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Conformal vector fields on Kaehler manifolds

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Abstract It is known that a conformal vector field on a compact Kaehler manifold is a Killing vector field. In this paper, we are interested in finding conditions under which a conformal vector field on a non-compact Kaehler manifold is Killing. First we prove that a harmonic analytic conformal vector field on a $2n$-dimensional Kaehler manifold ($n \neq 2$) of constant nonzero scalar curvature is Killing. It is also shown that on a $2n$-dimensional Kaehler Einstein manifold ($n > 1$) an analytic conformal vector field is either Killing or else the Kaehler manifold is Ricci flat. In particular, it follows that on non-flat Kaehler Einstein manifolds of dimension greater than two, analytic conformal vector fields are Killing.

Keywords Kaehler manifolds · Euclidean complex space form · Ricci curvature · Analytic vector fields · Conformal vector field · Harmonic vector fields

Mathematics Subject Classification (2000) 53C15 · 53A30

1 Introduction

Conformal vector fields are important objects on a space and have been studied quite extensively on Riemannian manifolds (cf. [3,4,6–8,10–12]). However, conformal vector fields on a Kaehler manifold have not been studied that extensively. Recall that a vector field u on a Kaehler manifold (M, J, g) is said to be analytic vector field if

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$$\pounds_u J = 0$$

where $\pounds_u$ is the Lie-derivative with respect the vector field u; and a vector field ξ on the Kaehler manifold (M, J, g) is said to be a conformal vector field if

$$\pounds_\xi g = 2\rho g$$

where ρ is a smooth real valued function on M called the potential function. On a compact Kaehler manifold of dimension greater than two, a conformal vector field is Killing (cf. [9]), however non-Killing conformal vector fields on non-compact Kaehler manifold are in abundance. For example, consider the Euclidean space C^n of dimension $2n$, which is a Kaehler manifold with natural canonical complex structure J, the vector field $\xi = \psi + J\psi$ is an analytic conformal vector field on the Kaehler manifold $(C^n, J, \langle,\rangle)$ which is not Killing, where ψ is the position vector field and $\langle,\rangle$ is the Euclidean metric on C^n. If ∇ is the Levi–Civita connection on $(C^n, J, \langle,\rangle)$, then we have

$$\nabla_X \xi = X + JX, \quad X \in \mathfrak{X}(C^n)$$

where $\mathfrak{X}(C^n)$ is the Lie algebra of smooth vector fields on C^n. This conformal vector field ξ on $(C^n, J, \langle,\rangle)$ satisfies $\Delta\xi = 0$, where Δ is the Laplacian operator acting on smooth vector fields on $(C^n, J, \langle,\rangle)$, that is ξ is a harmonic analytic conformal vector field on $(C^n, J, \langle,\rangle)$.

In this paper, we are interested in finding conditions under which a conformal vector field on non-compact Kaehler manifold is Killing. The main results of this paper are the following:

Theorem 1 *Let (M, J, g) be a Kaehler manifold of constant scalar curvature $S \neq 0$ and* $\dim M \neq 4$. *Then a harmonic analytic conformal vector field on M is Killing.*

Theorem 2 *Let (M, J, g) be a $2n$-dimensional Kaehler Einstein manifold $(n > 1)$. If ξ is an analytic conformal vector field on M, then either ξ is Killing vector field or else (M, J, g) is Ricci flat.*

2 Preliminaries

Let (M, J, g) be a $2n$-dimensional Kaehler manifold with complex structure J and Hermitian metric g. We denote by ∇ the Levi–Civita connection and by $\mathfrak{X}(M)$ the Lie-algebra of smooth vector fields on M. Then we have

$$\nabla_X JY = J\nabla_X Y, \quad g(JX, JY) = g(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.1}$$

The curvature tensor field R and the Ricci tensor field Ric of a Kaehler manifold (M, J, g) satisfy

$$R(JX, JY; JZ, JW) = R(X, Y; Z, W), \quad Ric(JX, JY) = Ric(X, Y). \tag{2.2}$$

$X, Y, Z, W \in \mathfrak{X}(M)$. A smooth vector field u on a Kaehler manifold (M, J, g) is said to be an analytic vector field if

$$(\pounds_u J)(X) = 0, \quad X \in \mathfrak{X}(M) \tag{2.3}$$

where $\pounds_u$ is Lie derivative with respect to u. Using the fact that on a Kaehler manifold the complex structure J satisfies the integrability condition, it is easy to deduce that if u is an analytic vector field, then the vector field $\overline{u} = Ju$ is also analytic vector field. A smooth vector field $\xi \in \mathfrak{X}(M)$ on a Kaehler manifold (M, J, g) is said to be a conformal vector field if

$$\left(\pounds_\xi g\right)(X, Y) = 2\rho g(X, Y), \quad X, Y \in \mathfrak{X}(M) \tag{2.4}$$

for a smooth function $\rho : M \to R$, where $\pounds_\xi$ is the Lie derivative with respect to ξ. We call the smooth function ρ associated with conformal vector field ξ in above definition the **potential function** of ξ. We have seen in the introduction an example of a vector field on the Kaehler manifold $(C^n, J, \langle,\rangle)$, which is both analytic as well as conformal. However, there are examples of analytic vector fields which are not conformal, for example consider a non-Einstein steady Kaehler Ricci soliton (M, J, g) (cf. [2]), which satisfies

$$H_f(X, Y) + Ric(X, Y) = 0, \quad X, Y \in \mathfrak{X}(M)$$

for a smooth function $f : M \to R$, where H_f is the Hessian of the function f. Then using second equation in (2.2), we conclude that the vector field $u = \nabla f$, the gradient of the smooth function f, is an analytic vector field which is not a conformal vector field.

If ξ is a conformal vector field on a Kaehler manifold (M, J, g) and η is the 1- form dual to ξ, we define a skew-symmetric tensor field ϕ of type $(1, 1)$ on M by

$$d\eta(X, Y) = 2g(\phi X, Y), \quad X, Y \in \mathfrak{X}(M).$$

Then using Kozul's formula, it is easy to deduce the following:

Lemma 2.1 (*cf.* [4]) *Let ξ be a conformal vector field on a Kaehler manifold (M, J, g) with potential function ρ. Then*

$$\nabla_X \xi = \rho X + \phi X, \quad X \in \mathfrak{X}(M).$$

It immediately follows from above Lemma that if ξ is a conformal vector field on a Kaehler manifold (M, J, g), then the curvature tensor of the Kaehler manifold satisfies

$$R(X, Y)\xi = X(\rho)Y - Y(\rho)X + (\nabla_X \phi)(Y) - (\nabla_Y \phi)(X), \quad X, Y \in \mathfrak{X}(M) \tag{2.5}$$

where $(\nabla_X \phi)(Y) = \nabla_X \phi Y - \phi(\nabla_X Y)$. Consequently, choosing a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$, and using the fact that ϕ is skew-symmetric, $g(\phi e_i, e_i) = 0$, in the Eq. 2.5, we get the following relation

$$Ric(X, \xi) = -(2n-1)X(\rho) - g\left(X, \sum_{i=1}^{2n} \left(\nabla_{e_i}\phi\right)(e_i)\right).$$

The Ricci operator Q is a symmetric $(1, 1)$ tensor field defined by $Ric(X, Y) = g(Q(X), Y)$, $X, Y \in \mathfrak{X}(M)$. Thus above equation gives

$$Q(\xi) = -(2n-1)\nabla\rho - \sum_{i=1}^{2n} \left(\nabla_{e_i}\phi\right)(e_i), \tag{2.6}$$

where $\nabla\rho$ is the gradient of the potential function ρ.

On a Riemannian manifold (M, g) with Levi–Civita connection ∇, for a smooth function $f : M \to R$, the Hessian operator H_f of the function f is defined by (cf. [5])

$$H_f(X, Y) = g(\nabla_X \nabla f, Y), \quad X, Y \in \mathfrak{X}(M)$$

The Laplacian Δf of the smooth function f is related to the Hessian H_f by $\Delta f =$ **Trace**H_f. Recently Garcia-Rio and others [8] have initiated the study of the Laplacian operator $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined on a Riemannian manifold (M, g) by

$$\Delta X = \sum_{i=1}^{n} \left(\nabla_{e_i}\nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X\right)$$

where ∇ is the Levi–Civita connection and $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. This operator is self adjoint elliptic operator with respect to the inner product $\langle , \rangle$ on the set $\mathfrak{X}^C(M)$ of compactly supported vector fields in $\mathfrak{X}(M)$, defined by

$$\langle X, Y\rangle = \int_M g(X, Y), \quad X, Y \in \mathfrak{X}^C(M)$$

A vector field X is said to be harmonic if $\Delta X = 0$. We shall denote by Δ both the Laplacian operators, the one acting on smooth functions on M as well as that acting on the smooth vector fields.

For a smooth vector field $X \in \mathfrak{X}(M)$ on a Kaehler manifold we shall use the notation $\overline{X} = JX$. Now we shall prepare following Lemmas which will be used as tools in proving our results.

Lemma 2.2 *Let ξ be an analytic conformal vector field on a Kaehler manifold (M, J, g). Then the tensor field ϕ in Lemma 2.1 satisfies $J \circ \phi = \phi \circ J$ and the covariant derivative of the vector field $\overline{\xi}$ is given by*

$$\nabla_X \overline{\xi} = AX + \rho JX, \quad X \in \mathfrak{X}(M)$$

where $A = J \circ \phi$ *is a symmetric* $(1, 1)$*-tensor field.*

Proof Since ξ is analytic we have

$$\nabla_{JX}\xi = J\nabla_X \xi, \quad X \in \mathfrak{X}(M) \tag{2.7}$$

which together with Lemma 2.1 gives $\phi \circ J = J \circ \phi$ and this also proves that the tensor field $A = J \circ \phi$ is symmetric. The expression for the covariant derivative of $\overline{\xi}$ then follows from Lemma 2.1. □

Lemma 2.3 *Let* ξ *be an analytic conformal vector field on a* $2n$*-dimensional Kaehler manifold* (M, J, g)*. Then the tensor field* A *in Lemma* 2.2 *satisfies*

$$\sum_{i=1}^{2n} \left(\nabla_{e_i} A\right)(e_i) = \Delta\overline{\xi} - J\nabla\rho$$

where $\{e_1, \ldots, e_{2n}\}$ *is a local orthonormal frame on* M *and* $\nabla\rho$ *is the gradient of the potential function* ρ*.*

Proof It follows immediately that

$$(\nabla_X A)(Y) = J(\nabla_X \phi)(Y) \tag{2.8}$$

and by Lemma 2.1 that

$$\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y}\xi = X(\rho)Y + (\nabla_X \phi)(Y)$$

This equation together with (2.8) gives

$$\sum_{i=1}^{2n} \left(\nabla_{e_i} A\right)(e_i) = J(\Delta\xi - \nabla\rho)$$

which together with the fact that $\Delta\overline{\xi} = J\Delta\xi$ (as J is parallel with respect to the Levi–Civita connection) proves the Lemma. □

Now using Eqs. 2.5 and 2.8 we immediately get the following.

Lemma 2.4 *Let* ξ *be an analytic conformal vector field on a Kaehler manifold* (M, J, g)*. Then the tensor field* A *in Lemma* 2.2 *satisfies*

$$(\nabla_X A)(Y) - (\nabla_Y A)(X) = J\{R(X, Y)\xi + Y(\rho)X - X(\rho)Y\}, \quad X, Y \in \mathfrak{X}(M).$$

Now for an analytic conformal vector field ξ on a Kaehler manifold (M, J, g) define a smooth function $h : M \to R$ by $h =$**Trace**A called the **trace** of the analytic conformal vector field ξ. Thus to an analytic conformal vector field ξ on a Kaehler manifold (M, J, g), there are naturally associated two smooth functions, the potential function ρ and the trace h. Then using Lemma 2.4, the symmetry of A and the property of the curvature tensor $R(X, Y)JZ = JR(X, Y)Z$, we immediately obtain the following

$$X(h) = -Ric(X, \overline{\xi}) - g(X, J\nabla\rho) + g\left(X, \sum_{i=1}^{2n} \left(\nabla_{e_i} A\right)(e_i)\right), \quad X \in \mathfrak{X}(M)$$

which gives

$$\nabla h = -Q(\overline{\xi}) - J\nabla\rho + \sum_{i=1}^{2n} \left(\nabla_{e_i} A\right)(e_i) \tag{2.9}$$

for a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$ on the Kaehler manifold (M, J, g). Now using the Eq. 2.9 in Lemma 2.3, we get

Lemma 2.5 *Let ξ be an analytic conformal vector field on a Kaehler manifold (M, J, g). Then*

$$\nabla h = \Delta\overline{\xi} - Q(\overline{\xi}) - 2J\nabla\rho.$$

On a Kaehler manifold, it is well known that $Q \circ J = J \circ Q$ holds, that is $Q(\overline{\xi}) = J(Q(\xi))$. Thus using the Eqs. 2.6 and 2.8, we arrive at

$$\sum_{i=1}^{2n} \left(\nabla_{e_i} A\right)(e_i) = -Q(\overline{\xi}) - (2n-1)J\nabla\rho$$

which together with Eq. 2.9 gives

Lemma 2.6 *Let ξ be an analytic conformal vector field on a $2n$-dimensional Kaehler manifold (M, J, g). Then*

$$\nabla h = -2Q(\overline{\xi}) - 2nJ\nabla\rho$$

Finally combining Lemmas 2.5 and 2.6, we have

$$-Q(\overline{\xi}) - 2(n-1)J\nabla\rho = \Delta\overline{\xi}$$

which together with the facts $J\Delta\xi = \Delta\overline{\xi}$ and $JQ(\xi) = Q(\overline{\xi})$ gives

Lemma 2.7 *Let ξ be an analytic conformal vector field on a Kaehler manifold (M, J, g). Then*

$$\Delta\xi + Q(\xi) = -2(n-1)\nabla\rho.$$

3 Proof of Theorem-1

Let (M, J, g) be a $2n$-dimensional Kaehler manifold and ξ be a harmonic analytic conformal vector field on M. Then by Lemma 2.7, we have

$$Q(\xi) = -2(n-1)\nabla\rho \tag{3.1}$$

which gives

$$Q(\overline{\xi}) = -2(n-1)J\nabla\rho$$

Using the above equation in Lemma 2.5 together with $\Delta\overline{\xi} = J\Delta\xi = 0$, we arrive at

$$-J\nabla h = 2(n-2)\nabla\rho \tag{3.2}$$

We find the divergence of the vector field $J\nabla h$

$$div\,(J\nabla h) = -\sum_{i=1}^{2n} g(A_h(e_i), Je_i) \tag{3.3}$$

where A_h is the Hessian operator of the smooth function h related to the Hessian H_h by $H_h(X, Y) = g(A_h(X), Y)$, $X, Y \in \mathfrak{X}(M)$. Since A_h is symmetric, we can choose a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$ of M that diagonalizes A_h and consequently the Eq. 3.3 gives $div\,(J\nabla h) = 0$. Thus taking divergence in Eq. 3.2 and using $div\,(J\nabla h) = 0$, we arrive at

$$(n-2)\Delta\rho = 0.$$

Since $n \neq 2$, we have $\Delta\rho = 0$. Thus by Eq. 3.1, we conclude that

$$div(Q\xi) = 0. \tag{3.4}$$

We find the divergence of $Q(\xi)$ as

$$div(Q\xi) = \sum_{i=1}^{2n} \left[g(\nabla_{e_i}\xi, Q(e_i)) + g(\xi, \left(\nabla_{e_i} Q\right)(e_i))\right] \tag{3.5}$$

Now using the well known fact that (cf. [1])

$$\sum_{i=1}^{2n} \left(\nabla_{e_i} Q\right)(e_i) = \frac{1}{2}\nabla S = 0,$$

where S is the scalar curvature (which is constant in our case), we see that the Lemma 2.1 is necessary to compute

$$div(Q\xi) = \rho S + \sum_{i=1}^{2n} g(\phi e_i, Qe_i)$$

Since Q is symmetric, we can choose a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$ that diagonalizes Q and consequently, the sum in above equation vanishes. So we obtain

$$div(Q\xi) = \rho S$$

Combining this last equation with Eq. 3.4 together with the assumption $S \neq 0$ in the hypothesis of the Theorem-1, we get $\rho = 0$ and this proves that ξ is Killing finishing the proof of Theorem-1.

Remark The question as to what happens in the case when dim $M = 4$ that is excluded in Theorem-1, is interesting and worthy of an answer. Also can one prove that a Kaehler manifold of constant zero scalar curvature admitting a harmonic analytic conformal vector field is necessarily isometric to the Euclidean complex space form $(C^n, J, \langle, \rangle)$? These are two important questions related to harmonic analytic conformal vector fields on a Kaehler manifold of constant scalar curvature.

4 Proof of Theorem-2

Let (M, J, g) be a $2n$-dimensional Kaehler Einstein manifold, $n > 1$ and ξ be an analytic conformal vector field on M. Then we have

$$Q(\xi) = \frac{S}{2n}\xi$$

where S is the constant scalar curvature of M and consequently Lemma 2.6 gives

$$\nabla h = -\frac{S}{n}\bar{\xi} - 2nJ\nabla\rho \quad \text{and} \quad J\nabla h = 2n\nabla\rho + \frac{S}{n}\xi \tag{4.1}$$

We use Lemma 2.1, in taking the divergence of equations in (4.1) and use the facts $div(\bar{\xi}) = h$, $div(\xi) = 2n\rho$ and $div(J\nabla\rho) = div(J\nabla h) = 0$, to get

$$\Delta h = -\frac{S}{n}h \quad \text{and} \quad \Delta\rho = -\frac{S}{n}\rho \tag{4.2}$$

(Note that $div(J\nabla\rho) = \sum_{i=1}^{2n} g(\nabla_{e_i} J\nabla\rho, e_i) = -\sum_{i=1}^{2n} g(\nabla_{e_i}\nabla\rho, Je_i) = -\sum_{i=1}^{2n} g(A_\rho(e_i), Je_i) = 0$, as one could choose a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$ that diagonalizes the symmetric operator A_ρ). Now, taking covariant derivative in first equation in (4.1) with respect to $X \in \mathfrak{X}(M)$, we get

$$A_h X = -\frac{S}{n}AX - \frac{S}{n}\rho JX - 2nJA_\rho X, \quad X \in \mathfrak{X}(M)$$

where A_ρ and A_h are the Hessian operators of the potential function ρ and the trace h of ξ respectively. The above equation gives

$$(\nabla_X A_h)(Y) = -\frac{S}{n}(\nabla_X A)(Y) - \frac{S}{n}X(\rho)JY - 2nJ\left(\left(\nabla_X A_\rho\right)(Y)\right), \tag{4.3}$$

$X, Y \in \mathfrak{X}(M)$. On a Riemannian manifold (M, g) for a smooth function $f : M \to R$ and a local orthonormal frame $\{e_1, \ldots, e_n\}$, $n = \dim M$, the following is known (cf. [5], with a sign difference in the definition of the Laplacian operator)

$$\sum_{i=1}^{n}\left(\nabla_{e_i} A_f\right)(e_i) = Q(\nabla f) + \nabla(\Delta f). \tag{4.4}$$

Taking a local orthonormal frame $\{e_1, \ldots, e_{2n}\}$ on the Kaehler manifold (M, J, g), and summing the Eq. 4.3 over this frame and using Eqs. 4.2 and 4.4, we arrive at

$$\frac{S}{2n}\nabla h - \frac{S}{n}\nabla h = -2nJ\left(\frac{S}{2n}\nabla\rho - \frac{S}{n}\nabla\rho\right) - \frac{S}{n}J\nabla\rho - \frac{S}{n}\sum_{i=1}^{2n}\left(\nabla_{e_i} A\right)(e_i).$$

Substituting the sum from Eq. 2.9 in above equation, we arrive at

$$\frac{S}{2n}\left(\nabla h + \frac{S}{n}\overline{\xi} - 2(n-2)J\nabla\rho\right) = 0$$

which together with Lemma 2.6, gives

$$(n-1)SJ\nabla\rho = 0.$$

Thus either $S = 0$ that is M is Ricci flat or else $\nabla\rho = 0$. In the second case we have $\Delta\rho = 0$, which together with Eq. 4.2 implies that $S\rho = 0$, that is for non-Ricci flat Kaehler manifold, $\rho = 0$ and this proves that ξ is Killing.

As a direct consequence of the Theorem-2, we have:

Corollary *Let (M, J, g) be a $2n$-dimensional non-Ricci flat Kaehler Einstein manifold ($n > 1$). If ξ is an analytic conformal vector field on M, then ξ is Killing.*

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