D_t w + a_3(t) D_x^3 w + a_2(t,x,u) D_x^2 w + a_1(t,x,u) D_x w \\
&\quad + [a_2(t,x,u) - a_2(t,x,\tilde{u})] D_x^2 \tilde{v} + [a_1(t,x,u) - a_1(t,x,\tilde{u})] D_x \tilde{v} \\
&\quad + [a_0(t,x,u) - a_0(t,x,\tilde{u})] = 0.
\end{aligned}
$$

This means that w is a solution of

$$
\tilde{P}(t,x,u,D_t,D_x) w(t,x) = \tilde{f}(t,x,u,\tilde{u},\tilde{v}),
$$

where $\tilde{P}(t,x,u,D_t,D_x) := P(t,x,u,D_t,D_x) - a_0(t,x,u)$ and

$$
\begin{aligned}
\tilde{f}(t,x,u,\tilde{u},\tilde{v}) &:= [a_2(t,x,u) - a_2(t,x,\tilde{u})] D_x^2 \tilde{v} \\
&\quad + [a_1(t,x,u) - a_1(t,x,\tilde{u})] D_x \tilde{v} + [a_0(t,x,u) - a_0(t,x,\tilde{u})].
\end{aligned}
$$

Since $u, \tilde{u}, \tilde{v} \in C([0,T_0]; H^s)$ we have that $\tilde{f} \in C([0,T_0]; H^{s-2})$ and, from (3.27) and $w(0,x) = 0$:

$$
\|w(t,\cdot)\|_{s-2} \le \sqrt{c_2} e^{\frac{1}{2}(3+\tilde{C}(1+C_r))T_0} \sqrt{T_0} \sup_{t\in[0,T_0]} \|\tilde{f}\|_{s-2} \tag{3.28}
$$

with

$$
\begin{aligned}
\|\tilde{f}\|_{s-2} &\le \|(a_2(t,x,u) - a_2(t,x,\tilde{u})) D_x^2 \tilde{v}\|_{s-2} + \|(a_1(t,x,u) - a_1(t,x,\tilde{u})) D_x \tilde{v}\|_{s-2} \\
&\quad + \|a_0(t,x,u) - a_0(t,x,\tilde{u})\|_{s-2}.
\end{aligned}
$$

Since $s - 2 > 1/2$ by assumption, then $H^{s-2}(\mathbb{R})$ is an algebra and

$$
\begin{aligned}
\|(a_2(t,x,u) - a_2(t,x,\tilde{u})) D_x^2 \tilde{v}\|_{s-2} &\le C_s \|a_2(t,x,u) - a_2(t,x,\tilde{u})\|_{s-2} \|D_x^2 \tilde{v}\|_{s-2} \\
&\le C_{s,r} \|u - \tilde{u}\|_{s-2}
\end{aligned} \tag{3.29}
$$

where $C_{s,r}$ is a positive constant depending on s and r, and more precisely

$$
C_{s,r} = C'_s \left(\sum_{\alpha+\beta\le[s]-1} \sup_{\substack{(t,x)\in[0,T_0]\times\mathbb{R} \\ |w|\le C_s r}} |D_x^\alpha D_w^{\beta+1} a_2(t,x,w)| \right) \|\tilde{v}\|_s
$$

for some $C'_s > 0$.

Analogously, up to changing the constant $C_{s,r}$,

$$
\|(a_1(t,x,u) - a_1(t,x,\tilde{u})) D_x \tilde{v}\|_{s-2} \le C_{s,r} \|u - \tilde{u}\|_{s-2} \tag{3.30}
$$

$$
\|a_0(t,x,u) - a_0(t,x,\tilde{u})\|_{s-2} \le C_{s,r} \|u - \tilde{u}\|_{s-2}. \tag{3.31}
$$

Substituting (3.29), (3.30) and (3.31) in (3.28) we have that

$$\|w\|_{s-2} \leq 3C_{s,r}\sqrt{c_2}e^{\frac{1}{2}(3+\tilde{C}(1+C_r))T_0}\sqrt{T_0}\sup_{t\in[0,T_0]}\|u-\tilde{u}\|_{s-2}. \tag{3.32}$$

We now choose $T^* \leq T_0$ sufficiently small so that

$$L := 3C_{s,r}\sqrt{c_2}e^{\frac{1}{2}(3+\tilde{C}(1+C_r))T^*}\sqrt{T^*} < 1,$$

and define

$$\begin{aligned} |||u|||_s &:= \sup_{t\in[0,T^*]}\|u(t,\cdot)\|_s\,, \\ B_r^* &:= \{u \in C([0,T^*];H^s) : \ |||u(t,\cdot)|||_s \leq r\}. \end{aligned}$$

Then (3.32) implies that $S: \ B_r^* \to B_r^*$ is a contraction with the $|||\cdot|||_{s-2}$ norm:

$$|||S(u)-S(\tilde{u})|||_{s-2} \leq L|||u-\tilde{u}|||_{s-2}, \qquad 0<L<1. \tag{3.33}$$

Define now recursively

$$\begin{cases} u_1 = S(u_0) \\ u_{n+1} = S(u_n), \quad n\geq 1. \end{cases}$$

From (3.33):

$$\begin{aligned} |||u_{n+1}-u_n|||_{s-2} &= |||S(u_n)-S(u_{n-1})|||_{s-2} \leq L|||u_n-u_{n-1}|||_{s-2} \\ &= L|||S(u_{n-1})-S(u_{n-2})|||_{s-2} \leq L^2|||u_{n-1}-u_{n-2}|||_{s-2} \\ &\leq \ldots \leq L^n|||u_1-u_0|||_{s-2}. \end{aligned}$$

Therefore,

$$\begin{aligned} |||u_{n+p}-u_n|||_{s-2} &\leq |||u_{n+p}-u_{n+p-1}|||_{s-2} + |||u_{n+p-1}-u_{n+p-2}|||_{s-2} \\ &\quad + \ldots + |||u_{n+1}-u_n|||_{s-2} \\ &\leq L^n(1+L+\ldots+L^{p-1})|||u_1-u_0|||_{s-2} \\ &\leq \frac{L^n}{1-L}|||u_1-u_0|||_{s-2}, \end{aligned}$$

so that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T^*];H^{s-2})$ and hence converges in $C([0,T^*];H^{s-2})$ to some $u \in C([0,T^*];H^{s-2})$. In particular, for every fixed $t \in [0,T^*]$,

$$u_n(t,\cdot) \to u(t,\cdot) \qquad \text{in } H^{s-2}. \tag{3.34}$$

At the same time, since $H^s(\mathbb{R})$ is a reflexive space and $\|u_n(t,\cdot)\|_s \le r$, by Kakutani's Theorem we have that there exists a subsequence $\{u_{n_h}\}_{h\in\mathbb{N}}$ which weakly converges in H^s to some $\tilde{u} \in H^s(\mathbb{R})$:

$$u_{n_h}(t,\cdot) \rightharpoonup \tilde{u}(t,\cdot) \qquad \text{in } H^s \tag{3.35}$$

and hence

$$\|\tilde{u}(t,\cdot)\|_s \le \liminf_{h\to+\infty} \|u_{n_h}(t,\cdot)\|_s. \tag{3.36}$$

From (3.34) and (3.35) we have that $u(t,\cdot) = \tilde{u}(t,\cdot) \in H^s(\mathbb{R})$.

Moreover, by (3.33):

$$|||S(u_n) - S(u)|||_{s-2} \le L|||u_n - u|||_{s-2} \to 0.$$

Therefore, as $n \to +\infty$:

$$u \leftarrow u_{n+1} = S(u_n) \to S(u) \qquad \text{in } C([0,T^*]; H^{s-2}),$$

so that $S(u) = u \in C([0,T^*]; H^s)$ and we have thus found a solution $u \in C([0,T^*]; H^s)$ of the Cauchy problem

$$\begin{cases} P(t,x,u,D_t,D_x)u(t,x) = f(t,x), & (t,x) \in [0,T^*] \times \mathbb{R} \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

Since (3.26) is satisfied with $v(t,\cdot) = u_{n_h}(t,\cdot)$, for $t \in [0,T^*]$, from (3.36) we have that

$$\|u(t,\cdot)\|_s^2 \le c_2 e^{(3+\tilde{C}(1+C_r))t} \left(\|u_0\|_s^2 + \int_0^t \|f(\tau,\cdot)\|_s^2 d\tau \right) \qquad \forall t \in [0,T^*]$$

which gives (1.9).

Uniqueness follows from (3.33).

The proof is thus complete. □

References

1. Agliardi, R., Zanghirati, L.: Cauchy problem for nonlinear p-evolution equations. Bull. Sci. Math. **133**, 406–418 (2009)
2. Ascanelli, A.: Quasilinear hyperbolic operators with Log-Lipschitz coefficients. J. Math. Anal. Appl. **295**, 70–79 (2004)
3. Ascanelli, A., Boiti, C.: Well-posedness of the Cauchy problem for p-evolution systems of pseudo-differential operators. J. Pseudo Differ. Oper. Appl. **4**(2), 113–143 (2013). doi:10.1007/s11868-013-0068-y
4. Ascanelli, A., Boiti, C.: Cauchy problem for higher order p-evolution equations. J. Diff. Equ. **255**, 2672–2708 (2013)

5. Ascanelli, A., Boiti, C., Zanghirati, L.: Well-posedness of the Cauchy problem for p-evolution equations. J. Differ. Equ. **253**, 2765–2795 (2012)
6. Ascanelli, A., Cicognani, M.: Schrödinger equations of higher order. Math. Nachr. **280**(7), 717–727 (2007)
7. Ascanelli, A., Cicognani, M., Colombini, F.: The global Cauchy problem for a vibrating beam equation. J. Differ. Equ. **247**, 1440–1451 (2009)
8. Cicognani, M., Colombini, F.: The Cauchy problem for p-evolution equations. Trans. Am. Math. Soc. **362**(9), 4853–4869 (2010)
9. Cicognani, M., Zanghirati, L.: The Cauchy problem for nonlinear hyperbolic equations with Levi condition. Bull. Sci. math. **123**, 413–435 (1999)
10. Cicognani, M., Reissig, M.: Degenerate Schrödinger Equations. Evol. Equ. Control Theory (2013) (in press)
11. Fefferman, C., Phong, D.H.: On positivity of pseudo-differential operators. Proc. Natl. Acad. Sci. USA **75**(10), 4673–4674 (1978)
12. Ichinose, W.: Some remarks on the Cauchy problem for Schrödinger type equations. Osaka J. Math. **21**, 565–581 (1984)
13. Kajitani, K., Baba, A.: The Cauchy problem for Schrödinger type equations. Bull. Sci. Math. **119**, 459–473 (1995)
14. Kumano-Go, H.: Pseudo-differential Operators. The MIT Press, Cambridge (1982)
15. Mizohata, S.: On the Cauchy problem. Notes and Reports in Mathematics in Science and Engineering, vol 3. Academic Press Inc., Orlando; Science Press, Beijing (1985)