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# Asymptotic profiles of steady Stokes and Navier–Stokes flows around a rotating obstacle

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**Abstract** We analyze the spatial anisotropic profiles at infinity of steady Stokes and Navier–Stokes flows around a rotating obstacle. It is shown that the Stokes flow is largely concentrated along the axis of rotation in the leading term and that a rotating profile can be found in the second term. The leading term for Navier–Stokes flow will be an adequate Landau solution. The proofs rely upon a detailed analysis of the associated fundamental solution tensor.

**Keywords** Asymptotic profile · Steady Stokes flow · Steady Navier–Stokes flow · Rotating obstacle

**Mathematics Subject Classification (2000)** 35Q30 · 35Q35 · 35B40 · 76D05 · 76D07

## 1 Introduction

In this survey article we find the leading profiles at infinity of steady Stokes and Navier–Stokes flows in the exterior of a rotating obstacle. Our results make clear that the axis of rotation plays an important role for both flows as preferred direction. Since

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the Navier–Stokes equations for incompressible fluids are rotationally invariant, without loss of generality, the axis of rotation may be assumed to be the $x_3$-axis so that the angular velocity of the obstacle is given by $\omega = ae_3$, where $a \in \mathbb{R}\setminus\{0\}$ is a constant and $e_3 = (0, 0, 1)^T$. By using a coordinate system attached to the rotating obstacle, as in [2,8,12], one can reduce the original Navier–Stokes problem to an equivalent one in a fixed exterior domain $D \subset \mathbb{R}^3$, where its boundary $\partial D$ is assumed to be smooth. We will address steady flow (in the reference frame) which obeys

$$-\Delta u - (\omega \times x) \cdot \nabla u + \omega \times u + \nabla p + u \cdot \nabla u = 0, \qquad \mathrm{div} u = 0 \tag{1.1}$$

in $D$ subject to boundary conditions

$$u = \omega \times x \quad (x \in \partial D), \qquad u \to 0 \quad (|x| \to \infty), \tag{1.2}$$

where $u(x) = (u_1, u_2, u_3)^T$ and $p(x)$ denote the velocity and pressure, respectively, of the fluid. Note that the boundary condition on the surface $\partial D$ is the usual no-slip one since $\omega \times x = a(-x_2, x_1, 0)^T$ is the rotating velocity of the obstacle.

By [9] and [5] we know that the problem (1.1)–(1.2) possesses a unique solution that enjoys $u(x) = O(1/|x|)$ as $|x| \to \infty$ provided the angular velocity $\omega$ is sufficiently small. The asymptotic stability of this steady flow has been proved by [10] and [14]. One of the purposes of this article is to find its leading term which decays exactly at the rate $1/|x|$ so that the remaining term decays faster. It is proved that the leading term is given by a member of the family of $(-1)$-homogeneous solutions, found first by Landau [17] and resumed by Šverák [18], for the usual Navier–Stokes equation

$$-\Delta u + \nabla p + u \cdot \nabla u = 0, \qquad \mathrm{div} u = 0 \qquad (x \in \mathbb{R}^3\setminus\{0\}). \tag{1.3}$$

Note that, for (1.3), $(-1)$-homogeneity is equivalent to self-similarity. It is proved in [18] that the family of solutions constructed by Landau covers all self-similar solutions of (1.3). Each member of this family is parameterized by a vector about which it is axisymmetric, see Sect. 3. For the leading term of the flow under consideration, this vectorial parameter is parallel to the angular velocity $\omega$. Therefore, this leading term satisfies also (1.1) in $\mathbb{R}^3\setminus\{0\}$ since the additional two terms vanish, cf. (3.5) below. This study is inspired by the recent work [15] due to Korolev and Šverák, in which the leading term of the usual exterior Navier–Stokes flow for the case $\omega = 0$ is provided; it is given by another member of the same family as above and possesses symmetry about the axis whose direction (the vectorial parameter mentioned above) is the net force (3.4) of the given flow.

We remark that the leading profile is the Oseen fundamental solution (without effect of nonlinearity) when the obstacle is translating with constant velocity, see for instance [4], on account of better decay properties outside the wake region behind the obstacle. In the case where both translation and rotation of the obstacle are taken into account, a wake region was still found by [11] (see also [16] for the linearized problem); in this case as well, very probably, the leading term comes from the Stokes flow unlike the purely rotating problem discussed in this article.

The first step toward analysis of Navier–Stokes flow should be the study of the associated Stokes problem, that is, the problem (1.1)–(1.2) in which $u \cdot \nabla u$ is neglected. Even for this linear problem, it is no longer clear what the leading term of the Stokes flow is, because we have the term $(\omega \times x) \cdot \nabla u$ with variable coefficient. In addition, the anisotropic decay structure arising from the effect of rotation is interesting in itself and it must be observed at the level of the linear problem. The other purpose (closely related to the previous one) is to derive such a structure from the asymptotic representation of Stokes flow for $|x| \to \infty$. We will look not only for the leading term ($\sim 1/|x|$) but also for the second one ($\sim 1/|x|^2$). It turns out that the leading profile is given by the third column vector of the usual Stokes fundamental solution tensor (2.2). The reason why the third one is selected is that the axis of rotation is the $x_3$-axis. This points out the important role of the axis of rotation and helps to find the leading term of the Navier–Stokes flow explained above. The second term of the Stokes flow is also interesting because it includes the rotating profile $e_3 \times x$. The proof relies upon a detailed analysis of the fundamental solution of the Eq. (2.7) below.

In the next section we provide the asymptotic representation of the Stokes flow at infinity. The final section is devoted to finding the leading term of the Navier–Stokes flow. The complete proof of the results given here will be found in [6,7].

## 2 Stokes flow

In this section we consider the Stokes problem

$$-\Delta u - (\omega \times x) \cdot \nabla u + \omega \times u + \nabla p = f, \qquad \mathrm{div} u = 0 \qquad (x \in D) \tag{2.1}$$

subject to (1.2) and derive an asymptotic representation for $|x| \to \infty$ of the solution. The results in [5,13] suggest that the optimal rate of decay of the solution to (2.1) is $1/|x|$ in general even though the external force has good properties such as, for instance, $f = \mathrm{div} F$ with $F \in C_0^\infty(D)^{3\times 3}$. Theorem 2.1 below provides its rigorous explanation when we look at the leading term. For the sake of simplicity to catch the profile, the external force is of the form $f = \mathrm{div} F$ with $F \in C_0^\infty(\overline{D})^{3\times 3}$, the restriction of $F \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$ to $\overline{D}$ (although divergence form is not needed, see [6]).

In the following, let the pair of $(E_{St}, Q_{St})$,

$$E_{St}(x) = \frac{1}{8\pi}\left(\frac{1}{|x|}\mathbb{I} + \frac{x \otimes x}{|x|^3}\right), \qquad Q_{St}(x) = \nabla\left(\frac{-1}{4\pi|x|}\right) = \frac{x}{4\pi|x|^3}, \tag{2.2}$$

denote the usual Stokes fundamental solution, where $\mathbb{I}$ is the $3 \times 3$ unity matrix and $x \otimes x = (x_i x_j)_{1\le i,j\le 3}$. Moreover, let $\nu$ be the exterior unit normal to the boundary $\partial D$, and

$$T = T(u, p) = \nabla u + (\nabla u)^T - p\mathbb{I} \tag{2.3}$$

be the Cauchy stress tensor. Then

$$\nu\cdot(T+F)=\left((\nu\cdot(T+F))',(\nu\cdot(T+F))_3\right)^T=\left(\sum_j (T_{ij}+F_{ij})\nu_j\right)_{1\le i\le 3}$$

where we used the decomposition $z=(z',z_3)^T,\ z'=(z_1,z_2)$, for $z\in\mathbb{R}^3$.

**Theorem 2.1** *Let $\omega=ae_3$ with $a\in\mathbb{R}\backslash\{0\}$. Given $f=\mathrm{div}F$ with $F\in C_0^\infty(\overline{D})^{3\times3}$, let $(u,p)$ be the solution to (2.1) subject to (1.2). Then it has the representation*

$$\begin{aligned} u(x)&=U_{1st}(x)+U_{2nd}(x)+\left(1+\frac{1}{|a|}\right)O\left(\frac{1}{|x|^3}\right),\\ p(x)&=P_{1st}(x)+O\left(\frac{1}{|x|^3}\right)\end{aligned}\tag{2.4}$$

*for $|x|\to\infty$ with*

$$\begin{aligned} U_{1\mathrm{st}}(x)&=\frac{1}{8\pi}\int\limits_{\partial D}(\nu\cdot(T+F))_3\,d\sigma_y\left(\frac{e_3}{|x|}+\frac{x_3x}{|x|^3}\right)\\ &=E_{St}(x)\left(\int\limits_{\partial D}(\nu\cdot(T+F))_3\,d\sigma_y\right)e_3,\end{aligned}\tag{2.5}$$

$$\begin{aligned} U_{2\mathrm{nd}}(x)&=\frac{1}{8\pi|x|^3}\begin{pmatrix}\alpha&-\beta&0\\ \beta&\alpha&0\\ 0&0&\alpha\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}-\frac{3\,(x\otimes x)}{8\pi|x|^5}\begin{pmatrix}\frac{\alpha'}{2}x_1\\ \frac{\alpha'}{2}x_2\\ \alpha_3x_3\end{pmatrix}\\ &=\frac{1}{8\pi|x|^3}\left(\beta(e_3\times x)+\left(\alpha_3-\frac{\alpha'}{2}\right)\frac{|x'|^2-2x_3^2}{|x|^2}x\right),\end{aligned}\tag{2.6}$$

$$P_{1\mathrm{st}}(x)=\int\limits_{\partial D}\{(\nu\cdot(\Delta u))\,y-p\nu+\nu\cdot F\}\,d\sigma_y\cdot Q_{St}(x).$$

*Finally,*

$$\begin{aligned}\alpha&=-\int\limits_{\partial D}y\cdot(\nu\cdot(T+F))d\sigma_y+\int\limits_D tr\,F\,dy=\alpha'+\alpha_3,\\ \alpha'&=-\int\limits_{\partial D}y'\cdot(\nu\cdot(T+F))'d\sigma_y+\int\limits_D(F_{11}+F_{22})dy,\end{aligned}$$

$$\alpha_3 = -\int_{\partial D} y_3(\nu\cdot(T+F))_3 d\sigma_y + \int_D F_{33} dy,$$

$$\beta = e_3\cdot\int_{\partial D} y\times(\nu\cdot(T+F))d\sigma_y + \int_D (F_{12}-F_{21})dy.$$

We set

$$N = \int_{\partial D} \nu\cdot(T(u,p)+F)\,d\sigma,$$

the total net force exerted on the boundary $\partial D$ by the fluid and the force term $F$.

From (2.5) we conclude that $e_3\cdot N$ is sufficient to control the rate of decay of $u(x)$, while all components of $N$ are needed for the case $\omega = 0$ to do so. In the second term (2.6) the first part of the coefficient $\beta$ of the rotating profile $e_3\times x$ is the third component of

$$\int_{\partial D} y\times(\nu\cdot(T(u,p)+F))\,d\sigma_y,$$

which stands for the total torque exerted on the boundary $\partial D$.

We may consider the case of homogeneous boundary condition $u|_{\partial D} = 0$ since the original boundary condition (1.2) can be reduced to this case by subtracting a suitable auxiliary function, see [6]. So the proof is based on the potential representation formula (2.10) below of the solution in terms of the fundamental solution of the Eq. (2.7) in the whole space. First of all we find a useful explicit representation of the fundamental solution. We say that the pair of $3\times 3$-matrix $\Gamma(x,y)$ and column vector $Q(x,y)$ is the fundamental solution of the equation

$$-\Delta u - (\omega\times x)\cdot\nabla u + \omega\times u + \nabla p = f, \qquad \mathrm{div} u = 0 \qquad (x\in\mathbb{R}^3) \qquad (2.7)$$

if the volume potentials

$$u(x) = \int_{\mathbb{R}^3}\Gamma(x,y)f(y)\,dy, \qquad p(x) = \int_{\mathbb{R}^3} Q(x,y)\cdot f(y)\,dy$$

solve (2.7) for all $f = (f_1, f_2, f_3)^T \in C_0^\infty(\mathbb{R}^3)^3$. Since div $[(\omega\times x)\cdot\nabla u - \omega\times u] = (\omega\times x)\cdot\nabla\mathrm{div}\,u = 0$, we obtain $\Delta p = \mathrm{div} f$, so that

$$Q(x,y) = \nabla_y\frac{1}{4\pi|x-y|} = Q_{St}(x-y),$$

see (2.2). Set

$$G(x,t) = (4\pi t)^{-3/2} e^{-|x|^2/(4t)}$$

which is the heat kernel in $\mathbb{R}^3$. Then we see that

$$\Gamma^0(x,y) = \int_0^\infty O(at)^T G(O(at)x - y, t)\, dt \tag{2.8}$$

is a fundamental solution of the operator $-\Delta - (\omega \times x) \cdot \nabla + \omega \times$, where

$$O(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The additional part arising from the pressure is given by

$$\begin{aligned}\Gamma^1(x,y) &= -\int_0^\infty \int_0^s \nabla_x \nabla_y [G(O(at)x - y, s)]\, dt ds \\ &= \int_0^\infty (4\pi s)^{-3/2} \int_0^s e^{-|O(at)x - y|^2/(4s)} \\ &\quad \cdot \left\{ \frac{\left(x - O(at)^T y\right) \otimes (O(at)x - y)}{4s^2} - \frac{1}{2s} O(at)^T \right\} dt ds, \end{aligned} \tag{2.9}$$

where $z \otimes w = (z_i w_j)_{1 \le i,j \le 3}$; for details see [6].

**Proposition 2.1** *Let $\omega = ae_3$. Then the pair $\{\Gamma(x,y), Q_{St}(x-y)\}$ with*

$$\Gamma(x,y) = \Gamma^0(x,y) + \Gamma^1(x,y)$$

*is a fundamental solution of the Eq. (2.7), where $\Gamma^0(x,y)$, $\Gamma^1(x,y)$ and $Q_{St}(x)$ are given by (2.8), (2.9) and (2.2), respectively.*

We next introduce the following potential representation formula (2.10) of the solution $u(x)$ to (2.1) in terms of the fundamental solution $\Gamma(x,y)$.

**Proposition 2.2** *Let $f \in C_0^\infty(\overline{D})^3$, the restriction of $f \in C_0^\infty(\mathbb{R}^3)^3$ to $\overline{D}$. Then the solution $(u,p)$ to (2.1) with $u|_{\partial D} = 0$ can be represented as*

$$u(x) = \int_D \Gamma(x,y) f(y)\, dy + \int_{\partial D} \Gamma(x,y)\, (\nu \cdot T(u,p))\, (y)\, d\sigma_y, \tag{2.10}$$

$$p(x) = \int_D Q_{St}(x-y)\cdot f(y), dy + \int_{\partial D} \frac{\nu\cdot(\nabla p - f)(y)}{4\pi|x-y|}\, d\sigma_y \\ - \int_{\partial D} \nu\cdot Q_{St}(x-y)p(y)\, d\sigma_y. \tag{2.11}$$

*Here, $Q_{St}(x)$ is given by (2.2) and the formula (2.11) for the pressure holds true even for the boundary condition $u|_{\partial D} = \omega\times x$.*

For the proof, we employ the following Green formula (2.12) associated with the Stokes equation with rotation effect: if div $u$ = div $v = 0$, then

$$\int_W [v\cdot\{\Delta u + (\omega\times x)\cdot\nabla u - \omega\times u - \nabla p\} \\ -u\cdot\{\Delta v - (\omega\times x)\cdot\nabla v + \omega\times v + \nabla q\}]\, dx \\ = \int_{\partial W} [\nu\cdot\{v\cdot T(u,p) - u\cdot T(v,-q)\} + \nu\cdot(\omega\times x)(u\cdot v)]\, d\sigma, \tag{2.12}$$

where $W\subset\mathbb{R}^3$ is any bounded domain with smooth boundary $\partial W$.

We fix $R>0$ such that $f(y)=0$ for $|y|\ge R$. In view of (2.10), we may assume $|y|\le R$ and $|x|\ge 2R$. Our task is now to find leading terms $\Phi_1(x)\sim 1/|x|$ and $\Phi_2(x,y)\sim 1/|x|^2$ so that the fundamental solution is represented as

$$\Gamma(x,y) = \Phi_1(x) + \Phi_2(x,y) + \left(1+\frac{1}{|a|}\right) O\left(\frac{1}{|x|^3}\right) \tag{2.13}$$

for $|x|\to\infty$. The last term means that

$$|\Gamma(x,y) - \{\Phi_1(x)+\Phi_2(x,y)\}| \le \left(1+\frac{1}{|a|}\right)\frac{C_R}{|x|^3}$$

for $|x|\ge 2R\ge 2|y|$, where $C_R>0$ is independent of $a\in\mathbb{R}\backslash\{0\}$. Let $\Gamma^0(x,y)$ and $\Gamma^1(x,y)$ be as in (2.8) and (2.9); then, one can show the following propositions.

**Proposition 2.3** *For $|y|\le R$ and $|x|\to\infty$ we have*

$$\Gamma^0(x,y) = \Phi_1^0(x) + \Phi_2^0(x,y) + \left(1+\frac{1}{|a|}\right) O\left(\frac{1}{|x|^3}\right)$$

*where*

$$\Phi_1^0(x) = \frac{1}{4\pi|x|}\, e_3\otimes e_3, \\ \Phi_2^0(x,y) = \frac{1}{8\pi|x|^3}\begin{pmatrix} x'\cdot y' & (e_3\times x)\cdot y & 0 \\ -(e_3\times x)\cdot y & x'\cdot y' & 0 \\ 0 & 0 & 2x_3y_3 \end{pmatrix}. \tag{2.14}$$

**Proposition 2.4** *For $|y| \leq R$ and $|x| \to \infty$ one has*

$$\Gamma^1(x, y) = \Phi_1^1(x) + \Phi_2^1(x, y) + \left(1 + \frac{1}{|a|}\right) O\left(\frac{1}{|x|^3}\right)$$

*with*

$$\Phi_1^1(x) = \frac{1}{8\pi|x|^3} \begin{pmatrix} 0 & 0 & x_1x_3 \\ 0 & 0 & x_2x_3 \\ 0 & 0 & -|x'|^2 \end{pmatrix},$$

$$\Phi_2^1(x, y) = \frac{-1}{8\pi|x|^3} \left\{ x \otimes y + \begin{pmatrix} x_1y_1 & x_1y_2 & 0 \\ x_2y_1 & x_2y_2 & 0 \\ 0 & 0 & 2x_3y_3 \end{pmatrix} \right\} + \Psi(x, y),$$

*where*

$$\Psi(x, y) = \frac{3x}{8\pi|x|^5} \otimes \left\{ \frac{|x'|^2}{2} (y', 0)^T + x_3^2 (0, 0, y_3)^T \right\}. \tag{2.15}$$

It follows from Propositions 2.3 and 2.4 that (2.13) holds with $\Phi_1 = \Phi_1^0 + \Phi_1^1$ and $\Phi_2 = \Phi_2^0 + \Phi_2^1$, i.e.,

$$\Phi_1(x) = \frac{1}{8\pi|x|^3} \begin{pmatrix} 0 & 0 & x_1x_3 \\ 0 & 0 & x_2x_3 \\ 0 & 0 & |x|^2 + x_3^2 \end{pmatrix}, \tag{2.16}$$

$$\Phi_2(x, y) = \frac{-1}{8\pi|x|^3} \{x \otimes y - (e_3 \times x) \otimes (e_3 \times y)\} + \Psi(x, y),$$

where $\Psi(x, y)$ is given by (2.15). This combined with (2.10) provides the desired asymptotic representation (2.4) of $u(x)$.

For the proof of Proposition 2.3, we use the following elementary decay estimate based on the oscillating terms $\cos at$, $\sin at$.

**Lemma 2.1** *Let $a \in \mathbb{R}\backslash\{0\}$, $m > 2$ and $c > 0$. Given $n \in \mathbb{N}$ arbitrarily, there is a constant $K = K(n, m, c) > 0$ such that*

$$\left| \int_0^\infty \begin{pmatrix} \cos at \\ \sin at \end{pmatrix} t^{-m/2} e^{-c|x|^2/t} \, dt \right| \leq \frac{K}{|a|^n |x|^{2n+m-2}} \tag{2.17}$$

*for $x \in \mathbb{R}^3\backslash\{0\}$.*

In the rest of this section, we give the proof of Proposition 2.3 and omit that of Proposition 2.4 which is much more complicated although the idea is more or less the same, see [6] for the details.

*Proof of Proposition 2.3* We employ the Taylor formula (with respect to $y$)

$$e^{-|O(at)x-y|^2/(4t)} = e^{-|x|^2/(4t)} + e^{-|x|^2/(4t)} \frac{(O(at)x)\cdot y}{2t} + \frac{1}{2} e^{-|O(at)x-\theta y|^2/(4t)} y^T \frac{(O(at)x-\theta y)\otimes(O(at)x-\theta y)-2t\mathbb{I}}{4t^2} y \tag{2.18}$$

with some $\theta \in (0,1)$. We decompose $\Gamma^0(x,y)$ as

$$\Gamma^0(x,y) = \Gamma^{01}(x,y) + \Gamma^{02}(x,y) + \Gamma^{03}(x,y)$$

correspondingly to (2.18). By (2.17) ($n = 1,\ m = 3$) we find

$$\Gamma^{01}(x,y) = \Phi_1^0(x) + \frac{1}{|a|}\, O\left(\frac{1}{|x|^3}\right),$$

where $\Phi_1^0(x)$ is as in (2.14). Concerning

$$\Gamma^{02}(x,y) = \int_0^\infty O(at)^T (4\pi t)^{-3/2} e^{-|x|^2/(4t)} \frac{(O(at)x)\cdot y}{2t}\, dt,$$

we note

$$(O(at)x)\cdot y = (x'\cdot y')\cos at + ((e_3\times x)\cdot y)\sin at + x_3 y_3$$

to find

$$\frac{(O(at)x)\cdot y}{2}\, O(at)^T = \frac{1}{4} A(x,y) + \mathcal{R}(x,y,t)$$

with

$$A(x,y) = \begin{pmatrix} x'\cdot y' & (e_3\times x)\cdot y & 0 \\ -(e_3\times x)\cdot y & x'\cdot y' & 0 \\ 0 & 0 & 2x_3y_3 \end{pmatrix}$$

where the remainder $\mathcal{R}$ consists of the oscillating terms $\cos kat$ and $\sin kat$, $k = 1, 2$ and its degree is one with respect to $x$. By (2.17) ($n = 1,\ m = 5$) we are led to

$$\Gamma^{02}(x,y) = \frac{1}{8\pi|x|^3} A(x,y) + \frac{1}{|a|} O\left(\frac{1}{|x|^4}\right).$$

Finally, it is easy to see that $\left|\Gamma^{03}(x,y)\right| \le C/|x|^3$ without using (2.17). □

## 3 Navier–Stokes flow

By Theorem 2.1 we know what kind of effect on the profile the rotation of the obstacle causes. For the Stokes flow, the leading term is the third column vector of the usual Stokes fundamental solution (2.2) and possesses

- (i) symmetry about the axis of rotation ($x_3$-axis);
- (ii) $(-1)$-homogeneity.

Furthermore, the important quantity which controls the rate of decay is $e_3 \cdot N$, where

$$N = \int_{\partial D} \nu \cdot T(u,p)\, d\sigma \tag{3.1}$$

is the net force (case $F = 0$ in Theorem 2.1) and $T(u,p)$ is given by (2.3). Thus, it is reasonable to expect that the leading term $U$ of the Navier–Stokes flow for (1.1)–(1.2) still keeps the properties (i), (ii) above and solves

$$-\Delta U - (\omega \times x)\cdot \nabla U + \omega \times U + \nabla P + U \cdot \nabla U = (e_3 \cdot N)\, e_3 \delta, \quad \mathrm{div} U = 0 \tag{3.2}$$

in $\mathcal{D}'(\mathbb{R}^3)$, where $\delta$ denotes the Dirac measure at 0.

The present section concludes that this conjecture is correct. Here, we should note the relation

$$e_3 \cdot N = e_3 \cdot \widetilde{N} \tag{3.3}$$

where

$$\widetilde{N} = \int_{\partial D} \nu \cdot \{T(u,p) - u \otimes u\}\, d\sigma; \tag{3.4}$$

this is a consequence of $u|_{\partial D} = \omega \times x$ together with $e_3 \cdot (\omega \times x) = 0$. Note also that for all vector fields which are symmetric about the $x_3$-axis

$$(e_3 \times x)\cdot \nabla U - e_3 \times U = 0. \tag{3.5}$$

In fact, because such vector fields must be of the form

$$U = (V(r,x_3)\cos\theta,\, V(r,x_3)\sin\theta,\, U_3(r,x_3))^T$$

in cylindrical coordinates $r, \theta, x_3$, we see that $(e_3 \times x) \cdot \nabla U = \partial_\theta U = e_3 \times U$. Further, (3.5) holds in $\mathcal{D}'(\mathbb{R}^3)$ when $U \sim 1/|x|$ around $x = 0$. Thus the candidate above for the leading term solves (3.7) with $k = e_3 \cdot N$ and, due to [18], it must be a member of the family of Landau solutions explained below.

Let $b \in \mathbb{R}^3$ be a prescribed vector, that we call the Landau parameter. Then, among nontrivial smooth solutions of (1.3), Landau [17] found an exact solution, called the Landau solution, which satisfies:

- axisymmetry about $\mathbb{R}b$;
- the homogeneity

$$u(x) = \frac{1}{|x|}\, u\left(\frac{x}{|x|}\right), \qquad p(x) = \frac{1}{|x|^2}\, p\left(\frac{x}{|x|}\right);$$

- $-\Delta u + \nabla p + u \cdot \nabla u = b\delta$ in $\mathcal{D}'(\mathbb{R}^3)$.

When $b$ is parallel to $e_3$, the Landau solution is of the form

$$\begin{cases} u_1(x) = 2\frac{x_1(cx_3-|x|)}{|x|(c|x|-x_3)^2}, & u_2(x) = 2\frac{x_2(cx_3-|x|)}{|x|(c|x|-x_3)^2}, \\ u_3(x) = 2\frac{c|x|^2-2x_3|x|+cx_3^2}{|x|(c|x|-x_3)^2}, & p(x) = 4\frac{cx_3-|x|}{|x|(c|x|-x_3)^2} \end{cases} \tag{3.6}$$

with parameter $c \in (-\infty, -1) \cup (1, \infty)$, and it satisfies

$$-\Delta u + \nabla p + u \cdot \nabla u = ke_3\delta, \qquad \mathrm{div} u = 0 \tag{3.7}$$

in $\mathcal{D}'(\mathbb{R}^3)$, where $k$ is given by

$$k = k(c) = \frac{8\pi c}{3(c^2-1)}\left(2 + 6c^2 - 3c(c^2-1)\log\frac{c+1}{c-1}\right). \tag{3.8}$$

For this calculation we refer to [3]. Since the function $k(\cdot)$ is monotonically decreasing on each of $(-\infty, -1)$ and $(1, \infty)$, and fulfills

$$k(c) \to 0 \quad (|c| \to \infty); \quad k(c) \to -\infty \quad (c \to -1); \quad k(c) \to \infty \quad (c \to 1),$$

for every $\tilde{k} \in \mathbb{R}\backslash\{0\}$ there is a unique $c \in (-\infty, -1) \cup (1, \infty)$ such that $k(c) = \tilde{k}$. When $\tilde{k} = 0$, we may understand $(u, p) = (0, 0)$ as the solution (3.6) with $|c| \to \infty$.

Now, given a smooth solution $(u, p)$ of the Navier–Stokes problem (1.1)–(1.2), we take $N$ and $\widetilde{N}$ as in (3.1) and (3.4). Let $(U, P)$ be the Landau solution with the Landau parameter

$$b = (e_3 \cdot N)e_3 = (e_3 \cdot \widetilde{N})e_3,$$

see (3.3). Namely, $(U, P)$ is given by (3.6) with $c$ which is determined by $k(c) = e_3 \cdot N$ with $k(\cdot)$ as in (3.8); it is the trivial solution in case $e_3 \cdot N = 0$. Since $U$ is symmetric about the $x_3$-axis, we have (3.5) so that $(U, P)$ solves (3.2) in $\mathcal{D}'(\mathbb{R}^3)$. Now we are in a position to state our main result.

**Theorem 3.1** *Let $\omega = ae_3$ with $a \in \mathbb{R}\backslash\{0\}$. For each $q_0 \in (3/2, 3)$ there exists a constant $\eta = \eta(q_0) > 0$ such that if $u$ is a smooth solution to (1.1)–(1.2) and satisfies*

$$\sup_{x\in D} |x||u(x)| + |e_3 \cdot N| \leq \eta, \tag{3.9}$$

*then, for every $q \in (q_0, 3)$, we have*

$$u - U|_D \in L_q(D), \qquad \|u - U\|_{L_q(D)} \leq C(|a|^{-3/q+1} + 1) \tag{3.10}$$

*with some $C = C(q) > 0$, where $U$ is the Landau solution as explained above and satisfies (3.2).*

This theorem tells us that the remainder $u - U$ possesses better summability suggesting the pointwise decay $1/|x|^2$ at infinity; in this sense, the Landau solution $U$ is the leading term of the small solution $u$. Because the leading term of the usual Navier–Stokes flow ($\omega = 0$) found by Korolev-Šverák [15] is different (it is the Landau solution with $b = \widetilde{N}$), it is reasonable that the remainder $u - U$ in our case possesses singular behavior for $a \to 0$ as in (3.10).

To give a sketch of the proof of Theorem 3.1 we take $R > 0$ so that $\mathbb{R}^3\backslash D \subset B_{R-2} = \{|x| < R - 2\}$ and choose a cut-off function $\phi \in C_0^\infty(B_R)$ such that $\phi(x) = 1$ for $|x| \leq R-1$. Given a smooth solution $(u, p)$ of (1.1)–(1.2) satisfying $u(x) = O(1/|x|)$ for $|x| \to \infty$, we set

$$\widetilde{u} = (1 - \phi)u + S[u \cdot \nabla\phi], \qquad \widetilde{p} = (1 - \phi)p;$$

here $S$ denotes the Bogovskii operator ([1]) in the bounded domain $A_R = \{R - 2 < |x| < R\}$ such that $S[u \cdot \nabla\phi] \in C_0^\infty(A_R)$. To apply the operator $S$ note that $\int_{A_R} u \cdot \nabla\phi\, dx = 0$ on account of $u|_{\partial D} = \omega \times x$ together with $\int_{\partial D} \nu \cdot (\omega \times x) = 0$. The pair $(\widetilde{u}, \widetilde{p})$ obeys

$$-\Delta\widetilde{u} - (\omega \times x) \cdot \nabla\widetilde{u} + \omega \times \widetilde{u} + \nabla\widetilde{p} + \widetilde{u} \cdot \nabla\widetilde{u} = g, \quad \mathrm{div}\widetilde{u} = 0 \quad (x \in \mathbb{R}^3)$$

with some $g \in C_0^\infty(A_R)$ which satisfies

$$\int_{\mathbb{R}^3} g(x)\, dx = \widetilde{N}, \tag{3.11}$$

where $\widetilde{N}$ is the net force (3.4). We note that (3.11) follows only from the structure of the equation; in other words, we don't need any exact form of $g$.

Let $(U, P)$ be the Landau solution for $b = (e_3 \cdot N)e_3$. To regularize $(U, P)$ around $x = 0$, one may follow the same manner using $\phi$ and $S$ as above to define the pair $(\widetilde{U}, \widetilde{P})$; then, it enjoys

$$-\Delta\widetilde{U} - (\omega \times x) \cdot \nabla\widetilde{U} + \omega \times \widetilde{U} + \nabla\widetilde{P} + \widetilde{U} \cdot \nabla\widetilde{U} = h, \quad \mathrm{div}\widetilde{U} = 0 \quad (x \in \mathbb{R}^3)$$

with some $h \in C_0^\infty(A_R)$ which satisfies

$$\int_{\mathbb{R}^3} h(x)\,dx = (e_3 \cdot N)e_3. \tag{3.12}$$

Therefore, $(v, \pi) = (\widetilde{u} - \widetilde{U}, \widetilde{p} - \widetilde{P})$ should obey

$$\begin{aligned} -\Delta v - (\omega \times x)\cdot \nabla v + \omega \times v + \nabla \pi + v \cdot \nabla \widetilde{u} + \widetilde{U} \cdot \nabla v = g - h,\\ \mathrm{div} v = 0, \end{aligned} \tag{3.13}$$

in $\mathbb{R}^3$. For a moment we may regard (3.13) as a linear problem for the unknown $v$ and get the Propositions 3.1, 3.2 below. In what follows, for $1 < q < \infty$, we denote by $\|\cdot\|_q$ and by $\|\cdot\|_{q,\infty}$ the norms of $L_q(\mathbb{R}^3)$ and $L_{q,\infty}(\mathbb{R}^3)$ (weak-$L_q$ space), respectively. Note that both $\widetilde{u}$ and $\widetilde{U}$ are in $L_{3,\infty}(\mathbb{R}^3)$.

**Proposition 3.1** *There is a constant $\gamma > 0$ such that the solution of the problem (3.13) in the class $v \in L_{3,\infty}(\mathbb{R}^3)$ is unique provided*

$$\|\widetilde{u}\|_{3,\infty} + \|\widetilde{U}\|_{3,\infty} \le \gamma.$$

**Proposition 3.2** *For each $q_0 \in (3/2, 3)$ there is a constant $\widetilde{\gamma}(q_0) \in (0, \gamma]$ such that the problem (3.13) possesses a solution*

$$v \in L_{q_0,\infty}(\mathbb{R}^3) \cap L_{3,\infty}(\mathbb{R}^3) \tag{3.14}$$

*subject to*

$$\|v\|_q \le C(|a|^{-3/q+1} + 1) \quad \textit{for all} \quad q \in (q_0, 3)$$

*with some $C = C(q) > 0$ provided*

$$\|\widetilde{u}\|_{3,\infty} + \|\widetilde{U}\|_{3,\infty} \le \widetilde{\gamma}(q_0). \tag{3.15}$$

These propositions yield Theorem 3.1. In fact, it is obvious that (3.9) implies (3.15) when we take a suitable constant $\eta = \eta(q_0) > 0$; then, we see that $\widetilde{u} - \widetilde{U}$ is in the class (3.14) and also enjoys

$$\widetilde{u} - \widetilde{U} \in L_q(\mathbb{R}^3), \qquad \|\widetilde{u} - \widetilde{U}\|_q \le C(|a|^{-q/3+1} + 1)$$

for all $q \in (q_0, 3)$ because the only solution is $\widetilde{u} - \widetilde{U}$ by Proposition 3.1. Since $u - U = \widetilde{u} - \widetilde{U}$ for $|x| \ge R$, we obtain (3.10).

Propositions 3.1 and 3.2 are easily obtained by employing the estimate in weak-$L_q$ spaces (due to [5]) for weak solutions to the linear whole space problem

$$-\Delta u - (\omega \times x)\cdot \nabla u + \omega \times u + \nabla p = f, \qquad \mathrm{div} u = 0 \qquad (x \in \mathbb{R}^3)$$

and by the expansion (2.13) of the fundamental solution $\Gamma(x, y)$ with (2.16). We omit the details [7] but mention that the crucial point is the following lemma, which tells us the reason why good summability properties at infinity can be deduced in Proposition 3.2.

**Lemma 3.1** *The function*

$$v_0(x) = \int_{\mathbb{R}^3} \Gamma(x, y)(g - h)(y)\, dy$$

*satisfies*

$$\begin{aligned} v_0(x) &= O\left(\frac{1}{|x|^2}\right) \qquad as\ |x| \to \infty, \\ \|v_0\|_{q,\infty} &\le C(|a|^{-3/q+1} + 1) \qquad for\ \forall q \in [3/2, 3] \end{aligned} \tag{3.16}$$

*with some* $C = C(q) > 0$.

*Proof* Let $|x| \ge 2R$. Since $g - h \in C_0^\infty(A_R)$, it follows from (2.13) with (2.16) that

$$v_0(x) = e_3 \cdot \int_{\mathbb{R}^3} (g - h)(y)\, dy\, \frac{1}{8\pi}\left(\frac{e_3}{|x|} + \frac{x_3 x}{|x|^3}\right) + \left(1 + \frac{1}{|a|}\right) O\left(\frac{1}{|x|^2}\right).$$

We collect (3.3), (3.11) and (3.12) to find

$$e_3 \cdot \int_{\mathbb{R}^3} (g - h)(y)\, dy = e_3 \cdot \widetilde{N} - e_3 \cdot N = 0,$$

which proves (3.16) for $q = 3/2$. This combined with $\|v_0\|_{3,\infty} \le C$ completes the proof. □

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