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A fixed point theorem for contractions of rational type in partially ordered metric spaces

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Abstract The purpose of this paper is to present a fixed point theorem due to Dass and Gupta (Indian J Pure Appl Math 6:1455–1458, 1975) in the context of partially ordered metric spaces.

Keywords Fixed point · Partially ordered set

Mathematics Subject Classification 47H10

1 Introduction

In [1], Dass and Gupta proved the following fixed point theorem.

Theorem 1 *Let (X, d) be a complete metric space and $T : X \to X$ a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx, Ty) \leq \frac{\alpha\, d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta\, d(x, y)$$

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for all $x, y \in X$.

Then T *has a unique fixed point.*

The aim of this paper is to give a version of Theorem 1 in the context of partially ordered metric spaces.

Existence of fixed point in partially ordered metric spaces has been considered recently by many authors (see, [2–13], for example).

2 Main result

Definition 1 Let $(X, \leq)$ be a partially ordered set and $T\colon X \to X$. T is said to be a nondecreasing mapping if for $x, y \in X$

$$x \leq y \Rightarrow Tx \leq Ty.$$

Theorem 2 *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space. Let* $T\colon X \to X$ *be a continuous and nondecreasing mapping such that there exists* $\alpha, \beta \geq 0$ *with* $\alpha + \beta < 1$ *satisfying*

$$d(Tx, Ty) \leq \frac{\alpha\, d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta\, d(x, y) \quad \text{for } x, y \in X \text{ with } x \leq y. \tag{1}$$

If there exist $x_0 \in X$ *such that* $x_0 \leq Tx_0$ *then* T *has a fixed point.*

Proof If $Tx_0 = x_0$ then the proof is finished.

Suppose that $x_0 < Tx_0$. Since T is a nondecreasing mapping, by using induction, we obtain

$$x_0 < Tx_0 \leq T^2 x_0 \leq \ldots \leq T^n x_0 \leq T^{n+1} x_0 \leq \ldots.$$

Put $x_{n+1} = Tx_n$.

If there exists $n \geq 1$ such that $x_{n+1} = x_n$ then $x_{n+1} = Tx_n = x_n$, and x_n is a fixed point of T and the proof is finished.

Suppose that $x_{n+1} \neq x_n$ for $n \geq 0$.

Since $x_{n+1} \leq x_n$ for any $n \in \mathbb{N}$, using the contractive condition (1), we have for $n \geq 1$

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \frac{\alpha\, d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)} + \beta\, d(x_{n-1}, x_n) \\ &= \frac{\alpha\, d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta\, d(x_{n-1}, x_n), \end{aligned}$$

and, the last inequality implies

$$(1 - \alpha)\, d(x_n, x_{n+1}) \leq \beta\, d(x_{n-1}, x_n) \quad \text{for any } n \in \mathbb{N},$$

or, equivalently,

$$d(x_n, x_{n+1}) \leq \frac{\beta}{1-\alpha} d(x_{n-1}, x_n) \quad \text{for any } n \in \mathbb{N}.$$

Using mathematical induction, we have

$$d(x_n, x_{n+1}) \leq \left(\frac{\beta}{1-\alpha}\right)^n d(x_0, x_1) \quad \text{for any } n \in \mathbb{N}.$$

Notice that $r = \dfrac{\beta}{1-\alpha} < 1$.

Moreover, for $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \ldots + d(x_{m-1}, x_m) \\ &\leq (r^n + \ldots + r^{m-1}) d(x_0, x_1) \\ &< \frac{r^n}{1-r} d(x_0, x_1). \end{aligned}$$

Letting $n, m \to \infty$ and, since $r < 1$, we obtain $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

This proves that (x_n) is a Cauchy sequence.

Since (X, d) is a complete metric space, $\lim_{n\to\infty} x_n = x$ for certain $x \in X$.

The continuity of T gives us

$$Tx = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x.$$

Therefore, x is a fixed point.

This finishes the proof. □

In the sequel, we will prove that Theorem 2 is still valid for T not necessarily continuous, assuming the following hypothesis in X:

$$\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \to x \text{ then } x_n \leq x \text{ for all } n \in \mathbb{N}. \tag{2}$$

Theorem 3 *If in Theorem 2 we replace the condition of continuity of T by (2), the same conclusion holds.*

Proof In fact, following the proof of Theorem 2, we only have to check that x is a fixed point.

Since (x_n) is a nondecreasing sequence in X and $x_n \to x$ then, by (2), we have $x_n \leq x$ for all $n \in \mathbb{N}$.

Using the contractive condition, for any $n \in \mathbb{N}$, we get

$$\begin{aligned} d(x_{n+1}, Tx) &= d(Tx_n, Tx) \\ &\leq \frac{\alpha\, d(x, Tx)[1 + d(x_n, Tx_n)]}{1 + d(x_n, x)} + \beta\, d(x_n, x) \\ &= \frac{\alpha\, d(x, Tx)[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x)} + \beta\, d(x_n, x)\,. \end{aligned}$$

Taking into account that if $x_n \to x$ then $d(x_n, x_{n+1}) \xrightarrow[n\to\infty]{} 0$, letting $n \to \infty$ in the last inequality, it follows

$$d(x, Tx) \leq \alpha\, d(x, Tx)$$

and since $\alpha < 1$, this is imposible unless that $d(x, Tx) = 0$ this proves that $Tx = x$.
Thus, the proof is complete. □

Now, we present an example where it can be appreciated that assumptions in Theorem 2 do not guarantee the uniqueness of the fixed point. This example appears in [9].

Example 1 Let $X = \{(1, 0)\ \ (0, 1)\} \subset \mathbb{R}^2$ and consider the usual order given by

$$(x, y) \leq (z, t) \Leftrightarrow x \leq z \text{ and } y \leq t.$$

$(X, \leq)$ is a partially ordered set whose different elements are not comparable. Besides, (X, d_2), where d_2 is the euclidean distance, is a complete metric space. The identity map $T(x, y) = (x, y)$ is obviously continuous and nondecreasing and the contractive condition appearing in Theorem 2 is satisfied since elements in X are only comparable to themselves. Moreover $(1, 0) \leq T(1, 0)$ and T has two fixed points.

In what follows, we present a sufficient condition for the uniqueness of the fixed point in Theorems 2 and 3.

The condition is:

$$\text{for } x, y \in X \quad \text{there exists a lower bound.} \tag{3}$$

Theorem 4 *Adding assumption* (3) *to the hypotheses of Theorem* 2 *(or Theorem* 3*) we obtain uniqueness of the fixed point.*

Proof Suppose that $x, y \in X$ are fixed point of T. We distinguish two cases:

Case 1: x and y are comparable.
Suppose $x \leq y$ (the same argument works for $y \leq x$). By using the contractive condition, we get

$$\begin{aligned} d(x, y) &= d(Tx, Ty) \\ &\leq \frac{\alpha\, d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta\, d(x, y) \\ &= \beta\, d(x, y)\,. \end{aligned}$$

Since $\beta < 1$, this is only posible when $d(x, y) = 0$. Thus $x = y$.
Case 2: x and y are not comparable.
By (3), there exists $z \in X$ with $z \leq x$ and $z \leq y$.
Since $z \leq x$, the nondecreasing character of T gives us

$$T^n z \leq T^n x = x \quad \text{for any } n \in \mathbb{N}.$$

By using the contractive condition, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d(T^n z, x) &= d(T^n z, T^n x) \\ &\leq \frac{\alpha\, d(T^{n-1}x, T^n x)[1 + d(T^{n-1}z, T^n z)]}{1 + d(T^{n-1}z, T^{n-1}x)} + \beta\, d(T^{n-1}z, T^{n-1}x) \\ &= \frac{\alpha\, d(x, x)[1 + d(T^{n-1}z, T^n z)]}{1 + d(T^{n-1}z, x)} + \beta\, d(T^{n-1}z, x) \\ &= \beta\, d(T^{n-1}z, x)\,. \end{aligned}$$

Using mathematical induction, we obtain

$$d(T^n z, x) \leq \beta^n\, d(z, x)$$

and, since $\beta < 1$, $\lim_{n\to\infty} d(T^n z, x) = 0$.
This means that $\lim_{n\to\infty} T^n z = x$.
Using a similar argument, we can get $\lim_{n\to\infty} T^n z = y$.
Finally, the uniqueness of the limit gives us $x = y$.
This finishes the proof. □

3 Some remarks

Remark 1 In [9] instead condition (3), the authors use the following weaker condition:

For $x, y \in X$ there exists $z \in X$ which is comparable to x and y. (4)

We have not been able to prove Theorem 4 using condition (4).

The reason is that the contractive condition appearing in Theorem 2 is not symmetric in the following sense, it adopts distinct forms depending $x \leq y$ or $y \leq x$.

Remark 2 If in Theorems 2, 3 and 4, we put $\alpha = 0$, then Theorems 2.1, 2.2 and 2.3 of [9] are obtained.

If in Theorems 2, 3 and 4, β is equal to zero, we have the following fixed point theorem in ordered metric spaces.

Theorem 5 *Let* $(X, \leq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space. Let* $T : X \to X$ *be a nondecreasing mapping such that there exists* $\alpha \in [0, 1)$ *satisfying*

$$d(Tx, Ty) \leq \frac{\alpha\, d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} \quad \textit{for } x, y \in X \textit{ with } x \leq y. \tag{5}$$

Suppose also that either T is continuous or X satisfies condition (2).

If there exist $x_0 \in X$ such that $x_0 \leq Tx_0$ then T has a fixed point.

Besides, if $(X, \leq)$ satisfies condition (3), then the fixed point is unique.

Remark 3 If in Theorems 2, 3 and 4, $\alpha = 0$ and $0 < \beta < \frac{1}{3}$ then the contractive condition (1) appearing in Theorem 2 implies that

$$d(Tx, Ty) \leq \gamma\, [d(x, Tx) + d(y, Ty)] \quad \text{for } x, y \in X \text{ with } x \leq y,$$

where $\gamma \in (0, \frac{1}{2})$ (this is a Kannan type condition).

In fact, since $\alpha = 0$ and $0 < \beta < \frac{1}{3}$, the condition (1) takes the form

$$d(Tx, Ty) \leq \beta\, d(x, y) \quad \text{for } x, y \in X \text{ with } x \leq y.$$

By using the triangular inequality, for $x, y \in X$ with $x \leq y$ we have

$$d(Tx, Ty) \leq \beta\, d(x, y) \leq \beta\, [d(x, Tx) + d(y, Ty)] + \beta\, d(Tx, Ty),$$

and from this inequality it follows

$$d(Tx, Ty) \leq \frac{\beta}{1 - \beta} [d(x, Tx) + d(y, Ty)].$$

From $0 < \beta < \frac{1}{3}$ it is easily proved that $\gamma = \frac{\beta}{1-\beta} \in \left(0, \frac{1}{2}\right)$.

Remark 4 If $\operatorname{diam} X \leq 1$ and $2\alpha + \beta < 1$ then the contractive condition (1) appearing in Theorem 2 implies the following Reich type condition:

$$d(Tx, Ty) \leq \alpha\, d(x, Tx) + \alpha\, d(y, Ty) + \beta\, d(x, y) \quad \text{for } x, y \in X \text{ with } x \leq y.$$

In fact, for $x, y \in X$ with $x \leq y$ we have

$$\begin{aligned} d(Tx, Ty) &\leq \frac{\alpha\, d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta\, d(x, y) \\ &\leq \alpha\, d(y, Ty)[1 + d(x, Tx)] + \beta\, d(x, y) \\ &\leq \alpha\, d(y, Ty) + \alpha\, d(y, Ty) d(x, Tx) + \beta\, d(x, y). \end{aligned}$$

Since $\operatorname{diam} X \leq 1$, $d(y, Ty) \leq 1$, and therefore, the last inequality implies that

$$d(Tx, Ty) \leq \alpha\, d(y, Ty) + \alpha\, d(x, Tx) + \beta\, d(x, y).$$

In the sequel, we present an example where Theorem 2 can be applied and it cannot be treated by Theorem 1.

Example 2 Let $X = \{(0, 1)\ (1, 0)\ (1, 1)\}$ and we consider in X the partial order given by $\mathcal{R} = \{(x, x) : x \in X\}$. Notice that elements in X are only comparable to themselves. Moreover, (X, d_2), where d_2 is the euclidean distance, is a complete metric space.

Let $T : X \to X$ be defined by $T(0, 1) = (1, 0)\ T(1, 0) = (0, 1)$ and $T(1, 1) = (1, 1)$.

Obviously, T is continuous and nondecreasing, and the contractive condition appearing in Theorem 2 is satisfied since elements in X are only comparable to themselves.

As $(1, 1) \leq T(1, 1)$, Theorem 2 says us that T has a fixed point [this fixed point is $(1, 1)$].

On the other hand, for $x = (0, 1),\ y = (1, 1)$, we have

$$\begin{aligned} d_2(Tx, Ty) &= d_2(x, y) = 1 \\ d_2(x, Tx) &= \sqrt{2} \\ d_2(y, Ty) &= 0 \end{aligned}$$

and the contractive condition appearing in Theorem 1 is not satisfied since

$$\begin{aligned} 1 = d_2(Tx, Ty) &\leq \frac{\alpha\, d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta\, d(x, y) \\ &= \frac{\alpha \cdot 0[1 + \sqrt{2}]}{1 + 1} + \beta \cdot 1 = \beta \cdot 1 = \beta. \end{aligned}$$

Therefore, this example cannot be treated by Theorem 1.

Moreover, notice that in this example we have uniqueness of the fixed point and $(X, \leq)$ does not satisfy condition (3). This proves that condition (3) is not necessary condition for the uniqueness of the fixed point.

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