Ann Univ Ferrara (2011) 57:109–120
DOI 10.1007/s11565-011-0116-6

Growth and approximation of generalized biaxially symmetric potentials in several complex variables

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Received: 13 February 2011 / Accepted: 25 February 2011 / Published online: 20 March 2011
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Abstract The paper deals with the aspect of the growth of entire functions of several complex variables. The growth of the entire functions with respect to each of the variables separately has been studied by defining partial order and partial type. Finally, we have studied the growth and polynomial approximation of entire function generalized biaxially symmetric potential with respect to each of the variables separately.

Keywords Partial order · Partial type · Transfinite diameter · Extremal function · Hypersurface · Lagrange interpolation polynomial

Mathematics Subject Classification (2010) 41D10

1 Introduction

Let E be bounded closed set in the space $\mathbb{C}^n$ of n complex variables $\tilde{z} = (z_1, z_2, \ldots, z_n)$ and $\|f\|_E = \sup\left\{|f(\tilde{z})| : \tilde{z} \in E\right\}$ for a function f defined and bounded on E.

Suppose B is a complex Banach space with the norm $\|.\|$. Let $f : \mathbb{C}^n \mapsto B$ be an entire function. As in [11], let us denote $A_v(E)$ the set of all polynomials p of degree $\leq v$ such that $\|p\|_E \leq 1$. Again, we define the extremal function [10]

$$\phi(\tilde{z}) \equiv \phi(\tilde{z}, E) = \lim_{v\to\infty} \left\{\sup\left\{|p(\tilde{z})|^{1/v} : p \in A_v(E)\right\}\right\}, \quad \tilde{z} \in \mathbb{C}^n.$$

This work was done in the memory of Prof. H.S. Kasana, Senior Associate, ICTP, Trieste, Italy.

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It follows from the above definition that $\phi(\tilde{z}) \geq 1$ for $\tilde{z} \in \mathbb{C}^n$ and $\phi(\tilde{z}) = 1$ on E. Write

$$M_E(\overline{r}, f) = \sup\left\{\|f(\tilde{z})\| : \tilde{z} \in E_{\overline{r}}\right\}, \quad \overline{r} = (r_1, r_2, \ldots, r_n) > (1, 1, \ldots, 1),$$
$$M(\overline{r}, f) = \sup\left\{\|f(\tilde{z})\| : \|\tilde{z}\| = \overline{r}\right\}, \quad \overline{r} > (0, 0, \ldots, 0),$$

where

$$E_{\overline{r}} = \left\{\tilde{z} \in \mathbb{C}^n : \phi(\tilde{z}, E) = \overline{r}\right\}, \quad \overline{r} > (1, 1, \ldots, 1)$$

such that $\phi(\tilde{z}, E)$ is locally bounded.

Moreover, if $E = B(\overline{1}) = \left\{\tilde{z} \in \mathbb{C}^n : \|\tilde{z}\| = \overline{1}\right\}$, then $M_E(\overline{r}, f) = M(\overline{r}, f)$.

The order and type of an entire function $f : \mathbb{C}^n \mapsto B$ will be defined as in case of $n = 1$.

Definition 1.1 We call $\rho \equiv \rho(f)$ the order of f if

$$\rho(f) = \limsup_{\overline{r}\to\infty} \frac{\log\log M(\overline{r}, f)}{\log \overline{r}}, \quad 0 \leq \rho(f) \leq \infty. \tag{1.1}$$

For $0 < \rho(f) < \infty$, we say that f is of the type $\sigma \equiv \sigma(f)$, if

$$\sigma(f) = \limsup_{\overline{r}\to\infty} \frac{\log M(\overline{r}, f)}{\overline{r}^{\rho}}. \tag{1.2}$$

If, $M(\overline{r}, f)$ is replaced by $M_E(\overline{r}, f)$, in (1.1) and (1.2), then we get ρ_E and σ_E, the E-order and E-type, respectively, of f. It is proved in [11] that ρ_E is independent of E.

The growth of a function $f : \mathbb{C}^n \mapsto B$ as determined by its maximum modulus function $M(\overline{r}, f)$, can be studied in several different ways. Thus, to measure the growth of f with respect to all the variables simultaneously, Winiarski [11] introduced the concept of E-order and E-type. For a comprehensive view of growth of f, the concept of system of associated orders, system of associated types and their hyper surfaces are also introduced in [11]. In some cases the characterization of growth parameter of function f with respect to all variables is not possible, so the growth estimation with respect to one variable, keeping others fixed is needed. In this paper, we have studied this aspect of the growth of entire functions in several complex variables. The growth of entire functions with respect to each of the variables, separately is studied by defining partial order of a function.

Let us define the partial order but first, let

$$\mathbb{R}^n_+ = \left\{(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n : r_i \geq 0, i = 1, 2, \ldots, n\right\}$$
$$\mathbb{I}^n = \left\{(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n : 0 \leq r_i < \infty, i = 1, 2, \ldots, n\right\}.$$

Definition 1.2 A set $E^* \subset \mathbb{R}^n$ is called a complete domain in $\mathbb{R}^n$, if together with $\bar{r}^0 = (r_1^0, r_2^0, \ldots, r_n^0) \in E$, it contains all points $\bar{r} = (r_1, r_2, \ldots, r_n)$ with $0 \le r_1 \le r_i^0$ and contains no point of the closure of its complement in $\mathbb{R}^n_+$.

Definition 1.3 A set $E^* \subset \mathbb{R}^n_+$ is said to be convex if for each pair of points X and Y in E^* the entire segment $\lambda X + \mu Y, \lambda, \mu \ge 0, \lambda + \mu = 1$ lies in E^*.

Definition 1.4 A function $\phi(\bar{r}) = \phi(r_1, r_2, \ldots, r_n)$ defined on $\mathbb{I}^n$ is said to be pluri-convex in $\log r_1, \ldots, \log r_n$ if $\phi(\bar{r})$ is a convex function of the variables $\log r_1, \ldots,$ $\log r_n$.

Let $f : \mathbb{C}^n \mapsto B$ be an entire function. If $\log M(\bar{r}, f)$ is pluri-convex function of $\log \bar{r}$, then f is said to have partial order $\overline{\rho}_i$ in the variable z_i, if for $\|z_j\| = r_j$ remaining fixed $j \ne 1$,

$$\overline{\rho}_i = \limsup_{r_i \to \infty} \frac{\log \log M(\bar{r}, f)}{\log r_i}. \tag{1.3}$$

We first show that $\overline{\rho}_i$ is well defined in the sense that its value given by the right-hand side of (1.3) does not depend on the values of other fixed variables. Consider the class $\mathcal{C}(E)$, consisting of all functions f defined and bounded on E for which the function $\log M(\bar{r}, f)$ is a pluri-convex function in $\log r_1, \ldots, \log r_n$.

Theorem 1.1 *Let $f \in \mathcal{C}(E)$. Then the partial $\overline{\rho}_i$ of f in the variable z_i, as given by* (1.3)*, is independent of the value of other fixed variables.*

Proof Let us assume that $i = n$. Set

$$\rho(r_1, \ldots, r_{n-1}) = \limsup_{r_n \to \infty} \frac{\log \log M(r_1, \ldots, r_{n-1}, r_n)}{\log r_n}.$$

Since the function $M(\bar{r}, f) \equiv M(r_1, \ldots, r_n)$ is monotonically increasing in each of the variables, the function $\rho(r_1, \ldots, r_n)$ is also monotonically increasing in each of the variables. Therefore, for every $\varepsilon > 0$, there exists $A_\varepsilon(r_1, \ldots, r_{n-1}) < \infty$ is such that $0 \le r_i < \infty, i = 1, 2, \ldots, n$,

$$\log M(\bar{r}, f) \le A_\varepsilon(r_1, \ldots, r_{n-1}) + r_n^{\rho(r_1, \ldots, r_{n-1}) + \varepsilon}. \tag{1.4}$$

In view of Definition 1.4, we have for every $\bar{t} = (t_1, \ldots, t_n)$ and $\bar{s} = (s_1, \ldots, s_n)$ in $\mathbb{I}^n_+$, and for all λ, μ with $0 \le \lambda, \mu \le 1$ and $\lambda + \mu = 1$,

$$\phi\left(1 - t_1^{-\lambda} s_1^{-\mu}, \ldots, 1 - t_n^{-\lambda} s_n^{-\mu}\right) \le \lambda\phi(t_1, \ldots, t_n) + \mu\phi(s_1, \ldots, s_n). \tag{1.5}$$

Since $f \in \mathcal{C}(E), \phi(\overline{r}) \equiv \log M(r_1, \ldots, r_n)$ satisfies (1.5). In (1.5), we set for $0 \le r_i \le r_i^{'} < \infty$,

$$t_i = r_i, \quad s_i = 1 - \left(\frac{r_i^{\lambda}}{r_i^{'}}\right)^{1/\mu}, \quad i = 1, 2, \ldots, n-1.$$
$$t_n = 1 - (r_n^{'})^{-1/\lambda}, \quad s_n = 0,$$

so that, on using (1.4), we get

$$\begin{aligned}
\log M\left(r_1^{'}, \ldots, r_{n-1}^{'}, r_n^{'}\right) &= \log M\left(1 - t_1^{-\lambda} s_1^{-\mu}, \ldots, 1 - t_n^{-\lambda} s_n^{-\mu}\right) \\
&\le \lambda \log M\left(r_1, \ldots, r_{n-1}, 1 - r_n^{'-1/\lambda}\right) \\
&\quad + \mu \log M\left(1 - \left(\frac{r_1^{\lambda}}{r_1^{'}}\right)^{1/\mu}, \ldots, 1 - \left(\frac{r_{n-1}^{\lambda}}{r_{n-1}^{'}}\right)^{1/\mu}, 0\right) \\
&\le A_{\varepsilon}(r_1, \ldots, r_{n-1}) + r_n^{'\partial(r_1, \ldots, r_{n-1}) + \varepsilon} \\
&\quad + \log M\left(1 - \left(\frac{r_1^{\lambda}}{r_1^{'}}\right)^{1/\mu}, \ldots, 1 - \left(\frac{r_{n-1}^{\lambda}}{r_{n-1}^{'}}\right)^{1/\mu}, 0\right).
\end{aligned}$$

Thus, for any λ satisfying $0 < \lambda < 1, \varepsilon > 0$ and $0 \le r_i, r_i^{'} < \infty$, we have

$$\limsup_{r_n^{'} \to \infty} \frac{\log \log M\left(r_1^{'}, \ldots, r_{n-1}^{'}, r_n^{'}\right)}{\log r_n^{'}} \le \frac{1}{\lambda}(\rho(r_1, \ldots, r_{n-1}) + \varepsilon).$$

Since $\varepsilon > 0$ and λ satisfying $0 < \lambda < 1$ are arbitrary, the above inequality gives

$$\rho\left(r_1^{'}, \ldots, r_{n-1}^{'}\right) \le \rho(r_1, \ldots, r_{n-1})$$

and since the numbers $r_i, r_i^{'}$ in $0 \le r_i, r_i^{'} < \infty$ are arbitrary, we get

$$\rho\left(r_1^{'}, \ldots, r_{n-1}^{'}\right) = \rho(r_1, \ldots, r_{n-1}).$$

There are simple relationships that exist between E-order and partial order ρ_i in one of the variables for a function $f \in \mathcal{C}(E)$.

Theorem 1.2 *Let $f \in \mathcal{C}(E)$ have E-order ρ_E, and let $\overline{\rho}_i$ denote the partial order of f with respect to the variable z_i, $i = 1 \ldots, n$. Then*

$$\rho_E \le \sum_{i=1}^{n} \overline{\rho}_i.$$

Proof Since $f \in \mathcal{C}(E)$, $\log M(r_1, \ldots, r_n) \equiv \log M(\overline{r}, f)$ is a convex function of the variables $\log M(r_1, \ldots, r_n)$, it follows that for $\lambda_i \geq 0$, $\lambda_1 + \cdots + \lambda_n = 1$ and $0 \leq t_{ij} < \infty$, $1 \leq i, j, \leq n$, we have the inequality

$$\log M\left(1 - t_{11}^{-\lambda_1} \ldots t_{1n}^{-\lambda_n}, \ldots, 1 - t_{n1}^{-\lambda_1} \ldots t_{nn}^{-\lambda_n}\right) \leq \sum_{i=1}^{n} \lambda_i \log M(t_{i1}, \ldots, t_{in}). \tag{1.6}$$

For $0 < r < \infty$, set

$$t_{i1} = 1 - r^{-1/\lambda_i} \quad \texttt{and} \quad t_{ij} = 0 \quad \texttt{for} \quad j \neq i.$$

Using inequality (1.6), we get

$$\log M(r, \ldots, r) \leq \sum_{i=1}^{n} \lambda_i \log M\left(0, \ldots, 0, 1 - r^{-1/\lambda_i}, 0, \ldots, 0\right).$$

Now, by definition of the partial order $\overline{\rho}_i$ with respect to one variable, for every $\varepsilon > 0$ and for every r in $0 < r < \infty$ there exists a constant $A(\varepsilon)$ such that

$$\log M(r, \ldots, r) \leq A(\varepsilon) + \sum_{i=1}^{n} r^{(\overline{\rho}_i + \varepsilon)/\lambda_i}.$$

Since $\lambda_i \geq 0$, and $\lambda_1 + \cdots + \lambda_n = 1$, we can choose, for $i = 1, 2, \ldots, n$,

$$\lambda_i = \frac{\overline{\rho}_i + \varepsilon}{\overline{\rho}_i + \cdots + \overline{\rho}_n + n\varepsilon},$$

so that the above estimate of $\log M(r, \ldots, r)$ becomes

$$\log M(r, \ldots, r) \leq A(\varepsilon) + nr^{\overline{\rho}_1 + \cdots + \overline{\rho}_n + n\varepsilon}.$$

Then, we have

$$\rho_E = \limsup_{r \to \infty} \frac{\log \log M(r, \ldots, r)}{\log r} \leq \overline{\rho}_1 + \cdots + \overline{\rho}_n.$$

For a function $f : \mathbb{C}^n \rightarrowtail B$, let $B_{\overline{\rho}} \equiv B_{\overline{\rho}}(f)$ be the set of all points $\overline{a} \in \mathbb{R}^n$ such that for $|\overline{r}| \to \infty$,

$$\log M(\overline{r}, f) < r_1^{a_1} + r_2^{a_2} + \cdots + r_n^{a_n}, \tag{1.7}$$

where $M(\overline{r}, f)$ is the maximum modulus of f defined as earlier. It is obvious that if $\overline{a}' \in B_{\overline{\rho}}$ the set $B_{\overline{\rho}}$ contains the entire hyper-octant $\overline{a} \in \mathbb{R}^n : a_i > \overline{a}_i'$, $i = 1, 2, \ldots, n$. The sets in $\mathbb{R}^n$ possessing the property are called octant-like.

It is also clear that if $\overline{a}^{'} \notin B_{\overline{\rho}}$, then any point $\overline{a} \in \mathbb{R}^n_+$ with $a_i \leq a_i^{'}$ is also not in $B_{\overline{\rho}}$. This hypersurface divides the hyper octant $\mathbb{R}^n_+$ into two parts, in one of which the inequality (1.7) holds, while in the other it is false. The hypersurface $S_{\overline{\rho}}$ characterizes the growth of the function $M(\overline{r}, f)$ and is called the hypersurface of associated order of f. If $\overline{\rho}_i > 0, i = 1, 2, \ldots, n$, then $\rho_1, \rho_2, \ldots, \rho_n$ forms a system of associated orders of f if and only if

$$\limsup_{|\overline{r}| \to \infty} \frac{\log \log M(\overline{r}, f)}{\log \left(r_1^{\rho_1} + \cdots + r_n^{\rho_n}\right)} = 1.$$

Theorem 1.3 *Let $S_{\overline{\rho}}(f)$ denote the hyper surface of associated orders of $f \in \mathcal{C}(E)$, and let $\overline{\rho}_i$ denote the partial order of f with respect to the variables z_i, $i = 1, 2, \ldots, n$. Then*

$$\overline{\rho}_i = \inf \rho_i,$$

where infimum is taken over all $\{\rho_1, \rho_2, \ldots, \rho_n\} \in S_{\overline{\rho}}(f)$.

Proof First, we may assume that $i = n$. By definition of the E-order ρ_E and the order $\overline{\rho}_n$ in the variables z_n, for any $\varepsilon > 0$ and r and r_n sufficiently large, we have

$$\log M(r, \ldots, r) < r^{\rho_E + \varepsilon},$$
$$\log M(0, \ldots, 0, r_n) < r_n^{\overline{\rho}_n + \varepsilon}.$$

Since $f \in \mathcal{C}(E)$, using the inequality (1.5) with $r = \max\{r_1, \ldots, r_n\}$, $t_i = 1 - r^{-1/\lambda}$, $s_i = 0$ for $i = 1, 2, \ldots, n-1$ and $t_n = 0$, $s_n = 1 - r_n^{-1/\mu}$ together with the monotonicity of the function $M(r_1, \ldots, r_n) \equiv M(\overline{r}, f)$ we get

$$\begin{aligned} \log M\,(r_1, \ldots, r_n) &\leq \log M(r, \ldots, r, r_n) \\ &\leq \lambda \log M\left(1 - r^{-1/\lambda}, \ldots 1 - r^{-1\lambda}, 0\right) \\ &\quad + \mu \log M\left(0, 0, \ldots, 0, 1 - r_n^{-1/\mu}, 0\right) \\ &< \lambda r^{(\rho_E + \varepsilon)\lambda} + \mu r_n^{(\overline{\rho}_n + \varepsilon)/\lambda} \\ &< \sum_{i=1}^{n-1} r_i^{(\rho_E + \varepsilon)\lambda} + r_n^{(\overline{\rho}_n + \varepsilon)\lambda}. \end{aligned} \tag{1.8}$$

So, by definition of the set $B_{\overline{\rho}}(f)$, note that

$$\left(\frac{\rho_E + \varepsilon}{\lambda}, \frac{\rho_E + \varepsilon}{\lambda}, \ldots, \frac{\rho_E + \varepsilon}{\lambda}, \frac{\overline{\rho}_n + \varepsilon}{\lambda}\right) \in B_{\overline{\rho}}(f).$$

But this implies that there exists a point $(\rho_1, \ldots, \rho_n)$ on the hypersurface $S_{\overline{\rho}}(f)$ such that

$$\rho_n \leq \frac{\overline{\rho}_n + \varepsilon}{\lambda}.$$

Since $\varepsilon > 0$ and λ in $0 < \lambda < 1$ are arbitrary, it follows that

$$\inf_{(\rho_1,\ldots,\rho_n)\in S_{\overline{\rho}}(f)} \{\rho_n\} \leq \overline{\rho}_n.$$

The reserve inequality is obviously true. This completes the proof.

2 GBASP

This section includes the studies of the equivalence of partial orders and partial types of entire function generalized biaxially symmetric potential (GBASP) and its associate.

The real valued GBASP $F_j^{\alpha,\beta}$ regular in the open unit hyper sphere $\sum^{\alpha,\beta} : x_j^2 + y_j^2 < 1, j = 1, 2, \ldots, n$ about the origin with respect to jth variable, keeping others fixed, can be expanded uniquely as

$$F_j^{(\alpha,\beta)}(x_j, y_j) = \sum_{|\overline{m}|=0}^{\infty} a_{\overline{m}} R_{\overline{m}}^{(\alpha,\beta)}(x_j, y_j), \quad \alpha > \beta > -1/2, \tag{2.1}$$

where $\overline{m} = (m_1, m_2, \ldots, m_n)$ and $|\overline{m}| = m_1 + m_2 + \cdots + m_n$. In terms of complete set

$$R_{\overline{m}}^{(\alpha,\beta)}(x_j, y_j) = \left(x_j^2 + y_j^2\right)^m \frac{P_{\overline{m}}^{(\alpha,\beta)}\left(x_j^2 - y_j^2/x_j^2 + y_j^2\right)}{P_{\overline{m}}^{(\alpha-\beta)}(1)}, \tag{2.2}$$

of biaxisymmetric harmonic polynomials. These even functions are classical solutions to the generalized biaxisymmetric potential equation

$$\left(\frac{\partial^2}{\partial x_j^2} + \frac{2\alpha+1}{x_j}\frac{\partial}{\partial x_j} + \frac{\partial^2}{\partial y_j^2} + \frac{2\beta+1}{y_j}\frac{\partial}{\partial y_j}\right) F_j^{(\alpha,\beta)} = 0 \tag{2.3}$$

subject to the Cauchy data

$$F_{x_j}^{(\alpha,\beta)}(0, y_j) = F_{y_j}^{(\alpha,\beta)}(x_j, 0) = 0,$$

along the singular lines in $\sum^{(\alpha,\beta)}$.

We can develop an operator mapping and its inverse as in [8] from Koornwinder's integral for Jacobi polynomials [1] with respect to jth variable keeping others fixed which associates each GBASP to a unique even analytic function $f : \mathbb{C}^n \rightarrowtail \mathbb{C}$. Thus, let

$$f(\overline{z}) = \sum_{|\overline{m}|=0}^{\infty} a_{\overline{m}} \overline{z}^{2\overline{m}}, \quad \overline{z} \in \mathbb{C}^n$$

be the unique associated even analytic function. The operator mapping of f onto GBASP

$$F_j^{(\alpha,\beta)}(x_j, y_j) = \sum_{|\overline{m}|=0}^{\infty} a_{\overline{m}} R_{\overline{m}}^{(\alpha,\beta)}(x_j, y_j), \quad \alpha > \beta > -1/2,$$

uniquely, is given by

$$\begin{aligned} F_j^{(\alpha,\beta)}(x_j, y_j) = K_{(\alpha,\beta)}(f) &= \int_0^1 \int_0^{\pi} f(\xi_i) d_{\mu_{(\alpha,\beta)}}(t, s), \\ \xi_j^2 &= x_j^2 - y_j^2 t^2 + 2i(x_j y_j t \cos s), \quad i \equiv iota \\ d_{\mu_{(\alpha,\beta)}} &= \nu_{\alpha,\beta}(1-t^2)^{\alpha-\beta-1} t^{2\beta+1} (\sin s)^{2\alpha} dt\, ds, \\ \nu_{\alpha,\beta} &= \frac{2\Gamma(\alpha+1)}{\Gamma(1/2)\Gamma(\alpha-\beta)\Gamma(\beta+1/2)}. \end{aligned}$$

The inverse operator $K_{\alpha,\beta}^{-1}$ is given by

$$\begin{aligned} &f(\overline{z}) = K_{\alpha,\beta}^{-1}\left(F_j^{(\alpha,\beta)}(x_j, y_j)\right) = \int_{-1}^{1} F_j\left[r_j \xi_j, r_i\left(1-\xi_j^2\right)\right] d\nu_{\alpha,\beta}\left(\overline{z}_i^2/r_j^2, \xi_j\right) \\ &d\nu_{\alpha,\beta}(\tau, \xi_j) = S_{\alpha,\beta}(\tau\xi_j)(1-\xi_j)^{\alpha}(1+\xi_j)^{\beta} d\xi_j. \end{aligned}$$

The kernel $S_{\alpha,\beta}$ is

$$\begin{aligned} &S_{\alpha,\beta} = \eta_{\alpha,\beta} \frac{(1-r)}{(1+\tau)^{\alpha+\beta+2}} F_1 \left\{\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}, \beta+1, \frac{2\tau(1+\xi_j)}{2}\right\}, \\ &\eta_{\alpha,\beta} = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)(\beta+1)}. \end{aligned}$$

The measures are normalized so that $K_{(\alpha,\beta)}^{-1}(1) = K_{(\alpha,\beta)}(1) = 1$. It is easy to prove that (see [3,4]) the GBASP $F_j^{(\alpha,\beta)}$ is regular in the hypersphere $\sum_{R_j}^{(\alpha,\beta)} : x_j^2 + y_j^2 < R_j^2$ if and only if its associate $f : \mathbb{C}^n \rightarrowtail \mathbb{C}$ is analytic in the polydisk $D_{R_j} : x_j^2 + y_j^2 < R_j^2$ with respect to the jth variable keeping others fixed. Here R_j are fixed real numbers for $j = 1, 2, \ldots, n$. On the singular axis $y_j = 0$, the identity

$$f(x_j + i0) = F_j^{(\alpha,\beta)}(x_j, 0), \quad \|x_j\| < R_j$$

can be continued analytically as

$$f(\overline{z}) = F_j^{(\alpha,\beta)}(\overline{z}, 0), \quad \overline{z} \in \mathbb{C}^n, \|z_j\| < R_j$$

via the law of Permanence of Functional Equations to recover the associate. All the above facts are summarized in the following result.

Theorem A *For each GBASP, $F_j^{(\alpha,\beta)}$ regular in the hypersphere $\sum_{R_j}^{(\alpha,\beta)}$ there is a unique $K_{(\alpha,\beta)}$ associated with the even function $f : \mathbb{C}^n \rightarrowtail \mathbb{C}$ analytic in the polydisk D_{R_j}, and conversely.*

McCoy [7] studied the growth and polynomial approximation of GBASP in single complex variable and obtained some results for GBASP of Sato index [9].

The maximum moduli of GBASP is defined as in complex function theory

$$M\left(\overline{r}, F_j^{(\alpha,\beta)}\right) = \max_{x_j^2+y_j^2=R_j^2} \left|F_j^{(\alpha,\beta)}(x_j, y_j)\right|, \quad j = 1, \ldots, n.$$

Let $f \in C(E)$, have partial orders $\overline{\rho}_j$, $j = 1, \ldots, n$. The partial type $\overline{T}_j$ of f with respect to the partial orders $\overline{\rho}_j$ is defined as

$$\overline{T}_j = \limsup_{r_j \to \infty} \frac{\log M(\overline{r}, f)}{r_j^{\overline{\rho}_j}}. \tag{2.4}$$

What follows next are the relations showing equivalence of partial orders and partial types for GBASP and associate.

Theorem 2.1 *Let $F_j^{(\alpha,\beta)}$ be real valued entire function GBASP defined on $\mathbb{C}^n$ with $K_{\alpha,\beta}$ associate $f : \mathbb{C}^n \rightarrowtail B$. Then the partial orders and partial types of $F_j^{(\alpha,\beta)}$ and f, respectively, are identical.*

Proof In view of Theorem A, we have

$$F_j^{(\alpha,\beta)}(x_j, y_j) = K_{(\alpha,\beta)}(f). \tag{2.5}$$

$$f(\overline{z}) = K_{(\alpha,\beta)}^{-1}\left(F_j^{(\alpha,\beta)}\right). \tag{2.6}$$

The non-negativity and the normalization of the measure (2.5) gives

$$M\left(\overline{r}, F_j^{(\alpha,\beta)}\right) \le M(\overline{r}, f). \tag{2.7}$$

and (2.6) leads to the estimates

$$\|f(\overline{z})\| \le M\left(\overline{r}, F_j^{(\alpha,\beta)}\right) N_{\alpha,\beta}(\tau_j), \quad \tau_j = \left(\frac{z_j}{r_j}\right)^2, \quad j = 1, \ldots, n$$

and

$$N_{\alpha,\beta}(\tau) = \max\left\{\eta_{\alpha,\beta}^{-1}\|S_{\alpha,\beta}(\tau_j, \xi_j)\|, -1 \leq \xi_j \leq 1\right\}.$$

However, for $z_j = \varepsilon r_j e^{i\theta}(\varepsilon \text{ real})$,

$$M(\varepsilon\overline{r}, f) \leq M\left(\overline{r}, F_j^{(\alpha,\beta)}\right) N_{\alpha,\beta}(\tau_j)$$

or

$$M(\overline{r}, f) \leq M\left(\varepsilon^{-1}\overline{r}, F_j^{(\alpha,\beta)}\right) N_{\alpha,\beta}(\varepsilon^2). \tag{2.8}$$

Using the definition of partial order and partial type of f together with (2.7) and (2.8), the proof is complete.

Using Theorem A as main plank, McCoy [7] studied the growth and polynomial approximation of GBASP in single complex variable and obtained some results for GBASP of Sato index [9] k. It has been noticed that when $\overline{f}(\overline{z}) = \lim_{\nu\to\infty} L_\nu(\overline{z})$, where L_ν is the Lagrange interpolation polynomial with nodes at extreme points of E, the type of $\overline{f}$ cannot be characterized by means of the measure of the Chebyshev best approximation to f on E by polynomials of degree $\leq \nu$ with respect to all variables. So, we have to consider the measures $e_{\overline{m}_j}(f, E)$, $\overline{m} = (m_1, \ldots, m_n)$ of the Chebyshev best approximation to f on $E = E_1 \times \cdots \times E_n$ by polynomials of the degree $\leq \overline{m}_j$ with respect to the jth variable, $j = 1, \ldots, n$, keeping others fixed, where E_j is a bounded closed set with a positive transfinite diameter $d_j \equiv d(E_j)$ in the complex z_j-plane. Our further objective is to study the growth and polynomial approximation of entire GBASP with respect to each of the variable separately.

3 Growth estimates of GBASP

In this section we discuss the necessary and sufficient conditions for an entire function GBASP $F_j^{(\alpha,\beta)}$ to be of finite partial order $\overline{\rho}_j$.

Theorem 3.1 *The entire function GBASP $F_j^{\alpha,\beta}$ is of finite partial order if and only if*

$$\overline{\mu}_j = \limsup_{\min(m_j)\to\infty} \frac{\log 2\overline{m}_j^{2\overline{m}_j}}{\log a_{2\overline{m}_j}}$$

is finite, and then the partial order $\overline{\rho}_j$ of $F_j^{(\alpha,\beta)}$ is equal to $\overline{\mu}_j$.

Proof By Theorem A, $F_j^{(\alpha,\beta)}$ is entire if and only if $f(\overline{z}) = \sum_{|\overline{m}|=0}^{\infty} a_{2\overline{m}}\overline{z}^{2\overline{m}}$ is an entire function. Since $f(\overline{z})$ is entire (see [6]) it follows that $F_j^{(\alpha,\beta)}$ is also an entire function. Also, by Theorem 2.1, the partial order of $f(\overline{z})$ and $F_j^{(\alpha,\beta)}$ are equal. Now, using Theorem IV of [2] for entire function $f(\overline{z})$ with respect to $j^{\overline{z}}$ with respect to jth variable keeping others fixed, the result is immediate.

Theorem 3.2 *If* $0 < \overline{\rho}_j < \infty$, *the entire GBASP* $F_j^{(\alpha,\beta)}$ *is of partial order* $\overline{\rho}_j$ *and partial type* $\overline{T}$, *if and only if*

$$e\overline{\rho}_j\overline{T}_j = \limsup_{\min\{m_j\}\to\infty} \left\{2\overline{m}_j^{2\overline{m}_j}(a_{2\overline{m}_j})^{\overline{\rho}_j}\right\}^{-2(m_1+m_2+\cdots+m_j)}.$$

Proof Using Theorem A, we conclude that the associate $f(\overline{z}) = \sum_{|\overline{m}|=0}^{\infty} a_{2\overline{m}}\overline{z}^{2\overline{m}}$, $\overline{z} \in \mathbb{C}^n$ is of same partial order $\overline{\rho}_j$ and partial type $\overline{T}_j$ as the GBASP $F_j^{(\alpha,\beta)}$. Applying Theorem V of [2] to the entire function $f(\overline{z})$ with respect to jth variable, the required result can be obtained after simple manipulations.

4 Approximation of GBASP

In this section we shall study the polynomial approximation of entire GBASP in $\mathbb{C}^n \to \mathbb{C}$. Let the Chebyshev norms be defined for $\mathcal{C}(E)$ and $F_j^{(\alpha,\beta)} \in \mathcal{C}\left(\partial E_j^{(\alpha,\beta)}\right)$ as

$$e_{2\overline{m}_j}(f) = \inf\left\{\|f - \pi_{2\overline{m}}\|, \pi_{2\overline{m}} \in k_{2\overline{m}}\right\},$$
$$\|f - \pi_{2\overline{m}}\| = \sup_{x\in E}\left\{|f(x) - \pi_{2\overline{m}}(x)|\right\}$$

and

$$e^*_{2\overline{m}_j}(F^{(\alpha,\beta)}) = \inf\left\{\left\|F_j^{(\alpha,\beta)} - Q_j^{(\alpha,\beta)}\right\|, Q_j^{(\alpha,\beta)} \in K_{2\overline{m}}^{(\alpha,\beta)}\right\},$$
$$\left\|F_j^{(\alpha,\beta)} - Q_j^{(\alpha,\beta)}\right\| = \sup_{x_j^2+y_j^2=R_j^2}\left|F_j^{(\alpha,\beta)}(x_j,y_j) - Q_j^{(\alpha,\beta)}(x_j,y_j)\right|.$$

The set $k_{2\overline{m}}$ contains all real polynomials of degree at most $2\overline{m}$, and the set $K_{2\overline{m}}^{(\alpha,\beta)}$ contains all real biaxisymmetric harmonic polynomials of degree at most $2\overline{m}$. The operators $K_{(\alpha,\beta)}$ and $K_{(\alpha,\beta)}^{-1}$ establish one-one equivalence of the set $k_{2\overline{m}}$ and $K_{2\overline{m}}^{(\alpha,\beta)}$.

Theorem 4.1 *The entire GBASP* $F_j^{(\alpha,\beta)}$ *is of finite partial order if and only if*

$$\mu^* = \limsup_{\min\{m_j\}\to\infty} \frac{\log 2\overline{m}^{2\overline{m}}}{\log e^*_{2\overline{m}_j}(F^{(\alpha,\beta)})}$$

is finite, and then the partial orders $\overline{\rho}_j$ *of* $F_j^{(\alpha,\beta)}$ *are equal to* μ_j^*.

Theorem 4.2 *If* $0 < \overline{\rho}_j < \infty$, *the entire function GBASP* $F_j^{(\alpha,\beta)}$ *is of partial order* $\overline{\rho}_j$ *and partial type* $\overline{T}_j$ *if and only if*

$$e\overline{\rho}_j\overline{T}_j d_j^{\overline{\rho}_j} = \limsup_{\min\{m_j\}\to\infty}\left[2\overline{m}^{2\overline{m}_j}\left(e^{*}_{2\overline{m}_j}\left(F_j^{(\alpha,\beta)}\right)\right)^{\overline{\rho}_j}\right]^{-2(m_1+m_2+\cdots+m_j)}.$$

Proofs of Theorems 4.1 and 4.2 follows using the reasoning of [5] for $(p, q) = (2, 1)$ with respect to jth variable keeping others fixed, as in the single variable case.

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