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Complex powers of classical SG-pseudodifferential operators

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Abstract Under a suitable ellipticity condition, we show that classical SG-pseudodifferential operators of nonnegative order possess complex powers. We show that the powers are again classical and derive an explicit formula for all homogeneous components.

Keywords Complex power · Weighted symbols · Noncompact manifolds

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1 Introduction

In his classic paper [21], Seeley in 1967 showed that a suitably parameter-elliptic pseudodifferential operator A of order $\mu > 0$ on a smooth closed manifold possesses *complex powers*. More precisely, one can define A^z for any $z \in \mathbb{C}$ (essentially by means of a Dunford integral, integrating the resolvent against λ^z) and show that A^z is a pseudodifferential operator of order μz. His results on complex powers were quite important for applications, e.g., for the study of eigenvalue asymptotics, index theory, or determinants of elliptic operators. Due to their importance, from there on complex powers for various classes of pseudodifferen-

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tial operators have been widely investigated in the literature. Without making any claim to completeness let us mention [1], [2], [8], [9], [13], [17], and [19].

In the present paper we shall investigate complex powers of a class of operators on $\mathbb{R}^n$, the so-called SG-pseudodifferential operators. These are operators $A = \mathrm{op}(a)$ with (matrix valued) symbols in $S^{\mu,m}(\mathbb{R}^n \times \mathbb{R}^n)$, i.e.

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leq C_{\alpha\beta}\, [x]^{m-|\beta|}[\xi]^{\mu-|\alpha|},$$

for all $\alpha,\beta \in \mathbb{N}_0^n$, where $[\cdot]$ denotes a smooth, positive function on $\mathbb{R}^n$ that coincides with the euclidean norm outside a ball. The investigation of this symbol class goes back to works of Parenti [16] and Cordes [4]. Based on an approach of Kumano-go [12], Schrohe in [20] has shown the existence of complex powers for operators of order $\mu, m \geq 0$ that are parameter-elliptic in a suitable sense. He also considers operators on so-called SG-manifolds which are, roughly speaking, generalizations of manifolds that at infinity have the structure of an opening cone.

A subspace of $S^{\mu,m}(\mathbb{R}^n \times \mathbb{R}^n)$ is that of all *classical* (occasionally also called *poly-homogeneous*) symbols, which have both in the variable and the covariable asymptotic expansions into components that are homogeneous of one step decreasing order. For details see Section 2, where we give an exposition of the corresponding calculus. Such symbols naturally arise in the parametrix construction of elliptic differential operators with polynomial coefficients as well as in the study of pseudodifferential operators on manifolds with singularities, see for example [7]. In [6] classical SG-Fourier Integral Operators have been considered.

The aim of this article is to show that the complex powers of a classical elliptic operator again are classical. This we shall derive in Section 3, giving also explicit formulas for all homogeneous components of the complex powers. It is worthwhile to point out that the formulas here obtained could be of some interest in connection with the study of the asymptotic behavior of eigenvalues of SG-classical operators in the spirit of [14].

While our approach follows that of Seeley, Kumano-go, Schrohe, let us mention that Guillemin [10] developed another method for the construction of complex powers for operators in a so-called Weyl algebra. Such an algebra is defined by certain axioms and comprises a generalization of pseudodifferential algebras on compact manifolds. Based on his results and those of Bucicovschi [3], Ammann, Lauter, Nistor, and Vasy in [1] develop an axiomatic approach to complex powers also for operators on noncompact manifolds. Their results on the existence of complex powers include ours for the special case of positive operators of order $m = 0$ and $\mu \geq 0$.

Our results also extend to operators on certain SG-manifolds. However, to keep the exposition short, we shall not go into details here.

2 The calculus for SG-pseudodifferential operators

We summarize the calculus for so-called *SG*-pseudodifferential operators. Besides the usual estimates in the covariable, the symbols of these operators have an analogous behaviour also in the variable. This additional control of the growth in the variable allows a calculus in weighted Sobolev spaces, including a concept of ellipticity which is equivalent to the Fredholm property. For more details we refer the reader to [11] or [7].

2.1 Symbol classes

In the following we set $\dot{\mathbb{R}}^n = \mathbb{R}^n \setminus \{0\}$ and let $[\cdot]$ denote a smoothed norm function, i.e. $x \mapsto [x]$ is smooth, positive, and $[x] = |x|$ for $|x| \geq 1$.

Definition 2.1 Let $\mu, m \in \mathbb{R}$. The space $S^{\mu,m} = S^{\mu,m}(\mathbb{R}^n \times \mathbb{R}^n)$ of symbols of order (μ, m) consists of all smooth functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ (or with values in matrices in case of systems) satisfying

$$\sup_{x\in\mathbb{R}^n,\,\xi\in\mathbb{R}^n} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)|[x]^{-m+|\beta|}[\xi]^{-\mu+|\alpha|} < \infty \tag{2.1}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. If $\mu, m \in \mathbb{C}$ we set $S^{\mu,m} = S^{\mathrm{Re}\,\mu,\mathrm{Re}\,m}$.

The expressions on the left-hand side of (2.1) define a countable system of semi-norms which give $S^{\mu,m}$ a Fréchet space structure. Similarly, we have spaces $S^m = S^m(\mathbb{R}^n_x)$ and $S^\mu = S^\mu(\mathbb{R}^n_\xi)$ of symbols depending only on one variable. Note that

$$S^{-\infty,-\infty} := \bigcap_{\mu,m\in\mathbb{R}} S^{\mu,m} = \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n),$$

the space of rapidly decreasing functions on $\mathbb{R}^n \times \mathbb{R}^n$.

A function $b : \dot{\mathbb{R}}^n \to E$ with values in a Fréchet space E is called *positively homogeneous* of order $z \in \mathbb{C}$, if b is smooth and

$$b(ty) = t^z b(y) \qquad \forall\, t > 0 \quad \forall\, y \neq 0.$$

The space of all such functions will be denoted by $S^{(z)}(\dot{\mathbb{R}}^n, E)$. If $E = \mathbb{C}$ we drop E from the notation. The canonical isomorphism with $\mathscr{C}^\infty(S^{n-1}, E)$, the smooth functions on the unit sphere, induces a Fréchet topology on $S^{(z)}(\dot{\mathbb{R}}^n, E)$.

A symbol $b \in S^z$ will be called *classical* of order $z \in \mathbb{C}$ if for each $j \in \mathbb{N}_0$ there exists a function $b_{(z-j)}$ which is positively homogeneous of degree $z - j$ such that, for any $N \in \mathbb{N}$,

$$b_N := b - \sum_{j=0}^{N-1} \chi b_{(z-j)} \in S^{z-N}.$$

Here, χ is an arbitrary zero excision function. The functions $b_{(z-j)}$ are uniquely determined and are called the *homogeneous components* of b. We will write S^z_{cl} for the space of all such symbols b. The maps $b \mapsto b_{(z-j)} : S^z_{\mathrm{cl}} \to S^{(z-j)}$ and $b \mapsto b_N : S^z_{\mathrm{cl}} \to S^{z-N}$ induce a Fréchet topology on S^z_{cl}.

We then define, for $m, \mu \in \mathbb{C}$,

$$S^{\mu,(m)}_{\mathrm{cl}} = S^{(m)}(\dot{\mathbb{R}}^n_x, S^\mu_{\mathrm{cl}}(\mathbb{R}^n_\xi)), \qquad S^{(\mu),m}_{\mathrm{cl}} = S^{(\mu)}(\dot{\mathbb{R}}^n_\xi, S^m_{\mathrm{cl}}(\mathbb{R}^n_x)). \tag{2.2}$$

Definition 2.2 Let $m,\mu\in\mathbb{C}$. The space $S_{\mathrm{cl}}^{\mu,m}=S_{\mathrm{cl}}^{\mu,m}(\mathbb{R}^n\times\mathbb{R}^n)$ consists of all symbols $a\in S^{\mu,m}$ having the following property: For each $j,k\in\mathbb{N}_0$ there exist (uniquely determined) functions

$$a_{(m-j)}\in S_{\mathrm{cl}}^{\mu,(m-j)},\qquad a^{(\mu-k)}\in S_{\mathrm{cl}}^{(\mu-k),m},$$

such that for each $N\in\mathbb{N}$

$$a-\sum_{j=0}^{N-1}\chi a_{(m-j)}\;\in\;S^{\mu,m-N},\qquad a-\sum_{k=0}^{N-1}\kappa a^{(\mu-k)}\;\in\;S^{\mu-N,m}.$$

Here, $\chi=\chi(x)$ and $\kappa=\kappa(\xi)$ are arbitrary zero excision functions.

The prototype of such symbols are finite linear combinations of symbols of the form $a_1(x)a_2(\xi)$ where both a_1 and a_2 are classical. In fact it can be shown that $S_{\mathrm{cl}}^{\mu,m}=S_{\mathrm{cl}}^{\mu}\,\widehat{\otimes}_\pi S_{\mathrm{cl}}^{m}$ (completed projective tensor product).

For each fixed ξ, the functions $x\mapsto a^{(\mu-k)}(x,\xi)$ belong to S_{cl}^{m}. Therefore, they have homogeneous components

$$(a^{(\mu-k)})_{(m-j)}\;\in\;S^{(\mu-k),(m-j)}:=S^{(m-j)}(\dot{\mathbb{R}}_x^n,S^{(\mu-k)}(\dot{\mathbb{R}}_\xi^n)).$$

Analogously, we can fix x and consider the homogeneous components $\big(a_{(m-j)}\big)^{(\mu-k)}$ of $\xi\mapsto a_{(m-j)}(x,\xi)$. As a matter of fact, the resulting components coincide and it is well-defined to set

$$a_{(m-j)}^{(\mu-k)}:=(a^{(\mu-k)})_{(m-j)}=\big(a_{(m-j)}\big)^{(\mu-k)}\;\in\;S^{(\mu-k),(m-j)}.\tag{2.3}$$

Lemma 2.1 *Let $a\in S_{\mathrm{cl}}^{\mu,m}$. Then, for any $N\geq 1$, $a-a_N\in S_{\mathrm{cl}}^{\mu-N,m-N}$ with*

$$a_N:=\sum_{j=0}^{N-1}\Big(\kappa a^{(\mu-j)}+\chi a_{(m-j)}\Big)-\kappa\chi\sum_{j,k=0}^{N-1}a_{(m-j)}^{(\mu-k)}.$$

Proof We first observe that

$$a-a_N=a-\sum_{j=0}^{N-1}\kappa a^{(\mu-j)}-\sum_{j=0}^{N-1}\chi\left(a_{(m-j)}-\kappa\sum_{k=0}^{N-1}a_{(m-j)}^{(\mu-k)}\right)$$

belongs to $S^{\mu-N,m}$ and, analogously, $a-a_N\in S^{\mu,m-N}$. This shows that

$$a-a_N\in S^{\mu-N,m}\cap S^{\mu,m-N}\subset S^{\mu-N/2,m-N/2}.$$

On the other hand, for every $M\geq 1$,

$$\begin{aligned}a_{M+1}-a_M\;=\;&\kappa\left(a^{(\mu-M)}-\chi\sum_{k=0}^{M-1}a_{(m-k)}^{(\mu-M)}\right)+\\&+\chi\left(a_{(m-M)}-\kappa\sum_{k=0}^{M-1}a_{(m-M)}^{(\mu-k)}\right)-\kappa\chi a_{(m-M)}^{(\mu-M)}\end{aligned}$$

belongs to $S^{\mu-M,m-M}$. We then can write

$$a-a_N=a-a_{2N}+a_{2N}-a_{2N-1}+\ldots+a_{N+1}-a_N$$

and this concludes the proof. □

In particular, $a \in S^{\mu,m}_{\text{cl}}$ belongs to $S^{\mu-1,m-1}_{\text{cl}}$ if and only if the homogeneous components $a^{(\mu)}$ and $a_{(m)}$ vanish identically.

2.2 Composition and ellipticity

We denote by $\mathscr{S}(\mathbb{R}^n)$ the rapidly decreasing functions on $\mathbb{R}^n$. For $a \in S^{\mu,m}$ we define the operator $\mathrm{op}(a) = a(x,D) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ by

$$[\mathrm{op}(a)u](x) = \int e^{ix\xi} a(x,\xi)\,\widehat{u}(\xi)\,đ\xi, \qquad x \in \mathbb{R}^n, \quad (đ\xi = (2\pi)^{-n}d\xi).$$

The class of *SG*-operators behaves well under composition:

Theorem 2.1 *Let $a \in S^{\mu_0,m_0}$ and $b \in S^{\mu_1,m_1}$ be given. Then there exists a unique symbol $a\#b \in S^{\mu_0+\mu_1,m_0+m_1}$ such that*

$$\mathrm{op}(a)\mathrm{op}(b) = \mathrm{op}(a\#b).$$

$a\#b$ is called the Leibniz product *of a and b. Explicitly, for each $N \in \mathbb{N}$,*

$$a\#b(x,\xi) = \sum_{|\alpha|=0}^{N-1} \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi) \ + r_N(x,\xi)$$

with a remainder $r_N \in S^{\mu_0+\mu_1-N,m_0+m_1-N}$ given by the expression

$$N \sum_{|\alpha|=N} \int_0^1 \frac{(1-\theta)^{N-1}}{\alpha!} \iint e^{iy\eta}\partial_\xi^\alpha a(x,\xi+\theta\eta) D_x^\alpha b(x+y,\xi)\,dy\,đ\eta\,d\theta$$

where the double integral is understood as an oscillatory integral.

Clearly, if both a and b are classical then so is $a\#b$. The homogeneous components are calculated according to the rule

$$(a\#b)^{(\mu_0+\mu_1-k)} = \sum_{k_0+k_1+|\alpha|=k} \frac{1}{\alpha!}\partial_\xi^\alpha a^{(\mu_0-k_0)} D_x^\alpha b^{(\mu_1-k_1)}$$

and similarly for the x-components. For the mixed components we have

$$(a\#b)^{(\mu_0+\mu_1-k)}_{(m_0+m_1-j)} = \sum_{\substack{k_0+k_1+|\alpha|=k \\ j_0+j_1+|\alpha|=j}} \frac{1}{\alpha!}\partial_\xi^\alpha a^{(\mu_0-k_0)}_{(m_0-j_0)} D_x^\alpha b^{(\mu_1-k_1)}_{(m_1-j_1)}.$$

Definition 2.3 A symbol $a \in S^{\mu,m}$ is called *elliptic*, if there exist constants $R \geq 0$ and $C > 0$ such that $a(x,\xi)$ is invertible for all $|(x,\xi)| \geq R$ and

$$|a(x,\xi)^{-1}| \leq C\,[x]^{-\mathrm{Re}\,m}[\xi]^{-\mathrm{Re}\,\mu} \qquad \forall\,|(x,\xi)| \geq R.$$

Setting $b_0(x,\xi) = \chi(x,\xi)a(x,\xi)^{-1}$ with an elliptic symbol $a \in S^{\mu,m}$ and a suitable zero excision function χ, we get a symbol $b_0 \in S^{-\mu,-m}$ such that both $a\#b_0 - 1$ and $b_0\#a - 1$ belong to $S^{-1,-1}$. Proceeding by the standard von Neumann series argument one obtains the following:

Theorem 2.2 *$a \in S^{\mu,m}$ is elliptic if and only if there exists a symbol $b \in S^{-\mu,-m}$ such that both $r_L := b\#a - 1$ and $r_R := a\#b - 1$ belong to $S^{-\infty,-\infty}$. If a is classical, then $b \in S_{\mathrm{cl}}^{-\mu,-m}$. The symbol b is called a* parametrix *of a.*

Note also that the operators with symbol from $S^{-\infty,-\infty}$ are exactly the integral operators with respect to Lebesgue measure on $\mathbb{R}^n$ whose kernel belongs to $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$.

Remark 2.1 If $a \in S^{\mu,m}$ is classical, then a is elliptic if and only if

a) $a^{(\mu)}(x,\xi)$ is invertible for all $x \in \mathbb{R}^n$ and all $\xi \neq 0$,
b) $a_{(m)}(x,\xi)$ is invertible for all $x \neq 0$ and all $\xi \in \mathbb{R}^n$, and
c) $a_{(m)}^{(\mu)}(x,\xi)$ is invertible for all $x \neq 0$ and all $\xi \neq 0$.

2.3 Sobolev spaces and Fredholm property

We define weighted Sobolev spaces

$$H^{s,\delta}(\mathbb{R}^n) = \{u \in \mathscr{S}'(\mathbb{R}^n) \mid [\cdot]^{\delta} u \in H^s(\mathbb{R}^n)\}, \qquad s,\delta \in \mathbb{R},$$

and equip them with the norm $\|u\|_{H^{s,\delta}(\mathbb{R}^n)} = \|\,[\cdot]^{\delta} u\|_{H^s(\mathbb{R}^n)}$. The standard properties of Sobolev spaces carry over to the weighted spaces. For example: $H^{s',\delta'}(\mathbb{R}^n) \hookrightarrow H^{s,\delta}(\mathbb{R}^n)$ for $s' \geq s$ and $\delta' \geq \delta$ (this embedding is compact provided $s' > s$ and $\delta' > \delta$); the dual space of $H^{s,\delta}(\mathbb{R}^n)$ can be identified with $H^{-s,-\delta}(\mathbb{R}^n)$ using the standard L_2-pairing. Note also that $\bigcap_{s,\delta \in \mathbb{R}} H^{s,\delta}(\mathbb{R}^n) = \mathscr{S}(\mathbb{R}^n)$.

Theorem 2.3 *Let $a \in S^{\mu,m}$ and $s,\delta \in \mathbb{R}$. Then* $\mathrm{op}(a)$ *has a continuous extension to*

$$\mathrm{op}(a) : H^{s,\delta}(\mathbb{R}^n) \longrightarrow H^{s-\mathrm{Re}\,\mu,\delta-\mathrm{Re}\,m}(\mathbb{R}^n).$$

From the existence of the parametrix one obtains the standard results of elliptic regularity and Fredholm property:

Theorem 2.4 *Let $a \in S^{\mu,m}$ be elliptic. Then*

a) $\mathrm{op}(a) : H^{s,\delta}(\mathbb{R}^n) \to H^{s-\mathrm{Re}\,\mu,\delta-\mathrm{Re}\,m}(\mathbb{R}^n)$ *is a Fredholm operator for any given $s,\delta \in \mathbb{R}$.*
b) If u is in a weighted Sobolev space and $\mathrm{op}(a)u \in H^{s,\delta}(\mathbb{R}^n)$, *then $u \in H^{s+\mathrm{Re}\,\mu,\delta+\mathrm{Re}\,m}(\mathbb{R}^n)$.*

In fact, also the converse of a) holds true: If $a \in S^{\mu,m}$ and $\mathrm{op}(a) : H^{s,\delta}(\mathbb{R}^n) \to H^{s-\mathrm{Re}\,\mu,\delta-\mathrm{Re}\,m}(\mathbb{R}^n)$ is a Fredholm operator for some given $s,\delta \in \mathbb{R}$, then a is elliptic. Using this, it is easy to derive the spectral invariance of pseudodifferential operators:

Proposition 2.1 *Let $a \in S_{(\mathrm{cl})}^{\mu,m}$ and suppose that* $\mathrm{op}(a) : H^{s,\delta}(\mathbb{R}^n) \to H^{s-\mathrm{Re}\,\mu,\delta-\mathrm{Re}\,m}(\mathbb{R}^n)$ *is invertible for some given $s,\delta \in \mathbb{R}$. Then* $\mathrm{op}(a)^{-1} = \mathrm{op}(c)$ *for some $c \in S_{(\mathrm{cl})}^{-\mu,-m}$.*

Proof By the preceding comment, a is elliptic and thus has a parametrix $b \in S^{-\mu,-m}_{(\mathrm{cl})}$. Then, with notation from Theorem 2.2,

$$\mathrm{op}(a)^{-1} = \mathrm{op}(b) - \mathrm{op}(r_L \# b) + \mathrm{op}(r_L)\mathrm{op}(a)^{-1}\mathrm{op}(r_R). \tag{2.4}$$

The third operator on the right-hand side maps each weighted Sobolev space into $\mathscr{S}(\mathbb{R}^n)$. This is also true for its L_2-adjoint. Hence it has a Schwartz kernel in $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$, and thus belongs to $S^{-\infty,-\infty}$. Therefore the right-hand side of (2.4) is an operator with symbol in $S^{-\mu,-m}_{(\mathrm{cl})}$. □

Another easy consequence of the existence of a parametrix is the following:

Proposition 2.2 *Let $\mu, m \geq 0$ and $a \in S^{\mu,m}$ be elliptic. The closure of the unbounded operator*

$$\mathrm{op}(a) : \mathscr{S}(\mathbb{R}^n) \subset H^{s,\delta}(\mathbb{R}^n) \longrightarrow H^{s,\delta}(\mathbb{R}^n)$$

is given by $\mathrm{op}(a)$ *acting on* $H^{s+\mu,\delta+m}(\mathbb{R}^n)$.

3 Complex powers of classical operators

In [20] it is shown, in particular, that under a suitable ellipticity assumption *SG*-pseudodifferential operators have complex powers. The aim of this section is to show that if the operators additionally are classical then so are the complex powers.

From now on let μ and m be fixed nonnegative reals and let Λ be a closed subsector of the complex plane with corner in the origin. We denote by C any positive constant independent of $\lambda \in \Lambda$, $x \in \mathbb{R}^n$, and $\xi \in \mathbb{R}^n$.

3.1 Λ-ellipticity

We recall the standard parameter-ellipticity condition for global symbols and then characterize it for classical symbols purely in terms of the principal homogeneous components.

Definition 3.1 $a \in S^{\mu,m}$ is called *Λ-elliptic* if there exist constants $C > 0$, $R \geq 0$ such that

$$\mathrm{spec}(a(x,\xi)) \cap \Lambda = \emptyset \qquad \forall\, |(x,\xi)| \geq R$$

and

$$|(\lambda - a(x,\xi))^{-1}| \leq C[x]^{-m}[\xi]^{-\mu} \qquad \forall\, \lambda \in \Lambda \quad \forall\, |(x,\xi)| \geq R.$$

Lemma 3.1 *A classical symbol $a \in S^{\mu,m}_{\mathrm{cl}}$ is Λ-elliptic if and only if there exists a constant $C \geq 0$ such that the following hold:*

a) $\mathrm{spec}(a^{(\mu)}(x,\omega)) \cap \Lambda = \emptyset$ *for all $x \in \mathbb{R}^n$, $|\omega| = 1$, and*

$$|(\lambda - a^{(\mu)}(x,\omega))^{-1}| \leq C[x]^{-m} \qquad \forall\, x \in \mathbb{R}^n \quad \forall\, |\omega| = 1 \quad \forall\, \lambda \in \Lambda,$$

b) $\operatorname{spec}(a_{(m)}(\theta,\xi))\cap\Lambda=\emptyset$ *for all* $\xi\in\mathbb{R}^n$, $|\theta|=1$, *and*

$$|(\lambda-a_{(m)}(\theta,\xi))^{-1}|\le C[\xi]^{-\mu}\qquad\forall\,\xi\in\mathbb{R}^n\quad\forall\,|\theta|=1\quad\forall\,\lambda\in\Lambda.$$

Proof First assume that a is Λ-elliptic. Then, for $|(x,\xi)|\ge R$ and $\lambda\in\Lambda$,

$$\begin{aligned}\lambda-a^{(\mu)}(x,\xi)&=(\lambda-a(x,\xi))\Big(1+(\lambda-a(x,\xi))^{-1}(a(x,\xi)-a^{(\mu)}(x,\xi))\Big)\\&=:(\lambda-a(x,\xi))(1+r(x,\xi,\lambda)).\end{aligned}$$

For $|\xi|$ large enough, we have $|r(x,\xi,\lambda)|<\frac{1}{2}$ uniformly in $x\in\mathbb{R}^n$ and $\lambda\in\Lambda$. This together with the ξ-homogeneity of $a^{(\mu)}$ implies the first property in a). Moreover, for $|\xi|$ large,

$$|(\lambda-a^{(\mu)}(x,\xi))^{-1}|\le 2|(\lambda-a(x,\xi))^{-1}|\le C[x]^{-m}|\xi|^{-\mu}$$

uniformly in x and λ. Again by homogeneity, this is equivalent to the second condition in a). In the same way one shows b).

Now assume the validity of a) and b). By homogeneity, a) is equivalent, for every $x\in\mathbb{R}^n$, $\xi\neq 0$ and $\lambda\in\Lambda$, to

$$|(\lambda-a^{(\mu)}(x,\xi))^{-1}|\le C[x]^{-m}|\xi|^{-\mu}. \tag{3.1}$$

Writing

$$\lambda-a(x,\xi)=(\lambda-a^{(\mu)}(x,\xi))\Big(1-(\lambda-a^{(\mu)}(x,\xi))^{-1}(a(x,\xi)-a^{(\mu)}(x,\xi))\Big)$$

one deduces similarly as above the existence of an $R\ge 0$ such that $a(x,\xi)$ has no spectrum in Λ for all $x\in\mathbb{R}^n$ and $|\xi|\ge R$ and that

$$|(\lambda-a(x,\xi))^{-1}|\le C[x]^{-m}[\xi]^{-\mu}\qquad\forall\,\lambda\in\Lambda\quad\forall\,x\in\mathbb{R}^n\quad\forall\,|\xi|\ge R.$$

Using property b) one obtains the same estimate but uniformly in $\lambda\in\Lambda$, $\xi\in\mathbb{R}^n$, and $|x|$ large enough. Thus a is Λ-elliptic. □

Proposition 3.1 *A classical symbol* $a\in S_{\mathrm{cl}}^{\mu,m}$ *is* Λ*-elliptic if and only if*

a) $\operatorname{spec}(a^{(\mu)}(x,\omega))\cap\Lambda=\emptyset$ *for all* $x\in\mathbb{R}^n$ *and* $|\omega|=1$,
b) $\operatorname{spec}(a_{(m)}(\theta,\xi))\cap\Lambda=\emptyset$ *for all* $\xi\in\mathbb{R}^n$ *and* $|\theta|=1$ *and*
c) $\operatorname{spec}(a_{(m)}^{(\mu)}(\theta,\omega))\cap\Lambda=\emptyset$ *for all* $|\theta|=1$ *and* $|\omega|=1$.

Proof Let us first assume that a)–c) are valid. By c) we have

$$M:=\sup_{\substack{\lambda\in\Lambda,\\|\theta|=|\omega|=1}}|(\lambda-a_{(m)}^{(\mu)}(\theta,\omega))^{-1}|<\infty.$$

Thus, by homogeneity,

$$|(\lambda-a_{(m)}^{(\mu)}(x,\omega))^{-1}|\le M|x|^{-m}\qquad\forall\,\lambda\in\Lambda\quad\forall\,x\neq 0\quad\forall\,|\omega|=1.$$

Therefore, by writing

$$\begin{aligned}&\lambda - a^{(\mu)}(x,\omega) = \\ &= (\lambda - a^{(\mu)}_{(m)}(x,\omega))\left(1 - (\lambda - a^{(\mu)}_{(m)}(x,\omega))^{-1}(a^{(\mu)}(x,\omega) - a^{(\mu)}_{(m)}(x,\omega))\right),\end{aligned}$$

we obtain the existence of $C, R \geq 0$ such that

$$|(\lambda - a^{(\mu)}(x,\omega))^{-1}| \leq C[x]^{-m} \qquad \forall\, \lambda \in \Lambda \quad \forall\, |x| \geq R \quad \forall\, |\omega| = 1.$$

For small $|x|$ this estimate holds anyway by a). This shows that a satisfies a) of Lemma 3.1, and analogously we deduce b) of Lemma 3.1 by b) and c).

Assume, vice versa, that a is Λ-elliptic, i.e. a) and b) of Lemma 3.1 are satisfied. We obviously only have to show c). However this works as before by writing

$$\begin{aligned}&\lambda - a^{(\mu)}_{(m)}(x,\omega) = \\ &= (\lambda - a^{(\mu)}(x,\omega))\left(1 + (\lambda - a^{(\mu)}(x,\omega))^{-1}(a^{(\mu)}(x,\omega) - a^{(\mu)}_{(m)}(x,\omega))\right),\end{aligned}$$

and using the homogeneity in x of $a^{(\mu)}_{(m)}$. $\square$

Remark 3.1 If $a \in S^{\mu,m}$ is Λ-elliptic we can always find $\widetilde{a} \in S^{\mu,m}$ that is Λ-elliptic with constant $R = 0$ and such that $a - \widetilde{a} \in S^{-\infty,-\infty}$. For example, in case Λ is centered around the negative real half axis, one can choose $\widetilde{a} := a + (L+1)\chi$ where χ is a zero excision function with $\chi(x,\xi) = 1$ for $|(x,\xi)| \leq R$, and $L = \max_{|(x,\xi)| \leq R} |a(x,\xi)|$.

3.2 A parametrix construction and complex powers

Let us summarize a parametrix construction for parameter-elliptic symbols. It may be found in [12] for symbols having uniform bounds in the x-variable and in [20] for *SG*-symbols. We assume, without loss of generality (see Remark 3.1 and Corollary 3.1.1, below), that $a \in S^{\mu,m}$ is Λ-elliptic with constant $R = 0$.

Lemma 3.2 *There exists a constant $c_0 \geq 1$ such that, for every (x,ξ),*

$$\mathrm{spec}(a(x,\xi)) \subset \Omega_{[x],[\xi]} := \{z \in \mathbb{C} \setminus \Lambda \mid \tfrac{1}{c_0}[x]^m[\xi]^\mu < |z| < c_0[x]^m[\xi]^\mu\}$$

and

$$|(\lambda - a(x,\xi))^{-1}| \leq C(|\lambda| + [x]^m[\xi]^\mu)^{-1} \qquad \forall (x,\xi),\ \lambda \in \mathbb{C} \setminus \Omega_{[x],[\xi]}$$

uniformly in x, ξ, and λ.

Proof According to the Λ-ellipticity assumption,we have

$$|a(x,\xi)^{-1}| \leq C[x]^{-m}[\xi]^{-\mu}$$

uniformly in x and ξ. Thus, when $|\lambda| \leq \frac{1}{2C}[x]^m[\xi]^\mu$,

$$\begin{aligned}|(\lambda - a(x,\xi))^{-1}| &= |a(x,\xi)^{-1}(1-a(x,\xi)^{-1}\lambda)^{-1}| \\ &\leq 2C[x]^{-m}[\xi]^{-\mu} \leq (1+2C)(|\lambda| + [x]^m[\xi]^\mu)^{-1}.\end{aligned}$$

As a is a symbol, $|a(x,\xi)| \leq C_1[x]^m[\xi]^\mu$ uniformly in x and ξ. Writing

$$|(\lambda - a)^{-1}| = |\lambda|^{-1}|(1-a/\lambda)^{-1}| \quad \text{for} \quad |\lambda| \geq 2C_1[x]^m[\xi]^\mu$$

as well as

$$|(\lambda - a)^{-1}| = |\lambda|^{-1}|1 + a(\lambda - a)^{-1}|$$

for $\lambda \in \Lambda \cap \{z \in \mathbb{C} \mid \frac{1}{2C}[x]^m[\xi]^\mu < |\lambda| < 2C_1[x]^m[\xi]^\mu\}$, we conclude the proof by setting $c_0 = 2\max\{C, C_1\}$. □

Note that there exists an $\varepsilon > 0$ such that

$$\Lambda_\varepsilon := \Lambda \cup \{z \mid |z| \leq \varepsilon\} \subset \mathbb{C} \setminus \Omega_{[x],[\xi]} \qquad \forall\, x, \xi \in \mathbb{R}^n. \tag{3.2}$$

Let us now define symbols b_{-k}, $k \in \mathbb{N}_0$, recursively by

$$b_0(x,\xi,\lambda) = (\lambda - a(x,\xi))^{-1},$$

and, for $k \geq 1$,

$$b_{-k}(x,\xi,\lambda) = \sum_{\substack{j+|\alpha|=k \\ j<k}} \frac{1}{\alpha!}(\partial_\xi^\alpha b_{-j})(x,\xi,\lambda)\,(D_x^\alpha a)(x,\xi)\, b_0(x,\xi,\lambda). \tag{3.3}$$

By induction, each $\partial_\xi^\alpha \partial_x^\beta b_{-k}$ is a finite linear combination of terms $b_0(\partial_\xi^{\alpha_1}\partial_x^{\beta_1} a) \cdot \ldots \cdot b_0(\partial_\xi^{\alpha_l}\partial_x^{\beta_l} a) b_0$ with $|\alpha_1| + \ldots + |\alpha_l| = |\alpha| + k$, $|\beta_1| + \ldots + |\beta_l| = |\beta| + k$, and $l \geq 2$ if $k \geq 1$. Therefore, when $k \geq 0$, one gets

$$|\partial_\xi^\alpha \partial_x^\beta b_{-k}(x,\xi,\lambda)| \leq C\,(|\lambda| + [x]^m[\xi]^\mu)^{-1}[x]^{-k-|\beta|}[\xi]^{-k-|\alpha|}, \tag{3.4}$$

while, when $k \geq 1$, one gets

$$|\partial_\xi^\alpha \partial_x^\beta b_{-k}(x,\xi,\lambda)| \leq C\,(|\lambda| + [x]^m[\xi]^\mu)^{-3}[x]^{2m-k-|\beta|}[\xi]^{2\mu-k-|\alpha|},$$

uniformly in $x, \xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{C} \setminus \Omega_{[x],[\xi]}$. In particular, $b_{-k}(\lambda) \in S^{-\mu-k,-m-k}$ uniformly for $\lambda \in \Lambda_\varepsilon$. Arguing as in the proof of the asymptotic summation for standard pseudodifferential symbols, we find a null sequence ε_k such that

$$b(x,\xi,\lambda) = b_0(x,\xi,\lambda) + \sum_{k=1}^{\infty} \chi(\varepsilon_k x, \varepsilon_k \xi)\, b_{-k}(x,\xi,\lambda) \tag{3.5}$$

defines a function satisfying estimates as (3.4) for $k = 0$ and, for any $N \in \mathbb{N}$,

$$\begin{aligned}&\left|\partial_\xi^\alpha \partial_x^\beta \left(b(x,\xi,\lambda) - \sum_{k=0}^{N-1} b_{-k}(x,\xi,\lambda)\right)\right| \leq \\ &\leq C\,(|\lambda| + [x]^m[\xi]^\mu)^{-3}[x]^{2m-N-|\beta|}[\xi]^{2\mu-N-|\alpha|}\end{aligned} \tag{3.6}$$

uniformly in $x, \xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{C} \setminus \Omega_{[x],[\xi]}$.

Theorem 3.1 *Let $b(\lambda)$ as above. Then both $[\lambda]\big((\lambda-a)\#b(\lambda)-1\big)$ and $[\lambda]\big(b(\lambda)\#(\lambda-a)-1\big)$ belong to $S^{-\infty,-\infty}$ uniformly in $\lambda\in\Lambda_\varepsilon$, cf.* (3.2). *In particular, the resolvent set of the unbounded operator*

$$A:=\mathrm{op}(a):H^{\mu,m}(\mathbb{R}^n)\subset L_2(\mathbb{R}^n)\longrightarrow L_2(\mathbb{R}^n)$$

contains all $\lambda\in\Lambda_\varepsilon$ of sufficiently large absolute value. Moreover, uniformly for large $\lambda\in\Lambda_\varepsilon$,

$$[\lambda]^2\big((\lambda-A)^{-1}-\mathrm{op}(b(\lambda)\big)\in S^{-\infty,-\infty}.$$

Proof By construction of $b(\lambda)$ and (3.6), for each $N\in\mathbb{N}$,

$$b(\lambda)\#(\lambda-a)\equiv\sum_{k=0}^{N-1}b_{-k}(\lambda)\#(\lambda-a)$$

modulo a remainder which is $O([\lambda]^{-1})$ in $S^{2\mu-N,2m-N}$. Then, by Theorem 2.1,

$$b(\lambda)\#(\lambda-a)\equiv\sum_{|\alpha|\le N-1}\sum_{k=0}^{N-1}\frac{1}{\alpha!}(\partial_\xi^\alpha b_{-k})(\lambda)D_x^\alpha(\lambda-a)$$

modulo a remainder of the same quality as above. Now we split the summation into two parts: One over those α and k with $|\alpha|+k\le N-1$ and the other over those with $N-1<|\alpha|+k\le 2(N-1)$ and $|\alpha|,k\le N-1$. The first summand then equals 1 by construction of the b_k. The second again is a remainder as before. Since N was arbitrary, it follows that $b(\lambda)$ is a left-parametrix. Similarly one can show that $b(\lambda)$ is a right-parametrix. Thus, writing $B(\lambda)=\mathrm{op}(b)(\lambda)$,

$$(\lambda-A)B(\lambda)=1-R_1(\lambda),\qquad B(\lambda)(\lambda-A)=1-R_2(\lambda),$$

with $[\lambda]R_j(\lambda)\in S^{-\infty,-\infty}$ uniformly in $\lambda\in\Lambda_\varepsilon$. Clearly, this induces the existence of the resolvent of A for large λ. Moreover, solving both equations for $(\lambda-A)^{-1}$, we obtain

$$(\lambda-A)^{-1}=B(\lambda)+B(\lambda)R_1(\lambda)+R_2(\lambda)(\lambda-A)^{-1}R_1(\lambda).$$

From this the final statement of the theorem follows. □

Corollary 3.1.1 *Let $a,\ \widetilde{a}\in S^{\mu,m}$ such that $\widetilde{a}$ is Λ-elliptic with constant $R=0$ and $a-\widetilde{a}\in S^{-\infty,-\infty}$, cf. Remark* 3.1. *Let $\widetilde{b}(\lambda)$ be the parametrix of $(\lambda-\widetilde{a})$ constructed as above. Then $\widetilde{b}(\lambda)$ is a parametrix of $(\lambda-a)$. In particular, the resolvent set of* $\mathrm{op}(a)$ *contains all $\lambda\in\Lambda_\varepsilon$ of sufficiently large absolute value and the resolvent differs from $\widetilde{b}(\lambda)$ by a remainder which is $O([\lambda]^{-2})$ in $S^{-\infty,-\infty}$.*

Proof That $\widetilde{b}(\lambda)$ is a parametrix of $(\lambda-a)$ follows from the identity

$$[\lambda]\big((\lambda-a)\#\widetilde{b}(\lambda)-1\big)=[\lambda]\big((\lambda-\widetilde{a})\#\widetilde{b}(\lambda)-1\big)+[\lambda]\big((\widetilde{a}-a)\#\widetilde{b}(\lambda)\big)$$

and the fact that $[\lambda]\widetilde{b}(\lambda)\in S^{\mu,m}$ uniformly in $\lambda\in\Lambda_\varepsilon$ which tell us that $[\lambda]\big((\lambda-a)\#\widetilde{b}(\lambda)-1\big)\in S^{-\infty,-\infty}$. The remaining statements then follow as in Theorem 3.1. □

The existence of the resolvent for large $|\lambda|$ in the sector is still not enough to define complex powers. In order to do this we need a stronger assumption. Namely, we assume that $a \in S^{\mu,m}$ (with μ, m nonnegative) is Λ-elliptic and that $A = \mathrm{op}(a)$ satisfies the following condition:

$$\begin{aligned} &\lambda - A \text{ is invertible for all } 0 \neq \lambda \in \Lambda \text{ and} \\ &\lambda = 0 \text{ is at most an isolated spectral point.} \end{aligned} \tag{A}$$

To consider an example, suppose that $a \in S^{\mu,m}$ with $\mu, m > 0$ is Λ-elliptic. Then A has compact resolvent due to the compact embedding of $H^{\mu,m}(\mathbb{R}^n)$ in $L_2(\mathbb{R}^n)$, hence the spectrum of A is discrete, consisting only of eigenvalues with corresponding eigenfunctions in $\mathscr{S}(\mathbb{R}^n)$. This means that only finitely many spectral points lie in Λ. One thus can find a subsector of Λ such that (A) holds for this new sector.

Remark 3.2 Under assumption (A), Theorem 3.1 is valid on the whole keyhole region Λ_ε with an arbitrarily small neighborhood of zero removed.

Let us define

$$A^z = \frac{1}{2\pi i}\int_{\partial\Lambda_\varepsilon} \lambda^z(\lambda - A)^{-1}\,d\lambda, \qquad \mathrm{Re}\,z < 0, \tag{3.7}$$

where $\partial\Lambda_\varepsilon$ is a parametrization of the boundary of Λ_ε, the circular part being traversed clockwise. The power $\lambda^z = e^{z\log\lambda}$ is determined by taking the main branch of the logarithm on the plane with the symmetry axis of Λ removed. We can use Theorem 3.1 to show that A^z, for $\mathrm{Re}\,z < 0$, is a pseudodifferential operator in $S^{\mu z, mz}$ with symbol $a(x,\xi,z)$ satisfying

$$a(x,\xi,z) \equiv \frac{1}{2\pi i}\int_{\partial\Omega_{[x],[\xi]}} \lambda^z b(x,\xi,\lambda)\,d\lambda \tag{3.8}$$

(for the definition of $\Omega_{[x],[\xi]}$ see Lemma 3.2) modulo a remainder in $S^{-\infty,-\infty}$ depending holomorphically on z with $\mathrm{Re}\,z < 0$. The definition of A^z is extended to arbitrary $z \in \mathbb{C}$ by $A^z := A^k A^{z-k}$ for any choice of a $k \in \mathbb{N}_0$ such that $\mathrm{Re}\,z - k < 0$. By using the composition formula for pseudodifferential operators we conclude that A^z is a pseudodifferential operator with symbol $a(x,\xi,z) \in S^{\mu z, mz}$ for any $z \in \mathbb{C}$. The symbol $a(x,\xi,z)$ depends smoothly on x,ξ and holomorphically on z. For arbitrary $s \in \mathbb{R}$, it satisfies the estimates

$$|\partial_z^k \partial_\xi^\alpha \partial_x^\beta a(x,\xi,z)| \leq C\,[x]^{sm-|\beta|}[\xi]^{s\mu-|\alpha|} \qquad \forall\, k \in \mathbb{N}_0, \quad \forall\, \alpha,\beta \in \mathbb{N}_0^n,$$

uniformly in $x,\xi \in \mathbb{R}^n$ and uniformly in $\{z \in \mathbb{C} \mid \mathrm{Re}\,z \leq s - \sigma\}$ for any $\sigma > 0$.

If A is invertible, the complex powers A^1, A^0, and A^{-1} coincide with A, the identity operator, and the inverse of A, respectively.

3.3 Classical symbols

Besides Λ-ellipticity and condition (A) we now assume additionally that $a \in S^{\mu,m}$ is a classical symbol. Inequality (3.1) can be used to prove an analog of Lemma 3.2 with a replaced by the the homogeneous component $a^{(\mu)}$, $\Omega_{[x],[\xi]}$ replaced by $\Omega_{[x],|\xi|}$ and valid for $(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$.

Proposition 3.2 *For every $i \geq 0$ there exist $b^{(-\mu-i)}$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta b^{(-\mu-i)}(x,\xi,\lambda)| \leq C(|\lambda| + [x]^m|\xi|^\mu)^{-1}[x]^{-|\beta|}|\xi|^{-i-|\alpha|} \tag{3.9}$$

uniformly in $x \in \mathbb{R}^n$, $\xi \neq 0$, and $\lambda \in \mathbb{C} \setminus \Omega_{[x],|\xi|}$,

$$b^{(-\mu-i)}(x,t\xi,t^\mu\lambda) = t^{-\mu-i}b^{(-\mu-i)}(x,\xi,\lambda), \qquad \forall\, t > 0, \tag{3.10}$$

and, for each $N \in \mathbb{N}$,

$$\left|\partial_\xi^\alpha \partial_x^\beta \big(b - \sum_{i=0}^{N-1} b^{(-\mu-i)}\big)(x,\xi,\lambda)\right| \leq C(|\lambda| + [x]^m|\xi|^\mu)^{-1}[x]^{-|\beta|}|\xi|^{-N-|\alpha|} \tag{3.11}$$

uniformly in $x \in \mathbb{R}^n$, $|\xi| \geq 1$, and $\lambda \in \mathbb{C} \setminus \Omega_{[x],|\xi|}$. Explicitly, these components are given by

$$b^{(-\mu)}(x,\xi,\lambda) = (\lambda - a^{(\mu)}(x,\xi))^{-1}, \tag{3.12}$$

and, for $i \geq 1$,

$$b^{(-\mu-i)} = \sum_{\substack{j+l+|\alpha|=i \\ j<i}} \frac{1}{\alpha!} (\partial_\xi^\alpha b^{(-\mu-j)})\,(D_x^\alpha a^{(\mu-l)})\, b^{(-\mu)}. \tag{3.13}$$

Proof Writing $(\lambda - a)^{-1} = (\lambda - a^{(\mu)})^{-1}\Big(1 - (\lambda - a^{(\mu)})^{-1}(a - a^{(\mu)})\Big)^{-1}$ and applying the formula $(1-q)^{-1} = \sum_{k=0}^{N-1} q^k + q^N(1-q)^{-1}$ to the second factor, one can derive that to $b_0 = (\lambda - a)^{-1}$ there exist components $b_0^{(-\mu-i)}$, $i \geq 0$, that satisfy an analog of (3.9), (3.10), and (3.11). Using expressions (3.3) and induction one can show that also each b_{-k}, with $k \geq 1$, has components $b_{-k}^{(-\mu-k-i)}$ with the same properties. From this follows, by means of (3.5), the existence of the components $b^{(-\mu-i)}$ to $b(\lambda)$ that satisfy (3.9), (3.10), and (3.11). It remains to show that the components $b^{(-\mu-i)}$ are really given by the recursion (3.12) and (3.13). However, this follows from Theorem 3.1, the composition formula stated in Theorem 2.1 and the homogeneity properties of $a^{(\mu-j)}$ and of $b^{(-\mu-i)}$. □

Replacing $a^{(\mu-l)}$ in (3.12), (3.13) by $a_{(m-l)}$, we obtain a sequence $b_{(-m-j)}$ of functions that are homogeneous of degree $-m-j$ in (x,λ) and a result analogous to Proposition 3.2 holds true (with the roles of x and ξ interchanged). The Λ-ellipticity assumption allows also the definition

$$b_{(-m)}^{(-\mu)}(x,\xi,\lambda) = (\lambda - a_{(m)}^{(\mu)}(x,\xi))^{-1}, \qquad x \neq 0,\ \xi \neq 0,\ \lambda \in \mathbb{C} \setminus \Omega_{|x|,|\xi|}.$$

If we then define recursively for $j+k \geq 1$

$$b^{(-\mu-k)}_{(-m-j)} = \sum_{\substack{j_0+j_1+|\alpha|=j \\ k_0+k_1+|\alpha|=k \\ j_0<j \text{ or } k_0<k}} \frac{1}{\alpha!} (\partial_\xi^\alpha b^{(-\mu-k_0)}_{(-m-j_0)})\,(D_x^\alpha a^{(\mu-k_1)}_{(m-j_1)})\, b^{(-\mu)}_{(-m)},$$

we can iterate the above procedure to show that, for each $N \in \mathbb{N}$,

$$\Big|\partial_\xi^\alpha \partial_x^\beta \big(b^{(-\mu-i)} - \sum_{j=0}^{N-1} b^{(-\mu-i)}_{(-m-j)}\big)\Big| \leq C(|\lambda| + |x|^m |\xi|^\mu)^{-1} |x|^{-N-|\beta|} |\xi|^{-i-|\alpha|},$$

and

$$\Big|\partial_\xi^\alpha \partial_x^\beta \big(b_{(-m-j)} - \sum_{i=0}^{N-1} b^{(-\mu-i)}_{(-m-j)}\big)\Big| \leq C(|\lambda| + |x|^m |\xi|^\mu)^{-1} |x|^{-j-|\beta|} |\xi|^{-N-|\alpha|},$$

for every $i \geq$, $j \geq 1$ and uniformly in $|x| \geq 1$, $|\xi| \geq 1$, $\lambda \in \mathbb{C} \setminus \Omega_{|x|,|\xi|}$. Proceeding as in the proof of Lemma 2.1 it is then straightforward to derive the following:

Proposition 3.3 *For each $N \in \mathbb{N}$ and any choice of zero excision functions $\chi = \chi(\xi)$ and $\kappa = \kappa(x)$,*

$$\Big|\partial_\xi^\alpha \partial_x^\beta \Big(b - \sum_{i=0}^{N-1} \chi b^{(-\mu-i)} - \sum_{j=0}^{N-1} \kappa b_{(-m-j)} + \sum_{j,k=0}^{N-1} \kappa\chi b^{(-\mu-k)}_{(-m-j)}\Big)(x,\xi,\lambda)\Big|$$
$$\leq C\,(|\lambda| + [x]^m [\xi]^\mu)^{-1} [x]^{-N-|\beta|} [\xi]^{-N-|\alpha|}$$

uniformly in $\lambda \in \mathbb{C} \setminus \Omega_{[x],[\xi]}$.

The following theorem is then an immediate consequence of the previous estimates.

Theorem 3.2 *If a is classical then $A^z \in S^{\mu z, mz}_{\mathrm{cl}}$ for any $z \in \mathbb{C}$. If $A^z = \mathrm{op}(a)(z)$ and* $\mathrm{Re}\, z < 0$*, then*

$$a^{(\mu z-k)} = \frac{1}{2\pi i} \int_{\partial\Omega_{[x],|\xi|}} \lambda^z b^{(-\mu-k)}(x,\xi,\lambda)\, d\lambda \;\in\; S^{(\mu z-k),m}, \tag{3.14}$$

$$a_{(mz-j)} = \frac{1}{2\pi i} \int_{\partial\Omega_{|x|,[\xi]}} \lambda^z b_{(-m-j)}\, d\lambda \;\in\; S^{\mu,(mz-j)},$$

$$a^{(\mu z-k)}_{(mz-j)} = \frac{1}{2\pi i} \int_{\partial\Omega_{|x|,|\xi|}} \lambda^z b^{(-\mu-k)}_{(-m-j)}(x,\xi,\lambda)\, d\lambda \;\in\; S^{(\mu z-k),(mz-j)}. \tag{3.15}$$

For the homogeneity of $a^{(\mu z-k)}(x,\xi,z)$ observe that

$$\int_{\partial\Omega_{[x],|t\xi|}} f(\lambda)\, d\lambda = t^\mu \int_{\partial\Omega_{[x],|\xi|}} f(t^\mu \lambda)\, d\lambda$$

for any $t > 0$ (analogously for the other homogeneous components).

Corollary 3.2.1 *For any $z \in \mathbb{C}$ we have*

$$a^{(\mu z)}(x,\xi,z) = (a^{(\mu)}(x,\xi))^z, \qquad a_{(mz)}(x,\xi,z) = (a_{(m)}(x,\xi))^z,$$

and

$$a^{(\mu z)}_{(mz)}(x,\xi,z) = (a^{(\mu)}_{(m)}(x,\xi))^z.$$

3.4 The case of commuting homogeneous components

As already observed in [21], the formulas for the homogeneous components of the complex powers become simpler, if one assumes that $a^{(\mu)}$ and $a_{(m)}$ commute with all the other homogeneous components $a^{(\mu-k)}$ and $a_{(m-j)}$, respectively. Note that then automatically $a^{(\mu)}_{(m)}$ commutes with any $a^{(\mu-k)}_{(m-j)}$. This clearly holds for scalar valued symbols and, in case of systems, for example if $a \equiv \operatorname{diag}(a_0, \ldots, a_0)$ modulo $S^{\mu-1,m-1}_{\mathrm{cl}}$ for a scalar valued symbol $a_0 \in S^{\mu,m}_{\mathrm{cl}}$.

By induction we can derive from (3.13) that, for $k \geq 1$,

$$b^{(-\mu-k)}(x,\xi,\lambda) = \sum_{l=1}^{2k} \gamma^{k;l}(x,\xi)\,(\lambda - a^{(\mu)}(x,\xi))^{-(l+1)}, \tag{3.16}$$

where each $\gamma^{k;l}$ is a linear combination of terms of the form

$$(\partial_\xi^{\alpha_1}\partial_x^{\beta_1} a^{(\mu-\nu_1)}) \cdot \ldots \cdot (\partial_\xi^{\alpha_l}\partial_x^{\beta_l} a^{(\mu-\nu_l)})$$

with $|\alpha_1| + \ldots + |\alpha_l| + \nu_1 + \ldots + \nu_l = k$ and $|\beta_1| + \ldots + |\beta_l| + \nu_1 + \ldots + \nu_l = k$. Note also that $\gamma^{k;l} \in S^{(\mu l-k)}_{ml}$. Similarly, we have

$$b_{(-m-j)}(x,\xi,\lambda) = \sum_{l=1}^{2j} \gamma_{j;l}(x,\xi)\,(\lambda - a_{(m)}(x,\xi))^{-(l+1)}. \tag{3.17}$$

Using the above recursion formula for the mixed homogeneous components one gets, for $j+k \geq 1$,

$$b^{(-\mu-k)}_{(-m-j)}(x,\xi,\lambda) = \sum_{l=1}^{2(j+k)} \gamma^k_{j;l}(x,\xi)\,(\lambda - a^{(\mu)}_{(m)}(x,\xi))^{-(l+1)}, \tag{3.18}$$

where each $\gamma^k_{j;l}$ is a linear combination of terms of the form

$$(\partial_\xi^{\alpha_1}\partial_x^{\beta_1} a^{(\mu-\nu_1)}_{(m-n_1)}) \cdot \ldots \cdot (\partial_\xi^{\alpha_l}\partial_x^{\beta_l} a^{(\mu-n_l)}_{(m-n_l)})$$

with $|\alpha_1|+\ldots+|\alpha_l|+\nu_1+\ldots+\nu_l = k$ and $|\beta_1|+\ldots+|\beta_l|+n_1+\ldots+n_l = j$. Then $\gamma^k_{j;l} \in S^{(\mu l-k)}_{(ml-j)}$. Inserting these formulas in (3.14) and (3.15) yields the following:

Corollary 3.2.2 *Assume* $\operatorname{Re} z < 0$. *If* $k \geq 1$, $j \geq 1$, *and* $j+k \geq 1$, *respectively, then*

$$a^{(\mu z-k)} = \sum_{l=1}^{2k} \frac{1}{l!}\gamma^{k;l}(x,\xi) z(z-1)\cdot\ldots\cdot(z-l+1)(a^{(\mu)}(x,\xi))^{z-l}$$

$$a_{(mz-j)} = \sum_{l=1}^{2j} \frac{1}{l!}\gamma_{j;l}(x,\xi) z(z-1)\cdot\ldots\cdot(z-l+1)(a_{(m)}(x,\xi))^{z-l}$$

$$a^{(\mu z-k)}_{(mz-j)} = \sum_{l=1}^{2(j+k)} \frac{1}{l!}\gamma^k_{j;l}(x,\xi) z(z-1)\cdot\ldots\cdot(z-l+1)(a^{(\mu)}_{(m)}(x,\xi))^{z-l}.$$

3.5 Holomorphic families

Let $b = b(\xi, z)$ be defined on $\mathbb{R}^n \times \mathbb{C}$, smooth in $\xi \in \mathbb{R}^n$ and holomorphic in $z \in \mathbb{C}$. We call b a holomorphic family of zero order symbols if $b(z) \in S^0_{\mathrm{cl}}(\mathbb{R}^n)$ for each z, all homogeneous components $b^{(-k)}(\xi, z)$ depend smoothly on $\xi \neq 0$ and holomorphically on $z \in \mathbb{C}$, and, for each $N \in \mathbb{N}_0$ and $\varepsilon > 0$,

$$\partial_z^p \Big(b - \sum_{k=0}^{N-1} \chi\, b^{(-k)} \Big)(\cdot, z) \in S^{-N+\varepsilon} \qquad \forall z \in \mathbb{C},\ p \in \mathbb{N}_0.$$

Similarly, we can define families that depend on more parameters, say (z, τ) with holomorphy in z and smoothness in τ.

Definition 3.2 Let $b = b(x, \xi, z)$ be defined on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{C}$, smooth in (x, ξ) and holomorphic in $z \in \mathbb{C}$. Then we call $b(z)$ a holomorphic family of zero order symbols if $b(z) \in S^{0,0}(\mathbb{R}^n \times \mathbb{R}^n)$ for each $z \in \mathbb{C}$ and

a) $b^{(-k)}(x, \xi, z)$ is a family of symbols in $S^0_{\mathrm{cl}}(\mathbb{R}^n_x)$ depending smoothly on $\xi \neq 0$ and holomorphically on $z \in \mathbb{C}$,
b) $b_{(-j)}(x, \xi, z)$ is a family of symbols in $S^0_{\mathrm{cl}}(\mathbb{R}^n_\xi)$ depending smoothly on $x \neq 0$ and holomorphically on $z \in \mathbb{C}$,
c) for each $N \in \mathbb{N}_0$ and $\varepsilon > 0$

$$\partial_z^p \Big(b - \sum_{k=0}^{N-1} \chi b^{(-k)} - \sum_{j=0}^{N-1} \kappa b_{(-j)} + \sum_{j,k=0}^{N-1} \kappa \chi b^{(-k)}_{(-j)} \Big)(\cdot, \cdot, z) \in S^{-N+\varepsilon, -N+\varepsilon}$$

for all $z \in \mathbb{C}$ and $p \in \mathbb{N}_0$. Here, $\chi = \chi(\xi)$ and $\kappa = \kappa(x)$ are arbitrary zero excision functions.

A *holomorphic symbol family* is a map $a : \mathbb{C} \to \cup_{\mu, m \in \mathbb{R}} S^{\mu, m}$ such that there exist entire functions $\mu(\cdot)$ and $m(\cdot)$ with $a(z) \in S^{\mu(z), m(z)}_{\mathrm{cl}}$ for all $z \in \mathbb{C}$, and $b(x, \xi, z) := [x]^{-m(z)} [\xi]^{-\mu(z)} a(x, \xi, z)$ is a holomorphic family of zero order symbols in the above sense.

Theorem 3.3 *If $a(x, \xi, z)$ is the symbol of the complex power A^z associated with $a \in S^{\mu, m}_{\mathrm{cl}}$, then $a(z) \in S^{\mu z, mz}_{\mathrm{cl}}$ is a holomorphic family.*

In fact, all the requested properties hold in view of what we proved in Sections 3.2 and 3.3, in particular, part c) of Definition 3.2 is a direct consequence of Proposition 3.3.

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