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The generalized cubic functional equation and the stability of cubic Jordan ∗-derivations

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Abstract In the current work, we obtain the general solution of the following generalized cubic functional equation

$$\begin{aligned} &f(x+my)+f(x-my) \\ &= 2\left(2\cos\left(\frac{m\pi}{2}\right)+m^2-1\right)f(x)-\frac{1}{2}\left(\cos\left(\frac{m\pi}{2}\right)+m^2-1\right)f(2x) \\ &\quad +m^2\{f(x+y)+f(x-y)\} \end{aligned}$$

for an integer $m \geq 1$. We prove the Hyers–Ulam stability and the superstability for this cubic functional equation by the directed method and a fixed point approach. We also employ the mentioned functional equation to establish the stability of cubic Jordan $*$-derivations on C^*-algebras and JC^*-algebras.

Keywords Banach algebra · Cubic derivation · Cubic functional equation · Hyers–Ulam stability · Superstability

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1 Introduction

The stability problem of functional equations originated from a question of Ulam [27] in 1940, concerning the stability of group homomorphisms: Let $(\mathcal{G}_1, .)$ be a group and let $(\mathcal{G}_2, *)$ be a metric group with the metric d. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $\varphi : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ satisfies the inequality $d(\varphi(s.t), \varphi(s) * \varphi(t)) < \delta$ for all $s, t \in \mathcal{G}_1$, then there exists a homomorphism $\psi : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ with $(\varphi(s), \psi(s)) < \epsilon$ for all $s \in \mathcal{G}_1$? In the other words, under what conditions there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equations arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In [13], Hyers gave the first affirmative answer to the question of Ulam for Banach spaces (see also [24]). During the last decades, various approaches to this problem have been studied by a number of authors (for example, see [4,9,11,14]).

The following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.1}$$

has been introduced by Jun and Kim in [19]. They found out the general solution and established the Hyers–Ulam stability for the functional equation (1.1). One can easily check that the function $f(x) = \alpha x^3$ satisfies (1.1). Thus, every solution of the cubic functional equation (1.1) is said to be a *cubic function*. In [19], it is also showed that a mapping f between real vector spaces $\mathcal{X}$ and $\mathcal{Y}$ is a solution of (1.1) if and only if there exists a unique mapping $C : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{Y}$ such that $f(x) = C(x, x, x)$ for all $x \in \mathcal{X}$, and C is symmetric for each fixed one variable and is additive for fixed two variables. The stability of the cubic functional equation (1.1) has been considered on different spaces by a number of authors (for instance, see [16,25,26]). Jun and Kim in [20] introduced the following cubic functional equation

$$f(x+2y) + f(x-2y) + 6f(x) = 4f(x+y) + 4f(x-y)$$

and they established the general solution and the Hyers–Ulam stability problem for it. Since $f(2x) = 8f(x)$, the last functional equation is equal to the following

$$f(x+2y) + f(x-2y) = 2f(x) - f(2x) + 4f(x+y) + 4f(x-y). \tag{1.2}$$

Some generalized cubic functional equations of (1.1) and (1.2) have been introduced in [17] and [15], respectively.

The stability of functional equations of $*$-derivations and of quadratic $*$-derivations with the Cauchy functional equation and the Jensen functional equation on Banach $*$-algebras has been investigated by Jang and Park in [18]. The superstability of $*$-derivations and of quadratic $*$-derivations on C^*-algebras is proved as well. Some results of [18] on a quadratic $*$-derivation are modified in [6]. Recall that a functional equation is called *superstable* if any approximate solution to the functional equation is a its exact solution. Recently, Yang et al. investigated the stability of cubic $*$-derivations on Banach $*$-algebras in [28]. They also proved the stability and the superstability of cubic $*$-derivations on a Banach $*$-algebra with a left bounded approximate identity. In [23], the stability and the superstability of $*$-derivations associated with the Cauchy functional equation and the Jensen functional equation via a fixed point method are investigated.

The Hyers–Ulam stability and the superstability for the functional equation (1.1) by using a fixed point theorem under certain conditions on Banach algebras has been studied by Bodaghi et al. in [3]. This method which is different from the "*direct method*", initiated by Hyers in 1941, had been applied by Cădariu and Radu for the first time in [7]. In other words, they employed this fixed point method to the investigation of the Cauchy functional equation [8] and for the quadratic functional equation [7] (for more applications of this method, see [2,5,11,12,22]).

In this paper we consider the following functional equation which is somewhat different from (1.1) and (1.2):

$$\begin{aligned} f(x+my)+f(x-my) &= 2\left(2\cos\left(\frac{m\pi}{2}\right)+m^2-1\right)f(x)-\frac{1}{2}\left(\cos\left(\frac{m\pi}{2}\right)+m^2-1\right)f(2x)\\ &\quad +m^2\{f(x+y)+f(x-y)\} \end{aligned} \tag{1.3}$$

where m is an integer and $m \geq 2$. Note that when $m = 2$, we have the equation (1.2).

Here our purpose is to find out the general solution and to prove the Hyers–Ulam stability problem and the superstability for the equation (1.3). The stability of cubic Jordan $*$-derivations on C^*-algebras and JC^*-algebras are established as well.

2 Solution of Equation (1.3)

We firstly solve the equation of (1.3) as follows:

Theorem 2.1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be real vector spaces. Then a mapping $f : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfies the functional equation* (1.2) *if and only if it satisfies the functional equation* (1.3). *Therefore, every solution of the functional equation* (1.3) *is also a cubic mapping.*

Proof Assume that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfies the functional equation (1.2). Putting $x = y = 0$ in (1.2), we have $f(0) = 0$. Let $y = 0$ in (1.2) to get $f(2x) = 8f(x)$ for all $x \in \mathcal{X}$. Setting $x = 0$ in (1.2) and using this fact that $f(2y) = 8f(y)$, we have $f(-y) = -f(y)$. Letting $y = x$ in (1.2), we obtain $f(3x) = 27f(x)$ for all $x \in \mathcal{X}$. By the same method, we get $f(kx) = k^3 f(x)$ for all positive integers k. Replacing x by $x + y$ and $x - y$ in (1.2) respectively, and applying (1.2), we obtain

$$f(x+3y)+f(x-3y)=16f(x)-4f(2x)+9\{f(x+y)+f(x-y)\}.$$

Similar to the above, we can deduce that

$$f(x+4y)+f(x-4y)=34f(x)-8f(2x)+16\{f(x+y)+f(x-y)\}.$$

Using the above method, we get

$$f(x+my)+f(x-my)=a_m f(x)-b_m f(2x)+m^2\{f(x+y)+f(x-y)\}$$

in which

$$\begin{cases} a_m = -a_{m-2}+4(m-1)^2, & a_2=2, a_3=16 \\ b_m = -b_{m-2}+(m-1)^2, & b_2=1, b_3=4. \end{cases}$$

Solving the above recurrence equations, we have

$$a_m = 2\left(2\cos\left(\frac{m\pi}{2}\right)+m^2-1\right) \quad \text{and} \quad b_m=\frac{1}{2}\left(\cos\left(\frac{m\pi}{2}\right)+m^2-1\right)$$

for all $x, y \in \mathcal{X}$ and each positive integer $m \geq 3$.

Conversely, suppose that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfies the functional equation (1.3) for any positive integer $m \geq 3$. So f satisfies (1.3) for each $k \geq m$, in particular for $k = m(m-1)$. Hence for each $x, y \in \mathcal{X}$, we have

$$\begin{aligned} &f(x+m(m-1)y)+f(x-m(m-1)y) \\ &= 2\left(2\cos\left(\frac{m\pi}{2}\right)+m^2-1\right)f(x)-\frac{1}{2}\left(\cos\left(\frac{m\pi}{2}\right)+m^2-1\right)f(2x) \\ &\quad + m^2\{f(x+(m-1)y)+f(x-(m-1)y)\} \end{aligned} \tag{2.1}$$

On the other hand,

$$\begin{aligned} &f(x+(m^2-m)y)+f(x-(m^2-m)y) \\ &= 2\left(2\cos\left(\frac{m^2-m}{2}\pi\right)+(m^2-m)^2-1\right)f(x) \\ &\quad -\frac{1}{2}\left(\cos\left(\frac{m^2-m}{2}\pi\right)+(m^2-m)^2-1\right)f(2x) \\ &\quad +(m^2-m)^2\{f(x+y)+f(x-y) \end{aligned} \tag{2.2}$$

for all $x, y \in \mathcal{X}$. Using (2.1) and (2.2), we get

$$\begin{aligned} &m^2\{f(x+(m-1)y)+f(x-(m-1)y)\} \\ &= 2\left(2\cos\left(\frac{m(m-1)}{2}\pi\right)+m^2(m-1)^2-2\cos\left(\frac{m\pi}{2}\right)-m^2\right)f(x) \\ &\quad -\frac{1}{2}\left(\cos\left(\frac{m(m-1)}{2}\pi\right)+m^2(m-1)^2-\cos\left(\frac{m\pi}{2}\right)-m^2\right)f(2x) \\ &\quad +m^2(m-1)^2\{f(x+y)+f(x-y)\} \\ &= 2m^2(2\cos\left(\frac{m-1}{2}\pi\right)+(m-1)^2-1)f(x)+4\left(\cos\left(\frac{m(m-1)}{2}\pi\right)-\cos\left(\frac{m\pi}{2}\right)\right. \\ &\quad \left.-m^2\cos\left(\frac{m-1}{2}\pi\right)\right)f(x) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}m^2\left(\cos\left(\frac{m-1}{2}\pi\right)+(m-1)^2-1\right)f(2x)-\frac{1}{2}\left(\cos\left(\frac{m(m-1)}{2}\pi\right)-\cos\left(\frac{m\pi}{2}\right)\right.\\
&\left.-m^2\cos\left(\frac{m-1}{2}\pi\right)\right)f(2x)+m^2(m-1)^2\{f(x+y)+f(x-y)\}\\
=\;&2m^2\left(2\cos\left(\frac{m-1}{2}\pi\right)+(m-1)^2-1\right)f(x)+4\left(\cos\left(\frac{m(m-1)}{2}\pi\right)-\cos\left(\frac{m\pi}{2}\right)\right.\\
&\left.-m^2\cos\left(\frac{m-1}{2}\pi\right)\right)f(x)\\
&-\frac{1}{2}m^2\left(\cos\left(\frac{m-1}{2}\pi\right)+(m-1)^2-1\right)f(2x)-4\left(\cos\left(\frac{m(m-1)}{2}\pi\right)-\cos\left(\frac{m\pi}{2}\right)\right.\\
&\left.-m^2\cos\left(\frac{m-1}{2}\pi\right)\right)f(x)+m^2(m-1)^2\{f(x+y)+f(x-y)\}\\
=\;&2m^2\left(2\cos\left(\frac{m-1}{2}\pi\right)+(m-1)^2-1\right)f(x)-\frac{1}{2}m^2\left(\cos\left(\frac{m-1}{2}\pi\right)\right.\\
&\left.+(m-1)^2-1\right)f(2x)\\
&+m^2(m-1)^2\{f(x+y)+f(x-y)\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&f(x+(m-1)y)+f(x-(m-1)y)\\
&\quad=2\left(2\cos\left(\frac{m-1}{2}\pi\right)+(m-1)^2-1\right)f(x)-\frac{1}{2}\left(\cos\left(\frac{m-1}{2}\pi\right)+(m-1)^2-1\right)f(2x)\\
&\quad+\;(m-1)^2\{f(x+y)+f(x-y)\}.
\end{aligned}
$$

Consequently, f satisfies the functional equation (1.2). Note that f satisfies (1.3) for the case $m=3$ and we have used the fact $f(2x)=8f(x)$ for all $x\in\mathcal{X}$ in the process of the proof (refer to beginning of the proof). □

Let m be an integer such that $m\geq 2$. We use the abbreviation for the given mapping $f:\mathcal{X}\longrightarrow\mathcal{Y}$ as follows:

$$
\begin{aligned}
\mathcal{D}_m f(x,y):=\;&f(x+my)+f(x-my)\\
&-2\left(2\cos\left(\frac{m\pi}{2}\right)+m^2-1\right)f(x)\\
&+\frac{1}{2}\left(\cos\left(\frac{m\pi}{2}\right)+m^2-1\right)f(2x)-m^2\{f(x+y)+f(x-y)\}
\end{aligned}
$$

for all $x,y\in\mathcal{X}$.

From now on, we assume that $\mathcal{X}$ is a normed real linear space with norm $\|.\|_{\mathcal{X}}$ and $\mathcal{Y}$ is a real Banach space with norm $\|.\|_{\mathcal{Y}}$. Now, we are going to prove the stability of the cubic functional equation (1.3).

Theorem 2.2 *Let α be a real number and let $f:\mathcal{X}\longrightarrow\mathcal{Y}$ be a mapping for which there exists a function $\phi:\mathcal{X}\times\mathcal{X}\longrightarrow[-\alpha,\infty)$ such that*

$$\widetilde{\phi}(x) := \sum_{k=0}^{\infty} \frac{1}{8^k}\phi(2^k x, 0) < \infty, \lim_{k\to\infty} \frac{1}{8^k}\phi(2^k x, 2^k y) = 0 \tag{2.3}$$

and

$$\|\mathcal{D}_m f(x, y)\|_{\mathcal{Y}} \leq \alpha + \phi(x, y) \tag{2.4}$$

for all $x, y \in \mathcal{X}$, where m is an integer with $m \geq 2$. Then there exists a unique cubic mapping $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\|f(x) - \mathcal{T}(x)\|_{\mathcal{Y}} \leq \frac{2\alpha}{7(\cos(\frac{m\pi}{2}) + m^2 - 1)} + \frac{\widetilde{\phi}(x)}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)} \tag{2.5}$$

for all $x \in \mathcal{X}$.

Proof Putting $y = 0$ in (2.4), we have

$$\left\| \frac{1}{2}\left(\cos\left(\frac{m\pi}{2}\right)+m^2-1\right) f(2x)-4\left(\cos\left(\frac{m\pi}{2}\right)+m^2-1\right) f(x)\right\|_{\mathcal{Y}} \leq \alpha+\phi(x, 0)$$

for all $x \in \mathcal{X}$. Thus

$$\left\|\frac{1}{8}f(2x) - f(x)\right\|_{\mathcal{Y}} \leq \frac{\alpha}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)} + \frac{\phi(x, 0)}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)} \tag{2.6}$$

for all $x \in \mathcal{X}$. Replacing x by $2x$ in (2.6) and continuing this method, we get

$$\left\|\frac{f(2^n x)}{8^n} - f(x)\right\|_{\mathcal{Y}} \leq \frac{1}{4}\sum_{k=0}^{n-1} \frac{\alpha}{8^k(\cos(\frac{m\pi}{2})+m^2-1)} + \frac{1}{4}\sum_{k=0}^{n-1} \frac{\phi(2^k x, 0)}{8^k(\cos(\frac{m\pi}{2})+m^2-1)}. \tag{2.7}$$

On the other hand, we can use induction to obtain

$$\left\|\frac{f(2^n x)}{8^n} - \frac{f(2^m x)}{8^m}\right\|_{\mathcal{Y}} \leq \frac{1}{4}\sum_{k=m}^{n-1} \frac{\alpha}{8^k(\cos(\frac{m\pi}{2})+m^2-1)} + \frac{1}{4}\sum_{k=m}^{n-1} \frac{\phi(2^k x, 0)}{8^k(\cos(\frac{m\pi}{2})+m^2-1)} \tag{2.8}$$

for all $x \in \mathcal{X}$, and $n > m \geq 0$. Thus the sequence $\left\{\frac{f(2^n x)}{8^n}\right\}$ is Cauchy by (2.3) and (2.8). Completeness of $\mathcal{Y}$ allows us to assume that there exists a mapping $\mathcal{T}$ so that

$$\lim_{n\to\infty} \frac{f(2^n x)}{8^n} = \mathcal{T}(x). \tag{2.9}$$

Taking the limit as n tends to infinity in (2.7) and applying (2.9), we can see that the inequality (2.5) holds. Now, we replace x, y by $2^n x, 2^n y$, respectively in (2.4), then

$$\frac{1}{8^n}\left\|\mathcal{D}_m f(2^n x, 2^n y)\right\|_{\mathcal{Y}} \leq \frac{1}{8^n}\alpha + \frac{\phi(2^n x, 2^n y)}{8^n} \qquad (x, y \in \mathcal{X}).$$

Letting the limit as $n \to \infty$, we obtain $\mathcal{D}_m \mathcal{T}(x, y) = 0$ for all positive integers $m \geq 2$ and all $x, y \in \mathcal{X}$. Hence, by Theorem 2.1, it indicates that $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{Y}$ is a cubic mapping. Now, let $\mathcal{T}' : \mathcal{X} \longrightarrow \mathcal{Y}$ be another cubic mapping satisfying (2.5). Then we have

$$\begin{aligned}
\left\|\mathcal{T}(x) - \mathcal{T}'(x)\right\|_{\mathcal{Y}} &= \frac{1}{8^n}\|\mathcal{T}(2^n x) - \mathcal{T}'(2^n x)\|_{\mathcal{Y}} \\
&\leq \frac{1}{8^n}(\|\mathcal{T}(2^n x) - f(2^n x)\|_{\mathcal{Y}} + \|f(2^n x) - \mathcal{T}'(2^n x)\|_{\mathcal{Y}}) \\
&\leq \frac{1}{8^n}\left[\frac{4\alpha}{7(\cos(\frac{m\pi}{2}) + m^2 - 1)} + \frac{\widetilde{\phi}(x)}{2(\cos(\frac{m\pi}{2}) + m^2 - 1)}\right] \\
&= \frac{4\alpha}{7(\cos(\frac{m\pi}{2}) + m^2 - 1)8^n} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{8^{n+k}}\phi(2^{n+k}x, 0) \\
&= \frac{4\alpha}{7(\cos(\frac{m\pi}{2}) + m^2 - 1)8^n} + \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{8^k}\phi(2^k x, 0)
\end{aligned}$$

for all $x \in \mathcal{X}$. We immediately find the uniqueness of $\mathcal{T}$ by the preceding inequality as $n \to \infty$. This completes the proof. □

Corollary 2.3 *Let α, β, γ, r and s be nonnegative real numbers such that $s > 0$ and $r, s < 3$. Suppose that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping fulfilling*

$$\|\mathcal{D}_m f(x, y)\|_{\mathcal{Y}} \leq \alpha + \beta\|x\|_{\mathcal{X}}^r + \gamma\|y\|_{\mathcal{X}}^s \tag{2.10}$$

for all $x, y \in \mathcal{X}$, where m is an integer with $m \geq 2$. Then there exists a unique cubic mapping $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\|f(x) - \mathcal{T}(x)\|_{\mathcal{Y}} \leq \frac{2\alpha}{7(\cos(\frac{m\pi}{2}) + m^2 - 1)} + \frac{2\beta}{(\cos(\frac{m\pi}{2}) + m^2 - 1)(8 - 2^r)}\|x\|_{\mathcal{X}}^r, \tag{2.11}$$

for all $x \in \mathcal{X}$ and all $x \in \mathcal{X}\backslash\{0\}$ if $r < 0$.

Proof The result follows from Theorem 2.2 by setting $\phi(x, y) = \beta\|x\|_{\mathcal{X}}^r + \gamma\|y\|_{\mathcal{X}}^s$. □

We have the following result which is analogous to Theorem 2.2 for cubic functional equations. We bring its proof.

Theorem 2.4 *Suppose that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping for which there exists a function $\phi : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ such that*

$$\widetilde{\phi}(x) := \sum_{k=1}^{\infty} 8^k \phi\left(\frac{x}{2^k}, 0\right) < \infty, \lim_{k\to\infty} 8^k \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) = 0 \tag{2.12}$$

and

$$\|\mathcal{D}_m f(x, y)\|_{\mathcal{Y}} \leq \phi(x, y) \tag{2.13}$$

for all $x, y \in \mathcal{X}$, where m is an integer with $m \geq 2$. Then there exists a unique cubic mapping $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\|f(x) - \mathcal{T}(x)\|_{\mathcal{Y}} \leq \frac{\widetilde{\phi}(x)}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)} \tag{2.14}$$

for all $x \in \mathcal{X}$.

Proof Putting $y = 0$ in (2.13), we have

$$\left\|\frac{1}{2}\left(\cos\left(\frac{m\pi}{2}\right) + m^2 - 1\right) f(2x) - 4\left(\cos\left(\frac{m\pi}{2}\right) + m^2 - 1\right) f(x)\right\|_{\mathcal{Y}} \leq \phi(x, 0)$$

for all $x \in \mathcal{X}$. If we replace x by $\frac{x}{2}$ in the above inequality and divide both sides by $(\cos(\frac{m\pi}{2}) + m^2 - 1)$, we get

$$\left\|f(x) - 8f\left(\frac{x}{2}\right)\right\|_{\mathcal{Y}} \leq \frac{2}{(\cos(\frac{m\pi}{2}) + m^2 - 1)}\phi\left(\frac{x}{2}, 0\right).$$

Using triangular inequality and proceeding this way, we obtain

$$\left\|8^n f\left(\frac{x}{2^n}\right) - f(x)\right\|_{\mathcal{Y}} \leq \frac{1}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)} \sum_{k=1}^{n} 8^k \phi\left(\frac{x}{2^k}, 0\right) \tag{2.15}$$

for all $x \in \mathcal{X}$. If we show that the sequence $\left\{8^n f(\frac{x}{2^n})\right\}$ is Cauchy, then it will be convergent by the completeness of $\mathcal{Y}$. For this, we replace x by $\frac{x}{2^m}$ in (2.15) and then multiply both side by 8^m, we get

$$\begin{aligned}\left\|8^{m+n} f\left(\frac{x}{2^n}\right) - 8^m f\left(\frac{x}{2^m}\right)\right\|_{\mathcal{Y}} &\leq \frac{1}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)} \sum_{k=1}^{n} 8^{k+m} \phi\left(\frac{x}{2^{k+m}}, 0\right) \\ &= \frac{1}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)} \sum_{k=m+1}^{m+n} 8^k \phi\left(\frac{x}{2^k}, 0\right)\end{aligned}$$

for all $x \in \mathcal{X}$, and $n > m > 0$. Thus the mentioned sequence is convergent to the mapping $\mathcal{T}$, i.e.,

$$\mathcal{T}(a) := \lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right).$$

Now, similar to the proof of Theorem 2.2, we can complete the rest of the proof. □

Corollary 2.5 *Let β, γ, r and s be nonnegative real numbers such that $r, s \in (3, \infty)$. Suppose that $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping fulfilling*

$$\|\mathcal{D}_m f(x, y)\|_{\mathcal{Y}} \leq \beta\|x\|_{\mathcal{X}}^r + \gamma\|y\|_{\mathcal{X}}^s \tag{2.16}$$

for all $x, y \in \mathcal{X}$, where m is an integer with $m \geq 2$. Then there exists a unique cubic mapping $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{Y}$ such that

$$\|f(x) - \mathcal{T}(x)\|_{\mathcal{Y}} \leq \frac{\beta}{2(\cos(\frac{m\pi}{2}) + m^2 - 1)(2^r - 8)}\|x\|_{\mathcal{X}}^r \tag{2.17}$$

for all $x \in \mathcal{X}$.

Proof Taking $\phi(x, y) = \beta\|x\|_{\mathcal{X}}^r + \gamma\|y\|_{\mathcal{X}}^s$ in Theorem 2.4, we can obtain the desired result. □

3 A fixed point approach

In this section, we will investigate the stability of the given cubic functional equation (1.3) by using a fixed point theorem. First, we bring this theorem which is proved in [10]. This result plays a fundamental role to achieve our purpose in this section.

Theorem 3.1 (The fixed point alternative theorem) *Let (Δ, d) be a complete generalized metric space and $\mathcal{J} : \Delta \longrightarrow \Delta$ be a mapping with Lipschitz constant $L < 1$. Then, for each element $\alpha \in \Delta$, either $d(\mathcal{J}^n\alpha, \mathcal{J}^{n+1}\alpha) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that*

(i) *$d(\mathcal{J}^n\alpha, \mathcal{J}^{n+1}\alpha) < \infty$ for all $n \geq n_0$;*
(ii) *the sequence $\{\mathcal{J}^n\alpha\}$ is convergent to a fixed point β^* of $\mathcal{J}$;*
(iii) *β^* is the unique fixed point of $\mathcal{J}$ in the set $\Delta_1 = \{\beta \in \Delta : d(\mathcal{J}^{n_0}\alpha, \beta) < \infty\}$;*
(iv) *$d(\beta, \beta^*) \leq \frac{1}{1-L}d(\beta, \mathcal{J}\beta)$ for all $\beta \in \Delta_1$.*

Theorem 3.2 *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a mapping with $f(0) = 0$ and let $\varphi : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ be a function such that*

$$\|\mathcal{D}_m f(x, y)\|_{\mathcal{Y}} \leq \varphi(x, y) \tag{3.1}$$

for all $x, y \in \mathcal{X}$, where m is an integer with $m \geq 2$. If there exists a constant $M \in (0, 1)$, such that

$$\varphi(2x, 2y) \leq 8M\varphi(x, y) \tag{3.2}$$

for all $x, y \in \mathcal{X}$, *then there exists a unique cubic mapping* $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{Y}$ *such that*

$$\|f(x) - \mathcal{T}(x)\|_{\mathcal{Y}} \leq \frac{1}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)(1 - M)}\varphi(x, 0) \tag{3.3}$$

for all $x \in \mathcal{X}$.

Proof We wish to make the conditions of Theorem 3.1. We consider the set

$$\Delta = \{g : \mathcal{X} \longrightarrow \mathcal{Y} |\ g(0) = 0\}$$

and define the mapping $\mathfrak{D}$ on $\Delta \times \Delta$ as follows:

$$\mathfrak{D}(g, h) := \inf\{C \in (0, \infty) : \|g(x) - h(x)\|_{\mathcal{Y}} \leq C\varphi(x, 0), \quad (\forall x \in \mathcal{X})\},$$

if there exists such constant C, and $\mathfrak{D}(g, h) = \infty$, otherwise. Easily, we can see that $\mathfrak{D}(g, g) = 0$ and $\mathfrak{D}(g, h) = \mathfrak{D}(h, g)$, for all $g, h \in \Delta$. Also, for each $g, h, k \in \Delta$, we have

$$\begin{aligned} &\inf\{C \in (0, \infty) : \|g(x) - h(x)\|_{\mathcal{Y}} \leq C\varphi(x) \text{ for all } x \in \mathcal{X}\} \\ &\quad \leq \inf\{C \in (0, \infty) : \|g(x) - k(x)\|_{\mathcal{Y}} \leq C\varphi(x) \text{ for all } x \in \mathcal{X}\} \\ &\qquad + \inf\{C \in (0, \infty) : \|h(x) - k(x)\|_{\mathcal{Y}} \leq C\varphi(x) \text{ for all } x \in \mathcal{X}\}. \end{aligned}$$

Hence $\mathfrak{D}(g, h) \leq \mathfrak{D}(g, k) + \mathfrak{D}(k, h)$. If $\mathfrak{D}(g, h) = 0$, then for every fixed $x_0 \in \mathcal{X}$ we have $\|g(x_0) - h(x_0)\|_{\mathcal{Y}} \leq C\varphi(x_0)$ for all $C > 0$. This implies $g = h$. Let $\{g_n\}$ be a $\mathfrak{D}$-Cauchy sequence in Δ. Then $\mathfrak{D}(g_p, g_n) \to 0$ for all $n, p \in \mathbb{N}$. So $\|g_n(x) - g_p(x)\|_{\mathcal{Y}} \longrightarrow 0$ for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, there exists $g \in \Delta$ such that $g_n \xrightarrow{\mathfrak{D}} g$ in Δ. Therefore $\mathfrak{D}$ is a generalized metric on Δ and the metric space $(\Delta, \mathfrak{D})$ is complete. Here, we define the mapping $\mathcal{J} : \Delta \longrightarrow \Delta$ via

$$\mathcal{J}h(x) = \frac{1}{8}h(2x), \quad (x \in \mathcal{X}). \tag{3.4}$$

If $g, h \in \Delta$ such that $\mathfrak{D}(g, h) < C$, by definitions of $\mathfrak{D}$ and $\mathcal{J}$, we have

$$\left\| \frac{1}{8}g(2x) - \frac{1}{8}h(2x) \right\|_{\mathcal{Y}} \leq \frac{1}{8}C\varphi(2x, 0)$$

for all $x \in \mathcal{X}$. Using (3.2), we get

$$\left\| \frac{1}{8}g(2x) - \frac{1}{8}h(2x) \right\|_{\mathcal{Y}} \leq CM\varphi(x, 0)$$

for all $x \in \mathcal{X}$. The above inequality shows that $\mathfrak{D}(\mathcal{J}g, \mathcal{J}h) \leq M\mathfrak{D}(g, h)$ for all $g, h \in \Delta$. Hence, $\mathcal{J}$ is a strictly contractive mapping on Δ with a Lipschitz constant M. We now show that $\mathfrak{D}(\mathcal{J}f, f) < \infty$. Putting $y = 0$ in (3.1), we obtain

$$\left\|\frac{1}{8}f(2x)-f(x)\right\|_{\mathcal{Y}}\leq\frac{\varphi(x,0)}{4(\cos(\frac{m\pi}{2})+m^2-1)}$$

for all $x\in\mathcal{X}$. We conclude from the last inequality that

$$\mathfrak{D}(\mathcal{J}f,f)\leq\frac{1}{4(\cos(\frac{m\pi}{2})+m^2-1)}.$$

Theorem 3.1 implies that $\mathfrak{D}(\mathcal{J}^n g,\mathcal{J}^{n+1}g)<\infty$ for all $n\geq 0$, and thus in this theorem we have $n_0=0$. Consequently, the parts (iii) and (iv) of Theorem 3.1 hold on the whole Δ. Hence there exists a unique mapping $\mathcal{T}:\mathcal{X}\longrightarrow\mathcal{Y}$ such that $\mathcal{T}$ is a fixed point of $\mathcal{J}$ and that $\mathcal{J}^n f\to\mathcal{T}$ as $n\to\infty$. Thus

$$\lim_{n\to\infty}\frac{f(2^n x)}{8^n}=\mathcal{T}(x) \tag{3.5}$$

for all $x\in\mathcal{X}$, and so

$$d(f,\mathcal{T})\leq\frac{1}{1-M}d(\mathcal{J}f,f)\leq\frac{1}{4(\cos(\frac{m\pi}{2})+m^2-1)(1-M)}.$$

The above inequalities show that (3.3) is true for all $x\in\mathcal{X}$. Now, it follows from (3.2) that

$$\lim_{n\to\infty}\frac{\varphi(2^n x,2^n x)}{8^n}=0. \tag{3.6}$$

Substituting x and y by $2^n x$ and $2^n y$ respectively in (3.1), we get

$$\frac{1}{8^n}\|\mathcal{D}_m f(2^n x,2^n y)\|_{\mathcal{Y}}\leq\frac{\varphi(2^n x,2^n y)}{8^n}.$$

Taking the limit as $n\to\infty$, we obtain $\mathcal{D}_m\mathcal{T}(x,y)=0$ for all $m\geq 2$ for all $x,y\in\mathcal{X}$. It follows from Theorem 2.1 that $\mathcal{T}:\mathcal{X}\longrightarrow\mathcal{Y}$ is a cubic mapping which is unique. □

Corollary 3.3 *Let α,β,r and s be the nonnegative real numbers with $r,s<3$ and let $f:\mathcal{X}\longrightarrow\mathcal{Y}$ be a mapping such that*

$$\|\mathcal{D}_m f(x,y)\|_{\mathcal{Y}}\leq\alpha\|x\|_{\mathcal{X}}^r+\beta\|y\|_{\mathcal{X}}^s \tag{3.7}$$

for all $x,y\in\mathcal{X}$. Then there exists a unique cubic mapping $\mathcal{T}:\mathcal{X}\longrightarrow\mathcal{Y}$ satisfying

$$\|f(x)-\mathcal{T}(x)\|_{\mathcal{Y}}\leq\frac{2\alpha}{(\cos(\frac{m\pi}{2})+m^2-1)(8-2^r)}\|x\|_{\mathcal{X}}^r$$

for all $x\in\mathcal{X}$.

Proof Note that the inequality (3.7) implies that $f(0) = 0$. If we put $\varphi(x, y) = \alpha\|x\|_{\mathcal{X}}^r + \beta\|y\|_{\mathcal{X}}^r$ in Theorem 3.2, we obtain the desired result. □

In the upcoming result, we prove the superstability of cubic functional equations under some conditions.

Corollary 3.4 *Let r, α be nonnegative real numbers and s be a positive number with $r + s \neq 3$. If $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a mapping such that*

$$\|\mathcal{D}_m f(x, y)\|_{\mathcal{Y}} \leq \alpha\|x\|_{\mathcal{X}}^r\|y\|_{\mathcal{X}}^s \tag{3.8}$$

for all $x, y \in \mathcal{X}$, then f is a cubic mapping on $\mathcal{X}$.

Proof Putting $x = y = 0$ in (3.8), we get $f(0) = 0$. Again, if we put $y = 0$ in (3.8), then we have $f(2x) = 8f(x)$ for all $x \in \mathcal{X}$. It is easy to check that $f(2^n x) = 8^n f(x)$, and so $f(x) = \frac{f(2^n x)}{8^n}$ for all $x \in \mathcal{X}$ and $n \in \mathbb{N}$. Now, it follows from Theorem 3.2 that f is a cubic mapping when $\varphi(x, y) = \alpha\|x\|_{\mathcal{X}}^r\|y\|_{\mathcal{X}}^s$. □

4 Stability of cubic Jordan ∗-derivations

In this section, we prove the Hyers–Ulam stability of cubic Jordan ∗-derivations on real C^*-algebras and real JC^*-algebras.

Definition 4.1 Let $\mathcal{A}$ be a real C^*-algebra. A mapping $\mathfrak{D} : \mathcal{A} \longrightarrow \mathcal{A}$ is called a *cubic Jordan ∗-derivation* if $\mathfrak{D}$ is a cubic $\mathbb{R}$-homogeneous mapping, i.e., $\mathfrak{D}$ is cubic and $\mathfrak{D}(\lambda a) = \lambda^3\mathfrak{D}(a)$ for all $a \in \mathcal{A}$, $\lambda \in \mathbb{R}$, and

$$\mathfrak{D}(a^2) = a^3\mathfrak{D}(a) + \mathfrak{D}(a)(a^*)^3$$

for all $a \in \mathcal{A}$.

The mapping $\mathfrak{D}_x : \mathcal{A} \longrightarrow \mathcal{A}; a \mapsto a^3x - x(a^*)^3$ is a cubic Jordan ∗-derivation where x is a fixed element in $\mathcal{A}$.

A real C^*-algebra $\mathcal{A}$, endowed with the Jordan product $a \circ b := \frac{ab+ba}{2}$ on $\mathcal{A}$, is called a real *JC^*-algebra* (see [1,21]).

Definition 4.2 Let $\mathcal{A}$ be a real JC^*-algebra. A mapping $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is called a *cubic Jordan ∗-derivation* if δ is a cubic $\mathbb{R}$-homogeneous mapping and

$$\delta(a^2) = a^3 \circ \delta(a) + \delta(a) \circ (a^*)^3$$

for all $a \in \mathcal{A}$.

Theorem 4.3 *Let $\mathcal{A}$ be a real C^*-algebra. Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exists a function $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ such that*

$$\tilde{\phi}(x, y) := \sum_{k=0}^{\infty} \frac{1}{2^{3k}}\phi(2^k x, 2^k y) < \infty,$$

$$\|\mathcal{D}_m f(x, y)\| \leq \phi(x, y) \tag{4.1}$$

and

$$\|f(x^2) - x^3 f(x) - f(x)(x^*)^3\| \le \phi(x, x) \tag{4.2}$$

for all $x, y \in \mathcal{A}$, where m is an integer with $m \ge 2$. Also, if for each fixed $x \in \mathcal{A}$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathcal{A}$ is continuous, then there exists a unique cubic Jordan $*$-derivation $\mathfrak{D}$ on $\mathcal{A}$ such that

$$\|f(x) - \mathfrak{D}(x)\| \le \frac{\tilde{\phi}(x, 0)}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)} \tag{4.3}$$

for all $x \in \mathcal{A}$.

Proof We consider the mapping $\mathfrak{D}(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^{3n}}$ for all $x \in \mathcal{A}$. Similar to the proof of Theorem 2.2, we can prove that the inequality (4.3) holds in which $\mathfrak{D}$ is unique. Suppose that $\mathcal{F}$ is any continuous linear functional on $\mathcal{A}$ and x is a fixed element in $\mathcal{A}$. Define the mapping $h : \mathbb{R} \to \mathbb{R}$ via $h(t) = \mathcal{F}[\mathfrak{D}(tx)]$ for each $t \in \mathbb{R}$. It is easy to see that h is a cubic mapping. Under the hypothesis that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, the mapping h is the pointwise limit of the sequence of continuous mappings $\{h_n\}$ in which $h_n(t) = \frac{\mathcal{F}(2^n tx)}{2^{3n}}$, where $n \in \mathbb{N}$ and $t \in \mathbb{R}$. So, h is a continuous mapping and has the form $h(t) = t^3 h(1)$ for all $t \in \mathbb{R}$. Therefore

$$\mathcal{F}[\mathfrak{D}(tx)] = h(t) = t^3 h(1) = t^3 \mathcal{F}[\mathfrak{D}(x)] = \mathcal{F}[t^3 \mathfrak{D}(x)].$$

Since $\mathcal{F}$ is an arbitrary continuous linear functional on $\mathcal{A}$, we have $\mathfrak{D}(tx) = t^3 \mathfrak{D}(x)$ for all $t \in \mathbb{R}$ and $x \in \mathcal{A}$. Now, replacing x by $2^n x$, in (4.2), we get

$$\left\| \frac{f(2^n x \cdot 2^n x)}{2^{6n}} - \frac{x^3 f(2^n x)}{2^{3n}} - \frac{f(2^n x)(x^*)^3}{2^{3n}} \right\| \le \frac{\phi(2^n x, 2^n x)}{2^{6n}} \le \frac{\phi(2^n x, 2^n x)}{2^{3n}}$$

for all $x \in \mathcal{A}$. Thus we have

$$\|\mathfrak{D}(x^2) - x^3 \mathfrak{D}(x) - \mathfrak{D}(x)(x^*)^3\| \le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n x)}{2^{3n}} = 0.$$

Therefore $\mathfrak{D}$ is a cubic Jordan $*$-derivation on $\mathcal{A}$, as required. □

The following theorem is analogous to Theorem 4.3 for cubic Jordan $*$-derivations. Since the proof is similar, it is omitted.

Theorem 4.4 *Let $\mathcal{A}$ be a real C^*-algebra. Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exists a function $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ satisfying (4.1), (4.2) and*

$$\tilde{\phi}(x, y) := \sum_{k=1}^{\infty} 2^{3k} \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) < \infty$$

for all $x, y \in \mathcal{A}$, where m is an integer with $m \geq 2$. Also, if for each fixed $x \in \mathcal{A}$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathcal{A}$ is continuous, then there exists a unique cubic Jordan $*$-derivation $\mathfrak{D}$ on $\mathcal{A}$ such that

$$\|f(a) - \mathfrak{D}(x)\| \leq \frac{\tilde{\phi}(x,0)}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)}$$

for all $x \in \mathcal{A}$.

Corollary 4.5 *Let $\mathcal{A}$ be a real C^*-algebra and α, r be positive real numbers with $r \neq 3$. Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping such that*

$$\|\mathcal{D}_m f(x,y)\| \leq \alpha(\|x\|^r + \|y\|^r)$$

and

$$\|f(x^2) - x^3 f(x) - f(x)(x^*)^3\| \leq 2\alpha\|x\|^r$$

for all $x, y \in \mathcal{A}$, where m is an integer with $m \geq 2$. Also, if for each fixed $x \in \mathcal{A}$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathcal{A}$ is continuous, then there exists a unique cubic Jordan $*$-derivation $\mathfrak{D}$ on $\mathfrak{A}$ satisfying

$$\|f(x) - \mathfrak{D}(x)\| \leq \frac{\alpha}{(\cos(\frac{m\pi}{2}) + m^2 - 1)|4 - 2^{r-1}|} \;\|x\|^r \tag{4.4}$$

for all $x \in \mathcal{A}$.

Proof Putting $\phi(a,b) = \alpha(\|x\|^r + \|y\|^r)$ and applying Theorems 4.3 and 4.4, we get the inequality (4.4). □

Here and subsequently, we assume that $\mathcal{A}$ is a real JC^*-algebra. We show the Hyers–Ulam stability of cubic Jordan $*$-derivations on $\mathcal{A}$.

Theorem 4.6 *Suppose that $f : \mathcal{A} \longrightarrow \mathcal{A}$ is a mapping for which there exists a function $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ such that*

$$\tilde{\phi}(x,y) := \sum_{k=0}^{\infty} \frac{1}{2^{3k}} \phi(2^k x, 2^k y) < \infty,$$
$$\|\mathcal{D}_m f(x,y)\| \leq \phi(x,y) \tag{4.5}$$

and

$$\|f(x^2) - x^3 \circ f(x) - f(x) \circ (x^*)^3\| \leq \phi(x,x) \tag{4.6}$$

for all $x, y \in \mathcal{A}$, where m is an integer with $m \geq 2$. Also, if for each fixed $x \in \mathcal{A}$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to $\mathcal{A}$ is continuous, then there exists a unique cubic Jordan $*$-derivation $\mathfrak{D}$ on $\mathcal{A}$ satisfying

$$\|f(x) - \mathfrak{D}(x)\| \leq \frac{\tilde{\phi}(x, 0)}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)}$$

for all $x \in \mathcal{A}$.

Proof The proof is similar to the proof of Theorem 4.3. □

Similar to Theorem 4.4, we have the following result and we omit its proof.

Theorem 4.7 *Suppose that* $f : \mathcal{A} \longrightarrow \mathcal{A}$ *is a mapping for which there exists a function* $\phi : \mathcal{A} \times \mathcal{A} \longrightarrow [0, \infty)$ *satisfying* (4.5), (4.6) *and*

$$\tilde{\phi}(x, y) := \sum_{k=1}^{\infty} 2^{3k} \phi\left(\frac{x}{2^k}, \frac{y}{2^k}\right) < \infty$$

for all $x, y \in \mathcal{A}$, *where* m *is an integer with* $m \geq 2$. *Also, if for each fixed* $x \in \mathcal{A}$ *the mapping* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *to* $\mathcal{A}$ *is continuous, then there exists a unique cubic Jordan* $*$*-derivation* $\mathfrak{D}$ *on* $\mathcal{A}$ *satisfying*

$$\|f(a) - \mathfrak{D}(x)\| \leq \frac{\tilde{\phi}(x, 0)}{4(\cos(\frac{m\pi}{2}) + m^2 - 1)}$$

for all $x \in \mathcal{A}$.

Corollary 4.8 *Let* α, r *be positive real numbers with* $r \neq 3$. *Suppose that* $f : \mathcal{A} \longrightarrow \mathcal{A}$ *is a mapping such that*

$$\|\mathcal{D}_m f(x, y)\| \leq \alpha(\|x\|^r + \|y\|^r)$$

and

$$\|f(x^2) - x^3 \circ f(x) - f(x) \circ (x^*)^3\| \leq 2\alpha\|x\|^r$$

for all $x, y \in \mathcal{A}$, *where* m *is an integer with* $m \geq 2$. *Also, if for each fixed* $x \in \mathcal{A}$ *the mapping* $t \mapsto f(tx)$ *from* $\mathbb{R}$ *to* $\mathcal{A}$ *is continuous, then there exists a unique cubic Jordan* $*$*-derivation* $\mathfrak{D}$ *on* $\mathfrak{A}$ *satisfying*

$$\|f(x) - \mathfrak{D}(x)\| \leq \frac{\alpha}{(\cos(\frac{m\pi}{2}) + m^2 - 1)|4 - 2^{r-1}|} \|x\|^r$$

for all $x \in \mathcal{A}$.

Proof We can obtain the result by letting $\phi(x, y) = \alpha(\|x\|^r + \|y\|^r)$ in Theorems 4.6 and 4.7. □

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