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Multistationarity and Bistability for Fewnomial Chemical Reaction Networks

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Abstract

Bistability and multistationarity are properties of reaction networks linked to switch-like responses and connected to cell memory and cell decision making. Determining whether and when a network exhibits bistability is a hard and open mathematical problem. One successful strategy consists of analyzing small networks and deducing that some of the properties are preserved upon passage to the full network. Motivated by this, we study chemical reaction networks with few chemical complexes. Under mass action kinetics, the steady states of these networks are described by fewnomial systems, that is polynomial systems having few distinct monomials. Such systems of polynomials are often studied in real algebraic geometry by the use of Gale dual systems. Using this Gale duality, we give precise conditions in terms of the reaction rate constants for the number and stability of the steady states of families of reaction networks with one non-flow reaction.

Keywords Chemical reaction networks · Multistationarity and bistability · Fewnomial systems · Gale duality · Real algebraic geometry · Steady states of dynamical systems

1 Introduction

Bistability, that is, the existence of two asymptotically stable steady states together with an unstable steady state, is a key property of dynamical systems that provides an explanation of switch-like behavior in real systems. In particular, bistability and more generally multistability are linked to cell decision making and differentiation, explaining the coexistence of different states in cells with identical genetic material

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(Ferrell 2012; Laurent and Kellershohn 1999; Xiong and Ferrell 2003). Although the term bistability is widespread in molecular biology, determining whether and when a mathematical model exhibits bistability is a highly nontrivial task. This task is further complicated by the high number of variables and unknown parameters present in the models. Since bistability requires multistationarity, that is, the existence of more than one steady state, much effort has been centered around the easier (but still complex) problem of determining whether and when a network displays multistationarity.

One of the successful strategies to address multistationarity and bistability focuses on studying a smaller, but related, model and then seeks to lift these properties to the larger full model. To effectively implement this strategy, one requires a catalog of small networks that are multistationary or bistable. We contribute to this catalog through a detailed analysis of networks with one non-flow reaction.

Specifically, we consider the mathematical framework in which the evolution of the concentration of species in a chemical reaction network is modeled by means of a system of polynomial ordinary differential equations (ODEs). This system has the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_\kappa(x), \qquad x \in \mathbb{R}_{\geq 0}^n,$$

where $x = (x_1, \ldots, x_n)$ is the vector of concentrations, $\kappa \in \mathbb{R}_{>0}^s$ is the vector of reaction rate constants, and $f_\kappa(x) = [f_{\kappa,1}(x), \ldots, f_{\kappa,n}(x)]^T \in (\mathbb{R}[x_1, \ldots, x_n])^n$ is a vector of n variable polynomials. Due to $\mathbb{R}$-linear relations among the polynomials $f_{\kappa,i}(x)$, $i = 1, \ldots, n$, the dynamics of this ODE system are often confined to linear subspaces defined by relations

$$Wx = c, \tag{1}$$

where $c \in \mathbb{R}^d$ is determined by the initial conditions and W is a matrix in $\mathbb{R}^{d \times n}$. These equations are called conservation laws.

In this setting, the positive steady states of the network are the elements of the set

$$S_{\kappa,c} = \{x \in \mathbb{R}_{>0}^n \mid f_\kappa(x) = 0, \ Wx = c\}.$$

Multistationarity then refers to the existence of a choice of the parameters κ and c such that the set $S_{\kappa,c}$ contains at least two elements. The question of deciding *whether* a network exhibits multistationarity is essentially solved, and one can employ one of several existing methods, see for example Millán et al. (2012), Conradi and Flockerzi (2012), Ellison et al. (2012), Feliu and Wiuf (2013a), Donnell et al. (2014). On the other hand, deciding for *what* parameter values the network exhibits multistationarity, deciding *how many* elements the set $S_{\kappa,c}$ can contain, and deciding whether bistability arises remain open (and hard) problems.

As already discussed, it can be fruitful to have a detailed understanding of smaller networks contained within the larger network. In particular, it has been shown that after introducing inflow/outflow reactions (Craciun and Feinberg 2006) or adding the so-called intermediates to a reaction network (Feliu and Wiuf 2013b), the maximum number of elements in the sets $S_{\kappa,c}$ can only increase. Similar relations are found for subnetworks or embedded networks (Joshi and Shiu 2013). Motivated by this, in Joshi

([2013]) a characterization of multistationarity was given for generic reaction networks with one 'arbitrary' reaction and inflow/outflow reactions for all species; however, neither the actual number of positive steady states nor the existence of bistability was determined. In this work, we expand upon this characterization by counting the possible number of steady states when the non-flow reaction is irreversible and by exploring the parameter region with respect to the cardinality of $S_{\kappa,c}$. In particular, we determine that these networks have at most three positive steady states. Additionally, we also determine which of these networks exhibits bistability.
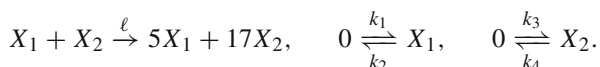
An additional goal of this paper is to present a strategy that can enhance our understanding of multistationarity, when the number of complexes appearing in the network is small. In principle, using cylindrical algebraic decompositions one can determine the region of multistationarity and the number of elements of $S_{\kappa,c}$, see for example, the book Basu et al. ([2007]). In practice, however, computing cylindrical algebraic decompositions is often unfeasible due to the high computational cost in relation to the degree, number of variables, and number of parameters in the polynomial systems being studied. In light of this, case-specific approaches [sometimes in conjunction with partial results from Conradi et al. ([2017])] are often employed.

Here, we study the number of steady states by finding a *Gale dual system* (Bihan and Sottile [2008], [2007]; Sottile [2011]). Let $l > 0$ be an integer and consider a square system of n polynomial equations with n variables, $n + l + 1$ monomials, and having a finite number of solutions. A Gale dual system is a new system of l equations in l variables, together with a cone, such that the solutions of the new system in the cone are in one-to-one correspondence with the *positive* solutions of the original system. Hence, it can be advantageous to pass to the Gale dual system when l is small, i.e., when the original system has few monomials. Such systems are often called *fewnomial* systems.

Moving to the Gale dual system is particularly advantageous when $l = 1$, as is the case for the networks in Joshi ([2013]) with one non-flow irreversible reaction. The Gale dual system in this case is a single polynomial in one variable, and the constraining cone is simply an interval on the real line. In this setting, classical methods such as the (generalized) Descartes' rule of signs, Sturm sequences, or real analytical techniques in $\mathbb{R}$, can be applied to study the solutions in terms of the coefficients (which correspond to the unknown reaction rate constants).

Using Gale dual systems, we also study a second family of networks from Joshi ([2013]), constructed similarly to the case above, but where the non-flow reaction is reversible. In this case $l = 2$, but with an appropriate choice of a Gale dual system, the problem is again reduced to the study of the roots of a single-variable polynomial in an interval. The resulting polynomial in this case has a more complicated structure. However, we determine the number of solutions in terms of the reaction rate constants in some special cases. Based on these results, we conjecture that the maximum number of positive steady states is also three for this second family of networks. We now give an example which illustrates our approach.

Example 1.1 Consider the chemical reaction network

$$X_1 + X_2 \xrightarrow{\ell} 5X_1 + 17X_2, \qquad 0 \underset{k_2}{\overset{k_1}{\rightleftharpoons}} X_1, \qquad 0 \underset{k_4}{\overset{k_3}{\rightleftharpoons}} X_2.$$

In this case, there are no conservation laws. Hence, setting $\kappa = [\ell, k_1, k_2, k_3, k_4]$, we have that $S_\kappa = \{x \in \mathbb{R}^2_{>0} \mid 4\ell x_1 x_2 - k_2 x_1 + k_1 = 0, \ 16\ell x_1 x_2 - k_4 x_2 + k_3 = 0\}$. The system has two variables and $n + l + 1 = 2 + 1 + 1 = 4$ monomials. Hence, a Gale dual system consists of a one-variable polynomial and a real interval. A Gale dual system is

$$p(y) = 64\ell^2 y^2 + (-k_2 k_4 + 16 k_1 \ell + 4 k_3 \ell) y + k_1 k_3, \qquad y > 0.$$

The number of solutions to this system agrees with the number of elements in S_κ. Since $p(y)$ has degree 2, there are at most two solutions. From a Sturm sequence for $p(y)$, we obtain the set of constraints which $k_1, k_2, k_3, k_4, \ell \in \mathbb{R}_{>0}$ must satisfy for $p(y)$ to have two positive roots:

$$k_2 k_4 - 16 k_1 \ell - 4 k_3 \ell > 0, \quad k_2^2 k_4^2 - (32 k_1 + 8 k_3) k_2 k_4 \ell$$
$$+ (256 k_1^2 - 128 k_1 k_3 + 16 k_3^2) \ell^2 > 0.$$

The parameters $\kappa = [1, 33602, 15447, 35984, 8034]$ satisfy the inequalities above. With this choice of parameters, and with $\omega = \sqrt{423113764748297}$, the positive steady states are:

$$S_\kappa = \left\{ \left[\frac{20749149 - \omega}{82384}, \frac{20617917 - \omega}{10712} \right], \left[\frac{20749149 + \omega}{82384}, \frac{20617917 + \omega}{10712} \right] \right\}$$
$$\cong \{[2.2, 4.5], [501.5, 3845]\}.$$

The paper is organized as follows. In Sect. 2, we give a brief overview of the mathematical study of chemical reaction networks, recall the notion of embedded networks, and introduce Gale dual systems. In Sect. 3, we study the family of networks from Joshi (2013) with one irreversible non-flow reaction. In Sect. 3.1, we give criteria for the reaction network to have zero, one, two, or three positive steady states and show that there can be no more than three steady states. In Sect. 3.2, we investigate the stability of the steady states discussed in Sect. 3.1. Finally, in Sect. 4, we modify the family of reaction networks discussed in Sect. 3 to include a reversible reaction and obtain results regarding the possible number and stability of steady states in certain cases.

2 Background

We begin this section with a brief review of some elements of chemical reaction network theory. Following this, we will give an overview of the construction of Gale dual systems.

2.1 Reaction Networks and Embedded Networks

Informally, a reaction network is a collection of reactions between linear combinations of species in a set $\{X_1, \ldots, X_n\}$:

$$r_j: \sum_{i=1}^{n} a_{i,j} X_i \longrightarrow \sum_{i=1}^{n} b_{i,j} X_i, \qquad j = 1, \ldots, s, \tag{2}$$

such that $a_{i,j}, b_{i,j} \in \mathbb{Z}_{\geq 0}$. A *reaction rate constant* $k_j > 0$ is associated with each reaction and is typically written as a label of the reaction. We let x_i denote the concentration of X_i. The concentrations of the species over time are modeled by the following system of ODEs:

$$\frac{dx_i}{dt} = f_{k,i}(x), \quad \text{where} \quad f_{k,i}(x) = \sum_{j=1}^{s}(b_{i,j} - a_{i,j})k_j \prod_{\ell=1}^{n} x_\ell^{a_{\ell,j}}, \qquad i = 1, \ldots, n. \tag{3}$$

This ODE system arises from the assumption of mass action kinetics. Throughout this paper, and without further reference, we will assume that *all* reaction networks are equipped with mass action kinetics.

The positive steady states of the network (2) are the positive solutions $x \in \mathbb{R}^n_{>0}$ to the polynomial system $f_{k,i}(x) = 0$, for $i = 1, \ldots, n$. Given a network with reactions as in (2), an *embedded network* is obtained by removing a subset of species. Specifically, if we fix a subset $\mathcal{X} = \{X_{i_1}, \ldots, X_{i_\omega}\}$ of $\{X_1, \ldots, X_n\}$, the embedded network on the subset $\mathcal{X}$ has reactions

$$\sum_{\nu=1}^{\omega} a_{i_\nu,j} X_{i_\nu} \longrightarrow \sum_{\nu=1}^{\omega} b_{i_\nu,j} X_{i_\nu},$$
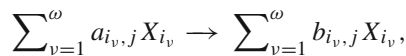
for all j such that the two ends of the reaction are distinct. Repeated reactions are considered only once. For more on chemical reaction network theory see, for example, Feinberg (1980), Gunawardena (2003).

In this paper, we study reaction networks which contain inflow reactions $0 \xrightarrow{\kappa_i} X_i$ and outflow reactions $X_i \xrightarrow{c_i} 0$, for all i. In particular, each equation in (3) contains a linear term $-c_i x_i$ and an independent term κ_i. It follows that there are no $\mathbb{R}$-linear relations among the $f_{k,i}(x)$ that hold for all $k \in \mathbb{R}^s$. Hence, these networks have no conservation laws, see (1). For a reaction network without conservation laws, a steady state x^* is said to be *non-degenerate* if the Jacobian matrix of $f_k = [f_{k,1}, \ldots, f_{k,n}]$ evaluated at x^* is non-singular.

For reaction networks with inflow and outflow reactions, we can apply the results in Joshi and Shiu (2013) which relate the possible numbers of positive steady states of the given network to the possible numbers of steady states for the embedded networks. We state a simplified version of Theorem 4.2 in Joshi and Shiu (2013) for the case where the network has mass action kinetics (the original theorem of Joshi and Shiu (2013) is valid in greater generality). See the recent results in Shiu and de Wolff (2018) for relaxing the assumption on non-degeneracy of the steady states.

Proposition 2.1 (Theorem 4.2 Joshi and Shiu 2013) *Let N be a reaction network with inflow and outflow reactions for all its species, and let N' be an embedded network. If for some choice of reaction rate constants, N' admits L positive non-degenerate steady states, then there exist reaction rate constants such that N also admits L positive non-degenerate steady states. Further, if among the L positive non-degenerate steady*

states of N', R are asymptotically stable and $L - R$ unstable, then this is also the case for N, for some choice of reaction rate constants.

Another result of Joshi and Shiu (2013) which we employ in § 4 concerns the process of making an irreversible reaction (i.e., a reaction $y \to y'$ for which $y' \to y$ is not in the network) into a reversible reaction by adding the missing reverse reaction. Again, we state a simplified version of the result here.

Proposition 2.2 (Theorem 3.1 Joshi and Shiu 2013) *Let N' be a reaction network containing an irreversible reaction $y \to y'$, and let N be the network obtained by adding $y' \to y$ to N'. If for some choice of reaction rate constants, N' admits L positive non-degenerate steady states, then there exist reaction rate constants such that N also admits L positive non-degenerate steady states. Further, if among the L positive non-degenerate steady states of N', R are asymptotically stable and $L - R$ unstable, then this is also the case for N, for some choice of reaction rate constants.*

2.2 Gale Duality

We now give a brief overview of the construction of Gale dual polynomial systems. Given a square system of polynomial equations, this construction allows us to obtain a new *Gale dual* system whose solutions are in bijective correspondence with the positive real solutions to the original system. We will see that in some cases the Gale dual system is simpler to study than the original system. For the interested reader, more details are given in "Appendix A," see also the book Sottile (2011).

Throughout this paper, we will use standard multinomial notation, that is, if we are working with (Laurent) polynomials in variables $x_1, \ldots, x_n$ we will write the (Laurent) monomial $x_1^{w_1} \cdots x_n^{w_n}$ as x^w for $w = [w_1, \ldots, w_n]^T \in \mathbb{Z}^n$. Similarly, for a matrix $W \in \mathbb{Z}^{n \times m}$ with column vectors $w^{(1)}, \ldots, w^{(m)}$, we will write $x^W = [x^{w^{(1)}}, \ldots, x^{w^{(m)}}]^T$. To construct a Gale dual system, we will compute Gale duals of matrices. A matrix Q is *Gale dual* to a matrix A if the columns of Q form a basis for the kernel of A, so that $\mathrm{Col}(Q) = \ker(A)$ and $A \cdot Q = 0$. If A is an integer matrix, we, additionally, require that the Gale dual matrix Q is also an integer matrix.

In this subsection, we study the positive real solutions to a polynomial system with n equations and n variables:

$$f_1(x_1, \ldots, x_n) = \cdots = f_n(x_1, \ldots, x_n) = 0, \quad x_i \in \mathbb{R}_{>0}. \tag{4}$$

In the notation defined above, the system (4) can be written as

$$Cx^W = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix} = 0 \tag{5}$$

where x^W consists of the $(n + l + 1)$ unique monomials (possibly including the monomial 1 if there are constant terms) which appear in the polynomials $f_1, \ldots, f_n$

defining the system (4). We adopt the convention that C is an $n \times (n+l+1)$ matrix and that W is an $n \times (n+l+1)$ integer matrix. Further, we may assume, without loss of generality, that 1 is the last monomial in the vector x^W. In particular, if this were not the case, since we only consider solutions with nonzero coordinates, we could simply divide the system (5) by the given last monomial to obtain a system of the same form with 1 as the last entry in the new vector x^W.

Assume that $l > 0$ and that the system (5) has a finite number of solutions. This implies that C and W both have maximal rank n. Let $Q = [q_{i,j}]$ be an $(n+l+1)\times(l+1)$ integer matrix Gale dual to W chosen so that its last column is $[0, \ldots, 0, 1]^T \in \ker(W)$, and let $D = [d_{i,j}]$ be an $(n+l+1) \times (l+1)$ matrix Gale dual to C chosen to have its last row given by the vector $[0, 0, \ldots, 1]$. Note that the choice of the last row of D can be achieved by column operations. A vector $x \in \mathbb{R}^n_{>0}$ is a solution to (5) if and only if the vector $x^W = [x^{w^{(1)}}, \ldots, x^{w^{(n+l)}}, 1]$ belongs to $\ker(C)$, or equivalently, if and only if there exists $y_1, \ldots, y_l$ such that

$$x^W = D \cdot [y_1, \ldots, y_l, 1]^T. \tag{6}$$

For a column vector $z \in \mathbb{R}^{n+l+1}_{>0}$, we have that

$$x^W = z \text{ for some } x \in \mathbb{R}^n_{>0}$$
$$\Leftrightarrow \quad W^T \log(x) = \log(z) \text{ for some } x \in \mathbb{R}^n_{>0} \quad \Leftrightarrow \quad \log(z) \in \operatorname{im}(W^T)$$
$$\Leftrightarrow \quad Q^T \log(z) = 0 \quad \Leftrightarrow \quad \log(z^Q) = 0 \quad \Leftrightarrow \quad z^Q = [1, 1, \ldots, 1]^T.$$

Since W has full rank, z and x determine each other from the equation $x^W = z$. Using this with $z = D \cdot [y_1, \ldots, y_l, 1]^T$ in (6), positive solutions to (5) are in one-to-one correspondence with vectors $[y_1, \ldots, y_l] \in \mathbb{R}^l$ which satisfy the $l + 1$ equations

$$\prod_{i=1}^{n+l+1} d_i(y)^{q_{i,1}} = 1, \quad \ldots \quad \prod_{i=1}^{n+l+1} d_i(y)^{q_{i,l+1}} = 1, \quad \text{such that } d_i(y) > 0, \tag{7}$$

where the $d_i(y)$ are the linear forms in $\mathbb{R}[y_1, \ldots, y_l]$ defined by the rows of $D \cdot [y_1, \ldots, y_l, 1]^T$. That is,

$$d_i(y) := (D \cdot [y_1, \ldots, y_l, 1]^T)_i = d_{i,l+1} + \sum_{r=1}^{l} d_{i,r} y_r, \quad \text{for } i = 1, \ldots, n+l+1. \tag{8}$$

Note that $d_{n+l+1}(y) = 1$ by our choice of the last row of D and that the last equation in (7) is simply $1 = 1$ by our choice of the last column of Q. Hence, the solutions of the system (5) are in one-to-one correspondence with solutions of the system of l equations given by

$$\prod_{i=1}^{n+l} d_i(y)^{q_{i,1}} = 1, \quad \ldots \quad \prod_{i=1}^{n+l} d_i(y)^{q_{i,l}} = 1, \quad \text{such that } d_i(y) > 0. \tag{9}$$

Additionally, this correspondence preserves the *scheme* structure of the two systems, and in particular the multiplicity of the solutions is preserved. For details see "Appendix A." We summarize this in the following proposition.

Proposition 2.3 *Let $l > 0$, $w^{(i)} \in \mathbb{Z}^n$, and define the matrix $W = [w^{(1)}, \ldots, w^{(n+l)}, 0]$. Consider the system of n (Laurent) polynomials in variables $x_1, \ldots, x_n$ given by*

$$C \cdot x^W = 0,$$

where $C \in \mathbb{R}^{n \times (n+l+1)}$. Let $D \in \mathbb{R}^{(n+l+1) \times n}$ be a matrix with last row $[0, \ldots, 0, 1]$ which is Gale dual to C, and let $Q = [q_{i,j}] \in \mathbb{Z}^{(n+l+1) \times n}$ be a matrix with last column $[0, \ldots, 0, 1]^T$ which is Gale dual to W. There is a one-to-one, multiplicity preserving, correspondence between the set of solutions $x \in \mathbb{R}^n_{>0}$ to $C \cdot x^W = 0$ and the set of solutions to the system of l equations
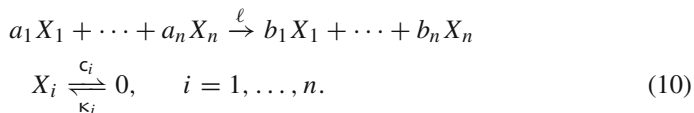
$$\prod_{i=1}^{n+l} d_i(y)^{q_{i,1}} = 1, \quad \ldots \quad \prod_{i=1}^{n+l} d_i(y)^{q_{i,l}} = 1, \quad \text{such that} \quad d_i(y) > 0$$

in the variables $y_1, \ldots, y_l$, where $d_i(y)$ is as in (8).

We apply Gale duality to systems of polynomial equations defining the steady states of a network. The fact that the Gale dual system preserves the multiplicity of solutions implies that the solutions to the Gale dual system of multiplicity one correspond to non-degenerate steady states.

3 Networks with One Non-flow Irreversible Reaction

In this section, we apply Gale duality to study the number of positive steady states of a particular family of reaction networks, originally introduced in Joshi (2013) (see also Joshi and Shiu 2013, 2015, 2017). Specifically, consider a reaction network in n species $X_1, \ldots, X_n$ of the form

$$a_1 X_1 + \cdots + a_n X_n \xrightarrow{\ell} b_1 X_1 + \cdots + b_n X_n$$
$$X_i \underset{\kappa_i}{\overset{c_i}{\rightleftharpoons}} 0, \qquad i = 1, \ldots, n. \tag{10}$$

Each of these networks consists of inflow and outflow reactions with reaction rate constants κ_i and c_i, respectively, and a non-flow reaction with reaction rate constant ℓ. Using the multinomial notation $x^a = x_1^{a_1} \cdots x_n^{a_n}$, the system of ODEs (3) is:

$$\frac{dx_i}{dt} = (b_i - a_i)\ell x^a - c_i x_i + \kappa_i, \qquad i = 1, \ldots, n.$$

Therefore, the polynomial system we are interested in studying is

$$(b_i - a_i)\ell x^a - c_i x_i + \kappa_i = 0, \qquad i = 1, \ldots, n. \tag{11}$$

Note that a_i, b_i are fixed, while the parameters ℓ, c_i, and κ_i can vary. In Joshi (2013), it was shown that this system has at least two positive solutions for some positive ℓ, c_i, and κ_i if and only if

$$\sum_{b_i > a_i} a_i > 1.$$

We will now use Gale duality to determine the precise number of positive solutions as well as to understand what parameter regions contain at least two solutions. Let $\text{sgn}(i) = \text{sgn}(b_i - a_i)$ denote the sign function applied to $b_i - a_i$, that is

$$\text{sgn}(i) := \begin{cases} 1 & \text{if } b_i > a_i \\ -1 & \text{if } a_i > b_i \\ 0 & \text{if } a_i = b_i. \end{cases} \tag{12}$$

We note that we can eliminate the dependence of (11) on $b_1, \ldots, b_n$ and ℓ (up to a term for the sign of $(b_i - a_i)$) by dividing the ith equation by the absolute value of $(b_i - a_i)\ell$ whenever $b_i \neq a_i$. With this in mind, we may rewrite (11) as:

$$\text{sgn}(i)x^a - c_i x_i + k_i = 0, \qquad i = 1, \ldots, n, \tag{13}$$

where $c_i := c_i$ and $k_i := \kappa_i$ if $b_i = a_i$, and

$$c_i := \frac{c_i}{|b_i - a_i|\ell} > 0 \ \text{ and } \ k_i := \frac{\kappa_i}{|b_i - a_i|\ell} > 0 \ \text{ whenever } \ b_i \neq a_i.$$

We can write (13) in the form $C \cdot x^W = 0$ where $x^W = [x_1, \ldots, x_n, x^a, 1]^T$, and

$$C = \begin{bmatrix} -c_1 & \cdots & 0 & \text{sgn}(1) & k_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -c_n & \text{sgn}(n) & k_n \end{bmatrix}, \quad W = \begin{bmatrix} 1 & \cdots & 0 & a_1 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & a_n & 0 \end{bmatrix}.$$

Now, compute a Gale dual system to (13) using Proposition 2.3 (see also §2.2). In the notation of Proposition 2.3 applied to the system (13), we have that $l = 1$ and hence any Gale dual system depends on one variable y. One can easily check that the following choices of Gale dual matrices $D \in \mathbb{R}^{(n+2)\times 2}$ and $Q \in \mathbb{Z}^{(n+2)\times 2}$ satisfy the requirements of Proposition 2.3:

$$D = \begin{bmatrix} \frac{\text{sgn}(1)}{c_1} & \frac{k_1}{c_1} \\ \vdots & \vdots \\ \frac{\text{sgn}(n)}{c_n} & \frac{k_n}{c_n} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} a_1 & 0 \\ \vdots & \vdots \\ a_n & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{14}$$

In the notation of Sect. 2.2, we have

$$d_i(y) = \frac{k_i}{c_i} + \frac{\text{sgn}(i)}{c_i}y, \quad i = 1, \ldots, n, \qquad d_{n+1}(y) = y.$$

Hence, the Gale dual system to (13) in $\mathbb{R}[y]$ is given by

$$\frac{1}{y}\prod_{i=1}^{n}\left(\frac{\text{sgn}(i)y + k_i}{c_i}\right)^{a_i} = 1 \quad \text{where} \ (\text{sgn}(i)y + k_i) > 0, \ \text{for all } i \text{ and } y > 0. \quad (15)$$

We rewrite this as

$$g(y) = \prod_{i=1}^{n}(\text{sgn}(i)y + k_i)^{a_i} - \kappa y = 0, \quad \text{for} \ \ y \in (0, k_-), \ \ \text{with } \kappa = c^a, \quad (16)$$

where

$$k_- := \begin{cases} +\infty & \text{if } \text{sgn}(i) \geq 0 \text{ for all } i = 1, \ldots, n, \\ \min\{k_i \mid \text{sgn}(i) = -1\} & \text{otherwise.} \end{cases} \quad (17)$$

Similarly, we define

$$k_+ := \begin{cases} -\infty & \text{if } \text{sgn}(i) \leq 0 \text{ for all } i = 1, \ldots, n, \\ \min\{k_i \mid \text{sgn}(i) = 1\} & \text{otherwise.} \end{cases} \quad (18)$$

Remark 3.1 The indices i in (15) for which $\text{sgn}(i) = 0$ (that is, $b_i = a_i$) or for which $a_i = 0$ play no role in the number of solutions of the equation (15). It is clear from (11) that if $\text{sgn}(i) = 0$, then $x_i = \kappa_i/c_i$ at steady state. Replacing ℓ with $\ell(\kappa_i/c_i)^{a_i}$ and removing X_i yields a new reaction network of the form (10) with $n - 1$ species . In this way, we can remove all species with $\text{sgn}(i) = 0$ and can assume, without loss of generality, that $\text{sgn}(i) \neq 0$ for all i.

By Proposition 2.3, there is a bijection between positive steady states of the network (10) and the roots of g in the interval $(0, k_-)$. Given this, we study the roots of g for varying values of the parameters κ, and k_i for $\text{sgn}(i) \neq 0$, and deduce the possible number of positive steady states of (10). Subsequently, we consider the original steady state equations and the stability of the steady states. In particular, we determine which networks have the capacity for *bistability*, that is, the existence of three positive steady states, two asymptotically stable, and one unstable.

3.1 Number of Steady States

In light of Remark 3.1, we may assume, without loss of generality, that $\text{sgn}(i) \neq 0$. By defining

$$h(y) = \prod_{i=1}^{n} (\text{sgn}(i)y + k_i)^{a_i}, \tag{19}$$

we may rewrite (16) as

$$g(y) = h(y) - \kappa y = 0, \quad \text{where } y \in (0, k_-). \tag{20}$$

From (20), we see that the solutions to the Gale dual system correspond to the intersection points of the polynomial $h(y)$ with the line κy in the interval $(0, k_-)$. Observe that the polynomial $h(y)$ has only real roots occurring at $-\text{sgn}(i)k_i$ of multiplicity a_i and recall that $\kappa > 0$. Let

$$a_+ := \sum_{\text{sgn}(i)=1} a_i, \quad \text{and} \quad a_- := \sum_{\text{sgn}(i)=-1} a_i. \tag{21}$$

Our analysis of the number of roots of g can be broken into three cases:

  (i) h has no negative real roots ($a_+ = 0$),
 (ii) h has negative real roots but no positive real roots ($a_+ > 0$ and $a_- = 0$),
(iii) h has both positive and negative real roots ($a_+ > 0$ and $a_- > 0$).

In the first case, we will see that there must be exactly one positive steady state, in the second case that there are at most two positive steady states, and in the third we will see that there are at most three positive steady states.

Remark 3.2 In Bihan and Dickenstein (2016), the authors study the positive real solutions of systems of n polynomial equations in n variables which have $n + 1 + 1$ monomials (i.e., when $l = 1$ in the notation of §2.2). By Theorem 3.3 of Bihan and Dickenstein 2016, there are at most four positive real solutions to the system of steady-state equations (11) of the family of networks considered in this section.

We start with a simple lemma that applies to the polynomial $h(y)$; the proof is included for completeness.

Lemma 3.3 *Let* $p(y) = \lambda \prod_{i=1}^{n} (y - \alpha_i)^{a_i}$ *be an arbitrary polynomial with only real roots occurring at* $\alpha_1 < \cdots < \alpha_n$. *Then, the derivative* $p'(y)$ *has only real roots; specifically, there is one root of multiplicity one in each interval* (α_i, α_{i+1}) *for* $i = 1, \ldots, n - 1$, *and the remaining roots occur at* α_i *with multiplicity* $a_i - 1$ *for* $i = 1, \ldots, n$.

Proof We know that $p'(y)$ is a polynomial of degree $(a_1 + \cdots + a_n) - 1$ and that for $j = 1, \ldots, n, \alpha_j$ is a root of $p'(y)$ of multiplicity $a_j - 1$. This gives $(a_1 - 1) + \cdots + (a_n - 1)$ roots of $p'(y)$. Now by Rolle's theorem, $p'(y)$ must have a root in each of the intervals (α_r, α_{r+1}), for $r = 1, \ldots, n - 1$, since $p(\alpha_j) = 0$ for all $j = 1, \ldots, n$. Since there are $n - 1$ such intervals, we have found all $(a_1 + \cdots + a_n) - 1$ roots of $p'(y)$. $\quad\square$

3.1.1 No Negative Real Roots

The first case listed above is straightforward to analyze.

Theorem 3.4 *If $a_+ = 0$, then the Gale dual system* (20), $g(y) = 0$ *for $y \in (0, k_-)$, has exactly one solution of multiplicity one. Equivalently, the network* (10) *has exactly one positive non-degenerate steady state for all choices of parameters ℓ, c_i, and κ_i for all i.*

Proof If $a_- = 0$, then h is a constant function with $h(0) > 0$. If $a_- > 0$, then h has only positive real roots and k_- is the smallest root. By Lemma 3.3 applied to h, h' has no root smaller than k_-. Since $h(0) > 0$ and $h(k_-) = 0$, we conclude that h is strictly decreasing and positive in the interval $(0, k_-)$. In both cases, h must meet any line through the origin with positive slope at exactly one point (with multiplicity one) in the interval $(0, k_-)$. □

3.1.2 No Positive Real Roots and At Least One Negative Real Root

Considering the case $a_+ > 0$ and $a_- = 0$, it follows that $b_i > a_i \geq 0$ and hence that $\text{sgn}(i) = 1$ for all i and $k_- = +\infty$. From (16), we see immediately that $g(y)$ has no positive roots if

$$k^a \cdot \sum_{i=1}^{n} \frac{a_i}{k_i} \geq \kappa,$$

since in this case all nonzero coefficients of g are positive. If this inequality does not hold, then by the Descartes' rule of signs, we conclude that $g(y)$ has at most two positive roots.

Theorem 3.5 *Suppose $a_- = 0$ and $a_+ > 0$. Then, for any choice of the parameters κ, k_i the Gale dual system* (20), $g(y) = 0$ *for $y > 0$, has*

- *either no solution or one solution of multiplicity one, if $a_+ = 1$;*
- *either no solution or two solutions (counted with multiplicity), if $a_+ > 1$.*

In particular, the reaction network (10) *admits at most two positive non-degenerate steady states if $a_+ > 1$ and at most one otherwise.*

Proof If $a_+ = 1$, then h is a line that intersects the line κy in at most one point with multiplicity one. Since $h(0) > 0$, by choosing κ larger than the slope of h the two lines intersect, and by choosing it smaller, the two lines do not intersect in $(0, +\infty)$.

Assume now $a_+ > 1$. Then, $\lim_{y \to +\infty} g(y) = +\infty$ and since $g(0) > 0$, then g has either zero or two positive roots (with multiplicity). The polynomial h has no positive roots, and h' also has no positive roots by Lemma 3.3. Since $h(y)$ tends to $+\infty$, $h(y)$ is increasing and positive for all $y > 0$. Further, by applying Lemma 3.3 to h', we conclude that h has no positive inflection points and $h''(y) > 0$ for $y > 0$.

For any choice of k_i, there exists a value κ' such that h and $\kappa' y$ intersect in exactly one point, tangentially. To see this, note that the slope of the tangent line to h at a
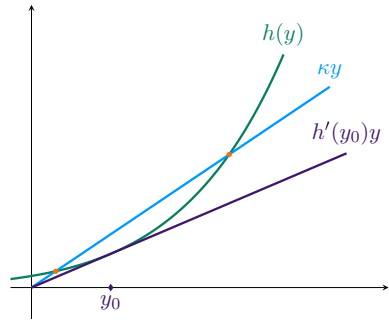
point y increases toward infinity as y increases. Thus, for y large enough, the tangent line intersects the vertical axis at a negative value. Since the tangent line at $y = 0$ intersects at a positive value, namely $h(0)$, then by the continuity of h', there exists $y_0 > 0$ such that the intercept of tangent line is 0. The slope of this line is κ'. Since $h''(y_0) > 0$, any $\kappa > \kappa'$ gives a choice of parameters for which the two curves intersect transversely at two points. $\qquad\square$

The proof of the previous theorem is constructive. Choose k_i arbitrarily and solve the equation $h(y_0) = h'(y_0)y_0$ for $y_0 > 0$. If we then choose κ larger than $h'(y_0)$, this gives two positive non-degenerate steady states. For example, let $a = [3, 4, 2, 2]$ with $b_i > a_i$. Choose $k = [2, 1, 8, 6]$, then $h(y) = (y + 2)^3 (y + 1)^4 (y + 8)^2 (y + 6)^2$. To find a value of κ such that the Gale dual system $g(y) = h(y) - \kappa y$ has two positive solutions, we first find the positive root of the polynomial $h(y_0) - h'(y_0)y_0$. We obtain (approximately) $y_0 \simeq 0.189$ and $h'(y_0) \simeq 284908$. Hence, setting $\kappa > 284908$ will yield two positive solutions. If we choose $\kappa = 450000$, the two positive solutions of $g(y) = 0$ are (approximately) $y \simeq 0.058$ and $y \simeq 0.47$, as shown in Fig. 1.

3.1.3 At Least One Negative and One Positive Real Root

Assume that $a_+ > 0$ and $a_- > 0$, so that h has both positive and negative roots. Then, k_- is the first positive root of h. Since $g(k_-) < 0$ and $g(0) > 0$, we must have at least one solution and must have an odd number of solutions (counted with multiplicity). We show that in fact we have at most three solutions in this case and that this number can be achieved for appropriate choices of k_i and κ.

Theorem 3.6 *Suppose that* $a_+ > 0$ *and* $a_- > 0$. *Consider the Gale dual system* $g(y) = 0$ *for* $y \in (0, k_-)$ *as in* (20). *We have that:*

(i) *There are at most three solutions if* $a_+ > 1$, *and exactly one otherwise.*
(ii) *There are three solutions if and only if* h' *has a root* ξ *in* $(0, k_-)$, h'' *has a root* y^* *in* $(0, \xi)$, *and* $t_h(0) \leq 0$ *where* $t_h(y)$ *is the tangent line to* h *at* y^*.
(iii) *The polynomial* h' *has a root* ξ *in* $(0, k_-)$ *and* h'' *has a root* y^* *in* $(0, \xi)$ *if and only if*

$$\gamma = \sum_{j=1}^{n} \frac{\mathrm{sgn}(j)a_j}{k_j} > 0 \quad \text{and} \quad \theta = \left(\gamma^2 - \sum_{j=1}^{n} \frac{a_j}{(k_j)^2} \right) > 0.$$

(iv) *When $a_+ > 1$, there exist k_i and κ such that there are exactly three solutions of multiplicity one.*

In particular, the reaction network (10) *admits at most three positive non-degenerate steady states if $a_+ > 1$ and at most one otherwise.*

By Lemma 3.3, h' has exactly one root ξ of multiplicity one in the interval $(-k_+, k_-)$. So $h''(\xi) \neq 0$ and ξ is a local extremum. Since $h(0) > 0$ and $h(k_-) = 0$, it follows that ξ is a local maximum, $h''(\xi) < 0$, and h decreases in the interval (ξ, k_-). This gives that h and κy intersect in at least one point in the interval $(0, k_-)$.

First, we prove (i). Consider the case $a_+ = 1$. Then, h has exactly one negative real root of multiplicity 1 and ξ is the smallest root of h'. By Lemma 3.3 applied to h', h'' has no root smaller than ξ, and hence h has no inflection point smaller than ξ. Therefore, the function h in the interval $(0, k_-)$ is positive, and either strictly decreasing, or increasing with $h'' < 0$ up to $\xi > 0$ and afterward decreasing. In both cases, we say that h has property †; the intersection of such a function with a line through the origin κy is transversal and consists of exactly one point. Indeed, in the first case, a strictly decreasing function and a strictly increasing function intersect. In the second case, if h and κy intersect in $(0, \xi)$ then, since h is concave in this region, there can be only one intersection; further, there can be no intersections in $[\xi, k_-)$ since h is decreasing in this interval. Similarly, if h and κy do not intersect in $(0, \xi)$, then they must intersect in $[\xi, k_-)$, but can intersect no more than once since h is decreasing and κy is increasing in (ξ, k_-).

Now assume $a_+ > 1$, then by Lemma 3.3, h' has at least one more root ξ' smaller than ξ. Therefore, h'' has one root y^* in the interval (ξ', ξ). If $y^* \leq 0$, then h again has property † and intersects the line through the origin κy in exactly one point in the interval $(0, k_-)$. If $y^* > 0$, then a line through the origin may intersect h in at most three points, and at least one. See Fig. 2.

We now prove (ii). Assume that $\xi, y^* > 0$ and consider the tangent line to h at y^*, given by $t_h(y) = h(y^*) + h'(y^*)(y - y^*)$. Since $h'(y)$ increases for $y < y^*$ and $h'(y)$ decreases for $y^* < y < \xi$, then
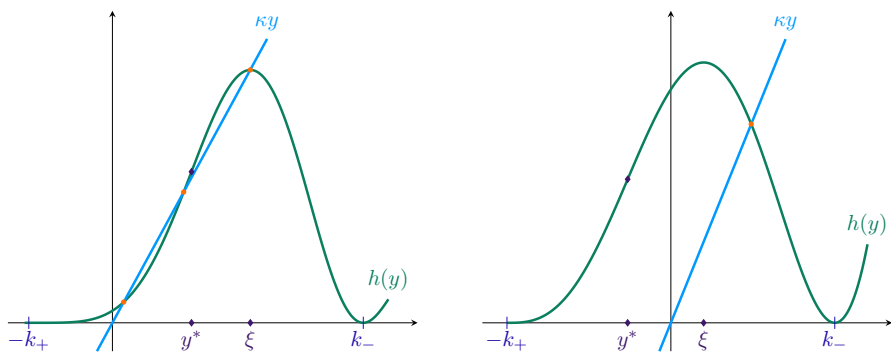
Fig. 2 Plots of the intersection of h and the line κy in the interval $(-k_+, k_-)$ for $a_+ > 1$ and $a_- > 0$. Let ξ be the local maximum of h in the interval $(0, k_-)$. For h and κy to meet at more than one point, h must have an inflection point $y^* \in (0, \xi)$ (Color figure online)

$$h(y) - t_h(y) > 0, \quad \text{for } -k_+ < y < y^*, \quad \text{and} \quad h(y) - t_h(y) < 0, \quad \text{for } y^* < y < k_-,$$
$$\tag{22}$$

and the only intersection of the line t_h with h in the interval $(0, k_-)$ occurs at y^*. Let L be any line through $(y^*, h(y^*))$ with positive slope. If the slope of L is larger than $h'(y^*)$, then it intersects h only at y^*. On the other hand, if $L(y)$ has positive slope smaller than $h'(y^*)$ and if $L(0) < h(0)$, then L intersects h in three points. Therefore, if $t_h(0) \leq 0$, then by decreasing the slope of the line $L = t_h$ until $L(0) = 0$, we obtain a choice of κ such that κy intersects h at three points. This shows the reverse implication of (ii).

To prove the forward implication, assume that h and κy intersect at three points in the interval $(0, k_-)$. Then, h does not have property †, which gives that $\xi \in (0, k_-)$ and h'' has a root y^* in $(0, \xi)$. Let $t_h(y)$ be the tangent line to h at y^*. We show that if $t_h(0) > 0$, then h and κy intersect in one point in $(0, k_-)$, yielding a contradiction. If the line κy meets h in the interval $(0, y^*]$, then it also meets $t_h(y)$ in that interval, and hence κ is greater than the slope of $t_h(y)$. Using (22), it follows that κy and $h(y)$ intersect once in $(0, k_-)$. If h and κy only intersect in the interval (y^*, k_-), then intersection can only occur in one point, since h has property † in (y^*, ξ). This concludes the proof of (ii).

Now we prove (iii). In the interval $(-k_+, k_-)$, we may write

$$h'(y) = h(y) \sum_{j=1}^{n} \frac{\operatorname{sgn}(j) a_j}{\operatorname{sgn}(j) y + k_j},$$

$$h''(y) = h'(y) \sum_{j=1}^{n} \frac{\operatorname{sgn}(j) a_j}{\operatorname{sgn}(j) y + k_j} - h(y) \sum_{j=1}^{n} \frac{a_j}{(\operatorname{sgn}(j) y + k_j)^2}$$

$$= h(y) \left(\left(\sum_{j=1}^{n} \frac{\operatorname{sgn}(j) a_j}{\operatorname{sgn}(j) y + k_j} \right)^2 - \sum_{j=1}^{n} \frac{a_j}{(\operatorname{sgn}(j) y + k_j)^2} \right). \tag{23}$$

Since $h' > 0$ in $(-k_+, \xi)$ and $h' < 0$ in (ξ, k_-), ξ belongs to $(0, k_-)$ if and only if $h'(0) > 0$. Using that $h'(0) = h(0) \cdot \gamma$ and that $h(0) > 0$, we obtain that $\xi \in (0, k_-)$ if and only if $\gamma > 0$. Similarly, assume now that $\xi \in (0, k_-)$. Recall that by Lemma 3.3, if h'' has a root $y^* \in (-k_+, \xi)$, then it has multiplicity one. Since $h''(\xi) < 0$, it follows that h'' has a root in the interval $(0, \xi)$ if and only if $h''(0) > 0$. By (23), we have $h''(0) = h(0) \cdot \theta$, which gives that h'' has a root $y^* \in (0, \xi)$ if and only if $\theta > 0$. This concludes the proof of (iii).

We now prove (iv). We can reduce to the case $n = 2$ by setting $k_i = k_+$ for all i such that $\operatorname{sgn}(i) = 1$, and setting $k_i = k_-$ for all i such that $\operatorname{sgn}(i) = -1$. With this substitution, we may now study the system:

$$g(y) = (y + k_+)^{a_+} (-y + k_-)^{a_-} - \kappa y = 0, \quad 0 < y < k_-.$$

We wish to show that we may choose k_-, k_+, and κ such that g changes sign three times in the interval $(0, k_-)$. To do this, we will consider the values $g(k_+)$ and $g(2k_+)$.

We have that

$$g(k_+) = 2^{a_+} k_+^{a_+} (k_- - k_+)^{a_-} - \kappa k_+, \quad \text{and} \quad g(2k_+) = 3^{a_+} k_+^{a_+} (k_- - 2k_+)^{a_-} - 2\kappa k_+.$$

Since $g(0) > 0$ and $g(k_-) < 0$, it suffices to show that there exists a choice of k_-, k_+, and κ such that $2k_+ < k_-$, $g(k_+) < 0$ and $g(2k_+) > 0$, or equivalently, that $2k_+ < k_-$ and

$$2^{a_+} k_+^{a_+ - 1} (k_- - k_+)^{a_-} < \kappa < \tfrac{3^{a_+}}{2} k_+^{a_+ - 1} (k_- - 2k_+)^{a_-}. \tag{24}$$

The inequality $2^{a_+} k_+^{a_+ - 1} (k_- - k_+)^{a_-} < \tfrac{3^{a_+}}{2} k_+^{a_+ - 1} (k_- - 2k_+)^{a_-}$ is equivalent to

$$\frac{k_- - k_+}{k_- - 2k_+} < \alpha, \quad \text{with} \quad \alpha = \left(\tfrac{3^{a_+}}{2^{1+a_+}} \right)^{\frac{1}{a_-}}. \tag{25}$$

Since $a_+ > 1$ we have that $\alpha > 1$. For a fixed k_+, it is clear that we can make $\frac{k_- - k_+}{k_- - 2k_+}$ arbitrarily close to one for k_- large enough, and in particular less that α. It follows that there exist k_- and k_+, such that $2k_+ < k_-$, and the inequality (25) holds; hence, we can choose κ such that the inequality (24) holds. □

Example 3.7 Let $a = [3, 4, 2, 2]$ and suppose that $b_1 > 3$, $b_2 > 4$, $b_3 = b_4 = 1$. Then, $h(y) = (y + k_1)^3 (y + k_2)^4 (-y + k_3)^2 (-y + k_4)^2$. By Theorem 3.6 to obtain three solutions to the Gale dual system, we must have that:

$$\gamma = \tfrac{3}{k_1} + \tfrac{4}{k_2} - \tfrac{2}{k_3} - \tfrac{2}{k_4} > 0 \quad \text{and} \quad \theta = \gamma^2 - \left(\tfrac{3}{k_1^2} + \tfrac{4}{k_2^2} + \tfrac{2}{k_3^2} + \tfrac{2}{k_4^2} \right) > 0.$$

Choosing $k = [4, 2, 4, 6]$ gives $k_- = 4$, $\gamma = \tfrac{23}{12} > 0$ and $\theta = \tfrac{145}{144} > 0$; in this case, the one root of h' in $(0, 4)$ is $\xi = 2$ and the inflection point in the interval $(0, 2)$ is $y^* \simeq 0.917$. The tangent line to h at y^* is (approximately):

$$t_h(y) = 1.985169547 \cdot 10^6 y + 293252.134, \quad \text{with} \quad t_h(0) > 0.$$

Hence, any choice of κ gives exactly one intersection of κy and h in the interval $(0, 4)$. This is illustrated in the left-hand plot in Fig. 3. On the other hand, choosing $k = [4, 2, 6, 7]$ gives $k_- = 6$, $\gamma = \tfrac{179}{84} > 0$ and $\theta = \tfrac{7699}{7056} > 0$; in this case, the one root of $h'(y)$ in $(0, 6)$ is $\xi \simeq 3.1075$ and the inflection point in the interval $(0, \xi)$ is $y^* \simeq 1.73$. The tangent line to h at y^* is (approximately):

$$t_h(y) = 1.379464137 \times 10^7 y - 5.42408176 \times 10^6, \quad \text{with} \quad t_h(0) < 0.$$

From this, we see that setting $\kappa = \frac{h(y^*)}{y^*} = 1.065687498 \times 10^7$ gives exactly three intersections of κy and h in the interval $(0, 6)$. This is illustrated in the right-hand plot in Fig. 3. The three points are (approximately): $y \simeq 0.3099, \ 1.7286, \ 2.8531$.

A sufficient condition to obtain three positive steady states of (11) is given in Proposition 3.8.
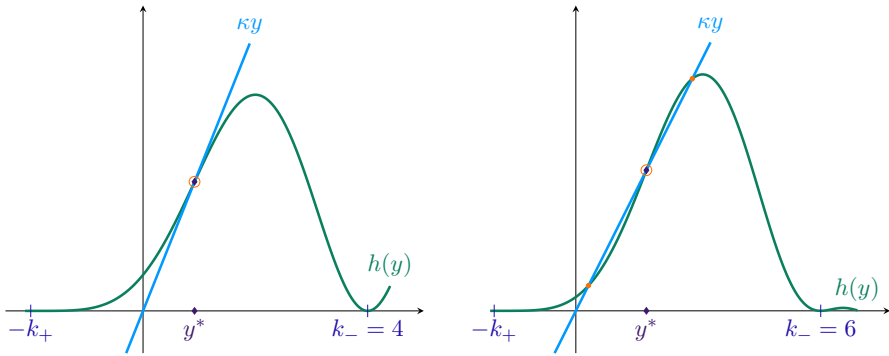
Fig. 3 Plots of the intersection of h and the κy in the interval $(-k_+, k_-)$ for different choices of k_i and κ. See Example 3.7 (Color figure online)

Proposition 3.8 *Assume that $n \geq 3$, $a_+ > 0$, and $a_- > 0$. If there exist $k_\mu < k_\nu < k_-$ with $\mathrm{sgn}(\mu) = \mathrm{sgn}(\nu) = 1$, $a_\mu \geq 1$, $a_\nu \geq 1$, and*

$$2^{a_\mu} k_\mu^{a_\mu - 1} \prod_{\substack{i=1 \\ i \neq \mu}}^n \left(\mathrm{sgn}(i) k_\mu + k_i \right)^{a_i} \quad < \kappa < 2^{a_\nu} k_\nu^{a_\nu - 1} \prod_{\substack{i=1 \\ i \neq \nu}}^n \left(\mathrm{sgn}(i) k_\nu + k_i \right)^{a_i}$$

for some $\kappa > 0$,

then the Gale dual system (16) has exactly three solutions.

Proof Let g be as in (16), we have that $g(0) > 0$, $g(k_\mu) < 0$, $g(k_\nu) > 0$, and $g(k_-) < 0$. Hence, g has at least three roots in the interval $(0, k_-)$; the conclusion follows by Theorem 3.6(i). □

3.2 Stability of the Steady States

We now study the asymptotic stability of the steady states of the reaction network (10). For any number of species n, the embedded network on any subset of species $\{X_{i_1}, \ldots, X_{i_s}\}$ is of the same type, (10), with the number of species equal to $s \leq n$. In light of this, we first study the case where $n = 1$; in this case all signs agree. We then study the case where $n = 2$ and both nonzero signs occur. Finally, we extend our conclusions to arbitrary n using Proposition 2.1 and the auxiliary networks described in Lemma 3.12.

Proposition 3.9 *Let $n = 1$ and consider the stability of the steady states of (10).*

- *If $a_1 > b_1$, the only positive steady state is asymptotically stable.*
- *If $a_1 < b_1$ and there is one positive steady state of multiplicity one, then it is asymptotically stable. If there are two positive steady states, then the smallest is asymptotically stable and the largest is unstable.*

Proof The ODE system of the network (10) is

$$\frac{dx_1}{dt} = f(x_1), \quad \text{with } f(x_1) = (b_1 - a_1)\ell x_1^{a_1} - c_1 x_1 + \kappa_1.$$

A steady state x_1^* is a solution to the equation $f(x_1^*) = 0$. It is well known that if $f'(x_1^*) < 0$, then the steady state is asymptotically stable, and if $f'(x_1^*) > 0$, then it is unstable. Since $f(0) > 0$, if the network has one positive steady state x_1^*, of multiplicity one, then necessarily $f'(x_1^*) < 0$ and thus x_1^* is asymptotically stable. If the network has two positive steady states (which must each be of multiplicity one), then the smallest steady state satisfies $f'(x_1^*) < 0$ and is asymptotically stable, and the largest satisfies $f'(x_1^*) > 0$ and it is unstable. □

Proposition 3.10 *Let $n = 2$ with $0 < a_1 < b_1$, $a_2 > b_2$ and consider the stability of the steady states of the reaction network* (10).

- *If there is only one positive steady state of multiplicity one, it is asymptotically stable.*
- *If $c_1 < c_2$ and there are three positive steady states, then two are asymptotically stable and the other is unstable.*

Proof The ODE system is of the form

$$\frac{dx_1}{dt} = \alpha_1 x_1^{a_1} x_2^{a_2} - c_1 x_1 + \kappa_1, \qquad \frac{dx_2}{dt} = -\alpha_2 x_1^{a_1} x_2^{a_2} - c_2 x_2 + \kappa_2$$

with $\alpha_1 = (b_1 - a_1)\ell > 0$ and $\alpha_2 = (a_2 - b_2)\ell > 0$. The associated Jacobian matrix is

$$J(x) = \begin{bmatrix} a_1\alpha_1 x_1^{a_1-1} x_2^{a_2} - c_1 & a_2\alpha_1 x_1^{a_1} x_2^{a_2-1} \\ -a_1\alpha_2 x_1^{a_1-1} x_2^{a_2} & -a_2\alpha_2 x_1^{a_1} x_2^{a_2-1} - c_2 \end{bmatrix}.$$

Consider a steady state $x = (x_1, x_2)$. Since we have only two variables, if the determinant of $J(x)$ is negative, then the steady state is unstable, and if the determinant is positive and the trace is negative, then the steady state is asymptotically stable Perko (2001). Compute the trace and determinant of $J(x)$:

$$\det(J(x)) = -x_1^{a_1-1} x_2^{a_2-1}(a_1\alpha_1 c_2 x_2 - a_2\alpha_2 c_1 x_1) + c_1 c_2,$$
$$\text{Tr}(J(x)) = x_1^{a_1-1} x_2^{a_2-1}(a_1\alpha_1 x_2 - a_2\alpha_2 x_1) - c_1 - c_2.$$

Rewriting the two steady-state equations, we see that for our steady state x we have that $x_2 = \frac{-\alpha_2 c_1 x_1 + \alpha_1 \kappa_2 + \alpha_2 \kappa_1}{\alpha_1 c_2}$ and hence

$$f(x_1) = \alpha_1 x_1^{a_1} \left(\frac{-\alpha_2 c_1 x_1 + \kappa_2 \alpha_1 + \alpha_2 \kappa_1}{\alpha_1 c_2} \right)^{a_2} - c_1 x_1 + \kappa_1 = 0. \qquad (26)$$

If the network has one positive steady state x of multiplicity one, since $f(0) > 0$, we must have $f'(x_1) < 0$. If the network admits three steady states of multiplicity one, then two of them satisfy $f'(x_1) < 0$ and the other $f'(x_1) > 0$. Hence, it is enough to show that if $f'(x_1) < 0$, then the steady state is asymptotically stable, and if $f'(x_1) > 0$, then the steady state is unstable. Let $x = (x_1, x_2)$ be a positive steady state. Using (26), the derivative of f at x_1 is

$$f'(x_1) = a_1\alpha_1 x_1^{a_1-1} x_2^{a_2} - a_2\alpha_2 x_1^{a_1} x_2^{a_2-1} \tfrac{c_1}{c_2} - c_1 = \tfrac{-\det(J(x))}{c_2}.$$

Thus, $f'(x_1)$ and $\det(J(x))$ have opposite signs at a steady state (x_1, x_2). It follows that if $f'(x_1) > 0$, then the determinant is negative and the steady state is unstable. On the other hand, if $f'(x_1) < 0$ this implies $\det(J(x)) > 0$ and we need to show that the trace is negative for the steady state to be asymptotically stable. Suppose that $\det(J(x)) > 0$. We have that
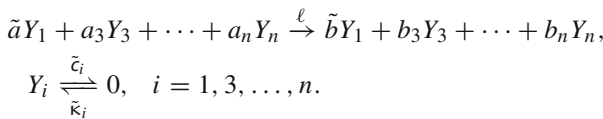
$$\alpha_1 a_1 x_1^{a_1-1} x_2^{a_2} < \frac{a_2\alpha_2 c_1 x_1^{a_1} x_2^{a_2-1}}{c_2} + c_1, \quad \text{giving,}$$

$$\mathrm{Tr}(J(x)) < \frac{x_1^{a_1} x_2^{a_2-1} a_2\alpha_2 (c_1 - c_2)}{c_2} - c_2.$$

The last inequality shows that $\mathrm{Tr}(J(x))$ is negative if $c_1 < c_2$. $\qquad\square$

Remark 3.11 Note that the condition $c_1 < c_2$ does not affect the capacity of the network to have multiple steady states. Specifically, while the parameters c_i do appear in κ, see (16), and while the value of κ does play a role in determining if multiple steady states may be achieved, requiring that $c_1 < c_2$ does not constrain the values which κ can take.

Before stating the main theorem of this section, we prove the following lemma.

Lemma 3.12 *Let N be a reaction network as in (10) with $n \geq 2$ and assume $a_1 < b_1$, $a_2 < b_2$. Let $\tilde{a} = a_1 + a_2$ and $\tilde{b} = b_1 + b_2$ and consider the reaction network $\tilde{N}$ of type (10) with $n - 1$ species $Y_1, Y_3 \ldots, Y_n$ and reactions*

$$\tilde{a}Y_1 + a_3 Y_3 + \cdots + a_n Y_n \xrightarrow{\ell} \tilde{b}Y_1 + b_3 Y_3 + \cdots + b_n Y_n,$$

$$Y_i \underset{\tilde{\kappa}_i}{\overset{\tilde{c}_i}{\rightleftharpoons}} 0, \quad i = 1, 3, \ldots, n.$$

Let $\beta_i = \frac{b_i - a_i}{\tilde{b} - \tilde{a}}$ and define

$$c_i = \begin{cases} \beta_i \tilde{c}_1 & \text{for } i = 1, 2 \\ \tilde{c}_i & \text{for } i = 3, \ldots, n \end{cases} \quad \text{and} \quad \kappa_i = \begin{cases} \beta_i \tilde{\kappa}_1 & \text{for } i = 1, 2 \\ \tilde{\kappa}_i & \text{for } i = 3, \ldots, n \end{cases}.$$

If $\tilde{y} = (y_1, y_3, \ldots, y_n)$ is a steady state of $\tilde{N}$ for the reaction rate constants ℓ, $\tilde{c}_i$, $\tilde{\kappa}_i$, for $i = 1, 3, \ldots, n$, then $y^* = (y_1, y_1, y_3, \ldots, y_n)$ is a steady state of N for the reaction rate constants ℓ, c_i, and κ_i.

Proof For $i \geq 3$, since $\tilde{y}$ is a steady state of $\tilde{N}$ we have

$$0 = (b_i - a_i)\ell y_1^{\tilde{a}} y_3^{a_3} \cdots y_n^{a_n} - \tilde{c}_i y_i + \tilde{\kappa}_i = (b_i - a_i)\ell y_1^{a_1} y_1^{a_2} y_3^{a_3} \cdots y_n^{a_n} - c_i y_i + \kappa_i.$$

Hence, the equations $\frac{dx_i}{dt} = 0$ hold for N at the point y^*. Further, we know that $0 = (\tilde{b} - \tilde{a})\ell y_1^{\tilde{a}} y_3^{a_3} \cdots y_n^{a_n} - \tilde{c}_1 y_1 + \tilde{\kappa}_1$. For $i = 1, 2$, the steady-state equation for N evaluated at y^* is

$$(b_i - a_i)\ell y_1^{a_1} y_1^{a_2} y_3^{a_3} \cdots y_n^{a_n} - c_i y_i + \kappa_i$$
$$= \beta_i\left((\tilde{b} - \tilde{a})\ell y_1^{\tilde{a}} y_3^{a_3} \cdots y_n^{a_n} - \tilde{c}_1 y_1 + \tilde{\kappa}_1\right) = 0.$$

$\square$

Using Propositions 2.1, 3.9, 3.10 and Lemma 3.12, we obtain the following theorem.

Theorem 3.13 *Consider a reaction network of the type* (10).

(i) *There exists a choice of rate parameters such that the network has at least one positive steady state which is asymptotically stable.*
(ii) *If $a_+ > 1$ and $a_- = 0$, then there exist rate parameter values such that there is one asymptotically stable positive steady state and one unstable positive steady state.*
(iii) *If $a_+ > 1$ and $a_- > 0$, then there exist rate parameter values such that the network has two asymptotically stable positive steady states and one unstable positive steady state.*

Proof Part (i) follows immediately from Propositions 3.9 and 2.1 by considering any embedded network with one species. We now prove (ii). If there exists i such that $a_i > 1$, then the statement is a consequence of Propositions 3.9 and 2.1, by considering the embedded network on X_i. If $a_i \leq 1$ for all i, then we proceed as follows. Assume for simplicity $a_1 = a_2 = 1$ and recall that $b_1 > a_1$, $b_2 > a_2$. Consider the embedded network N on $\{X_1, X_2\}$ and the network $\tilde{N}$ with one species Y as constructed in Lemma 3.12 with $\tilde{a} = a_1 + a_2 = 2$ and $\tilde{b} = b_1 + b_2$. Choose reaction rate constants ℓ, $\tilde{c}_1$, $\tilde{\kappa}_1$ such that $\tilde{N}$ admits two positive non-degenerate steady states. Let y^* be one such steady state. Then, (y^*, y^*) is a positive steady state of N for the choice of reaction rate constants given in Lemma 3.12. We show now that if y^* is asymptotically stable (respectively unstable), then so is (y^*, y^*). Consider $M(y^*) = 2(\tilde{b} - 2)\ell y^* - \tilde{c}_1$. If $M(y^*) < 0$, then y^* is asymptotically stable, and if $M(y^*) > 0$, then y^* is unstable. Now, consider the equations $f_i(x) = (b_i - 1)\ell x_1 x_2 - c_i x_i + \kappa_i$, for $i = 1, 2$, defining the ODE system for N. By letting $\lambda = \frac{(b_1 - 1)(b_2 - 1)}{(\tilde{b} - 2)^2} > 0$, the determinant and trace of the Jacobian matrix J of (f_1, f_2) evaluated at (y^*, y^*) are

$$\det(J(y^*, y^*)) = -(b_1 - 1)\ell c_2 y^* - (b_2 - 1)\ell c_1 y^* + c_1 c_2 = -\lambda \tilde{c}_1 M(y^*),$$
$$\text{Tr}(J(y^*, y^*)) = (b_1 - 1)\ell y^* + (b_2 - 1)\ell y^* - c_1 - c_2 = (\tilde{b} - 2)\ell y^* - \tilde{c}_1 < M(y^*),$$

where we use that $\tilde{b} > \tilde{a} = 2$. It follows that if $M(y^*) > 0$, then y^* is unstable and so is (y^*, y^*). If $M(y^*) < 0$, then y^* is asymptotically stable and (y^*, y^*) is as well, since $\det(J(y^*, y^*)) > 0$ and $\mathrm{Tr}(J(y^*, y^*)) < 0$. This concludes the proof of (ii).

Now consider (iii). Let j such that $a_j > 0$ and $\mathrm{sgn}(j) = -1$. If $a_i > 1$ for some i such that $\mathrm{sgn}(i) = 1$, then we consider the embedded network on $\{X_i, X_j\}$. The result follows from Theorem 3.6 and Propositions 3.10 and 2.1. If $a_i \leq 1$ for all i such that $\mathrm{sgn}(i) = 1$, we proceed similarly to the proof of (ii). Assume $a_1 = a_2 = 1$, $a_1 < b_1, a_2 < b_2$ and $j = 3$. Consider the embedded network N on $\{X_1, X_2, X_3\}$, and let $\tilde{N}$ be the reaction network on the species Y_1, Y_3 defined as in Lemma 3.12 such that $\tilde{a} = 2$. By Proposition 2.1, it is enough to show that the statement holds for N. By Theorem 3.6 and Proposition 3.10, we can choose reaction rate constants ℓ, $\tilde{\kappa}_1$, $\tilde{\kappa}_3$ and $\tilde{c}_1 < \tilde{c}_3$ such that $\tilde{N}$ admits three positive non-degenerate steady states, two of which are asymptotically stable and the other unstable. By the proof of Proposition 3.10, the Jacobian matrix $\tilde{J}$ is such that $\det(\tilde{J})$ is negative when evaluated at the unstable steady state and is also such that $\det(\tilde{J}) > 0$ and $\mathrm{Tr}(\tilde{J}) < 0$ when evaluated at the asymptotically stable steady states. In particular, we have

$$\det(\tilde{J}(y_1, y_3)) = a_3(a_3 - b_3)\ell \tilde{c}_1 y_1^2 y_3^{a_3-1} - 2\ell(\tilde{b} - 2)\tilde{c}_3 y_1 y_3^{a_3} + \tilde{c}_1 \tilde{c}_3,$$
$$\mathrm{Tr}(\tilde{J}(y_1, y_3)) = -a_3(a_3 - b_3)\ell y_1^2 y_3^{a_3-1} + 2\ell(\tilde{b} - 2)y_1 y_3^{a_3} - \tilde{c}_1 - \tilde{c}_3.$$

Let (y_1, y_3) be a steady state of $\tilde{N}$ such that (y_1, y_1, y_3) is a steady state of N for the choice of reaction rate constants given in Lemma 3.12. The ODE system for N is defined by the polynomials $f_i(x) = (b_i - a_i)\ell \, x_1 x_2 x_3^{a_3} - c_i x_i + \kappa_i$ for $i = 1, 2, 3$ (recall that $a_1 = a_2 = 1$). The characteristic polynomial of the Jacobian matrix J of (f_1, f_2, f_3) evaluated at (y_1, y_1, y_3) is

$$\begin{aligned}
\chi(z) = {} & z^3 + \big(a_3(a_3 - b_3)\ell y_1^2 y_3^{a_3-1} - \ell(\tilde{b} - 2)y_1 y_3^{a_3} + c_1 + c_2 + c_3\big)z^2 \\
& + \big(a_3(a_3 - b_3)\ell(c_1 + c_2)y_1^2 y_3^{a_3-1} - \ell((b_2 - 1)c_1 + (b_1 - 1)c_2 \\
& + (\tilde{b} - 2)c_3)y_1 y_3^{a_3} + c_1 c_2 + c_1 c_3 + c_2 c_3\big)z \\
& + a_3(a_3 - b_3)\ell c_1 c_2 y_1^2 y_3^{a_3-1} - \ell((b_2 - 1)c_1 + (b_1 - 1)c_2)c_3 y_1 y_3^{a_3} + c_1 c_2 c_3.
\end{aligned}$$

Let ξ_2, ξ_1, and ξ_0 be the coefficients of the degree two, one, and zero terms of $\chi(z)$, respectively. Since ξ_0 has the opposite sign as that of the product of the eigenvalues of J, then if $\xi_0 < 0$ it follows that the steady state is unstable. Otherwise, we use the Routh–Hurwitz criterion for polynomials of degree 3; that is, if $\xi_2 > 0$, $\xi_0 > 0$, and $\xi_2\xi_1 > \xi_0$, then all eigenvalues have negative real part and hence the steady state is asymptotically stable.

Let $\lambda = \frac{(b_1-1)(b_2-1)}{(\tilde{b}-2)^2} > 0$ as above and note that since $\tilde{b} = b_1 + b_2$, we have $\lambda < \frac{1}{2}$. Using the definition of κ_i, c_i in Lemma 3.12, we have

$$\xi_0 = \lambda \tilde{c}_1 \Big(a_3(a_3 - b_3)\ell \tilde{c}_1 y_1^2 y_3^{a_3-1} - 2\ell(\tilde{b} - 2)\tilde{c}_3 y_1 y_3^{a_3} + \tilde{c}_1 \tilde{c}_3\Big) = \lambda \tilde{c}_1 \det(\tilde{J}(y_1, y_3)).$$
$$(27)$$

Thus, if $\det(\tilde{J}(y_1, y_3)) < 0$, then $\xi_0 < 0$ and (y_1, y_1, y_3) is unstable. Assume $\det(\tilde{J}(y_1, y_3)) > 0$ so that $\xi_0 > 0$, and also assume $\text{Tr}(\tilde{J}(y_1, y_3)) < 0$. It follows from the latter inequality that

$$\ell(\tilde{b} - 2)y_1 y_3^{a_3} < \tfrac{1}{2}\left(a_3(a_3 - b_3)\ell y_1^2 y_3^{a_3-1} + \tilde{c}_1 + \tilde{c}_3\right). \tag{28}$$

Using (28), $a_3 > b_3$, and $\lambda < \tfrac{1}{2}$, we have

$$\begin{aligned}
\xi_2 &= a_3(a_3 - b_3)\ell y_1^2 y_3^{a_3-1} - \ell(\tilde{b} - 2)y_1 y_3^{a_3} + \tilde{c}_1 + \tilde{c}_3 \\
&> \tfrac{1}{2}\left(a_3(a_3 - b_3)\ell y_1^2 y_3^{a_3-1} + \tilde{c}_1 + \tilde{c}_3\right) > \lambda\tilde{c}_1 > 0.
\end{aligned} \tag{29}$$

All that remains is to show $\xi_2\xi_1 > \xi_0$. We first find that

$$\xi_1 = a_3(a_3 - b_3)\ell\tilde{c}_1 y_1^2 y_3^{a_3-1} - \ell(\tilde{b} - 2)(2\lambda\tilde{c}_1 + \tilde{c}_3)y_1 y_3^{a_3} + \lambda(\tilde{c}_1)^2 + \tilde{c}_1\tilde{c}_3.$$
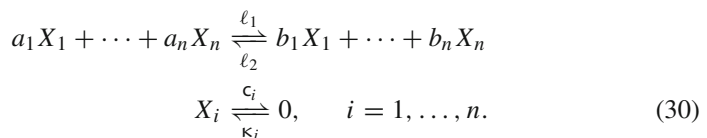
Since $2\lambda\tilde{c}_1 < \tilde{c}_1 < \tilde{c}_3$, we have that $2\lambda\tilde{c}_1 + \tilde{c}_3 < 2\tilde{c}_3$. It follows that $\xi_1 > \det(\tilde{J}(y_1, y_3))$, and in particular $\xi_1 > 0$. Combining this inequality with (27) and (29), we find

$$\xi_2\xi_1 - \xi_0 > \lambda\tilde{c}_1 \det(\tilde{J}(y_1, y_3)) - \lambda\tilde{c}_1 \det(\tilde{J}(y_1, y_3)) = 0.$$

This shows that the steady state (y_1, y_1, y_3) is asymptotically stable, proving (iii). □

4 Networks with One Non-flow Reversible Reaction

In this section, we apply Gale duality to study a modified version of the family of networks (10) analyzed in §3. These networks were originally introduced in Joshi (2013). In particular, we consider the following reaction networks with n species $X_1, \ldots, X_n$:

$$a_1 X_1 + \cdots + a_n X_n \underset{\ell_2}{\overset{\ell_1}{\rightleftharpoons}} b_1 X_1 + \cdots + b_n X_n$$

$$X_i \underset{\kappa_i}{\overset{c_i}{\rightleftharpoons}} 0, \qquad i = 1, \ldots, n. \tag{30}$$

The system of equations describing the steady states of the network (30) is:

$$(b_i - a_i)\ell_1 x^a - (b_i - a_i)\ell_2 x^b - c_i x_i + \kappa_i = 0, \quad \text{for } i = 1, \ldots, n. \tag{31}$$

As in §3, we let $\text{sgn}(i) = \text{sgn}(b_i - a_i)$. In the remaining sections, we will assume that $\text{sgn}(i) \neq 0$ for all i. This assumption is without loss of generality; see Remark 3.1.

Similar to (21), we let

$$a_+ = \sum_{\text{sgn}(i)=1} a_i, \quad b_+ = \sum_{\text{sgn}(i)=1} b_i, \quad a_- = \sum_{\text{sgn}(i)=-1} a_i, \quad \text{and} \quad b_- = \sum_{\text{sgn}(i)=-1} b_i.$$
(32)

In Joshi (2013), it was shown that the system (31) admits at least two positive solutions for some $\ell_1, \ell_2, c_i,$ and κ_i if and only if $a_+ > 1$ or $b_- > 1$. Note that the network (30) is obtained by making an irreversible non-flow reaction of the network (10) reversible. This construction applies to either the reaction with label ℓ_1 or the one with label ℓ_2. Hence, by Proposition 2.2 and the results in § 3 on the steady states of the network (10), we have:

- If either $a_+ > 1$ and $a_- = 0$ or $b_- > 1$ and $b_+ = 0$, then the network (30) admits at least two positive non-degenerate steady states.
- If either $a_+ > 1$ and $a_- > 0$ or $b_- > 1$ and $b_+ > 0$, then the network (30) admits at least three positive non-degenerate steady states.

In all other cases, there exist values of the reaction rate constants such that the network (30) has one positive non-degenerate steady state.

Conjecture 4.1 *Based on the analysis presented below, we conjecture that the maximum possible number of positive steady states of the reaction network* (30) *is three.*

Again, we will study the number of positive steady states by constructing a Gale dual system. Since, by assumption, $a_i \neq b_i$ for all i, we let

$$\alpha := \frac{\ell_2}{\ell_1}, \quad c_i := \frac{c_i}{|b_i - a_i|\ell_1}, \quad k_i := \frac{\kappa_i}{|b_i - a_i|\ell_1}.$$

We may rewrite the system (31) as

$$\text{sgn}(i)\left(x^a - \alpha\, x^b\right) - c_i x_i + k_i = 0, \quad \text{for } i = 1, \dots, n.$$
(33)

Using the monomial vector $x^W = [x_1, \dots, x_n, x^a, x^b, 1]^T$, the system of equations (33) can also be expressed in the form $C \cdot x^W = 0$, where

$$C = \begin{bmatrix} -c_1 & \cdots & 0 & \text{sgn}(1) & -\text{sgn}(1)\alpha & k_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -c_n & \text{sgn}(n) & -\text{sgn}(n)\alpha & k_n \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 1 & \cdots & 0 & a_1 & b_1 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & a_n & b_n & 0 \end{bmatrix}.$$
(34)

We now compute a Gale dual system to (33) using Proposition 2.3. We have that $l = 2$ and hence any Gale dual system depends on two variables, y_1 and y_2. One can check that the following choices of Gale dual matrices $D \in \mathbb{R}^{(n+3)\times 3}$ and $Q \in \mathbb{Z}^{(n+3)\times 3}$

satisfy the requirements stated in Proposition 2.3:

$$
D = \begin{bmatrix} \frac{\operatorname{sgn}(1)}{c_1} & 0 & \frac{k_1}{c_1} \\ \vdots & \vdots & \vdots \\ \frac{\operatorname{sgn}(n)}{c_n} & 0 & \frac{k_n}{c_n} \\ 1 & 1 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} a_1 & b_1 & 0 \\ \vdots & \vdots & \vdots \\ a_n & b_n & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{35}
$$

In the notation of §2.2, we have

$$
d_i(y_1, y_2) = \tfrac{k_i}{c_i} + \tfrac{\operatorname{sgn}(i)}{c_i} y_1, \quad i = 1, \dots, n, \quad d_{n+1}(y_1, y_2) = y_1 + y_2,
$$
$$
\text{and} \quad d_{n+2}(y_1, y_2) = \tfrac{1}{\alpha} y_2.
$$

Hence, the Gale dual system to (33) in $\mathbb{R}[y_1, y_2]$ is given by

$$
g_1(y_1, y_2) = g_2(y_1, y_2) = 0, \text{ for } y_1 > 0, \ y_2 > 0 \text{ and } (k_i + \operatorname{sgn}(i)y_1) > 0 \tag{36}
$$

where

$$
g_1(y_1, y_2) = \prod_{i=1}^{n} (\operatorname{sgn}(i)y_1 + k_i)^{a_i} - c^a(y_1 + y_2), \text{ and } g_2(y_1, y_2)
$$
$$
= \prod_{i=1}^{n} (\operatorname{sgn}(i)y_1 + k_i)^{b_i} - \tfrac{c^b}{\alpha} y_2.
$$

From the relation $g_2(y_1, y_2) = 0$, we have that

$$
y_2 = \frac{\alpha}{c^b} \prod_{i=1}^{n} (\operatorname{sgn}(i)y_1 + k_i)^{b_i}. \tag{37}
$$

After substituting y_2 into g_1 and scaling by a constant, we arrive at the following expression:

$$
\widetilde{g}_1(y_1) = c^{-a} \prod_{i=1}^{n} (\operatorname{sgn}(i)y_1 + k_i)^{a_i} - \alpha c^{-b} \prod_{i=1}^{n} (\operatorname{sgn}(i)y_1 + k_i)^{b_i} - y_1. \tag{38}
$$

Hence, we will study the positive steady states of the network using the system

$$
\widetilde{g}_1(y_1) = 0, \text{ for } y_1 > 0 \text{ and } (k_i + \operatorname{sgn}(i)y_1) > 0. \tag{39}
$$

Note that from (37) we see that the stated constraints on y_1 ensure that the constraint $y_2 > 0$ appearing in (36) is always satisfied. As in §3, we will study the number

of solutions of (39) by considering the different cases arising from the values of $a_+, a_-, b_+,$ and b_- in (32). Note that by symmetry, a system analogous to (39) is obtained by reversing the roles of a and b, and replacing α by α^{-1}.

4.1 Number of Steady States

As in (17), let k_- be the smallest value of k_i for which sgn(i) is negative (recall that we set $k_- = +\infty$ if there is no negative sign). With $\widetilde{g}_1$ as in (38) we study the solutions to $\widetilde{g}_1(y_1) = 0$ for $y_1 \in (0, k_-)$. Our analysis of the number of positive steady state of the network (30) focuses on the following two cases:

(i) either $a_+ > 1$ and $a_- = 0$, or $b_- > 1$ and $b_+ = 0$,
(ii) either $a_+ > 1$ and $a_- > 0$, or $b_- > 1$ and $b_+ > 0$.

Case (i) implies that either sgn$(i) = 1$ for all i, or sgn$(i) = -1$ for all i (for example, $a_- = 0$ implies that $a_i > b_i$ is not possible). In case (ii) both signs occur.

4.1.1 Case (i)

Due to symmetry, we assume, without loss of generality, that $a_+ > 1$ and $a_- = 0$. In this case, the constraints (sgn$(i)y_1 + k_i) > 0$, $y_1 > 0$ are simply given by $y_1 > 0$, so we want to determine the positive roots of $\widetilde{g}_1$.

Remark 4.2 A finite sequence of real-valued functions $f_1(y), \ldots, f_\ell(y)$ satisfies Descartes' rule of signs on an interval $(a, b) \subset \mathbb{R}$ if the number of zeros of the function $\mathfrak{c}_1 f_1(y) + \cdots + \mathfrak{c}_\ell f_\ell(y)$ in (a, b) (counted with multiplicity) is less than or equal to the number of sign changes in the sequence $\mathfrak{c}_1, \ldots, \mathfrak{c}_\ell$ for any choice of real constants $\mathfrak{c}_i$. For more see (Pólya and Szegö 1997, Part 5, 87–90).

Remark 4.3 Let $\omega_0, \ldots, \omega_\ell$ be arbitrary real numbers. A classical result of Runge (see, for example, (Dimitrov and Rafaeli 2009, p. 36)) states that the sequence of polynomials

$$1, \; y - \omega_0, \; (y - \omega_0)(y - \omega_1), \ldots, \; (y - \omega_0)(y - \omega_1)\cdots(y - \omega_\ell) \qquad (40)$$

satisfies Descartes' rule of signs (in the sense of Remark 4.2) for $y > \max(\omega_i \mid i = 0, \ldots, \ell)$; this also holds for any subsequence of this sequence.

We now show that the maximum number of positive roots of $\widetilde{g}_1$ is three.

Proposition 4.4 *If $a_+ > 1$ and $a_- = 0$, the equation*

$$\widetilde{g}_1(y_1) = c^{-a} \prod_{i=1}^{n} (y_1 + k_i)^{a_i} - \alpha c^{-b} \prod_{i=1}^{n} (y_1 + k_i)^{b_i} - y_1 = 0$$

has at most three positive solutions (counted with multiplicity), and there exist values of the parameters for which $\widetilde{g}_1$ has three distinct positive roots.

Hence, the network (30) admits three positive non-degenerate steady states when $a_+ > 1$ and $a_- = 0$ (by symmetry, this also holds when $b_- > 1$ and $b_+ = 0$).

Proof We have that $\text{sgn}(i) = 1$ for all i. Set $k_+ = \min(k_i \mid i = 1, \ldots, n)$. Let $1 \leq \nu \leq n$ be an integer such that $a_\nu > 0$ and consider the sequence of functions

$$1, \ y_1 + k_\nu, \ \gamma(y_1) = \prod_{i=1}^{n} (y_1 + k_i)^{a_i}, \ \omega(y_1) = \prod_{i=1}^{n} (y_1 + k_i)^{b_i}. \quad (41)$$

By Remark 4.3, it follows that the sequence of functions (41) satisfies Descartes' rule of signs on the open interval $y_1 > -k_+$. To see this, take $\omega_0 = -k_\nu$, $\omega_1 = \cdots = \omega_{a_1} = -k_1$, $\omega_{a_1+1} = \cdots = \omega_{a_1+a_2} = -k_2$, and so on (omitting one $\omega_j = -k_i$ when $i = \nu$) until we reach $\omega_{a_+-1} = -k_n$, then set $\omega_{a_+} = \cdots = \omega_{a_++(b_1-a_1)} = -k_1$ and so on until we reach $\omega_{b_+-1} = -k_n$. With this choice of ω, the sequence of functions (41) is a subsequence of that in (40) (recall that $b_i > a_i$ for all i). Rewrite $\widetilde{g}_1$ as

$$\widetilde{g}_1(y_1) = k_\nu - (y_1 + k_\nu) + c^{-a}\gamma(y_1) - \alpha c^{-b}\omega(y_1).$$

The sign sequence for $\widetilde{g}_1$ with respect to the sequence of functions (41) is $+, -, +, -$. Hence, by Remark 4.2 we have that $\widetilde{g}_1$ has at most three roots in the interval $(-k_+, \infty)$. In particular, $\widetilde{g}_1$ has at most three positive roots.

To show that three solutions are possible, set $x_1 = \cdots = x_n$, $k_1 = \cdots = k_n$, and $c_1 = \cdots = c_n$ in (33) and recall that $\text{sgn}(i) = 1$ for all i, meaning $b_+ > a_+$ (also note that $b_i - a_i$ is a fixed value, so we choose each κ_i and c_i appropriately to yield $k_1 = \cdots = k_n$ and $c_1 = \cdots = c_n$). Then, for all i, the equations in (33) have the form

$$\text{sgn}(i) \left(x_1^{a_1} \cdots x_n^{a_n} - \alpha \, x_1^{b_1} \cdots x_n^{b_n} \right) - c_i x_i + k_i = x_1^{a_+} - \alpha \, x_1^{b_+} - c_1 x_1 + k_1 = 0$$

for $i = 1, \ldots, n$. In particular, all steady-state equations (33) are the same for all i and are equal to

$$p(x_1) = k_1 - c_1 x_1 + x_1^{a_+} - \alpha x_1^{b_+} = 0. \quad (42)$$

The polynomial $\frac{-p(x_1)}{\alpha}$ is an arbitrary monic polynomial with real coefficients of fixed sign. Since $b_+ > a_+$, its Descartes sign sequence is $+, -, +, -$ (starting with the constant term); hence, there must exist values of the coefficients for which Descartes bound of three positive solutions is achieved. $\qquad \square$
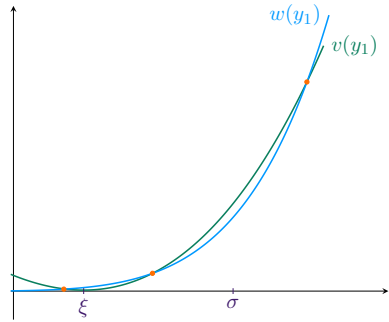
Let

$$v(y_1) = \prod_{i=1}^{n} (y_1 + k_i)^{a_i} - c^a \cdot y_1, \quad w(y_1) = \alpha c^{a-b} \prod_{i=1}^{n} (y_1 + k_i)^{b_i}. \quad (43)$$

To find the solutions to (38), we seek the points $y_1 > 0$ where v and w intersect. Note that

$$v(0) = k^a, \quad v'(0) = k^a \sum_{j=1}^{n} \frac{a_j}{k_j} - c^a, \quad \text{and} \ w(0) = \frac{\alpha k^b}{c^{b-a}}.$$

Fig. 4 The situation described in Proposition 4.5 (iii); c_i and k_i are chosen so that $v(0) > w(0)$ and $v'(0) < 0$, and α is chosen so that $v(\xi) < w(\xi)$ and $v(\sigma) > w(\sigma)$ where ξ is a positive root of v' and $\sigma > \xi$ (Color figure online)

Proposition 4.5 *Suppose that $a_i < b_i$ for all i. Let v and w be as in* (43) *and consider the solutions to $\widetilde{g}_1(y_1) = 0$ for $y_1 > 0$ with $\widetilde{g}_1$ as in* (38).

(i) *If $v(0) \leq w(0)$, then $\widetilde{g}_1$ has at most two positive roots.*
(ii) *If $v(0) > w(0)$, then $\widetilde{g}_1$ has at least one positive root.*
(iii) *Suppose that $v(0) > w(0)$ and $v'(0) < 0$. Let ξ be the positive root of v' and let $\sigma > \xi$. Then, $\widetilde{g}_1$ has exactly three positive roots if $v(\xi) < w(\xi)$ and $v(\sigma) > w(\sigma)$.*

Proof Let $k_+ = \min(k_i \mid i = 1, \ldots, n)$. By the proof of Proposition 4.4, we know that v and w intersect at most three times in the interval $(-k_+, \infty)$. Also note that $v(-k_+) > 0$ and $w(-k_+) = 0$. Hence, if $v(0) \leq w(0)$, then v and w intersect at least once in $(-k_+, 0]$, and hence at most twice in $(0, \infty)$; this proves (i).

Item (ii) follows immediately from the fact that $w(y_1)$ is larger than $v(y_1)$ for sufficiently large y_1.

Now consider (iii). Suppose that $v(0) > w(0)$ and $v'(0) < 0$. Since $v'(\theta) > 0$ for θ large, and since v' has at most one positive real root by Descartes' rule of signs, v' must have exactly one positive real root ξ. If $v(\xi) < w(\xi)$, then v and w intersect at least once in $(0, \xi)$. Similarly, if $v(\sigma) > w(\sigma)$ for some $\sigma > \xi$, then v and w must intersect at least once in (ξ, σ). Finally, we know that for ζ sufficiently large $v(\zeta) < w(\zeta)$; hence, v and w must intersect at least once in the interval (σ, ∞). Taken together this implies that v and w intersect exactly once in each of the intervals $(0, \xi)$, (ξ, σ), and (σ, ∞); this proves (iii). Note that the condition $v'(0) < 0$ is necessary since v' is strictly increasing for all positive y_1; hence, if $v'(0) > 0$, then v' has no positive roots. The situation described by (iii) is illustrated in Fig. 4. □

4.1.2 Case (ii)

We now consider the case where either $a_+ > 1$ and $a_- > 0$ or $b_- > 1$ and $b_+ > 0$ (note we must have $n \geq 2$). By computing a cylindrical algebraic decomposition, we have found that the maximum number of roots of $\widetilde{g}_1$ in $(0, k_-)$ is three for the following combinations of vectors a and b:

	$n = 2$						$n = 3$		
$[a_1, \ldots, a_n]$	[2, 1]	[2, 2]	[2, 2]	[2, 2]	[2, 2]	[2, 3]	[3, 1]	[2, 1, 1]	[2, 3, 1]
$[b_1, \ldots, b_n]$	[3, 0]	[4, 0]	[4, 1]	[5, 1]	[6, 1]	[3, 2]	[4, 0]	[3, 0, 0]	[3, 2, 0]

We have seen in §3 that for networks with one non-flow irreversible reaction, increasing n does not change the possible number of positive steady states of the network. In light of this, along with the results of §4.1.1 and the investigations using cylindrical algebraic decomposition, we conjecture that $\widetilde{g}_1$ admits up to three roots in $(0, k_-)$, see Conjecture 4.1.

We now give a result which uses Remark 4.2 to bound the number of roots of $\widetilde{g}_1$ in $(0, k_-)$.

Proposition 4.6 *Let $a_+ > 0$, and let $\widetilde{g}_1$ is as in (38). The number of roots of $\widetilde{g}_1$ in $(0, k_-)$ is at most $\max(a_-, b_+ + b_- - a_+) + 2$.*

Proof Since $a_+ > 0$, there exists an index ν such that $a_\nu > 0$. Let $\Omega = \max(a_-, b_+ + b_- - a_+)$. By Remark 4.3, it follows that the sequence of $\Omega + 3$ functions

$$1, \ y_1 + k_\nu, \ \prod_{\text{sgn}(i)=1} (y_1 + k_i)^{a_i}, \ y_1 \prod_{\text{sgn}(i)=1} (y_1 + k_i)^{a_i}, \ \ldots, y_1^\Omega \prod_{\text{sgn}(i)=1} (y_1 + k_i)^{a_i} \tag{44}$$

satisfies Descartes' rule of signs on for $y_1 > 0$. Let

$$\gamma(y_1) = c^{-a} \prod_{\text{sgn}(i)=-1} (-y_1 + k_i)^{a_i} - \alpha c^{-b} \prod_{\text{sgn}(i)=1} (y_1 + k_i)^{b_i - a_i} \prod_{\text{sgn}(i)=-1} (-y_1 + k_i)^{b_i},$$

and note that $\deg(\gamma) = \Omega$. With this notation $\widetilde{g}_1(y_1) = \prod_{\text{sgn}(i)=1} (y_1 + k_i)^{a_i} \gamma(y_1) - (y_1 + k_\nu) + k_\nu$. Expanding γ, we see that the polynomial $\widetilde{g}_1$ has at most $\Omega + 2$ sign changes when written in terms of the sequence of $\Omega + 3$ functions (44). □

By symmetry, we conclude that network (30) admits at most

$$\min(\max(a_-, b_+ + b_- - a_+), \max(b_+, a_+ + a_- - b_-)) + 2$$

positive non-degenerate steady states.

Remark 4.7 By Theorem 1 of Bates et al. 2007, the number of nonzero real solutions to the system (33) is no more than $n^2(e^4 + 3)$. Alternatively, by the Bernstein–Kushnirenko theorem, the number of nonzero real solutions to (33) is no more that $n! \text{vol}(\text{conv}(W))$, where $\text{vol}(\text{conv}(W))$ denotes the Euclidean volume of the convex hull of the columns of the matrix W in (34). These bounds are both much larger than the bound of three for the case (i) explored in §4.1.1 and are also often larger than the bound of Proposition 4.6. For example, consider the case where $n = 4$, $a = [2, 3, 3, 6]$, and $b = [1, 2, 4, 7]$. In this case, Proposition 4.6 tells us that there are at most $\max(a_-, b_+ + b_- - a_+) + 2 = 5 + 2 = 7$ positive solutions to (33), on the other hand $4^2(e^4 + 3) \simeq 876.57$, and $4! \text{vol}(\text{conv}(W)) = 40$.

4.2 Stability

In this section, we show that if a network of the type (30) admits three positive steady states, then it displays bistability. We start with a lemma on bistability.

Lemma 4.8 *Suppose that either $n = 1$ and $1 < a_1 < b_1$, or $n = 2$ and $a_1 = 1$, $a_2 = 1$ $\mathrm{sgn}(1) = \mathrm{sgn}(2) = 1$. If the network (30) has three positive steady states, then two of them are asymptotically stable and one is unstable.*

Proof First consider the case $n = 1$. The ODE system of the network (30) is

$$\frac{dx_1}{dt} = f(x_1), \quad \text{with } f(x_1) = (b_1 - a_1)\ell_1 x_1^{a_1} + (a_1 - b_1)\ell_2 x_1^{b_1} - c_1 x_1 + \kappa_1.$$

Since $f(0) > 0$, if the network has three positive steady states (which must be of multiplicity one), then the smallest and largest steady states satisfy $f'(x_1^*) < 0$ and are asymptotically stable, and the middle one satisfies $f'(x_1^*) > 0$ and is unstable.

Now, consider the case where $n = 2$, $a_1 = 1$ and $a_2 = 1$. The ODE system is of the form

$$\frac{dx_i}{dt} = \alpha_i \ell_1 x_1 x_2 - \alpha_i \ell_2 x_1^{b_1} x_2^{b_2} - c_i x_i + \kappa_i, \quad \text{for } i = 1, 2$$

where $\alpha_i = b_i - 1 > 0$. The associated Jacobian matrix is

$$J(x) = \begin{bmatrix} \alpha_1 \ell_1 x_2 - b_1 \alpha_1 \ell_2 x_1^{b_1-1} x_2^{b_2} - c_1 & \alpha_1 \ell_1 x_1 - b_2 \alpha_1 \ell_2 x_1^{b_1} x_2^{b_2-1} \\ \alpha_2 \ell_1 x_2 - b_1 \alpha_2 \ell_2 x_1^{b_1-1} x_2^{b_2} & \alpha_2 \ell_1 x_1 - b_2 \alpha_2 \ell_2 x_1^{b_1} x_2^{b_2-1} - c_2 \end{bmatrix}.$$

We have

$$\det(J(x)) = -c_2(\alpha_1 \ell_1 x_2 - b_1 \alpha_1 \ell_2 x_1^{b_1-1} x_2^{b_2}) - c_1(\alpha_2 \ell_1 x_1$$
$$- b_2 \alpha_2 \ell_2 x_1^{b_1} x_2^{b_2-1}) + c_1 c_2,$$
$$\mathrm{Tr}(J(x)) = (\alpha_1 \ell_1 x_2 - b_1 \alpha_1 \ell_2 x_1^{b_1-1} x_2^{b_2}) + (\alpha_2 \ell_1 x_1 - b_2 \alpha_2 \ell_2 x_1^{b_1} x_2^{b_2-1}) - c_1 - c_2.$$

A steady state of (30) satisfies the equation $x_2 = \frac{\alpha_2 c_1 x_1 - \alpha_2 \kappa_1 + \alpha_1 \kappa_2}{\alpha_1 c_2}$, and $f(x_1) = 0$ where

$$f(x_1) = \alpha_1 \ell_1 x_1 \left(\frac{\alpha_2 c_1 x_1 - \alpha_2 \kappa_1 + \alpha_1 \kappa_2}{\alpha_1 c_2} \right)$$
$$- \alpha_1 \ell_2 x_1^{b_1} \left(\frac{\alpha_2 c_1 x_1 - \alpha_2 \kappa_1 + \alpha_1 \kappa_2}{\alpha_1 c_2} \right)^{b_2} - c_1 x_1 + \kappa_1. \quad (45)$$

If the network has three positive steady states of multiplicity one, then two of them satisfy $f'(x_1) < 0$ and the other satisfies $f'(x_1) > 0$. Therefore, it is enough to show that if $f'(x_1) < 0$, then the steady state is asymptotically stable, and if $f'(x_1) > 0$,

then the steady state is unstable. Let $x = (x_1, x_2)$ be a positive steady state. Using (45), the derivative of f at x_1 is

$$f'(x_1) = \alpha_1 \ell_1 x_2 + \alpha_2 \ell_1 x_1 \frac{c_1}{c_2} - b_1 \alpha_1 \ell_2 x_1^{b_1-1} x_2^{b_2} - b_2 \alpha_2 \ell_2 x_1^{b_1} x_2^{b_2-1} \frac{c_1}{c_2}$$
$$- c_1 = \frac{-\det(J(x))}{c_2}.$$

It follows that if $f'(x_1) > 0$, then $\det(J(x)) < 0$ and the steady state is unstable. If $f'(x_1) < 0$, then $\det(J(x)) > 0$. In order to show that the steady state is asymptotically stable, we need to show that $\text{Tr}(J(x)) < 0$. From $\frac{dx_1}{dt} = \frac{dx_2}{dt} = 0$, we obtain

$$\alpha_1 \ell_1 x_1 x_2 = \alpha_1 \ell_2 x_1^{b_1} x_2^{b_2} + c_1 x_1 - \kappa_1, \quad \alpha_2 \ell_1 x_1 x_2 = \alpha_2 \ell_2 x_1^{b_1} x_2^{b_2} + c_2 x_2 - \kappa_2,$$

which gives

$$\begin{aligned} x_1 x_2 \text{Tr}(J(x)) &= x_2(\alpha_1 \ell_2 x_1^{b_1} x_2^{b_2} + c_1 x_1 - \kappa_1) - b_1 \alpha_1 \ell_2 x_1^{b_1} x_2^{b_2+1} \\ &\quad + x_1(\alpha_2 \ell_2 x_1^{b_1} x_2^{b_2} + c_2 x_2 - \kappa_2) - b_2 \alpha_2 \ell_2 x_1^{b_1+1} x_2^{b_2} - (c_1 + c_2)x_1 x_2 \\ &= -\kappa_1 x_2 + (1 - b_1)\alpha_1 \ell_2 x_1^{b_1} x_2^{b_2+1} \\ &\quad - \kappa_2 x_1 + (1 - b_2)\alpha_2 \ell_2 x_1^{b_1+1} x_2^{b_2} < 0. \end{aligned}$$

$\square$

Theorem 4.9 *If $a_+ > 1$ or $b_- > 1$, then there exists a choice of parameters such that the network* (30) *has two asymptotically stable positive steady states and one unstable positive steady state.*

Proof If either $a_+ > 1$ and $a_- > 0$ or $b_- > 1$ and $b_+ > 0$, then the statement follows from Proposition 2.2 and Theorem 3.13. Otherwise, the network has an embedded network of one of the types in Lemma 4.8. Hence, the statement follows from Propositions 2.1 and 4.4. $\square$

Appendix A: More on Gale Dual Systems

Let $\mathcal{V}$ be the zero dimensional subscheme of $(\mathbb{R}_{>0})^n$ defined by the system of n Laurent polynomials

$$Cx^W = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix} = 0, \tag{46}$$

with W and C having Gale dual matrices Q and D, respectively, chosen as in §2.2. Recall that W and C are $n \times (n+l+1)$ matrices, while Q and D are $(n+l+1) \times (l+1)$

matrices. We can now define a homomorphism of algebraic groups specified by the monomial map determined by the exponents of (46)

$$\varphi_W \colon (\mathbb{R}_{>0})^n \to (\mathbb{R}_{>0})^{n+l} \times \{1\} \subset \mathbb{P}^{n+l}_{\mathbb{R}}$$
$$\varphi_W \colon x \mapsto x^W = \left[x^{w_1} : \cdots : x^{w_{n+l}} : 1 \right]^T.$$

The homomorphism φ_W is dual to the homomorphism of free abelian groups $\iota_W \colon \mathbb{Z}^{l+n} \to \mathbb{Z}^n$ which maps the standard i^{th} basis vector of $\mathbb{Z}^{l+n}$ to the column vector w_i. Let $[z_1 : \cdots : z_{n+l+1}]$ be coordinates for $\mathbb{P}^{n+l}_{\mathbb{R}}$, then the polynomials f_i are the pullbacks of linear forms Λ_i under the monomial map φ_W, that is

$$f_i = \varphi_W^*(\Lambda_i), \qquad \text{where} \quad \Lambda_i = \sum_{j=1}^{n+l+1} c_{i,j} z_j. \tag{47}$$

The scheme $\mathcal{V}$ defined by (46) is the pullback of the linear space $L = V(\Lambda_1, \ldots, \Lambda_n) \subset \mathbb{P}^{n+l}_{\mathbb{R}}$. Let $\mathbb{Z}W$ denote the integer lattice spanned by the columns of W. Since $\mathbb{Z}W = \mathbb{Z}^n$, we have that the intersection $Y = L \cap \varphi_W((\mathbb{R}_{>0})^n)$ is proper (that is the intersection has the expected dimension) and the map φ_W defines a scheme theoretic isomorphism between $\mathcal{V}$ and Y; see, for example, Proposition 1.1 of Bihan and Sottile 2008.

Define the map $\psi_{\mathcal{V}} \colon \mathbb{R}^l \to \mathbb{P}^{l+n}_{\mathbb{R}}$ given by $\psi_{\mathcal{V}}(y) = [d_1(y) : \cdots : d_{n+l}(y) : 1]$ where the $d_i(y)$ are the linear forms in $\mathbb{R}[y_1, \ldots, y_l]$ defined by the rows of $D \cdot [y_1, \ldots, y_l, 1]^T$ as in (8). Note that by construction $\psi_{\mathcal{V}}$ is an isomorphism from $\mathbb{R}^l$ to the linear subspace L of $\mathbb{P}^{n+l}$. Hence, we have an isomorphism of schemes given by $\psi_{\mathcal{V}}^{-1} \circ \varphi_W$ so that $\mathcal{V} \cong Z = \psi_{\mathcal{V}}^{-1}(\varphi_W(\mathcal{V})) \subset \mathbb{R}^l$. The resulting isomorphic scheme Z is referred to as the *Gale dual scheme* of $\mathcal{V}$.

We now give the equations which describe Z. Every integer linear relation, $\sum_i \beta^{(i)} w_i = 0$ with $\beta^{(i)} \in \mathbb{Z}$, among column vectors w_i in W, corresponds to the Laurent monomial equality $\prod_{i=1}^{n+l} z_i^{\beta^{(i)}} = 1$ on $\varphi_W((\mathbb{R}_{>0})^n) \subset \mathbb{P}^{l+n}_{\mathbb{R}}$; here, we have that $z_i > 0$. Pulling this relation back under the map $\psi_{\mathcal{V}}$ gives the relation

$$\prod_{i=1}^{n+l} d_i(y)^{\beta^{(i)}} = 1 \quad \text{in} \ \ \mathbb{R}[y_1, \ldots, y_l] \ \ \text{with} \ \ d_i(y) > 0. \tag{48}$$

There is one such relation (48) for each row of Q giving the following system of l equations in $\mathbb{R}[y_1, \ldots, y_l]$

$$\prod_{i=1}^{n+l} d_i(y)^{q_{i,1}} = 1, \ldots, \prod_{i=1}^{n+l} d_i(y)^{q_{i,l}} = 1, \quad \text{such that} \ \ d_i(y) > 0. \tag{49}$$

By construction $Z \subset (\mathbb{R}_{>0})^l$ is the set of solutions to the system of equations (49) and $Z \cong \mathcal{V}$ as schemes. The system (49) is referred to as the *Gale dual system* of the original system (46). Hence, the one-to-one correspondence in Proposition 2.3 is an isomorphism of schemes.

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