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On stationary thermo-rheological viscous flows

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Abstract We study the system of equations describing a stationary thermoconvective flow of a non-Newtonian fluid. We assume that the stress tensor $\mathbf{S}$ has the form

$$\mathbf{S} = -P\mathbf{I} + \left(\mu(\theta) + \tau(\theta)|\mathbf{D}(\mathbf{u})|^{p(\theta)-2}\right)\mathbf{D}(\mathbf{u}),$$

where $\mathbf{u}$ is the vector velocity, P is the pressure, θ is the temperature and μ, p and τ are the given coefficients depending on the temperature. $\mathbf{D}$ and $\mathbf{I}$ are respectively the rate of strain tensor and the unit tensor. We prove the existence of a weak solution under general assumptions and the uniqueness under smallness conditions.

Keywords Non-Newtonian fluids · Nonlinear thermal diffusion equations · Heat and mass transfer

Mathematics Subject Classification (2000) 76A05 · 76D07 · 76E30 · 35G15

1 Introduction

In recent years various nonlinear constitutive relations for the stress tensor extending the classical Navier-Stokes problem have been proposed in the mathematical literature. In [17], Rajagopal and R*ůžič*ka have discussed mathematical models of

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electro-rheological fluids where in a growth condition the exponent depends on the electromagnetic field. These models lead to new interesting mathematical issues concerning existence, uniqueness and stability of flows that take into account also electrodynamical effects. First interesting results can be found in [18] and in [10].

For stationary solutions, where the problem is essentially uncoupled, the electromagnetic field being known, the growth exponent dependence became a given variable function $p(x)$, which under smoothness conditions yields regularity results for the respective weak solutions (see [1], [4]), similarly to the generalized Newtonian fluids [14] and [5].

Here we are interested on the analysis of steady flows of fluids with shear-dependent viscosity that are strongly influenced by the temperature field, rather than by an external electromagnetic field, the so-called thermo-rheological fluids. The two dimensional Stokes problem has been considered by Zhikov [22], where the coupling in the temperature equation was given by an energy dissipation term with a small parameter and was controlled by a Meyer's type estimate. Recently, a steady-state Boussinesq problem with a non-standard force in a non-linear feedback form evolving the temperature in the exponent of the velocity has been considered in [2].

In this work we consider two and three dimensional thermoconvective stationary flows. In Section 2, we introduce the temperature-velocity coupled problem and we recall some useful results on the generalized Orlicz-Lebesgue spaces $L^{p(x)}(\Omega)$ and Orlicz-Sobolev spaces $W^{1,p(x)}(\Omega)$. Exploiting the Hölder continuity of the temperature and the maximum principle, as in [19], [20], we prove in Section 3 the existence of at least one weak solution under general assumptions on the data, for the classical integrability condition $p(x) \geq p_* > 3N/(N+2)$, $(N = 2,3)$ on the convection terms. An interesting open question is to improve this lower bound to $2N/(N+2)$ as it was done recently for the case of constant exponents [13]. In Section 4, we show that weak solutions with a small amplitude in temperature and subject to sufficiently small external forces are in fact unique.

2 Governing equations and auxiliary results

2.1 Statement of the problem

Let Ω be a bounded open subset of $\mathbf{R}^N$, $N = 2,3$, with Lipschitz boundary Γ. We consider in Ω the following boundary value problem: find the functions $\theta(x)$, $\mathbf{u}(x) = (u_1, \ldots, u_N)$, $P(x)$ satisfying the equations

$$(\mathbf{u}\cdot\nabla)b(\theta) = \Delta\theta + g(x), \tag{2.1}$$

$$\theta(x) = \theta_1(x),\ x \in \Gamma = \partial\Omega. \tag{2.2}$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \operatorname{div}\left(\mu(\theta) + \tau(\theta)|\mathbf{D}(\mathbf{u})|^{p(\theta)-2}\right)\mathbf{D}(\mathbf{u}) - \nabla P + \mathbf{f}(x), \tag{2.3}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{2.4}$$

$$\mathbf{u}(x) = 0,\ x \in \Gamma = \partial\Omega. \tag{2.5}$$

In these equations $\mathbf{u}$, $\mathbf{P}$ and θ are respectively the velocity field, the pressure and the temperature of the fluid,

$$\mathbf{D}(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T)$$

is the tensor of rate of deformation, $\mathbf{f}(x)$ is the prescribed mass force. The coefficients p, μ and τ depend on the temperature θ. We assume that the given functions satisfy:

$$1 < p_* \le p(\theta), \tag{2.6}$$

$$0 \le \mu_* \le \mu(\theta),\ 0 \le \tau_* \le \tau(\theta), \tag{2.7}$$

$$\theta_1 \in H_0^1(\Omega)\cap C^\alpha(\overline{\Omega}),\ \alpha > 0;\ g \in L^s(\Omega),\ s > \frac{N}{2};\ \mathbf{f} \in W^{-1,p'_*}(\Omega), \tag{2.8}$$

$$b,\ \mu,\ \tau \in C^0(\mathbf{R}),\ \ p \in C^1(\mathbf{R}). \tag{2.9}$$

2.2 Classical functional spaces

We use the classical spaces of continuous functions in Ω, $C^\alpha(\overline{\Omega})$, $0 \le \alpha \le 1$, with the Hölder property for $0 < \alpha < 1$ and continuous differentiability for $\alpha = 1$.

The classical Sobolev spaces $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, for constant exponent $1 \le p < \infty$, are defined as usual with $W^{1,p}(\Omega) = H^1(\Omega)$ and $W_0^{1,p}(\Omega) = H_0^1(\Omega)$.

In addition, for vector valued functions we shall also use the notation

$$J^r(\Omega) = \{\mathbf{v} \in L^r(\Omega)^N : \int_\Omega \mathbf{v}\cdot\nabla\phi\, dx = 0, \forall\phi : \nabla\phi \in L^r(\Omega)^N\}, \tag{2.10}$$

$1 \le r < \infty$, for the generalized solenoidal vector fields.

For integrable functions we shall use the inverse Hölder's inequality

$$\left(\int_\Omega |g|^{\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}} \left(\int_\Omega |f|^q dx\right)^{\frac{1}{q}} \le \int_\Omega |f||g|\, dx \tag{2.11}$$

which is valid for any q constant, such that $0 < q < 1$.

2.3 Generalized Lebesgue spaces

We use the notations from [9]. Let Ω be a bounded open set of $\mathbf{R}^N$ and $p(x)$ be a measurable function on Ω such that

$$1 < p_* \le p(x) \le p^* < \infty,\ x \in \overline{\Omega}, \tag{2.12}$$

(p_* and p^* are some constants). By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$A_p(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty. \tag{2.13}$$

This is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} = \|f\|_{L^{p(\cdot)}} = inf\left\{\lambda > 0 : \, A_p\left(\frac{f}{\lambda}\right) \leq 1\right\}. \tag{2.14}$$

(see e.g.[8],[9],[11]). The following properties hold [12]:

i)

$$\|f\|_{p(\cdot)} < 1 \; (= 1; > 1) \Leftrightarrow A_p(f) < 1 \; (= 1; > 1); \tag{2.15}$$

ii)

$$\|f\|_{p(\cdot)} > 1 > \|f\|^{p_*}_{p(\cdot)} \leq A_p(f) \leq \|f\|^{p^*}_{p(\cdot)}; \tag{2.16}$$

iii)

$$\|f\|_{p(\cdot)} < 1 > \|f\|^{p^*}_{p(\cdot)} \leq A_p(f) \leq \|f\|^{p_*}_{p(\cdot)}; \tag{2.17}$$

iv)

$$\|f\|_{p(\cdot)} \to 0 \Leftrightarrow A_p(f) \to 0, \; \|f\|_{p(\cdot)} \to \infty \Leftrightarrow A_p(f) \to \infty. \tag{2.18}$$

2.4 Hölder, Sobolev and Korn's inequalities

The space $\left(L^{p(\cdot)}(\Omega), \, \|\cdot\|_{p(\cdot)}\right)$ is a separable, uniform convex Banach space and its conjugate space is $L^{q(\cdot)}(\Omega)$, where $1/q(x) + 1/p(x) = 1$. For any $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$ the following Hölder's inequality is valid ([8], [9], [12]) :

$$\int_\Omega |f(x)g(x)| \, dx \leq C_p \|f\|_{p(\cdot)} \|g\|_{q(\cdot)}, \;\; C_p = (1/p_* + 1/q_*). \tag{2.19}$$

The space $W^{1,\, p(\cdot)}(\Omega)$ is defined by

$$W^{1,\, p(\cdot)}(\Omega) = \left\{f(x) \in L^{p(\cdot)}(\Omega) : \, |\nabla f(x)| \in L^{p(\cdot)}(\Omega)\right\}$$

and $W_0^{1,\, p(\cdot)}(\Omega)$ is the closure of the set $C_0^\infty(\Omega)$ with respect to the norm of $W^{1,\, p(\cdot)}(\Omega)$ [23], [21], provided we assume p is a uniformly continuous function on Ω, such that:

$$|p(x_1) - p(x_2)| \leq \frac{C}{ln|\frac{1}{x_1 - x_2}|}, \quad for \;\; |x_1 - x_2| \; \leq \frac{1}{2}, \, \forall \, x_1, x_2 \in \overline{\Omega}, \tag{2.20}$$

with some constant $C > 0$.

In [23], [21], was proved that condition (2.20) guarantee that $C^\infty(\overline{\Omega})$ is dense in $W^{1,\, p(\cdot)}(\Omega)$. It means that Lavrentiev phenomenon is absent.

If $p(x)$ satisfies to (2.20), then the Sobolev embedding inequality [6], [9],[11]

$$\|f\|_{p^*,\Omega} \leq C \|\nabla f\|_{p,\Omega}, \;\; p^*(x) = p(x)N/(N - p(x)) \tag{2.21}$$

holds for any $f \in W_0^{1,p(\cdot)}(\Omega)$ with a some constant $C = C(N, sup_{\overline{\Omega}}\, p(x))$. If p_1, p_2 satisfy (2.20) then $L^{p_2(\cdot)}(\Omega) \subset L^{p_1(\cdot)}(\Omega)$, and the imbedding is continuous.

An important tool is the generalization of Korn's inequality for $p = p(x)$ [7], in a bounded domain $\Omega \subset \mathbf{R}^N$ with Lipschitz boundary. Let $p(x) > 1$ be a bounded exponent such that p and its conjugate exponent q satisfy (2.20). Then there exists $K > 0$, such that for all $\mathbf{v} \in \left(W_0^{1,p(\cdot)}(\Omega)\right)^N$, $div\, \mathbf{v} = 0$, there holds

$$||\nabla \mathbf{v}||_{L^{p(\cdot)}(\Omega)} \leq K||\mathbf{D}(\mathbf{v})||_{L^{p(\cdot)}(\Omega)}. \tag{2.22}$$

In particular, the inequalities depending on the uniform condition (2.20) also hold for an exponent that is Hölder continuous in $\overline{\Omega}$.

3 Existence theorem

3.1 Definition of weak solution

Definition 3.1 The pair $(\mathbf{u}, \theta)$ is said to be a *weak solution* of (2.1)-(2.5) if:
(i)

$$\mathbf{u} \in W_0^{1,\widetilde{p}}(\Omega) \cap J^1(\Omega), \;\; \theta - \theta_1 \in H_0^1(\Omega) \cap C^{\alpha}(\overline{\Omega}), \; \alpha > 0,$$

$$\widetilde{p} = \max(2,\, p_*), \; if \; \mu_* > 0, \; \tau_* \geq 0; \; \widetilde{p} = \; p_*, \; if \; \mu = 0, \; \tau_* > 0,$$

$$\int_{\Omega} \left(\mu(\theta)|\mathbf{D}(\mathbf{u})|^2 + \tau(\theta)|\mathbf{D}(\mathbf{u})|^{p(\theta)}\right) dx < \infty; \tag{3.1}$$

(ii) and for any test functions $\zeta \in H_0^1(\Omega)$

$$\int_{\Omega} (\nabla\theta - b(\theta)\mathbf{u}) \cdot \nabla\zeta dx = \int_{\Omega} g\zeta dx; \tag{3.2}$$

(iii) and for any test vector function $\mathbf{w} \in C_0^1(\Omega) \cap J^1(\Omega)$

$$\int_{\Omega} \left(\left(\mu(\theta) + \tau(\theta)|\mathbf{D}(\mathbf{u})|^{p(\theta)-2}\right) \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w}) - (\mathbf{u} \otimes \mathbf{u}) : \mathbf{D}(\mathbf{w})\right) dx \tag{3.3}$$

$$= \int_{\Omega} \mathbf{f}\mathbf{w} dx.$$

Theorem 3.1 *Let us assume that conditions (2.8)-(2.9) hold with* $N = 2, 3$ *and, additionally, that one of the conditions*

$$\mu = 0, \; 0 < \tau_*, \; \frac{3N}{N+2} < p_* \leq 2, \tag{3.4}$$

$$0 < \mu_*, \; 0 < \tau_*, \; 1 < p_* < \infty, \tag{3.5}$$

$$\tau_* = 0, \; 0 < \mu_*, \;\; 1 < p_* < p^* = \max_{[\theta_*, \theta^*]} \; p(\theta) \leq 2, \tag{3.6}$$

is fulfilled. Then the problem (2.1)-(2.5) has at last one weak solution $(\mathbf{u}, \theta)$.

Proof We will prove this theorem into several steps.

3.2 The auxiliary thermoconvective problem

For a given fixed $\mathbf{v} \in J^q(\Omega)$, with $q > N$ we consider first the auxiliary problem

$$(\mathbf{v} \cdot \nabla) b(\theta) = \Delta \theta + g(x), \tag{3.7}$$

$$\theta(x) = \theta_1(x),\ x \in \Gamma = \partial\Omega. \tag{3.8}$$

where b is supposed to be Lipschitz continuous.

The convective term for $\mathbf{w} \in J^q(\Omega)$ does not changes the weak maximum principle for (3.7), (3.8) in the form [16]

$$\theta_* \leq \theta(x) \leq \theta^*,\ x \in \Omega, \tag{3.9}$$

$$\theta_* = \min_{\Gamma}\ \theta_1(x) - \gamma||g||_{s,\Omega},\ \theta^* = \max_{\Gamma}\ \theta_1(x) + \gamma||g||_{s,\Omega}, \tag{3.10}$$

which yields an a priori estimate of θ in $L^\infty(\Omega)$, since the constant $\gamma > 0$ depends only on Ω and $s > N/2$.

Then, using Schauder's fixed point theorem (for instance in $L^2(\Omega)$), we show the existence of a solution θ, such that

$$\theta - \theta_1 \in H_0^1(\Omega) : \int_\Omega (\nabla\theta - b(\theta)\mathbf{v}) \cdot \nabla\zeta dx = \int_\Omega g\zeta dx,\ \forall \zeta \in H_0^1(\Omega). \tag{3.11}$$

The uniqueness follows by a well known comparison argument, since b is Lipschitz continuous, and (2.8) implies (see [16]) that θ is Hölder continuous in $\overline{\Omega}$ and satisfies the estimate

$$\|\theta\|_{C^{\alpha'}(\overline{\Omega})} + \|\theta\|_{H^1(\Omega)} \leq C\left(\|\mathbf{v}\|_{L^q(\Omega)},\ \|\theta_1\|_{H^1(\Omega)\cap C^\alpha(\overline{\Omega})}, ||g||_{L^s(\Omega)}\right), \tag{3.12}$$

for some $0 < \alpha' < \alpha$, provided $q > N$ and $s > N/2$.

Hence, by a standard continuous dependence argument, it is easy to conclude that if $\mathbf{v}_n \to \mathbf{v}$ in $J^q(\Omega)$ weakly (respectively strongly) then the corresponding solution $\theta_n = \theta(\mathbf{v}_n) \to \theta = \theta(\mathbf{v})$ weakly (respectively strongly) and in $C^\lambda(\overline{\Omega})$ strongly for any $0 \leq \lambda < \alpha' < 1$ (see for instance [19]).

Then and we have the following proposition:

Proposition 3.1 *For any $\mathbf{v} \in J^q(\Omega)$, with $q > N$, there exists a unique solution θ to (3.11), that, in addition, satisfies the estimates (3.9) and (3.12). Moreover the operator*

$$\Lambda : J^q(\Omega) \ni \mathbf{v} \to \theta \in C^\lambda(\overline{\Omega}) \tag{3.13}$$

is continuous for some $\lambda > 0$.

3.3 The auxiliary flow problem

Now for a fixed $\theta \in C^{\lambda}(\overline{\Omega})$, $0 < \lambda < 1$, and $\mathbf{v} \in J^{r}(\Omega)$ with $r \geq max(2, \frac{2\widetilde{p}}{N\widetilde{p}-\widetilde{p}-2N})$, we consider the auxiliary problem of finding $\mathbf{u} \in W_0^{1,\widetilde{p}}(\Omega) \cap J^1(\Omega)$ satisfying (3.1) and for any test vector function $\mathbf{w} \in C_0^1(\Omega) \cap J^1(\Omega)$

$$\int_{\Omega} \left(\mu(\theta) + \tau(\theta)|\mathbf{D}(\mathbf{u})|^{p(\theta)-2}\right) \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w}) \; - \; (\mathbf{u} \otimes \mathbf{v}) : \nabla \mathbf{w} \;\; dx \tag{3.14}$$

$$= \int_{\Omega} \mathbf{f}\mathbf{w} dx.$$

Here $\mu = \mu(\theta)$, $\tau = \tau(\theta)$ and $p = p(\theta)$ are known functions of $x \in \Omega$ and $p \in C^{\lambda}(\overline{\Omega})$.

Now we derive energy estimates for the velocity $\mathbf{u}$. Using definition (3.1), after some standard calculations we come to the energy relation

$$\int_{\Omega} \left(\mu(\theta)\,|\mathbf{D}(\mathbf{u})|^2 + \tau(\theta)|\mathbf{D}(\mathbf{u})|^{p(\theta)}\right) dx = \int_{\Omega} \mathbf{f}\,\mathbf{u}\, dx \equiv I. \tag{3.15}$$

Assuming $\mu_* > 0$, $\tau \geq 0$ and applying (2.22), (2.21) with $p = 2$, we can estimate the therm I in the following form

$$|I| \leq C\,\|f\|_{W^{-1,2}(\Omega)}\,\|\mathbf{D}(\mathbf{u})\|_{2,\Omega} \leq \int_{\Omega} \mu\,|\mathbf{D}(\mathbf{u})|^2\,dx + \frac{C}{\mu_*}\,\|f\|^2_{W^{-1,2}(\Omega)}, \tag{3.16}$$

and obtain the standard estimate

$$\int_{\Omega} \left(\mu_*\,|\mathbf{D}(\mathbf{u})|^2 + \tau_*|\mathbf{D}(\mathbf{u})|^{p(\theta)}\right) dx \leq \frac{C}{\mu_*}\,\|f\|^2_{W^{-1,2}(\Omega)} \equiv K_0. \tag{3.17}$$

It implies that

$$||\mathbf{u}||_q \leq C||\mathbf{D}(\mathbf{u})||_2 \leq C(K_0),\; q = \frac{2N}{N-2} > N = 2,3, \tag{3.18}$$

if $\mu_* > 0$, $\tau \geq 0$ and $N = 2,3$. If $\mu_* > 0$, $\tau_* > 0$ the estimate (3.17) implies that $\mathbf{u} \in \mathbf{W}^{1,p_*}(\Omega)$.

In the case $\mu = 0$, we evaluate the term I in the following form

$$|I| \leq C\,\|f\|_{W^{-1,\beta}(\Omega)}\,\|\mathbf{D}(\mathbf{u})\|_{p(\theta)},\; \beta = \frac{p_*}{p_*-1}. \tag{3.19}$$

Applying inequalities [12]:

$$\|\mathbf{D}(\mathbf{u})\|_{p(\theta)} \leq \left(\int_{\Omega} |\mathbf{D}(\mathbf{u})|^{p(\theta)} dx\right)^{\frac{1}{p^*}},\; if\;\|\mathbf{D}(\mathbf{u})\|_{p(\theta)} < 1, \tag{3.20}$$

$$\|\mathbf{D}(\mathbf{u})\|_{p(\theta)} \leq \left(\int_{\Omega} |\mathbf{D}(\mathbf{u})|^{p(\theta)} dx\right)^{\frac{1}{p_*}},\; if\;\|\mathbf{D}(\mathbf{u})\|_{p(\theta)} > 1, \tag{3.21}$$

and Young inequality we can to write

$$|I| \leq \begin{cases} \int_\Omega \tau |\mathbf{D}(\mathbf{u})|^{p(\theta)} dx + \frac{C}{\tau_*} \|f\|_{W^{-1,\beta}(\Omega)}^{\frac{p^*}{p^*-1}}, \; if \; \|\mathbf{D}(\mathbf{u})\|_{p(\theta)} < 1 \\ \int_\Omega \tau |\mathbf{D}(\mathbf{u})|^{p(\theta)} dx + \frac{C}{\tau_*} \|f\|_{W^{-1,\beta}(\Omega)}^{\frac{p_*}{p_*-1}} \; if \; \|\mathbf{D}(\mathbf{u})\|_{p(\theta)} > 1 \end{cases} . \tag{3.22}$$

Finally we obtain the estimate

$$\tau_* \int_\Omega |\mathbf{D}(\mathbf{u})|^{p(\theta)} dx \leq \frac{C}{\tau_*} \left(\|f\|_{W^{-1,\beta}(\Omega)}^{\frac{p^*}{p^*-1}} + \|f\|_{W^{-1,\beta}(\Omega)}^{\frac{p_*}{p_*-1}} \right) \equiv K_1. \tag{3.23}$$

Since by the assumption of Theorem 3.1 $\mu_* > 0$ or $\tau_* > 0$, the left hand side of (3.14) defines a strictly monotone and coercive operator on $W_0^{1,\widetilde{p}}(\Omega) \cap J^1(\Omega)$, ($\widetilde{p} = 2$ if $\mu_* > 0$ or $\widetilde{p} = p_*$ if $\mu_* = 0$), the unique solvability of problem (3.14) follows from standard results (see [22] for the case $\mathbf{v} \equiv \mathbf{0}$, $\mu = 0$ and [12] for the scalar case). We consider now the continuous dependence of the operator

$$N : J^r(\Omega) \times C^\lambda(\overline{\Omega}) \ni (\mathbf{v}, \theta) \to \mathbf{u} \in W_0^{1,\widetilde{p}}(\Omega) \cap J^s(\Omega) \tag{3.24}$$

where $\mathbf{u}$ is the solution to (3.14).

Proposition 3.2 *Let* $\mathbf{u}_n = \mathbf{u}(\mathbf{v}_n, \theta_n)$ *and* $\mathbf{u} = \mathbf{u}(\mathbf{v}, \theta)$ *denote the respective solution to (3.14) associated with the converging data* $\mathbf{v}_n \to \mathbf{v}_n$ *in* $J^r(\Omega)$*−weakly,* $r \geq max(2, \frac{2\widetilde{p}}{N\widetilde{p}-\widetilde{p}-2N})$ *and* $\theta_n \to \theta$ *in* $C^\lambda(\overline{\Omega})$.

Then we have

$$\mathbf{u}_n \to \mathbf{u} \; in \; W_0^{1,\widetilde{p}}(\Omega) \; weak \; and \; in \; J^s(\Omega) \; strongly \tag{3.25}$$

for any $s < N\widetilde{p}/(N - \widetilde{p})$ *if* $\widetilde{p} < N$ *or any* $s < \infty$ *if* $\widetilde{p} \geq N$, $N = 2, 3$.

Proof It is simple extension of Proposition 1 of [20] (and Lemma 3.1 of [22]). From the previous estimates, we have

$$\int_\Omega \left(\mu_* |\mathbf{D}(\mathbf{u}_n)|^2 + \tau_* |\mathbf{D}(\mathbf{u}_n)|^{\widetilde{p}} \right) dx \leq C \tag{3.26}$$

for a positive constant $C > 0$ depending on $\|f\|_{W^{-1,p'_*}(\Omega)}$ and $\frac{1}{\mu_*}$, if $\mu_* > 0$, or $\frac{1}{\tau_*}$, if $\tau_* > 0$, but independent of $\mathbf{v}_n$ and θ_n.

By Korn's inequality and Sobolev embeddings, (3.26) implies

$$\|\mathbf{u}_n\|_{L^s(\Omega)} \leq C \|\mathbf{D}(\mathbf{u_n})\|_{L^{\widetilde{p}}(\Omega)} \leq R, \tag{3.27}$$

for $s < N\widetilde{p}/(N - \widetilde{p})$ if $\widetilde{p} < N$ or any $s < \infty$ if $\widetilde{p} \geq N$, where $R > 0$ is a constant independent of $\mathbf{v}_n$ and θ_n.

Since $p(\theta_n) \to p(\theta)$ in $C^\lambda(\overline{\Omega})$ and $\mu(\theta_n) \to \mu(\theta)$ and $\tau(\theta_n) \to \tau(\theta)$ uniformly on $\overline{\Omega}$, as in Lemma 3.1 of of [22] our problem is regular and

$$\mathbf{u}_n \to \mathbf{u} \; in \; W_0^{1,\widetilde{p}}(\Omega) \; weakly \tag{3.28}$$

$$\left(\mu(\theta_n)\,|\mathbf{D}(\mathbf{u_n})|^2 + \tau(\theta_n)|\mathbf{D}(\mathbf{u_n})|^{\widetilde{p}} \right) \to \left(\mu(\theta)\,|\mathbf{D}(\mathbf{u_0})|^2 + \tau(\theta)|\mathbf{D}(\mathbf{u_0})|^{\widetilde{p}} \right) \tag{3.29}$$

in $L^1(\Omega)$ weakly.

By Minty's Lemma, $\mathbf{u_0}$ solves the limit problem (3.14) and , by uniqueness $\mathbf{u_0} = \mathbf{u} = \mathbf{u}(\mathbf{v}, \theta)$.

The strong convergence in $J^s(\Omega)$ follows by Rellich-Kondratchev compactness theorem. □

3.4 The proof of Theorem 3.1 and complements

We apply Schauder fixed point theorem in the following convex, closed, bounded subset of $J^s(\Omega)$:

$$B_R = \left\{ \mathbf{v} \in J^r(\Omega) : \|\mathbf{v}\|_{L^r(\Omega)} \leq R \right\} \tag{3.30}$$

with $r = 6$ if $N = 2$ or $r = 9/2$ if $N = 3$, where the constant $R > 0$ is given by the a priori estimate (3.27). We define $\widetilde{N} : B_R \to B_R$ by $\mathbf{u} = \widetilde{\mathbf{N}}(\mathbf{v}) = N(\mathbf{v}, \Lambda\mathbf{v})$, being N defined by (3.24) and Λ by (3.13). From Propositions (3.1) and (3.2), $\widetilde{N}$ is a well defined and completely continuous operator in $J^r(\Omega)$ and its fixed point

$$\mathbf{u} = \widetilde{N}(\mathbf{u}) \; with \; \theta = \Lambda\mathbf{u} \tag{3.31}$$

yields a solution to Theorem (3.1) in the case of b Lipschitz continuous.

Since the estimates (3.9) and (3.12) do not depend on the Lipschitz regularity of b, by uniform approximation of this function we obtain a solution $(\mathbf{u}, \theta)$ for b only continuous, concluding the proof of Theorem (3.1). □

Remark 3.1 The result of Theorem (3.1) may be extended to more general flows with constitutive laws of power type as in [20] for the Boussinesq-Stefan problem For instance, we could replace the Laplacian in (2.1) by the q-Laplacian $\Delta_q v = \nabla(|\nabla v|^{q-2}\nabla v)$, with $q > n/p_*$, or consider a two-phase Stefan problem with convection, and replace the Dirichlet boundary condition (2.2) for the temperature by a mixed boundary condition.

Remark 3.2 We may apply the Meyer's type estimate of [22] for the velocity in the following way. Consider the equation (2.3) in he form

$$\operatorname{div}\left(\left(\mu(\theta) + \tau(\theta)|\mathbf{D}(\mathbf{u})|^{p(\theta)-2} \right) \mathbf{D}(\mathbf{u}) - P\mathbf{I} + \mathbf{Z} \right) = 0, \tag{3.32}$$

with the tensor $\mathbf{Z}$ given by

$$\mathbf{Z} = \nabla F - (\mathbf{u} \otimes \mathbf{u}), \; and \; \Delta F = -\mathbf{f} \; in \; \Omega, \; F|_{\partial\Omega} = 0. \tag{3.33}$$

Since $\widetilde{p} > 3N/(N+2), (N = 2,3)$ we have $\mathbf{u} \otimes \mathbf{u} \in \mathbf{L}^r(\Omega)$ for $r < 3N/2(N-1)$, and if we assume $\mathbf{f} \in W^{-1,p'_*+\delta_1}(\Omega)$, for some

$\delta_1 \geq \widetilde{p} - 3N/(N+2) \equiv \delta_0 > 0$, it is not difficult to conclude that $\mathbf{Z} \in \mathbf{L}^q(\Omega)$ for some $q > p'_* + \delta_0$. By Theorem 3.1 of [22], if $\tau_* > 0$ we derive the estimate

$$\int_\Omega |D(\mathbf{u})|^{p(\theta)+\delta} dx \leq C \quad for\ some\ \delta > 0, \tag{3.34}$$

and if $\mu_* > 0$ we also obtain the Meyer's estimate

$$\int_\Omega |D(\mathbf{u})|^{2+\delta} dx \leq C \quad for\ some\ \delta > 0. \tag{3.35}$$

In particular, it is possible to extend Zhikov's existence result for $N = 2$ to the convective case with an additional coupling term in the temperature equation (2.1) by inserting the dissipation energy of the type

$$g = \lambda \left(\mu(\theta) \, |\mathbf{D}(\mathbf{u})|^2 + \tau(\theta) |\mathbf{D}(\mathbf{u})|^{p(\theta)} \right)$$

for sufficiently small $\lambda > 0$.

4 Uniqueness of weak solution

Let $(\theta_1, \mathbf{u}_1)$ and $(\theta_2, \mathbf{u}_2)$ be two different solutions to problem (2.1), (2.2), (2.3), (2.4), (2.5) and

$$\theta = (\theta_1 - \theta_2), \ \mathbf{u} = (\mathbf{u}_1 - \mathbf{u}_2). \tag{4.1}$$

The functions $(\theta, \mathbf{u})$ satisfy to the following problem

$$\int_\Omega (\nabla\theta - (b(\theta_1) - b(\theta_2))\mathbf{u}_1 - b(\theta_2)\mathbf{u}) \nabla\zeta dx = 0, \tag{4.2}$$

$$\int_\Omega ((\mathbf{u}_1\nabla)\mathbf{u}_1 - (\mathbf{u}_2\nabla)\mathbf{u}_2) \, \mathbf{w} dx = - \tag{4.3}$$

$$\int_\Omega \left(\mu \mathbf{D}(\mathbf{u}) + \tau \left(|\mathbf{D}(\mathbf{u}_1)|^{p(\theta_1)-2} \mathbf{D}(\mathbf{u}_1) - |\mathbf{D}(\mathbf{u}_2)|^{p(\theta_2)-2} \mathbf{D}(\mathbf{u}_2) \right) : \mathbf{D}(\mathbf{w}) \right) dx.$$

We will prove the uniqueness result only for close solutions. According to (3.9) and (3.10), we have

$$|\theta| = |(\theta_1 - \theta_2)| \leq \lambda = \theta^* - \theta_*. \tag{4.4}$$

We assume that

$$|p(\theta_1) - p(\theta_2)| \leq \max_{[\theta_*, \theta^*]} |p'(\theta)| \lambda \leq \delta_0, \tag{4.5}$$

with δ_0 given in (3.34). We use the notations

$$p_* = \min_{[\theta_*, \theta^*]} p(\theta), \ p^* = \max_{[\theta_*, \theta^*]} p(\theta),$$

assuming that

$$p^* - p_* \leq \delta = \max_{[\theta_*, \theta^*]} |p'(\theta)| \, \lambda \leq \delta_0. \tag{4.6}$$

The Meyer's type estimate (3.34) guarantee that

$$\mathbf{D}(\mathbf{u}_1), \mathbf{D}(\mathbf{u}_2), \mathbf{D}(\mathbf{w}) \in L^{p_*+\delta_0},$$

with δ_0 given in (3.34). Moreover all integrals in (4.3) will be bounded if p^*, p_* satisfy to (4.6). We only show how to prove the boundedness of the last integral, which is the main one. Using Hölder's inequality, we have

$$\left| \int_\Omega |\mathbf{D}(\mathbf{u}_j)|^{p(\theta_j)-2} \mathbf{D}(\mathbf{u}_1) : \mathbf{D}(\mathbf{w})\, dx \right|$$

$$\leq \left(\int_\Omega |\mathbf{D}(\mathbf{u}_j)|^{\frac{(p(\theta_j)-1)(p_*+\delta_0)}{p_*+\delta_0-1}} dx \right)^{\frac{p_*+\delta_0-1}{p_*+\delta_0-1}} \left(\int_\Omega |\mathbf{D}(\mathbf{w})|^{p_*+\delta_0} dx \right)^{\frac{1}{p_*+\delta_0}} \leq C,$$

$j = 1, 2,$ because

$$\frac{(p(\theta_j)-1)(p_*+\delta_0)}{p_*+\delta_0-1} \leq \frac{(p^*-1)(p_*+\delta_0)}{p_*+\delta_0-1} \leq p_*+\delta_0.$$

Let $\mathbf{v}_h$ be an average of the function $\mathbf{v}$ such that $||D(\mathbf{v}_h) - D(\mathbf{v})||_q \to 0$ as $h \to 0$, $(q = p_* + \delta_0)$. Putting $\zeta = (\theta_1 - \theta_2)$, $\mathbf{w} = (\mathbf{u}_1 - \mathbf{u}_2)_h$, to (4.3), after some standard calculations and a passage to the limit with respect to $h \to 0$, we get the relations

$$\int_\Omega |\nabla\theta|^2 dx = \int_\Omega ((b(\theta_1) - b(\theta_2))\mathbf{u}_1 - b(\theta_2)\mathbf{u})\nabla\theta dx, \tag{4.7}$$

$$I_0 \equiv \int_\Omega ((\mathbf{u}_1\nabla)\mathbf{u}_1 - (\mathbf{u}_2\nabla)\mathbf{u}_2)\,\mathbf{u} dx = - \tag{4.8}$$

$$\int_\Omega \left(\mu\mathbf{D}(\mathbf{u}) + \tau \left(|\mathbf{D}(\mathbf{u}_1)|^{p(\theta_1)-2}\mathbf{D}(\mathbf{u}_1) - |\mathbf{D}(\mathbf{u}_2)|^{p(\theta_2)-2}\mathbf{D}(\mathbf{u}_2) \right) \right) : \mathbf{D}(\mathbf{u}) dx$$

$$\equiv I_1 + I_2 + I_3,$$

where

$$I_1 = \int_\Omega \tau \left[|\mathbf{D}(\mathbf{u}_2)|^{p(\theta_2)-2}\mathbf{D}(\mathbf{u}_2) - |\mathbf{D}(\mathbf{u}_1)|^{p(\theta_2)-2}\mathbf{D}(\mathbf{u}_1) \right] : \mathbf{D}(\mathbf{u}) dx, \tag{4.9}$$

$$I_2 = \int_\Omega \tau \left[|\mathbf{D}(\mathbf{u}_1)|^{p(\theta_2)-2}\mathbf{D}(\mathbf{u}_1) - |\mathbf{D}(\mathbf{u}_1)|^{p(\theta_1)-2}\mathbf{D}(\mathbf{u}_1) \right] : \mathbf{D}(\mathbf{u}) dx, \tag{4.10}$$

$$I_3 = \int_\Omega \mu |\mathbf{D}(\mathbf{u})|^2 dx. \tag{4.11}$$

First we consider the case $\mu = const$, $\tau = const$.

Theorem 4.1 *Under conditions of Theorem (3.1) there exist positive constants* $\varepsilon^* > 0,\ \delta^* > 0,$ *(generally small), such that, the problem (2.1)-(2.5) has a unique weak solution* $(\mathbf{u}, \theta)$ *provided*

$$\theta^* - \theta_* \le \varepsilon^*,\ ||\mathbf{f}||_{W^{-1,r}(\Omega)} \le \delta^*,$$

$$\mu = const > 0,\ \tau = const > 0,\ and\ r > p'_* + \widetilde{p} - 3N/(N+2).$$

If $\mu = 0$ *and* $p^* \le 2$*, this assertion is valid if, in addition,*

$$p^* - p_* \le 2 - p_* \le \delta^*.$$

Proof First we consider the case μ and $\tau = const > 0,\ 2 \le p_*$.

Applying the inequality [12] for $\forall \xi, \eta \in \mathbf{R}^N,\ 2 \le p < \infty$,

$$\left(\frac{1}{2}\right)^p |\xi - \eta|^p \le \left[\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi - \eta)\right],$$

we obtain

$$\int_\Omega \tau 2^{-p^*} |\mathbf{D}(\mathbf{u})|^{p_2(\cdot)} dx \le I_1,\ p_2 = p(\theta_2), \tag{4.12}$$

therefore

$$\int_\Omega \left(\mu |\mathbf{D}(\mathbf{u})|^2 + \tau 2^{-p^*} |\mathbf{D}(\mathbf{u})|^{p_2(\cdot)}\right) dx \le I_1 + I_3 = I_0 - I_2. \tag{4.13}$$

Using the inequality for $0 \le \eta \le \infty,\ p^* \in [p_1, p_2]$,

$$\left|\eta^{p_2-1} - \eta^{p_1-1}\right| \le \eta^{p^*-1} |\ln \eta|\, |p_2 - p_1|, \tag{4.14}$$

and the Young inequality, we evaluate I_2 in the following way

$$\begin{aligned} |I_2| &\le \int_\Omega \tau |\mathbf{D}(\mathbf{u_1})|^{p^*-1} |ln|\mathbf{D}(\mathbf{u_1})||\ |\mathbf{D}(\mathbf{u})|\, |p_1 - p_2| dx \qquad (4.15)\\ &\le \frac{1}{2}\int_\Omega \tau 2^{-p^*} |\mathbf{D}(\mathbf{u})|^{p_2(\cdot)} dx + C\delta^{\frac{p^*}{p^*-1}} I_4,\\ I_4 &= \int_\Omega \left(|\mathbf{D}(\mathbf{u_1})|^{p^*(\cdot)-1} |ln|\mathbf{D}(\mathbf{u_1})||\right)^{p'_2(\cdot)} dx. \end{aligned}$$

with $\delta \ge |p_1 - p_2|$ given in (4.6) and some $C = C(p^*, p_*, \tau)$. Next we use the inequality

$$|\ln y| \le \frac{1}{\varepsilon}(y^\varepsilon + y^{-\varepsilon}),\ 0 < y < \infty,\ 0 < \varepsilon < \infty,$$

and evaluate I_4 in the following way

$$|I_4| \le \int_\Omega \left(|\mathbf{D}(\mathbf{u_1})|^{p^*(\cdot)-1+\varepsilon} + |\mathbf{D}(\mathbf{u_1})|^{p^*-1-\varepsilon}\right)^{p'_2(\cdot)} dx. \tag{4.16}$$

We choose ε and δ, such that,

$$\frac{(p^* - 1 \pm \varepsilon)p_2}{p_2 - 1} \le \frac{(p^* - 1 \pm \varepsilon)p_*}{p_* - 1} \le p_* + \delta_0, \tag{4.17}$$

$$p^* - p_* + \varepsilon \le \delta + \varepsilon \le \delta_0(\frac{p_* - 1}{p_*}). \tag{4.18}$$

Later on we fix some ε and δ satisfying to

$$\delta + \varepsilon = \max_{[\theta_*, \theta^*]} |p'(\theta)|\, \lambda + \varepsilon \le \delta_0(\frac{p_* - 1}{p_*}).$$

Finally, taking into consideration inequality (3.34), we obtain the estimate

$$|I_4| \le C, \tag{4.19}$$

where the constant C does not depend on δ. From (4.13), (4.15) and (4.19), it follows

$$\int_\Omega \left(\mu |\mathbf{D}(\mathbf{u})|^2 + \frac{\tau}{2} 2^{-p^*} |\mathbf{D}(\mathbf{u})|^{p_2(\cdot)} \right) dx \le |I_0| + C\delta^{\frac{p^*}{p^*-1}}. \tag{4.20}$$

As usual, following [15], the term I_0 can be transformed into the form

$$I_0 = \int_\Omega u_k \mathbf{u}_1 \cdot \mathbf{u}_{x_k} dx$$

and evaluated in the standard way

$$|I_0| \le C \|\mathbf{u}_1\|_{W^{1,2}} \|\mathbf{D}(\mathbf{u})\|_2^2 \le C \|\mathbf{f}\|_{W^{-1,2}} \|\mathbf{D}(\mathbf{u})\|_2^2. \tag{4.21}$$

Applying (3.17) and assuming $C \|\mathbf{f}\|_{W^{-1,2}} \le \mu/2$, ($\|\mathbf{f}\|_{W^{-1,2}}$ is small), we obtain

$$|I_0| \le \frac{\mu}{2} \|\mathbf{D}(\mathbf{u})\|_2^2. \tag{4.22}$$

Joining (4.20), (4.22) we find

$$\int_\Omega \left(\mu |\mathbf{D}(\mathbf{u})|^2 + \tau |\mathbf{D}(\mathbf{u})|^{p_2} \right) dx \le C\delta^{\frac{p^*}{p^*-1}}. \tag{4.23}$$

Returning to (4.2) we can write the estimate

$$\|\theta\|_{C^\alpha(\overline{\Omega})} \le C \left(||\mathbf{u}_1||_{q,\Omega} \max|\theta| + ||\mathbf{u}||_{q,\Omega} \right), \ (N = 2 < q < \infty). \tag{4.24}$$

Using (3.17) and assuming that

$$2 = N < q \le p_*, \ C||\mathbf{u}_1||_{q,\Omega} \le CK_0 < 1, \tag{4.25}$$

(to derive the last inequality we used the smallness of $\|\mathbf{f}\|_{W^{-1,2}}$) we obtain from (4.23), (4.24)

$$\|\theta\|_{C^\alpha(\overline{\Omega})} \le C\delta^{\frac{p^*}{p^*-1}} \le C (\max|\theta|)^{\frac{p^*}{p^*-1}}. \tag{4.26}$$

The last inequality implies that $\max|\theta| = 0$.

Now we consider the case $0 < \mu$, $0 < \tau$, $p^* \le 2$.

We use the following inequality [12] for $\forall \xi, \eta \in \mathbf{R}^N$, $1 < p < 2$,

$$(p-1)|\xi - \eta|^2 (|\xi|^p + |\eta|^p)^{\frac{p-2}{p}} \le \left[\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta \right)(\xi - \eta) \right].$$

Then the term I_1 given by (4.9) can be estimated below in the following way (compare with (4.12))

$$\Xi^2 \equiv (p_* - 1)\int_\Omega \tau|\mathbf{D}(\mathbf{u})|^2 \left(|\mathbf{D}(\mathbf{u}_1)|^{p_2} + |\mathbf{D}(\mathbf{u}_2)|^{p_2}\right)^{\frac{p_2-2}{p_2}} dx \le I_1. \tag{4.27}$$

Therefore we obtain the estimate

$$\int_\Omega \mu|\mathbf{D}(\mathbf{u})|^2 dx + \Xi^2 \le |I_0| + |I_2|. \tag{4.28}$$

Repeating (4.15), (4.16) with

$$\frac{(p^* - 1 \pm \varepsilon)p_2}{p_2 - 1} \le \frac{(p^* - 1 \pm \varepsilon)p_*}{p_* - 1} \le 2, \tag{4.29}$$

and using (4.21), (4.22), we obtain analogously to (4.23)

$$\int_\Omega \mu|\mathbf{D}(\mathbf{u})|^2 dx + \Xi^2 \le C\delta^{\frac{p^*}{p^*-1}}. \tag{4.30}$$

This completes the proof.

Now we consider the case $\mu = 0$, $N/2 < 3N/(N+2) < p_*$, $N < 4$. In this case the relations (4.13), (4.27), (4.28) yield

$$(p_* - 1)\int_\Omega \tau|\mathbf{D}(\mathbf{u})|^2 \left(|\mathbf{D}(\mathbf{u}_1)|^{p_2} + |\mathbf{D}(\mathbf{u}_2)|^{p_2}\right)^{\frac{p_2-2}{p_2}} dx \le I_1 \le |I_0| + |I_2|. \tag{4.31}$$

Using the inverse Hölder's inequality [3]

$$\left(\int_\Omega |g|^{\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}} \left(\int_\Omega |f|^q dx\right)^{\frac{1}{q}} \le \int_\Omega |f||g|\, dx,\ 0 < q = const < 1,$$

we get the estimate

$$(p_* - 1)\left(\int_\Omega |\mathbf{D}(\mathbf{u})|^{2q}\right)^{\frac{1}{q}} \left(\int_\Omega \left(|\mathbf{D}(\mathbf{u}_1)|^{p_2} + |\mathbf{D}(\mathbf{u}_2)|^{p_2}\right)^{\frac{(2-p_2)q}{p_2(1-q)}} dx\right)^{\frac{q-1}{q}} \le I_1. \tag{4.32}$$

Choosing q, such that,

$$\frac{(2-p_2)q}{(1-q)} \le p_* + \delta_0 \to q \le \frac{p_* + \delta_0}{2 + \delta_0} < 1, \tag{4.33}$$

and taking into consideration the inequalities (3.34), (4.31), (4.32), we have

$$\left(\int_\Omega |\mathbf{D}(\mathbf{u})|^{2q}\right)^{\frac{1}{q}} \le CI_1 \le C(|I_0| + |I_2|). \tag{4.34}$$

The term I_0 and I_2 were already estimated previously (with $p_* \le 2$). To estimate I_0 (with $2 \ge p_*$) we use the Korn's (2.22) and Sobolev embedding inequalities (2.21). We evaluate I_0 in the following way (compare with (4.21))

$$|I_0| \le \int_\Omega |u_k \mathbf{u}_1 \cdot \mathbf{u}_{x_k}| dx \le C\|\mathbf{u}_1\|_{\lambda_1} \|\mathbf{u}\|_{\lambda_2} \|\mathbf{D}(\mathbf{u})\|_{2q}, \tag{4.35}$$

with

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{2q}\leq 1,\ \lambda_2=\frac{Nqp_2}{N-qp_2},\ \lambda_1=\frac{N(p_*+\delta_0)}{N-(p_*+\delta_0)},$$

and $q=(p_*+\delta_0)/(2+\delta_0)$ (see 4.33). The last inequality leads to

$$\frac{1}{p_*+\delta_0}-\frac{2}{N}+\frac{1}{q}\leq\frac{1}{p_*+\delta_0}-\frac{2}{N}+\frac{2(2+\delta_0)}{(p_*+\delta_0)p_*}\leq 1.$$

Assuming $\|\mathbf{D}(\mathbf{u})\|_{2q}<1$ and applying (3.20), (3.21) we obtain the inequality

$$|I_0\,|\leq C\,\|(\mathbf{D}(\mathbf{u})_1)\|_{p_*}\,\|\mathbf{D}(\mathbf{u})\|_{2q}^2\leq C\,\|D(\mathbf{u}_1)\|_{p_*}\left(\int_\Omega|\mathbf{D}(\mathbf{u})|^q\right)^{\frac{2}{qp^*}},$$

or taking into consideration (4.34)

$$\left(\int_\Omega|\mathbf{D}(\mathbf{u})|^{2q}\right)^{\frac{1}{q}}\leq CI_1\leq C(\|(\mathbf{D}(\mathbf{u})_1)\|_{p_*}\left(\int_\Omega|\mathbf{D}(\mathbf{u})|^{2q}\right)^{\frac{2}{qp^*}}+|I_2|).$$

It follows

$$\left(\int_\Omega|\mathbf{D}(\mathbf{u})|^{2q}\right)^{\frac{1}{q}}\leq C|I_2|, \tag{4.36}$$

if $\|D(\mathbf{u}_1)\|_{p^*}$ and $\|D(\mathbf{u})\|_{p^*}$ are sufficiently small and $p^*\leq 2$. To evaluate I_2 we use an argument similar to (4.13), (4.15) and the Hölder's inequality (2.19). Then we obtain

$$|I_2|\leq\frac{\delta}{\varepsilon}C\,\|\mathbf{D}(\mathbf{u})\|_{2q}\left(\left\||\mathbf{D}(\mathbf{u}_1)|^{p_*-1+\varepsilon}\right\|_\beta+\left\||\mathbf{D}(\mathbf{u}_1)|^{p_*-1-\varepsilon}\right\|_\beta\right), \tag{4.37}$$

with q and β, such that,

$$\beta=\frac{2q}{2q-1},\ \frac{(p_*-1+\varepsilon)2q}{2q}\leq p_*+\delta_0. \tag{4.38}$$

The last inequality will be valid if

$$\frac{(p^*-1+\varepsilon)p_*q}{qp_*-1}\leq p_*+\delta_0, \tag{4.39}$$

or

$$\frac{p_*+\delta_0}{p_*(p_*-p^*+1-\varepsilon+\delta_0)}\leq q. \tag{4.40}$$

The conditions (4.33), (4.40) are compatible if p^*-p_* and ε are sufficiently small and

$$\frac{2}{1+\delta_0}<p_*\leq 2. \tag{4.41}$$

¿From now on we assume ε fixed. Then we have the inequality

$$\left(\int_\Omega|\mathbf{D}(\mathbf{u})|^{2q}dx\right)^{\frac{1}{q}}\leq C\delta\,\|\mathbf{D}(\mathbf{u})\|_{2q}.$$

Hence assuming that $\|D(\mathbf{u})\|_{qp_2} < 1$ and applying inequality (3.20) and Young inequality, we obtain

$$\left(\int_\Omega |\mathbf{D}(\mathbf{u})|^{2q} dx\right)^{\frac{1}{q}} \leq \frac{1}{2}\left(\int_\Omega |\mathbf{D}(\mathbf{u})|^{2q} dx\right)^{\frac{1}{q}} + C\delta^{\frac{p^*}{p^*-1}}.$$

Then, repeat the arguments (4.24)-(4.26), we may complete the proof of the theorem. □

Now we consider a general case in which $\mu(\theta), \tau(\theta)$ are given functions, assuming the derivatives are bounded

$$|\mu',\ \tau',\ b',\ p'| \leq \gamma. \tag{4.42}$$

In this case relation, (4.8) takes the form

$$I_0 \equiv \int_\Omega ((\mathbf{u}_1\nabla)\mathbf{u}_1 - (\mathbf{u}_2\nabla)\mathbf{u}_2)\,\mathbf{u}dx \tag{4.43}$$

$$= -\int_\Omega (\mu_1\mathbf{D}(\mathbf{u}_1) - \mu_2\mathbf{D}(\mathbf{u}_2)) : D\mathbf{u}dx$$

$$-\int_\Omega \left(\tau_1|\mathbf{D}(\mathbf{u}_1)|^{p_2-2}\mathbf{D}(\mathbf{u}_1) - \tau_2|\mathbf{D}(\mathbf{u}_2)|^{p_2-2}\mathbf{D}(\mathbf{u}_2)\right) : D\mathbf{u}dx$$

$$\equiv I_{11} + I_{21} + I_{22} + I_{31} + I_{32},$$

where

$$I_{11} = \int_\Omega \tau_1\left[|\mathbf{D}(\mathbf{u}_2)|^{p_2-2}\mathbf{D}(\mathbf{u}_2) - |\mathbf{D}(\mathbf{u}_1)|^{p_2-2}\mathbf{D}(\mathbf{u}_1)\right] : D\mathbf{u}dx,$$

$$I_{21} = \int_\Omega \tau_1\left[|\mathbf{D}(\mathbf{u}_1)|^{p_2-2}\mathbf{D}(\mathbf{u}_1) - |\mathbf{D}(\mathbf{u}_1)|^{p_2-2}\mathbf{D}(\mathbf{u}_1)\right] : D\mathbf{u}dx,$$

$$I_{22} = \int_\Omega (\tau_2 - \tau_1)|\mathbf{D}(\mathbf{u}_2)|^{p_2-2}\mathbf{D}(\mathbf{u}_2) : D\mathbf{u}dx,$$

$$I_{31} = \int_\Omega \mu_1|\mathbf{D}(\mathbf{u})|^2 dx,\ I_{32} = \int_\Omega (\mu_2 - \mu_1)\mathbf{D}(\mathbf{u}_2) : \mathbf{D}(\mathbf{u})dx,$$

and $\mu_i = \mu(\theta_i), \tau_i = \tau(\theta_i),\ p_i = p(\theta_i),\ i = 1,2$. Here the terms I_{11}, I_{31} are positive and (4.43) will be written like

$$I_{11} + I_{31} = -(I_0 + I_{21} + I_{22} + I_{32}).$$

The new terms I_{22}, I_{32} (in comparison with (4.13)) can be evaluated in the following way

$$|I_{22}| \leq \gamma\, max|\theta|\ ||D\mathbf{u}||_{p_2}||\mathbf{D}(\mathbf{u}_2)||_{p_2},$$

$$|I_{32}| \leq \gamma\, max|\theta|\ ||\mathbf{D}(\mathbf{u}_2)||_2||\mathbf{D}(\mathbf{u})||_2,\ (if\ \mu_* > 0).$$

Remark that $||\mathbf{D}(\mathbf{u}_2)||_{p_2}, ||\mathbf{D}(\mathbf{u}_2)||_2$ are small. By repeating the previous arguments, we can finally formulate our uniqueness theorem in the following form:

Theorem 4.2 *Under conditions of Theorem (3.1) there exist positive constants* $\varepsilon^* > 0$, $\delta^* > 0$, *such that, the problem (2.1)-(2.5) has a unique weak solution* $(\mathbf{u}, \theta)$ *provided*

$$\gamma,\ \theta^* - \theta_* \leq \varepsilon^*,\ ||\mathbf{f}||_{W^{-1,r}(\Omega)} \leq \delta^*,$$

$$0 < \mu_*, \tau_*, ,\ \textit{and}\ r > p'_* + \widetilde{p} - 3N/(N+2)$$

($\gamma > 0$ *given in (4.42)). If* $\mu = 0$ *and* $p^* \leq 2$*, this assertion is valid if, in addition,*

$$p^* - p_* \leq 2 - p_* \leq \delta^*.$$

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