

Survival Analysis of Stochastic Competitive Models in a Polluted Environment and Stochastic Competitive Exclusion Principle

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Received: 30 May 2009 / Accepted: 22 June 2010 / Published online: 20 August 2010
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Abstract Stochastic competitive models with pollution and without pollution are proposed and studied. For the first system with pollution, sufficient criteria for extinction, nonpersistence in the mean, weak persistence in the mean, strong persistence in the mean, and stochastic permanence are established. The threshold between weak persistence in the mean and extinction for each population is obtained. It is found that stochastic disturbance is favorable for the survival of one species and is unfavorable for the survival of the other species. For the second system with pollution, sufficient conditions for extinction and weak persistence are obtained. For the model without pollution, a partial stochastic competitive exclusion principle is derived.

Keywords Competitive model · Polluted environment · Stochastic disturbance · Stochastic competitive exclusion principle

1 Introduction

The question of the effects of toxicants and pollutants on ecological communities is interesting from both theoretical and practical points of view. For example, Nelson (1970) studied oil pollution in the sea, Jensen and Marshall (1982) analyzed the dumping of toxic waste in rivers and lakes, and Shukla et al. (1989) investigated the degradation of forests.

Recently, in a series of papers (Hallam et al. 1983a, 1983b; Hallam and Deluna 1984), Hallam and his coworkers studied the effects of toxicants on various ecosys-

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tems by utilizing mathematical models. From then on, many deterministic models were proposed and analyzed, for example (Chattopadhyay 1996; Freedman and Shukla 1991; Hallam and Ma 1986, 1987; He and Wang 2007, 2009; Hsu et al. 1995; Liu and Ma 1991; Luna and Hallam 1987; Ma et al. 1989; Shukla and Dubey 1996; Thomas et al. 1996). Particularly, Liu and Ma (1991) investigated the effects of toxicants on two species under the assumption that the capacity of the environment is so large that the change of toxicants in the environment that comes from uptake and egestion by the organisms can be neglected.

Those important and useful studies on deterministic models provide a great insight into the effects of the pollution, but in the real world, population dynamics is inevitably affected by environmental white noise which is an important component in an ecosystem (see, e.g. Gard 1984, 1986, 1988). The deterministic models assume that parameters in the systems are all deterministic irrespective environmental fluctuations. Hence, they have some limitations in mathematical modeling of ecological systems; besides, they are quite difficult to fitting data perfectly and to predict the future dynamics of the system accurately (Bandyopadhyay and Chattopadhyay 2005). May (2001) pointed out the fact that due to environmental noise, the birth rates, carrying capacity, competition coefficients, and other parameters involved in the system exhibit random fluctuation to a greater or lesser extent.

There are a few successful studies on stochastic models in a polluted environment (see, e.g. Gard 1992; Liu and Wang 2009, 2010; Samanta and Maiti 2004). Particularly, Gard (1992) proposed a stochastic model by introducing stochastic production rate. Under the assumption that the concentration of toxicant in the organism is a constant, the author obtained conditions for the existence of an invariant distribution on $(0, +\infty)$. Then Liu and Wang (2009, 2010) studied some stochastic single-species models in a polluted environment without the constant assumption. However, all of those results deal only with single-species models. As population does not exist alone in nature, it is of more biological significance to study the persistence and extinction problems of each population in systems of two or more interacting species subjected to toxicant. As far as we know, a very little amount of work has been done with the stochastic competitive model with toxicants effect, and little is known of the impact of random noise on the survival of species living in a polluted environment. Motivated by these, in the first part of this study, we shall propose and study two competitive models which account for random noise in a polluted environment.

The second part of this study devotes to investigating the stochastic competitive exclusion principle. Consider a deterministic competitive model

$$(M_0) : \begin{cases} \frac{dx_1}{dt} = x_1[r_{10} - a_{11}x_1 - a_{12}x_2], \\ \frac{dx_2}{dt} = x_2[r_{20} - a_{21}x_1 - a_{22}x_2], \end{cases}$$

where $x_i(t)$ is the population size of the i th species at time t , r_{i0} , which denotes the intrinsic growth rate of the i th population is a constant, a_{ij} , which measures the action of species j upon the growth rate of species i (in particular, a_{ii} represents the intraspecific competition coefficient of species i), is a positive constant. It is well known that, for model M_0 , there is a classical deterministic competitive exclusion

principle or Gause's Law of competitive exclusion (Hardin 1960). The Competitive exclusion principle, which can be regarded as a classical and fundamental theoretical development in community ecology tells us that, if both a_{12} and a_{21} are sufficiently large, then x_1 and x_2 competing for the same resources cannot stably coexist. The two competitors will always compete with each other which leads to either the extinction of one of the competitors or an evolutionary or behavioral shift towards a different ecological niche. If "competition is always favored" (i.e., both a_{12} and a_{21} are sufficiently small such that interspecific competition is lower than intraspecific competition), then the two competitors will stably coexist and obey

$$\lim_{t \rightarrow +\infty} x_1(t) = \frac{a_{22}r_{10} - a_{12}r_{20}}{a_{11}a_{22} - a_{12}a_{21}}, \quad \lim_{t \rightarrow +\infty} x_2(t) = \frac{a_{11}r_{20} - a_{21}r_{10}}{a_{11}a_{22} - a_{12}a_{21}}.$$

If one of a_{12} and a_{21} is small and another is large, for example, a_{12} is large and a_{21} is small, then x_1 will go to extinction and x_2 will survive, and moreover

$$\lim_{t \rightarrow +\infty} x_2(t) = r_{20}/a_{22}.$$

Since population dynamics is inevitably affected by environmental white noise, it is worth while to study the stochastic system to find out whether the disturbance affects competitive exclusion principle from the biological and ecological points of view. From our results, it is easy to see that, similar to the deterministic competitive exclusion principle, competitive coefficients also play a very important role in determining persistence, or extinction of populations in stochastic model, which are neglected by all relevant known references.

The rest of the paper is arranged as follows. In Sect. 2, we are going to develop two stochastic competitive models in a polluted environment. In Sect. 3, we will carry out the survival analysis for the first competitive system (model (SM_1)) and obtain some sufficient conditions for extinction, nonpersistence in the mean, weak persistence in the mean, strong persistence in the mean, and stochastic permanence. The threshold between weak persistence in the mean and extinction will be established for each species. In Sect. 4, we are going to carry out the survival analysis for the second competitive system (model (SM_2)). Sufficient conditions for extinction and weak persistence will be obtained; especially, the threshold between weak persistence and extinction will be established for the single-species case. In Sect. 5, a partial stochastic competitive exclusion principle will be derived. In Sect. 6, we shall work out some figures to illustrate the various theorems obtained in Sects. 3–5. The last section devotes to the conclusions and future directions of our research.

2 The Basic Models

From now on, unless otherwise specified, we shall always work on a given complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. $B_i(t)$ stands for a given standard Brownian motion defined on the probability space, $i = 1, 2$.

Our study relies on the assumption that the environment is of complete spatial homogeneity and there is no migration. The individual organisms in the population are assumed to be nongrowing. Let $C_0(t)$ stand for the concentration of toxicant in the organism at time t ; $C_E(t)$ is the concentration of toxicant in the environment at time t . The unit of C_0 is in terms of $m_0^{-1}m_T$, C_E is in terms of $m_E^{-1}m_T$, where m_0 denotes the units of mass of the organism, m_T represents mass of toxicant, and m_E stands for the units of the mass of the environment. Assume that the living organisms absorb environmental toxicants into their bodies, the dynamics of the population is affected by internal toxicant. Let b_i (in units: t^{-1}) and d_i (t^{-1}) be the birth rate and the death rate of the i th population, respectively, $i = 1, 2$.

A coupling between species and toxicant is obtained by assuming that the intrinsic growth rate of the i th population, $b_i - d_i$, is a linear function of concentration of toxicant present in the organism:

$$b_i - d_i = r_{i0} - r_{i1}C_0,$$

where r_{i0} has unit in terms of t^{-1} and r_{i1} has unit $t^{-1}m_0m_T^{-1}$.

We assume that the population dynamics is given as follows:

$$\frac{dx_1}{dt} = x_1[r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2], \quad (1)$$

$$\frac{dx_2}{dt} = x_2[r_{20} - r_{21}C_0(t) - a_{21}x_1 - a_{22}x_2], \quad (2)$$

where a_{ij} has unit in terms of $t^{-1}m_0^{-1}$, $i, j = 1, 2$. The model consisting of (1) and (2) as well as initial conditions

$$x_i(0) > 0, \quad C_0(0) = 0 \quad (3)$$

will be referred to as model (M).

As said above, population systems are often subject to environmental fluctuations. Generally speaking, such fluctuations could be modeled by a colored noise (see, e.g. Arnold 1974). It has been noted that (see, e.g. Arnold 1974; Braumann 2002; Øsendal 1998) if the colored noise is not strongly correlated, then we can approximate the colored noise by a white noise, and the approximation works quite well. Thus, many authors introduced the white noise into population dynamics to study the effect of the environmental fluctuations in population system. Currently, there are two main ways considered in the literatures to introduce the white noise into Lotka–Volterra system. One is to assume that the most sensitive parameter is the intrinsic growth rate r_{i0} for it is the parameter most influential in the regulation of the fate of young recruits after reproduction, a very sensitive phase in the life cycle. The other one is to assume that the noise affects the removal processes (which represent those related to $a_{ij}x_i x_j$) mainly. The studies (Beddington and May 1977; Braumann 2002, 2008; Gard 1992; Li and Mao 2009; Liu and Wang 2009, 2010; Pang et al. 2008; Rudnicki and Pichor 2007; Zhu and Yin 2009) are the former case while the investigations (Bahar and Mao 2004; Du and Sam 2006; Luo and Mao 2007; Mao et al. 2002, 2003; Mao 2005) are the latter type.

Firstly, let us consider the former way. Recall that the parameter r_{i0} represents the intrinsic growth rate. In practice, we usually estimate it by an average value plus errors. In general, by the well-known central limit theorem, the error term follows a normal distribution; thus, for short correlation time, we may replace the rate r_{i0} by $r_{i0} \rightarrow r_{i0} + \alpha_i \dot{B}_i(t)$, where $\dot{B}_i(t)$ is the white noise, and α_i is a positive constant representing the intensity of the white noise. Then (1) and (2) can be described by the Itô equations:

$$dx_1 = x_1[r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2]dt + \alpha_1x_1 dB_1(t), \tag{4}$$

$$dx_2 = x_2[r_{20} - r_{21}C_0(t) - a_{21}x_1 - a_{22}x_2]dt + \alpha_2x_2 dB_2(t). \tag{5}$$

Here, it is useful to point out that there is no limitation on the relationship between $B_1(t)$ and $B_2(t)$. The random noises $B_1(t)$ and $B_2(t)$ could be independent. They could also be correlated, which corresponds to the situation when the same factor (like an epidemic disease) influences both populations x_1 and x_2 . The model consisting of (4) and (5) as well as initial conditions (3) will be referred to as model (SM_1).

If we assume that the noise affects the parameter a_{ij} mainly, with $-a_{ij} \rightarrow -a_{ij} + \sigma_{ij} \dot{B}_{ij}(t)$, which results in a new stochastic form

$$dx_1 = x_1[r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2]dt + \sigma_{11}x_1^2 dB_{11}(t) + \sigma_{12}x_1x_2 dB_{12}(t), \tag{6}$$

$$dx_2 = x_2[r_{20} - r_{21}C_0(t) - a_{21}x_1 - a_{22}x_2]dt + \sigma_{21}x_1x_2 dB_{21}(t) + \sigma_{22}x_2^2 dB_{22}(t), \tag{7}$$

where σ_{ij} is the intensity of the white noise $\dot{B}_{ij}(t)$, $i, j = 1, 2$. The model consisting of (6) and (7) as well as (3) will be referred to as model (SM_2). Here, we use the Itô calculus not the Stratonovich calculus on the grounds that, on the one hand, models (SM_1) and (SM_2) “are approximations to age-structured populations, with populations growth taking place in discrete time steps” (see, e.g. Beddington and May 1977; Braumann 2007; Ludwig 1975); On the other hand, the specific feature of the Itô model of “not looking into the future” “is a reason for choosing the Itô interpretation in many cases, for example, in biology” (see, e.g. Øsendal 1998; Turelli 1977). However, our general conclusions are not dependent on the choice between the two calculi because there is an explicit connection between the two calculi (we refer the reader to Øsendal 1998, p. 36 for more details of this explicit connection). For more biological motivation on this type of modeling in population dynamics, we refer the reader to (Gard 1984, 1986, 1988).

Now, we are in the position to introduce the model of the concentration of environmental toxicants. For the details of the modeling process, we refer the reader to (Liu and Wang 2009). We suppose that $C_0(t)$ and $C_E(t)$ obey the following equations:

$$\frac{dC_0(t)}{dt} = a_1C_E(t) - p_1\theta\beta/a_1 - (l_1 + l_2)C_0(t), \tag{8}$$

$$\frac{dC_E(t)}{dt} = -hC_E(t) + u(t), \tag{9}$$

where the first two terms on the right in (8) denote the organismal net uptake of toxicant from the environment and the food chain, respectively; the third term represents the organismal net loss of toxicant due to metabolic processing and other causes. The parameters $a_1, p_1 (\leq a_1), \theta, \beta, l_1,$ and l_2 are positive constants, a_1 denotes environmental toxicant uptake rate per unit mass organism; p_1 , the uptake rate of toxicant in food per unit mass organism; θ , the concentration of toxicant in the resources; β , the average rate of food intake per unit mass organism. l_1 and l_2 are organismal net ingestion and deportation rates of toxicant, respectively. The positive constant h in (9) represents the loss rate of toxicant from the environment including processes such as biological transformation, chemical hydrolysis, volatilization, microbial degradation, and photosynthetic degradation. The exogenous rate of input of toxicant into the environment is represented by $u(t)$. Here, $u(t)$ is restricted by $0 \leq U_1 \leq u(t) \leq U_2 < +\infty$ for constants U_1 and U_2 for all $t \in [0, +\infty)$.

The system consisting of (4), (5), (8), and (9) as well as initial conditions $x_i(0) > 0, C_0(0) = C_E(0) = 0$ will be referred to as model (GSM_1). The system consisting of (6), (7), (8), and (9) as well as initial conditions $x_i(0) > 0, C_0(0) = C_E(0) = 0$ will be referred to as model (GSM_2). One aim of this research is to find the conditions on $u(t)$ such that population $x_i(t)$ modeled by (GSM_k) is persistent or extinct, $i, k = 1, 2$. From the standpoint of mathematics, the question is to investigate the asymptotic behavior of the solution for the nonautonomous stochastic system (GSM_k). Since (8) and (9) in (GSM_k) are linear in C_0 and C_E , respectively; they are explicitly solvable. Therefore, the key of our problem is to find the conditions for persistence and extinction of each species through $C_0(t)$ for the two-dimensional subsystem (SM_k).

For the sake of convenience and simplicity, we define the following notations:

$$\text{Community interaction matrix: } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$R_+ := [0, +\infty); \quad \vartheta := r_{11}(r_{20} - \alpha_2^2/2) - r_{21}(r_{10} - \alpha_1^2/2);$$

$$\Delta := \det A = a_{11}a_{22} - a_{12}a_{21};$$

$$\Delta_1 := a_{22}(r_{10} - \alpha_1^2/2) - a_{12}(r_{20} - \alpha_2^2/2);$$

$$\Delta_2 := a_{11}(r_{20} - \alpha_2^2/2) - a_{21}(r_{10} - \alpha_1^2/2);$$

$$\Phi_1 := a_{22}r_{11} - a_{12}r_{21}; \quad \Phi_2 := a_{11}r_{21} - a_{21}r_{11}.$$

The following definitions are commonly used (see, e.g. He and Wang 2007, 2009; Liu and Wang 2009) and we list them below.

Definition 1 1. The population $x(t)$ is said to go to extinction if $\lim_{t \rightarrow +\infty} x(t) = 0$.

- 2. The population $x(t)$ is said to be nonpersistent in the mean if $\lim_{t \rightarrow +\infty} \langle x(t) \rangle = 0$, where $\langle f(t) \rangle := \frac{1}{t} \int_0^t f(s) ds$;
- 3. The population $x(t)$ is said to be weakly persistent if $x^* > 0$, where $f^* := \limsup_{t \rightarrow +\infty} f(t)$.
- 4. The population $x(t)$ is said to be weakly persistent in the mean if $\langle x \rangle^* > 0$.

In this study, we also need the following definitions:

Definition 2 5. The population $x(t)$ is said to be strongly persistent in the mean if $\langle x \rangle_* > 0$, where $f_* := \liminf_{t \rightarrow +\infty} f(t)$.

6. Model (SM_1) is said to be stochastically permanent if for any $\varepsilon \in (0, 1)$, there exists a pair of positive constants $\beta = \beta(\varepsilon)$ and $\chi = \chi(\varepsilon)$ such that for any initial value $x(0) = (x_1(0), x_2(0)) \in R_+^2$, the solution obeys

$$\liminf_{t \rightarrow +\infty} P \{ |x(t)| := \sqrt{x_1^2(t) + x_2^2(t)} \leq \chi \} \geq 1 - \varepsilon,$$

$$\liminf_{t \rightarrow +\infty} P \{ |x(t)| \geq \beta \} \geq 1 - \varepsilon.$$

It follows from the above definitions that strong persistence in the mean implies weak persistence in the mean, weak persistence in the mean indicates weak persistence, extinction means nonpersistence in the mean. But generally, the reverse of the above reasoning is not true.

There are many methods to analysis deterministic system, such as Lyapunov functions, coincidence degree theory, Jacobian matrix, and so on. But there is lack of mathematical machinery available to analyze the stochastic system. One of the current approaches for studying stochastic system is to make use of Fokker–Planck equation (see, e.g. Gard 1992). However, on the one hand, (SM_i) is a nonautonomous two-dimensional stochastic model, which corresponding Fokker–Planck equation is not an ordinary differential equation but a partial differential equation, $i = 1, 2$. On the other hand, the uniform boundedness of x_1 and x_2 in deterministic model (M) (obviously, $x_i(t) < r_{i0}/a_{ii}$ if $x_i(0) < r_{i0}/a_{ii}$, $i = 1, 2$) is destroyed in model (SM_i) by stochastic disturbance. It is well known that boundedness is a very important property in the proof. For example, the successful study Liu and Ma (1991) is based on the fact that the solutions of their deterministic models are bounded. In this work, we mainly use Itô’s formula, the theory of stochastic equation, and Lyapunov functions to analyze the properties of system (SM_i) .

3 The Survival Analysis for Model SM_1

We have two fundamental assumptions to model (SM_1) .

Assumption 1 $r_{10} > \alpha_1^2/2$ and $r_{20} > \alpha_2^2/2$.

Assumption 2 $\Delta_1 > 0$ and $\Delta_2 > 0$.

The reasons why we assume $r_{10} > \alpha_1^2/2$, $r_{20} > \alpha_2^2/2$ and $\Delta_1 > 0$, $\Delta_2 > 0$ as well as the biological interpretations of these assumptions will be given in Remark 5. It is useful to point out that if $r_{10} > \alpha_1^2/2$ and $r_{20} > \alpha_2^2/2$, then $\Delta_1 > 0$ and $\Delta_2 > 0$ mean $\Delta > 0$.

First, let us prepare some lemmas.

Lemma 1 *For model (SM_1) , with any given initial value $x(0) = (x_1(0), x_2(0)) \in R_+^2$, there is an unique solution $x(t) = (x_1(t), x_2(t))$ on $t \geq 0$ and the solution will remain in R_+^2 with probability one.*

The proof of Lemma 1 can be found in Li and Mao (2009). There a more generalized result about the existence and uniqueness of the global solution was given.

Since both $C_0(t)$ and $C_E(t)$ in models (GSM_1) and (GSM_2) are concentrations of the toxicant, so the inequalities $0 \leq C_0(t) < 1$, $0 \leq C_E(t) < 1$ for all $t \in R_+$ must be satisfied to be realistic. In fact, by solving (8) and (9) we have the following lemma.

Lemma 2 *For models (GSM_1) and (GSM_2) , if $0 < a_1 + d_1\theta\beta/a_1 < l_1 + l_2$, $U_2 \leq h$, then $0 \leq C_0(t) < 1$, $0 \leq C_E(t) < 1$ for all $t \in R_+$.*

From now on, we impose $0 < a_1 + d_1\theta\beta/a_1 < l_1 + l_2$, $U_2 \leq h$ on systems (GSM_1) and (GSM_2) . The following two lemmas will play an important role in obtaining our results.

Lemma 3 *For model (SM_1) , we have $\{\ln x_i(t)/t\}^* \leq 0$, $i = 1, 2$.*

The proof of this and subsequent results are deferred until the [Appendix](#).

Lemma 4 *Suppose $x(t) \in C[\Omega \times R_+, R_+^0]$, where $R_+^0 := \{a|a > 0, a \in R\}$.*

(I) *If there are positive constants λ_0, T and $\lambda \geq 0$ such that*

$$\ln x(t) \leq \lambda t - \lambda_0 \int_0^t x(s) ds + \sum_{i=1}^n \beta_i B_i(t) \tag{10}$$

for $t \geq T$, where β_i is a constant, $1 \leq i \leq n$, then $\langle x \rangle^ \leq \lambda/\lambda_0$, a.s. (i.e., almost surely).*

(II) *If there are positive constants λ_0, T and $\lambda \geq 0$ such that*

$$\ln x(t) \geq \lambda t - \lambda_0 \int_0^t x(s) ds + \sum_{i=1}^n \beta_i B_i(t)$$

for $t \geq T$, where β_i is a constant, $1 \leq i \leq n$, then $\langle x \rangle_ \geq \lambda/\lambda_0$, a.s.*

Now, we are in the position to establish our threshold theorems.

Theorem 1 For species $x_1(x_2)$ modeled by (SM_1) , if $\langle C_0 \rangle_* > \mu(v)$, then $x_1(x_2)$ will go to extinction a.s., where

$$\mu := \begin{cases} (r_{10} - \alpha_1^2/2)/r_{11}, & \vartheta \leq 0, \\ \Delta_1/\Phi_1, & \vartheta \geq 0; \end{cases}$$

$$v := \begin{cases} \Delta_2/\Phi_2, & \vartheta \leq 0, \\ (r_{20} - \alpha_2^2/2)/r_{21}, & \vartheta \geq 0. \end{cases}$$

Theorem 2 If $\langle C_0 \rangle_* = \mu(v)$, then $x_1(x_2)$ will be nonpersistent in the mean a.s.

Theorem 3 If $\langle C_0 \rangle_* < \mu(v)$, then $x_1(x_2)$ will be weakly persistent in the mean a.s.

Remark 1 From Theorems 1–3, we know that $x_1(x_2)$ will go to extinction if and only if $\langle C_0 \rangle_* > \mu(v)$; $x_1(x_2)$ will be weakly persistent in the mean if and only if $\langle C_0 \rangle_* < \mu(v)$. At the same time, it is easy to see that the $\langle C_0 \rangle_* - \mu(\langle C_0 \rangle_* - v)$ is the threshold between weak persistence in the mean and extinction for $x_1(x_2)$. More precisely.

(a) If $\vartheta < 0$, then $v < \mu$. If $\langle C_0 \rangle_* > \mu$, then both x_1 and x_2 will go to extinction; If $v < \langle C_0 \rangle_* < \mu$, then x_1 will be weakly persistent in the mean and x_2 will go to extinction; If $\langle C_0 \rangle_* < v$, then both x_1 and x_2 will be weakly persistent in the mean. This shows that the ability of x_1 to resist toxicant is stronger than that of x_2 if $\vartheta < 0$. From a biological point of view, since $\vartheta < 0$, $r_{21}(r_{10} - \alpha_1^2/2) > r_{11}(r_{20} - \alpha_2^2/2)$, which implies that x_1 owns larger intrinsic growth rate and smaller dose-response parameter to the organismal toxicant concentration, then it is more possible for x_1 to survive.

(b) If $\vartheta = 0$, then $v = \mu$. If $\langle C_0 \rangle_* > \mu$, then both x_1 and x_2 will go to extinction; If $\langle C_0 \rangle_* < \mu$, then both x_1 and x_2 will be weakly persistent in the mean. This implies that the abilities of x_1 and x_2 to resist toxicant are equal if $\vartheta = 0$.

(c) If $\vartheta > 0$, then $v > \mu$. If $\langle C_0 \rangle_* > v$, then both x_1 and x_2 will go to extinction. If $\mu < \langle C_0 \rangle_* < v$, then x_1 will go to extinction and x_2 will be weakly persistent in the mean; If $\langle C_0 \rangle_* < \mu$, then both x_1 and x_2 will be weakly persistent in the mean. This means that the toxicant-resistant ability of x_2 is stronger than that of x_1 when $\vartheta > 0$. The biological interpretation is similar to the case $\vartheta < 0$.

In the above, we have established the threshold between weak persistence in the mean and extinction for species x_i modeled by (SM_1) , $i = 1, 2$. Now, we shall strengthen the conditions to give some other persistence results. Without loss of generality, in the following theorem, we suppose that $\langle C_0 \rangle_* < \mu$ and $\langle C_0 \rangle_* < v$.

Theorem 4 If $\lim_{t \rightarrow +\infty} \langle C_0(t) \rangle$ exists, and suppose that

$$\Delta_1 - \Phi_1 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle > 0, \quad \Delta_2 - \Phi_2 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle > 0,$$

then both species x_1 and x_2 represented by (SM_1) will be strongly persistent in the mean almost surely, and moreover

$$\lim_{t \rightarrow +\infty} \langle x_1(t) \rangle = \left(\Delta_1 - \Phi_1 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) / \Delta,$$

$$\lim_{t \rightarrow +\infty} \langle x_2(t) \rangle = \left(\Delta_2 - \Phi_2 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) / \Delta.$$

On the other hand, it is well known that in the study of deterministic population system, persistence which means that the population will survive forever, is one of important and interesting topics owing to its theoretical and practical significance. For stochastic system, we can study the stochastic persistence.

Theorem 5 *If $\min_{i=1,2} \{r_{i0} - r_{i1} \limsup_{t \rightarrow +\infty} C_0(t)\} > \max_{i=1,2} \{\alpha_i^2/2\}$, then model (SM_1) will be stochastically permanent.*

Remark 2 Liu and Ma (1991) studied the deterministic model (M) and obtained the following proposition.

Proposition 1 *Denote $\tilde{\vartheta} := r_{11}r_{20} - r_{21}r_{10}$, $\tilde{\Phi}_1 := a_{22}r_{11} - a_{12}r_{21}$, $\tilde{\Phi}_2 := a_{11}r_{21} - a_{21}r_{11}$, $\tilde{\Delta}_1 := a_{22}r_{10} - a_{12}r_{20} > 0$, $\tilde{\Delta}_2 := a_{11}r_{20} - a_{21}r_{10} > 0$, then for model M :*

(A) *If $\langle C_0 \rangle_* > \tilde{\mu}(\tilde{\nu})$, then x_1 (x_2) will go to extinction, where*

$$\tilde{\mu} := \begin{cases} r_{10}/r_{11}, & \tilde{\vartheta} \leq 0; \\ \tilde{\Delta}_1/\tilde{\Phi}_1, & \tilde{\vartheta} \geq 0. \end{cases}$$

$$\tilde{\nu} := \begin{cases} \tilde{\Delta}_2/\tilde{\Phi}_2, & \tilde{\vartheta} \leq 0; \\ r_{20}/r_{21}, & \tilde{\vartheta} \geq 0. \end{cases}$$

(B) *If $\langle C_0 \rangle_* = \tilde{\mu}(\tilde{\nu})$, then x_1 (x_2) will be nonpersistent in the mean;*

(C) *If $\langle C_0 \rangle_* < \tilde{\mu}(\tilde{\nu})$, then x_1 (x_2) will be weakly persistent in the mean.*

Comparing Theorems 1–3 with Proposition 1, we can find out an interesting fact: the stochastic disturbance of x_i is unfavorable for the survival of species x_i but is favorable for the survival of species x_j . On the other hand, strong persistence in the mean and stochastic persistence were not studied in Liu and Ma (1991).

Remark 3 Liu and Wang (2009) investigated a stochastic model of the form

$$dx(t) = x(t)[r_0 - r_1 C_0(t) - nx]dt + \alpha x dB_t,$$

where r_0, r_1, n , and α are positive constants. The authors claimed that:

- (i) If $r_0 - \alpha^2/2 < r_1 \langle C_0 \rangle_*$, then population $x(t)$ will go to extinction a.s.
- (ii) If $r_0 - \alpha^2/2 > r_1 \langle C_0 \rangle_*$, then population $x(t)$ will be weakly persistent in the mean a.s.
- (iii) If $r_0 - \alpha^2 > r_1 C_0^*$, then population $x(t)$ will be stochastically persistent.

It is easy to see that for model (SM_1) , if we only consider one species, then our Theorems 1 and 3 will become (i) and (ii), respectively. At the same time, our conditions of Theorem 5 are much weaker than (iii). Moreover, nonpersistence in the mean and strong persistence in the mean were not studied in Liu and Wang (2009).

4 The Survival Analysis for Model SM_2

In the previous section, we have carried out the survival analysis for model SM_1 , in this section we will turn to model SM_2 . To this end, let us prepare a lemma.

Lemma 5 *For model SM_2 , if $\sigma_{ii} > 0$ and $\sigma_{ij} \geq 0$ for $i, j = 1, 2$ and $i \neq j$, then with any given initial value $x(0) = (x_1(0), x_2(0)) \in R_+^2$, there is a unique solution $x(t) = (x_1(t), x_2(t))$ on $t \geq 0$ and the solution will remain in R_+^2 with probability one. Moreover, the solution $x(t)$ will be stochastically ultimately bounded, i.e., for any $\varepsilon \in (0, 1)$, there exists a positive constant $\chi = \chi(\varepsilon)$ such that for any initial value $x(0) = (x_1(0), x_2(0)) \in R_+^2$, the solution obeys $\liminf_{t \rightarrow +\infty} P\{|x(t)| \leq \chi\} \geq 1 - \varepsilon$.*

The proof of Lemma 5 is a slight modification of Mao et al. (2002, 2003, 2005). This lemma show that the presence of even a tiny amount of noise in removal process can not only suppress a potential population explosion but also make the population become stochastically ultimately bounded. Let us now study the persistence and extinction of the species. First let us consider the stochastic logistic population system

$$dx = x(r_0 - r_1 C_0(t) - ax) + \sigma x^2 dB(t), \quad x(0) = x_0 > 0, \tag{11}$$

where r_0, r_1, a and σ are all positive numbers. For system (11), we have

Theorem 6 *If $r_0 - r_1 \langle C_0 \rangle_* < 0$, then species $x(t)$ will go to extinction with probability one; If $r_0 - r_1 \langle C_0 \rangle_* > 0$, then species $x(t)$ will be weakly persistent almost surely.*

Remark 4 On the one hand, from Theorem 6, we know that $r_0 - r_1 \langle C_0 \rangle_*$ is the threshold between weak persistence and extinction for system (11). On the other hand, from Theorem 6, one can see that the stochastic noise on a has no impact on the persistence-extinction threshold of the species modeled by (11).

Now consider system (SM_2) .

Theorem 7 *Suppose that $\sigma_{ii} > 0$ and $\sigma_{ij} \geq 0$ for $i, j = 1, 2$ and $i \neq j$. For species x_i modeled by (SM_2) , if $\langle C_0 \rangle_* > r_{i0}/r_{i1}$, then x_i will go to extinction, $i = 1, 2$.*

Theorem 8 *Suppose that $\sigma_{ii} > 0$ and $\sigma_{ij} \geq 0$ for $i, j = 1, 2$ and $i \neq j$, if $\langle C_0 \rangle_* > r_{i0}/r_{i1}$ and $\langle C_0 \rangle_* < r_{j0}/r_{j1}$, $j \neq i$, $i, j = 1, 2$, then species x_i will go to extinction and x_j will be weakly persistent almost surely.*

5 Stochastic Competitive Exclusion Principle

In this section, we shall study the stochastic competitive model:

$$(SM_0) : \begin{cases} dx_1 = x_1[r_{10} - a_{11}x_1 - a_{12}x_2] dt + \alpha_1 x_1 dB_1(t), \\ dx_2 = x_2[r_{20} - a_{21}x_1 - a_{22}x_2] dt + \alpha_2 x_2 dB_2(t). \end{cases}$$

Model (SM_0) can be regarded as the model (SM_1) in the absence of toxicant, that is, let the parameters r_{11} and r_{21} in model (SM_1) be zero. The main aim of this section is to show that competitive coefficients play a very important role in determining persistence or extinction of populations in stochastic model. First, let us show the impact of stochastic perturbation on the persistence or extinction of species.

Theorem 9 For model SM_0 ,

(I) If

$$r_{i0} < \alpha_i^2/2,$$

then x_i will go to extinction almost surely, $i = 1, 2$.

(II) If

$$r_{i0} = \alpha_i^2/2,$$

then x_i will be nonpersistent in the mean almost surely, $i = 1, 2$.

(III) Suppose

$$[r_{10} - \alpha_1^2/2][r_{20} - \alpha_2^2/2] < 0.$$

If $r_{i0} - \alpha_i^2/2 < 0$, then x_i will go to extinction and x_j will be weakly persistent in the mean almost surely, moreover

$$\lim_{t \rightarrow +\infty} \langle x_j(t) \rangle = (r_{j0} - \alpha_j^2/2)/a_{jj}, \quad a.s., \quad i, j = 1, 2, i \neq j.$$

The following theorem tells us that competition coefficients play an important role in determining persistence or extinction of species for stochastic model, which can be regarded as a partial stochastic competitive exclusion principle.

Theorem 10 Suppose $r_{10} > \alpha_1^2/2, r_{20} > \alpha_2^2/2$.

(I) If $\Delta \geq 0$ (It is easy to see that $\Delta_1 < 0$ and $\Delta_2 < 0$ can not simultaneously hold in this case.)

(i) If $\Delta_i < 0$, then x_i will go to extinction almost surely, $i = 1, 2$;

(ii) If $\Delta_i > 0$, then x_i will be weakly persistent in the mean almost surely;

(iii) Moreover, if $\Delta_i > 0, \Delta_j < 0$, then

$$\lim_{t \rightarrow +\infty} \langle x_i(t) \rangle = (r_{i0} - \alpha_i^2/2)/a_{ii}, \quad a.s., \quad i, j = 1, 2, i \neq j.$$

Especially, if $\Delta_1 > 0$ and $\Delta_2 > 0$, then $\lim_{t \rightarrow +\infty} \langle x_1(t) \rangle = \Delta_1/\Delta, \lim_{t \rightarrow +\infty} \langle x_2(t) \rangle = \Delta_2/\Delta$ a.s.

(II) If $\Delta < 0$ (It is easy to see that $\Delta_1 > 0$ and $\Delta_2 > 0$ can not simultaneously hold in this case.)

(a) If $\Delta_i < 0, \Delta_j > 0$, then x_i will go to extinction and x_j will be weakly persistent in the mean almost surely, moreover,

$$\lim_{t \rightarrow +\infty} \langle x_j(t) \rangle = (r_{j0} - \alpha_j^2/2)/a_{jj} \quad \text{a.s., } i \neq j, i, j = 1, 2.$$

(b) If $\Delta_1 < 0, \Delta_2 < 0$, then x_1 and x_2 will not simultaneously go to extinction almost surely.

Remark 5 On the one hand, by virtue of Theorem 9, one can see that if $r_{i0} < \alpha_i^2/2$, then species x_i will go to extinction even without pollution, $i = 1, 2$. Consequently, Assumption 1 ($r_{10} > \alpha_1^2/2, r_{20} > \alpha_2^2/2$) is reasonable. Similarly, in view of Theorem 10, we know that, if $\Delta_i < 0$, then x_i in model SM_0 will go to extinction even without pollution when $\Delta \geq 0, i = 1, 2$, so Assumption 2 ($\Delta_1 > 0, \Delta_2 > 0$) is reasonable in most cases. From a biological point of view, $\Delta_1 > 0, \Delta_2 > 0$ means that the community consisting of two species is a stable biotic community before the introduction of a toxicant. By comparing the results of model (SM_1) with those of model (SM_0) , we can see that pollution is unfavorable for the survival of both species x_1 and x_2 , which accords with our expectation.

On the other hand, because of the limitations of experimental techniques, data is often described by temporal averages, so $\lim_{t \rightarrow +\infty} \langle x_i(t) \rangle$ will be useful.

Remark 6 Zhu and Yin (2009) studied the stochastic system in the Stratonovich sense

$$\begin{cases} dx_1 = x_1[r_{10}(\gamma(t)) - a_{11}x_1 - a_{12}x_2]dt + \alpha_1(\gamma(t))x_1 \circ dB_1(t), \\ dx_2 = x_2[r_{20}(\gamma(t)) - a_{21}x_1 - a_{22}x_2]dt + \alpha_2(\gamma(t))x_2 \circ dB_2(t) \end{cases} \quad (12)$$

where $\gamma(t)$ is a continuous-time Markov chain with a finite state space $\{1, 2, \dots, m\}$ and other parameters are as described above. Denote the solution of system (12) by $x(t) = (x_1(t), x_2(t))^T$, authors claimed that, if the Markov chain $\gamma(\cdot)$ is ergodic with stationary distribution $\pi = (\pi_1, \dots, \pi_m)$ and (12) satisfies the following conditions:

- (A) $a_{11} > 0, a_{12} > 0, a_{22} > 0, a_{21} > 0, \Delta = \det A = a_{11}a_{22} - a_{12}a_{21} > 0$.
- (B) For $\gamma = 1, 2, \dots, m, a_{22}r_{10}(\gamma) > a_{12}r_{20}(\gamma)$ and $a_{11}r_{20}(\gamma) > a_{21}r_{10}(\gamma)$.
- (C) $\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_i(s) ds$ exists a.s. for $i = 1, 2$.

Then

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_1(s) ds &= (a_{22}r_1 - a_{12}r_2)/\Delta, \\ \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t x_2(s) ds &= (a_{11}r_2 - a_{21}r_1)/\Delta, \end{aligned}$$

where $r_i = \sum_{\gamma=1}^m \pi_\gamma r_{i0}(\gamma), i = 1, 2$. From (iii) in Theorem 10, we can see that for model (SM_0) , the above conclusions are valid even without condition (C).

6 Numerical Simulations

In this section, we are going to use the Milstein method mentioned in Higham (2001) to substantiate the analytical findings.

For model (SM_1), consider the discretization equation:

$$\begin{cases} x_{k+1} = x_k + x_k[r_{10} - r_{11}C_0(k\Delta t) - a_{11}x_k - a_{12}y_k]\Delta t \\ \quad + \alpha_1 x_k \sqrt{\Delta t} \xi_k + \frac{\alpha_1^2}{2} x_k^2 (\xi_k^2 \Delta t - \Delta t), \\ y_{k+1} = y_k + y_k[r_{20} - r_{21}C_0(k\Delta t) - a_{21}x_k - a_{22}y_k]\Delta t \\ \quad + \alpha_2 y_k \sqrt{\Delta t} \eta_k + \frac{\alpha_2^2}{2} y_k^2 (\eta_k^2 \Delta t - \Delta t), \end{cases}$$

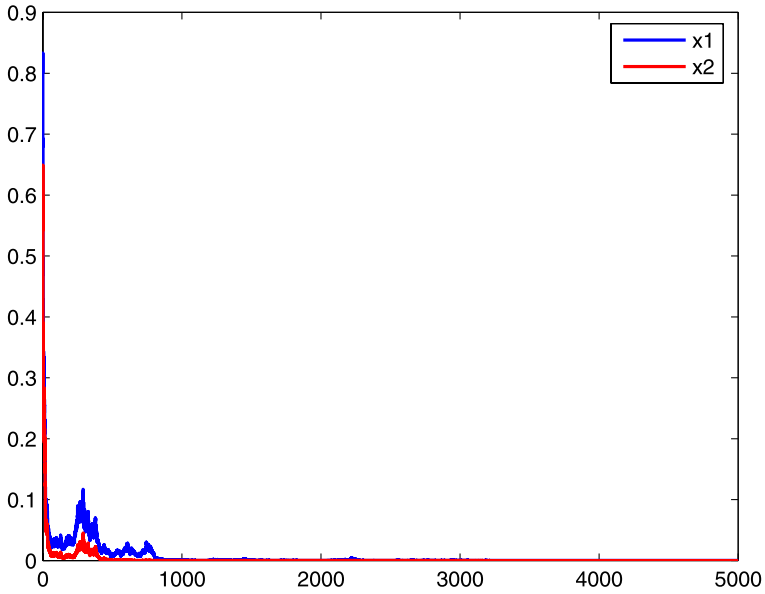
where ξ_k and η_k , $k = 1, 2, \dots, n$, are the Gaussian random variables $N(0, 1)$. It is useful to point out that the numerical method for model (SM_0) can be obtained by setting $r_{11} = r_{21} = 0$.

For model (SM_2), in order to avoid complicated notation, let $B_{i1}(t) = B_{i2}(t) = B_i(t)$ for $i = 1, 2$. Consider the discretization equation:

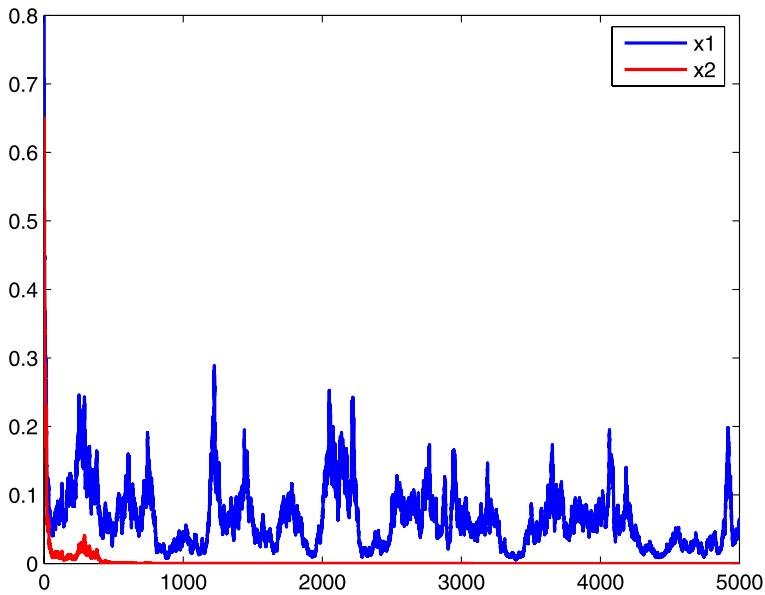
$$\begin{aligned} x_{k+1} &= x_k + x_k[r_{10} - r_{11}C_0(k\Delta t) - a_{11}x_k - a_{12}y_k]\Delta t \\ &\quad + \sigma_{11}x_k^2\sqrt{\Delta t}\xi_k + \sigma_{12}x_k y_k \sqrt{\Delta t}\xi_k \\ &\quad + 0.5[\sigma_{11}^2x_k^4 + \sigma_{12}^2x_k^2y_k^2 + 2\sigma_{11}\sigma_{12}x_k^3y_k](\xi_k^2\Delta t - \Delta t), \\ y_{k+1} &= y_k + y_k[r_{20} - r_{21}C_0(k\Delta t) - a_{21}x_k - a_{22}y_k]\Delta t \\ &\quad + \sigma_{21}y_k^2\sqrt{\Delta t}\eta_k + \sigma_{22}x_k y_k \sqrt{\Delta t}\eta_k \\ &\quad + 0.5[\sigma_{21}^2x_k^2y_k^2 + \sigma_{22}^2y_k^4 + 2\sigma_{21}\sigma_{22}x_k y_k^3](\eta_k^2\Delta t - \Delta t). \end{aligned}$$

Making use of the numerical simulation method given above and the help of Matlab software, choosing suitable parameters, we get simulations of the stochastic systems SM_1 , SM_2 , and SM_0 (in the following simulations, the step size $\Delta t = 0.001$).

As pointed out in Sect. 2, persistence or extinction of the species depends on the values of ν , μ , and $\langle C_0 \rangle_*$. In Fig. 1, we choose $r_{10} = 0.055$, $r_{20} = 0.05$, $r_{11} = 0.51$, $r_{21} = 0.5$, $\alpha_1^2 = \alpha_2^2 = 0.01$, $a_{11} = a_{22} = 0.21$ and $a_{21} = a_{12} = 0.1$ (from those conditions, it is easy to see that $\vartheta < 0$). The only difference between conditions of Fig. 1(A), Fig. 1(B), and Fig. 1(C) is that the representation of $C_0(t)$ is different. In Fig. 1(A), we choose $C_0(t) = 0.1 + 0.01 \sin t$. Then it is easy to obtain $\nu < \mu < \langle C_0 \rangle_*$. By Theorems 1–3, both x_1 and x_2 will go to extinction. Figure 1(A) confirms these. In Fig. 1(B), we choose $C_0(t) = 0.09 + 0.01 \sin t$. Then the conditions obey $\nu < \langle C_0 \rangle_* < \mu$. Making use of Theorems 1–3 again gives that x_1 will be weakly persistent in the mean and x_2 will go to extinction. These can be seen from Fig. 1(B). In Fig. 1(C), we choose $C_0(t) = 0.0001 + 0.00001 \sin t$. That is to say, the conditions satisfy $\langle C_0 \rangle_* < \nu < \mu$. In view of Theorems 1–3, we can get that both x_1 and x_2 will be weak persistent in the mean. See Fig. 1(C). When $\vartheta > 0$, some symmetrical figures can be obtained similarly.



(A)



(B)

Fig. 1 (Color online) Solutions of system (SM_1) for $r_{10} = 0.055, r_{20} = 0.05, r_{11} = 0.51, r_{21} = 0.5, \alpha_1^2 = \alpha_2^2 = 0.01, a_{11} = a_{22} = 0.21, a_{21} = a_{12} = 0.1, x_1(0) = 0.8, x_2(0) = 0.6, C_0(0) = 0$, step size $\Delta t = 0.001$. The horizontal axis this and the following figures represent the time t . **(A)** Is with $C_0(t) = 0.1 + 0.01 \sin t$; **(B)** Is with $C_0(t) = 0.09 + 0.01 \sin t$; **(C)** Is with $C_0(t) = 0.0001 + 0.00001 \sin t$

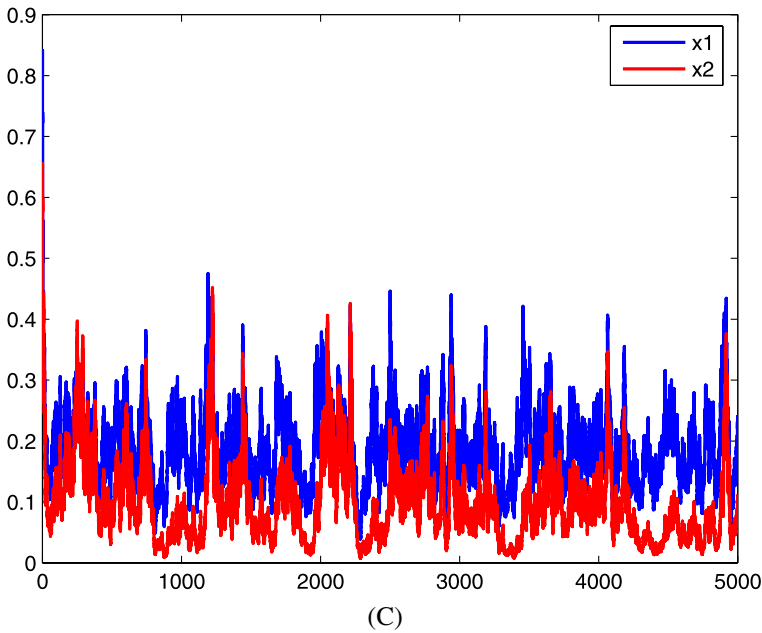


Fig. 1 (Continued)

In Fig. 2, we choose $r_{10} = 0.055, r_{20} = 0.05, \alpha_1^2 = \alpha_2^2 = 0.01, a_{11} = a_{22} = 0.21, a_{12} = a_{21} = 0.1, r_{11} = 0.51, r_{21} = 0.5$ and $C_0(t) = 0.05 + 0.001 \sin t$, then Theorem 4 indicates that both species x_1 and x_2 will be strongly persistent in the mean, and moreover,

$$\lim_{t \rightarrow +\infty} \langle x_1(t) \rangle = 0.0922, \quad \lim_{t \rightarrow +\infty} \langle x_2(t) \rangle = 0.0513.$$

Figure 2 confirms these.

In Fig. 3, we choose $r_{10} = 0.55, r_{20} = 0.48, \alpha_1^2 = 0.2, \alpha_2^2 = 0.1, a_{11} = a_{22} = 0.21, a_{12} = a_{21} = 0.1, r_{11} = r_{21} = 1, C_0(t) = 0.35 + 0.02 \sin t$, then Theorem 5 means that model (SM_1) will be stochastically permanent. See Fig. 3.

Let us now move to model (SM_2) .

In Fig. 4, we choose $r_{10} = 0.5, r_{20} = 0.4, \sigma_{ij}^2 = 2, i, j = 1, 2, a_{11} = a_{22} = 0.21, a_{21} = a_{12} = 0.3, C_0(t) = 0.55 + 0.01 \sin t$. The only difference between conditions of Fig. 4(A), Fig. 4(B), and Fig. 4(C) is that the values of r_{11} and r_{21} are different. In Fig. 4(A), we choose $r_{11} = r_{21} = 1$. Then it is easy to obtain that $r_{20}/r_{21} < r_{10}/r_{11} < \langle C_0 \rangle_*$. In view of Theorem 7, both x_1 and x_2 will go to extinction. Figure 4(A) confirms this. In Fig. 4(B), we choose $r_{11} = r_{21} = 0.8$. Then the conditions obey $r_{20}/r_{21} < \langle C_0 \rangle_* < r_{10}/r_{11}$. Making use of Theorem 8 gives that x_1 will be weakly persistent and x_2 will go to extinction. These can be seen from Fig. 4(B). In Fig. 4(C), we choose $r_{11} = 1, r_{21} = 0.7$. That is to say, the conditions satisfy $r_{10}/r_{11} < \langle C_0 \rangle_* < r_{20}/r_{21}$. By virtue of Theorem 8, we can get that x_1 will go to extinction and x_2 will be weakly persistent. See Fig. 4(C).

Now let us turn to the stochastic competitive exclusion principle.

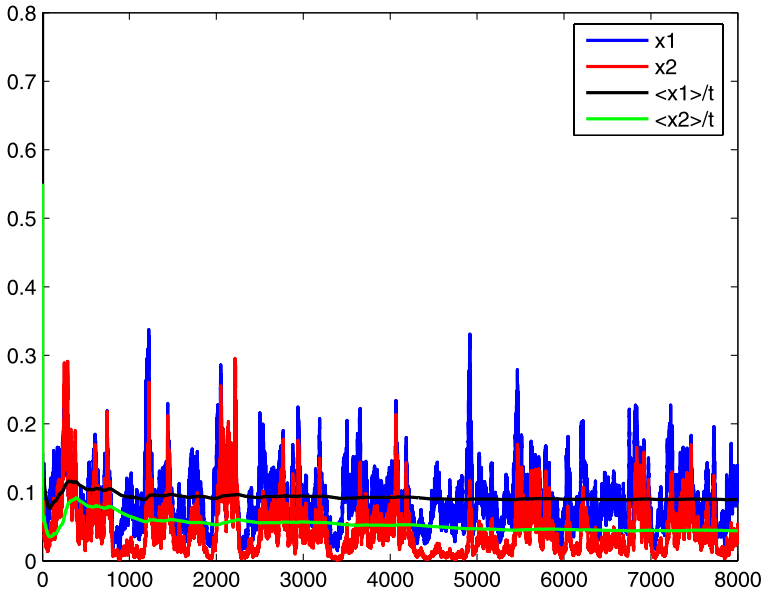


Fig. 2 (Color online) Solutions of system (SM_1) for $r_{10} = 0.055, r_{20} = 0.05, \alpha_1^2 = \alpha_2^2 = 0.01, a_{11} = a_{22} = 0.21, a_{12} = a_{21} = 0.1, r_{11} = 0.51, r_{21} = 0.5, C_0(t) = 0.05 + 0.001 \sin t, x_1(0) = 0.8, x_2(0) = 0.6, C_0(0) = 0$ and step size $\Delta t = 0.001$. This figure indicates that both species x_1 and x_2 will be strongly persistent in the mean, and moreover $\lim_{t \rightarrow +\infty} \langle x_1(t) \rangle = 0.0922, \lim_{t \rightarrow +\infty} \langle x_2(t) \rangle = 0.0513$

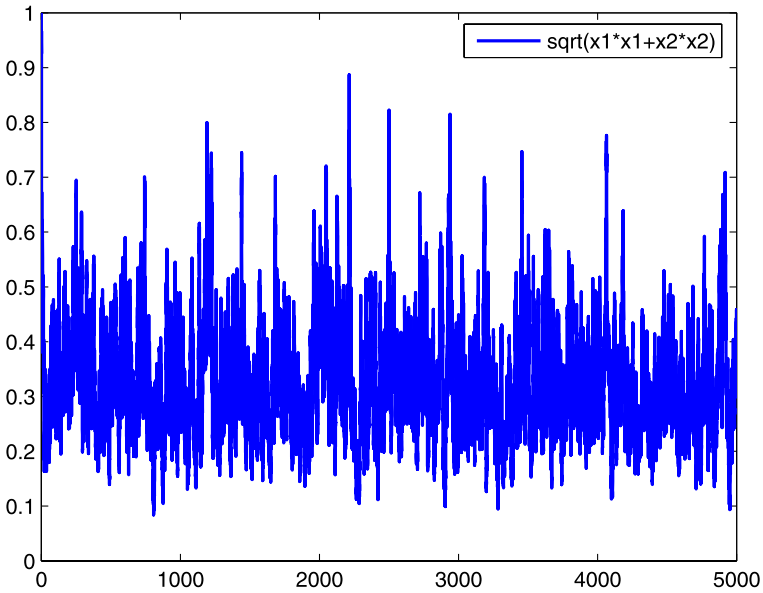


Fig. 3 (Color online) Solutions of system (SM_1) for $r_{10} = 0.55, r_{20} = 0.48, \alpha_1^2 = 0.2, \alpha_2^2 = 0.1, a_{11} = a_{22} = 0.21, a_{12} = a_{21} = 0.1, r_{11} = r_{21} = 1, C_0(t) = 0.35 + 0.02 \sin t, x_1(0) = 0.8, x_2(0) = 0.6, C_0(0) = 0$ and step size $\Delta t = 0.001$. This figure means that model (SM_1) will be stochastically permanent

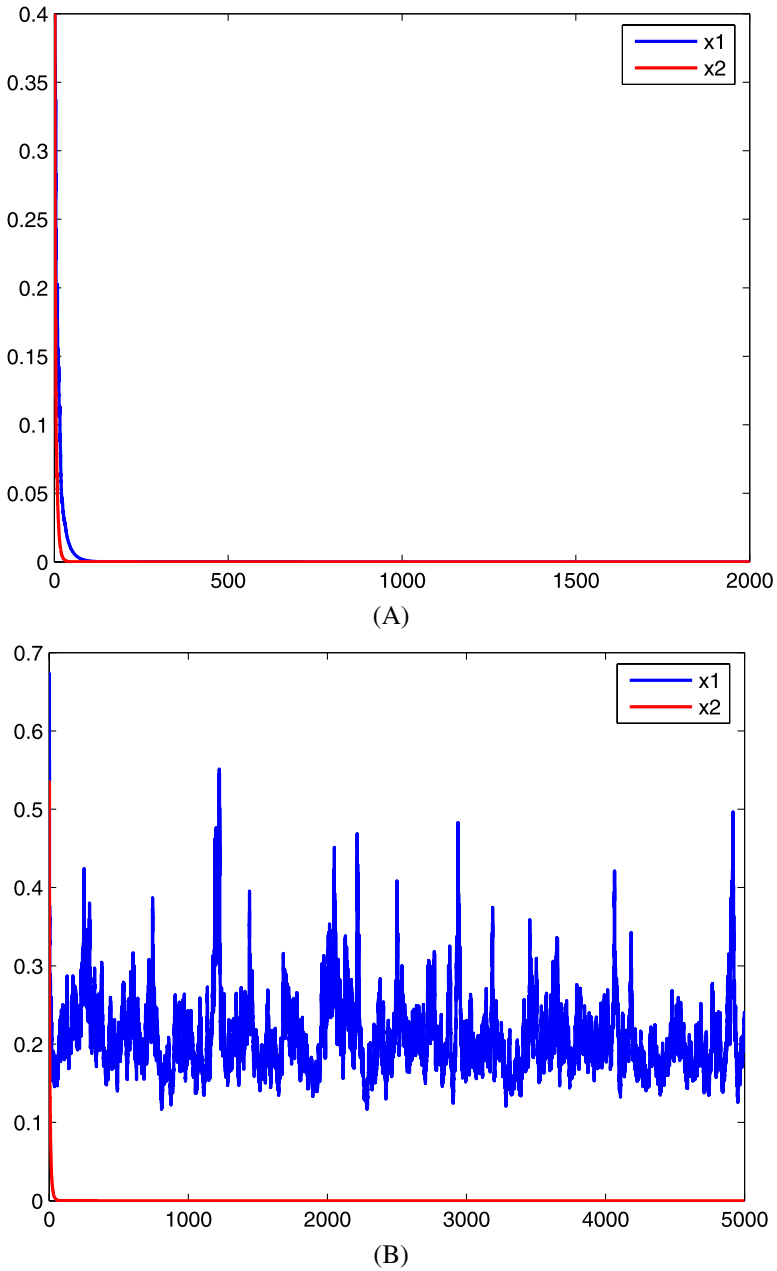


Fig. 4 (Color online) Solutions of system (SM_2) for $r_{10} = 0.5$, $r_{20} = 0.4$, $\sigma_{ij}^2 = 2$, $i, j = 1, 2$, $a_{11} = a_{22} = 0.21$, $a_{21} = a_{12} = 0.3$, $C_0(t) = 0.55 + 0.01 \sin t$, $x_1(0) = 0.6$, $x_2(0) = 0.5$, $C_0(0) = 0$ and step size $\Delta t = 0.001$. **(A)** Is with $r_{11} = r_{21} = 1$. **(B)** Is with $r_{11} = r_{21} = 0.8$. **(C)** Is with $r_{11} = 1$, $r_{21} = 0.7$

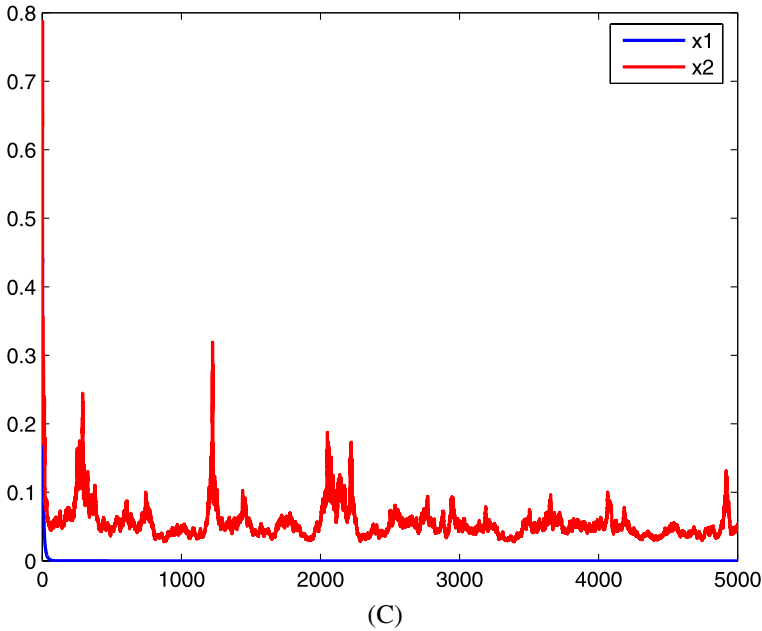


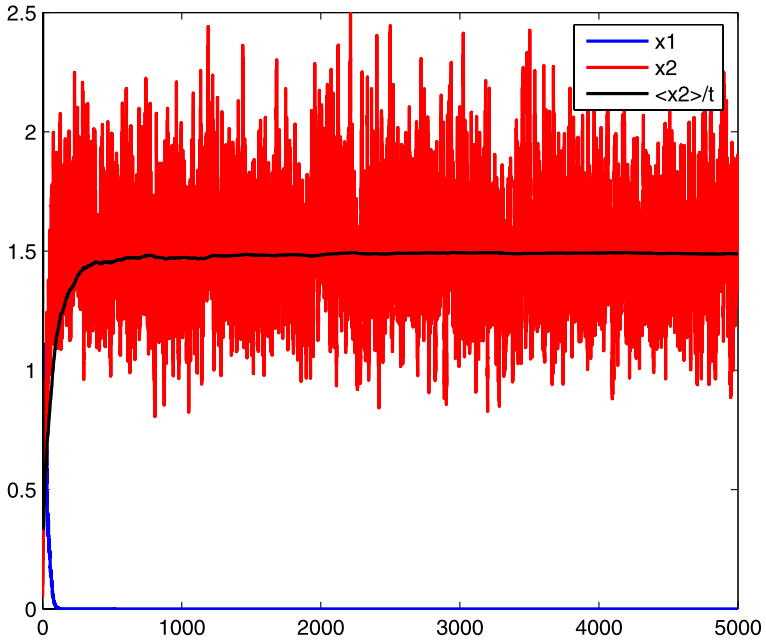
Fig. 4 (Continued)

In Fig. 5, we choose $r_{10} = 0.55, r_{20} = 0.5, \alpha_1^2 = \alpha_2^2 = 0.1, a_{11} = a_{22} = 0.3$. The only difference between conditions of Fig. 5(A), Fig. 5(B), Fig. 5(C) is that the values of a_{12} and a_{21} are different. In Fig. 5(A), we choose $a_{21} = 0.23, a_{12} = 0.38$, then $\Delta > 0, \Delta_1 < 0, \Delta_2 > 0$. In view of part (I) of Theorem 10, one can see that x_1 will go to extinction and x_2 will be weakly persistent in the mean as well as $\lim_{t \rightarrow +\infty} \langle x_1(t) \rangle = 1.5$. Figure 5(A) confirms these. In Fig. 5(B), we choose $a_{21} = 0.28, a_{12} = 0.32$, then $\Delta > 0, \Delta_1 > 0, \Delta_2 < 0$. Applying part (I) of Theorem 10 gives that x_1 will be weakly persistent in the mean and $\lim_{t \rightarrow +\infty} \langle x_1(t) \rangle = 1.667$ while x_2 will go to extinction. See Fig. 5(B). In Fig. 5(C), we choose $a_{21} = 0.23, a_{12} = 0.32$, then $\Delta > 0, \Delta_1 > 0, \Delta_2 > 0$. Using part (I) of Theorem 10 implies that both x_1 and x_2 will be weakly persistent in the mean and

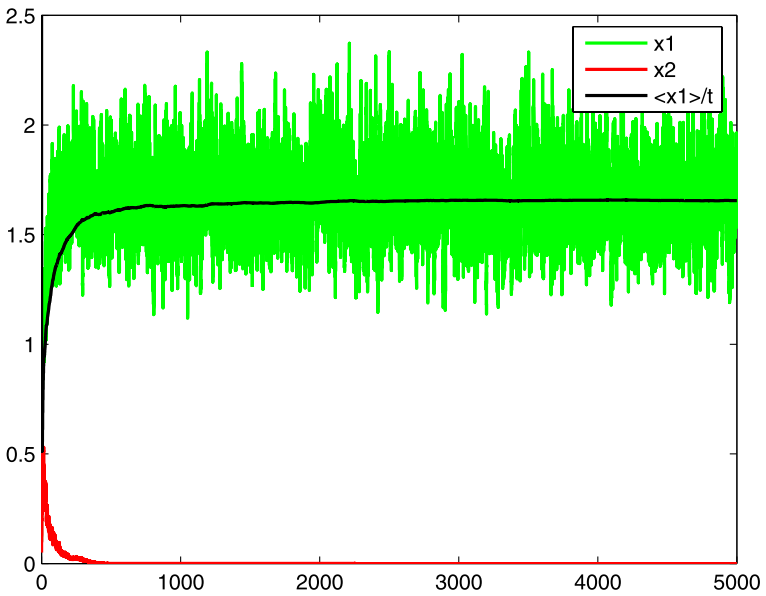
$$\lim_{t \rightarrow +\infty} \langle x_2(t) \rangle = 1.2195, \quad \lim_{t \rightarrow +\infty} \langle x_1(t) \rangle = 0.3659.$$

Figure 5(C) confirms these.

In Fig. 6, we choose $r_{10} = 0.55, r_{20} = 0.5, \alpha_1^2 = \alpha_2^2 = 0.1, a_{11} = 0.5, a_{22} = 0.7$. The only difference between conditions of Fig. 6(A), Fig. 6(B), and Fig. 6(C) is that the values of a_{12} and a_{21} are different. In Fig. 6(A), we choose $a_{21} = a_{12} = 0.6$, then $\Delta < 0, \Delta_1 > 0, \Delta_2 < 0$. Applying part (II) of Theorem 10 gives that x_1 will be weakly persistent in the mean and $\lim_{t \rightarrow +\infty} \langle x_1(t) \rangle = 1$ while x_2 will go to extinction. See Fig. 6(A). In Fig. 6(B), we choose $a_{21} = 0.4, a_{12} = 0.9$, then $\Delta < 0, \Delta_1 < 0, \Delta_2 > 0$. Making use of part (II) of Theorem 10 leads to that x_1 will go to extinction while x_2 will be weakly persistent in the mean and $\lim_{t \rightarrow +\infty} \langle x_2(t) \rangle = 0.643$. See Fig. 6(B). In Fig. 6(C), we choose $a_{21} = a_{12} = 0.9$, then $\Delta < 0, \Delta_1 < 0, \Delta_2 < 0$. In



(A)



(B)

Fig. 5 (Color online) Solutions of system (SM_0) for $r_{10} = 0.55$, $r_{20} = 0.5$, $\alpha_1^2 = \alpha_2^2 = 0.1$, $a_{11} = a_{22} = 0.3$, $x_1(0) = x_2(0) = 0.1$, $C_0(0) = 0$ and step size $\Delta t = 0.001$. **(A)** Is with $a_{21} = 0.23$, $a_{12} = 0.38$. **(B)** Is with $a_{21} = 0.28$, $a_{12} = 0.32$. **(C)** Is with $a_{21} = 0.23$, $a_{12} = 0.32$

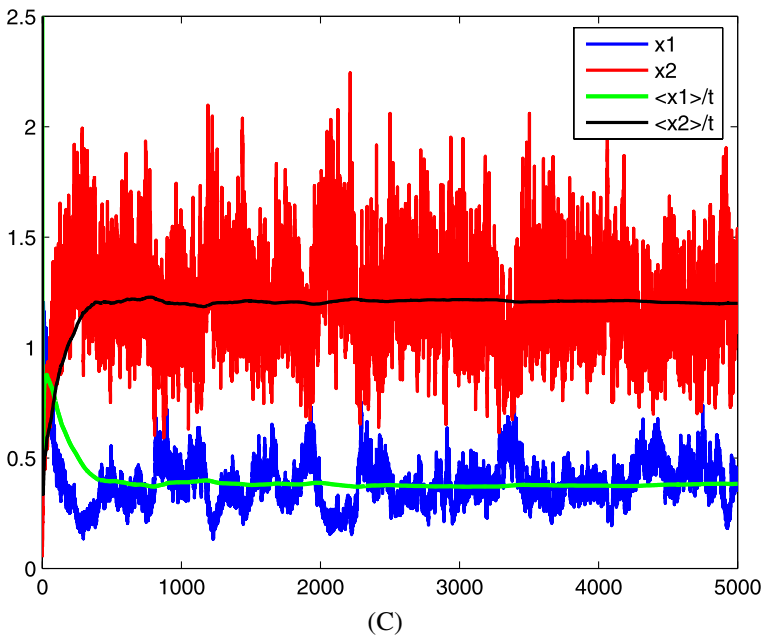


Fig. 5 (Continued)

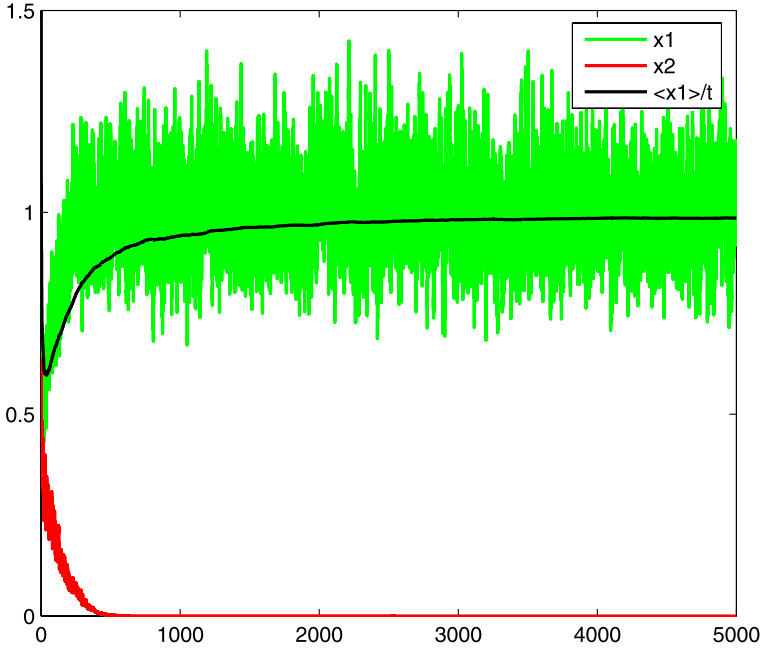
view of part (II) of Theorem 10, one can derive that x_1 and x_2 will not simultaneously go to extinction. Figure 6(C) confirms this. From Fig. 5 and Fig. 6, we can see that a_{ij} play a very important role in determining the persistence or extinction of species.

7 Conclusions and Future Directions

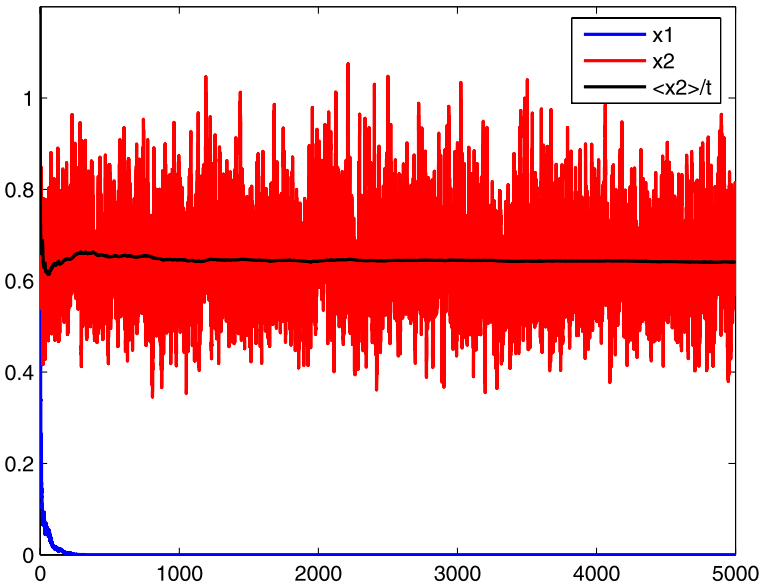
Stochastic competitive models of toxicant-population interaction and stochastic competitive exclusion principle were proposed and investigated. Our results have obvious biological significance.

Firstly, when considering the environmental pollution which may be related to the resources and health problems, it is an interesting topic to discuss the effect of different types of noise. In this study, we find that the stochastic perturbation on the production rate plays a very important role in determining persistence or extinction of the species. At the same time, Theorem 6 shows that the stochastic perturbation on the removal process has no impact on the persistence-extinction threshold of the species for system (11). Lemma 5 indicates that when we introduce the noise into the removal processes, the presence of even a tiny amount of noise can not only suppress a potential population explosion but also make the population become stochastically ultimately bounded.

Secondly, comparing our results for model (SM_1) with Proposition 1, it is easy to find out an interesting result: the stochastic disturbance of x_i is unfavorable for the survival of species x_i and is favorable for the survival of species x_j , $j \neq i$, $i, j = 1, 2$, which are neglected by all relevant known references. From a biological point of



(A)



(B)

Fig. 6 (Color online) Solutions of system (SM_0) for $r_{10} = 0.55, r_{20} = 0.5, \alpha_1^2 = \alpha_2^2 = 0.1, a_{11} = 0.5, a_{22} = 0.7, x_1(0) = x_2(0) = 0.6, C_0(0) = 0$ and step size $\Delta t = 0.001$. **(A)** Is with $a_{21} = a_{12} = 0.6$. **(B)** Is with $a_{21} = 0.4, a_{12} = 0.9$. **(C)** Is with $a_{21} = a_{12} = 0.9$

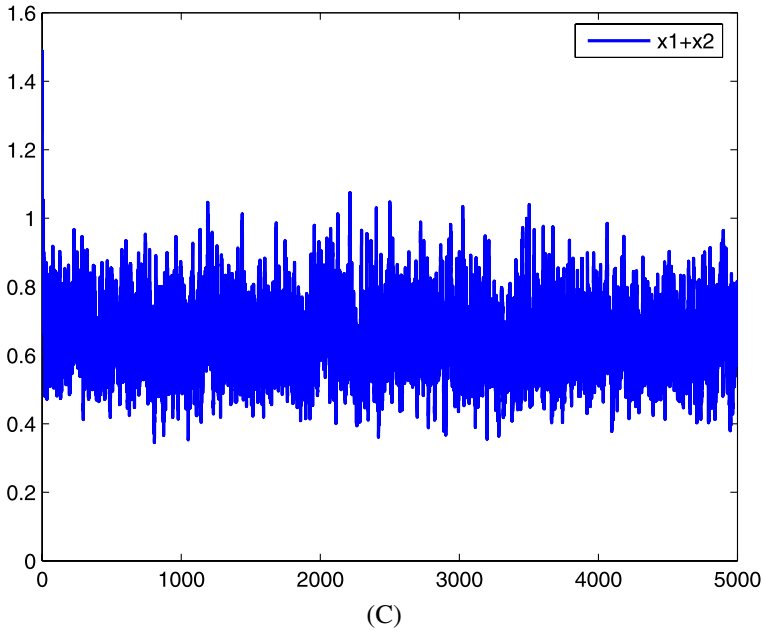


Fig. 6 (Continued)

view, this is understandable. Since model (SM_1) describes a two-species dynamical system, in which each individual competes with other for the limited resources in a polluted environment, and since the stochastic disturbance of x_i is unfavorable for the survival of x_i , then x_j will get more resources, in other words, stochastic disturbance of population x_i is favorable for the survival of x_j .

Thirdly, for model (SM_0) , from Theorem 10, we can find out that a_{ij} ($i, j = 1, 2$) play a very important role in determining whether x_i will survive or not. Theorems 9 and 10 tell us that both x_1 and x_2 will go to extinction if and only if $r_{10} < \alpha_1^2/2$ and $r_{20} < \alpha_2^2/2$. When $r_{10} \geq \alpha_1^2/2$ and $r_{20} \geq \alpha_2^2/2$, at least one species will survive, which species survive is highly dependent on the values of a_{12} and a_{21} . When $\Delta \geq 0$, if one of $a_{ij}, i \neq j$ is small and another is large such that $\Delta_1 \Delta_2 < 0$, for example, a_{12} is small and a_{21} is large such that $\Delta_1 > 0$ and $\Delta_2 < 0$, then x_1 will survive while x_2 will go to extinction; If “competition is always favored” (i.e., both $a_{ij}, i \neq j$ are sufficiently small such that $\Delta_1 > 0$ and $\Delta_2 > 0$), then x_1 and x_2 will coexist. When $\Delta < 0$, if $\Delta_i > 0$, then x_i will survive while x_j will go to extinction. In a word, the interaction rates a_{ij} play a very important role in determining persistence or extinction of the species for stochastic model.

Fourthly, it is well known that the threshold is very important for assessing the risk of extinction of species in systems exposed to toxicant. Then the threshold between weak persistence in the mean and extinction obtained for model (SM_1) as well as the threshold between weak persistence and extinction derived for model (11) are useful. For example, from the results for (SM_1) , we find that the extinction or persistence of species is dependent on the intensity of the white noises α_1^2 and α_2^2 , the mean stress measure in organisms $\langle C_0(t) \rangle$, the population intrinsic growth rates r_{10} and r_{20} , stress

response rates r_{i1} ($i = 1, 2$) and interaction rates a_{ij} ($i, j = 1, 2$). At the same time, from the results for system (11), we can observe that the extinction or persistence of species is only dependent on the mean stress measure in organisms $\langle C_0(t) \rangle$, the population intrinsic growth rate r_0 , stress response rates r_1 but is independent of the power of the white noise σ^2 and the initial population size x_0 as well as a .

Finally, a traditional approach to conserve a population x_i is to cut down the pollutant output $u(t)$, this study tells us that we have other ways to conserve a population: we can choose to reduce the intensity of the white noise α_i on the production rates or to enhance the intensity α_j , $i \neq j$; We can also choose to change the values of a_{ij} .

The conditions of Theorems 1–5 have some interesting biological meanings. Theorem 5, which measures the population size $x_i(t)$ directly, means that the whole population size will neither too small nor too large with large probability if the time is sufficiently large. Theorems 1–4 are based on the temporal average of the population size $x_i(t)$. Theorem 1 indicates that the population will go to extinction which is the worst case. Theorem 2 means that the population is bare. Theorems 3–4 admit the case that the population size is closed to zero even the time is sufficiently large. That is to say, the survival of species could be dangerous in reality. Thus, the survival condition of Theorem 5 are better than that of Theorems 1–4 in many cases. These are also the reasons why the temporal average of $C_0(t)$ is used in Theorems 1–4 whereas $\limsup_{t \rightarrow +\infty} C_0(t)$ is used in Theorem 5. At the same time, $\liminf_{t \rightarrow +\infty} \langle x(t) \rangle = 0$ is allowed in Theorem 3 but is not allowed in Theorem 4. That is to say the survival of Theorem 4 is better than Theorem 3, then we used $\liminf_{t \rightarrow +\infty} \langle C_0 \rangle$ in Theorem 3 and used $\lim_{t \rightarrow +\infty} \langle C_0 \rangle$ in Theorem 4.

Some interesting questions deserve further investigation. In Theorems 7 and 8, we obtained some sufficient conditions for extinction and weak persistence of model (SM_2). Since the persistence-extinction threshold is very useful in practice, then it is an important and interesting topic to study persistence-extinction threshold for model (SM_2). In fact, we also attempts to study those problems. Unfortunately, there are some technical obstacles that can not be overcome at present stage. On the other hand, one may propose some realistic but complex models. For example, when the population size is taken into account, a Langevin approach is more appropriate (we refer the reader to some literatures that derive Langevin equations from the underlying stochastic processes; see Alonso et al. (2007) for a SIR infectious disease model, see Cattiaux and Méléard (2009) for a Lotka–Volterra system, and see McKane and Newman (2005) for a predator-prey system. There the reader will find also methods to exactly simulate these stochastic systems; For more details of Langevin approach, see, e.g. Gillespie (2000)). In other words, one may consider the following model:

$$(SM_3) \quad \begin{cases} dx_1 = x_1[r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2]dt + \sqrt{\sigma_1 x_1} dB_1(t), \\ dx_2 = x_2[r_{20} - r_{21}C_0(t) - a_{21}x_1 - a_{22}x_2]dt + \sqrt{\sigma_2 x_2} dB_2(t). \end{cases}$$

For model (SM_3), similar to the proof given in Cattiaux and Méléard (2009), one can get that if the “balance condition $a_{12}\sigma_2 = a_{21}\sigma_1$ ” holds, then both populations x_1 and x_2 will “hit zero” in *finite* time with probability one (i.e., all species will die out in reality with probability one). Another example deserved further investigation is to incorporate Ornstein–Uhlenbeck process or the colored noise, such as

Markov chain, into the system. The motivation is that the population may suffer sudden-environmental changes, e.g., rain falls and changes in nutrition or food resources, etc.; frequently, the switching among different environments is memoryless and the waiting time for the next switch is exponentially distributed, then the sudden-environmental changes can be modeled by a continuous-time Markov chain (see, e.g. Luo and Mao 2007; Mao and Yuan 2006; Zhu and Yin 2009). It is also interesting to study stochastic predator-prey system and stochastic cooperative model in a polluted environment.

Since population does not exist alone in nature and many population models are inevitably affected by stochastic noise, it is more meaningful to study the persistence-extinction threshold of each species in systems of two or more interacting species subjected to toxicant. At the same time, because of the importance of competitive exclusion principle, the study of stochastic competitive exclusion principle is essential. The studies of those stochastic models are important and useful for better understanding of the real world. Owing to its theoretical and practical significance, stochastic population dynamics under the influence of pollution have deserved a lot of attention, but mainly in one dimension. The present paper is the first attempt, to our knowledge, of such a study in a multidimensional setting.

Acknowledgements The authors thank the referees for their very important and helpful comments and suggestions. The authors also thank the National Natural Science Foundation of P.R. China (No. 10701020 and No. 10971022).

Appendix

Proof of Lemma 3 Suppose that $y_i(t)$ is the solution of equation:

$$dy_i = y_i[r_{i0} - a_{ii}y_i]dt + \alpha_i y_i dB_i(t), \quad i = 1, 2. \tag{A.1}$$

Making use of comparison theorem for stochastic equations (Ikeda and Watanabe 1977) gives $x_i(t) \leq y_i(t)$. At the same time, (A.1) has explicit solution of the form:

$$y(t) = \frac{\exp[r_{i0}t - \alpha_i^2 t/2 + \alpha_i B_i(t)]}{1/y_i(0) + a_{ii} \int_0^t \exp[r_{i0}s - \alpha_i^2 s/2 + \alpha_i B_i(s)] ds}. \tag{A.2}$$

If $r_{i0} < \alpha_i^2/2$, then it follows from (A.2) that

$$x_i(t) \leq y_i(t) \leq y_i(0) \exp\{-t[\alpha_i^2/2 - r_{i0} - \alpha_i B_i(t)/t]\}.$$

Applying

$$\lim_{t \rightarrow +\infty} B(t)/t = 0 \tag{A.3}$$

yields $y_i^* \leq 0$ a.s. In other words, $\lim_{t \rightarrow +\infty} x_i(t) = 0$ a.s. Therefore,

$$\left[\frac{\ln\{x_i(t)\}}{t} \right]^* \leq 0 \quad a.s. \tag{A.4}$$

If $r_{i0} \geq \alpha_i^2/2$, then applying mean value theorem for integral gives

$$\begin{aligned} y_i(t) &\leq \frac{1}{a_{ii} \int_0^t \exp\{(r_{i0} - \alpha_i^2/2)(s - t) + \alpha_i(B_i(s) - B_i(t))\} ds} \\ &= \frac{1}{a_{ii} \exp(\alpha_i B_i(\tau) - \alpha_i B_i(t)) \int_0^t \exp\{(r_{i0} - \alpha_i^2/2)(s - t)\} ds} \\ &= \frac{r_{i0} - \alpha_i^2/2}{a_{ii}} \frac{1}{\exp(\alpha_i B_i(\tau) - \alpha_i B_i(t)) \{1 - \exp[-(r_{i0} - \alpha_i^2/2)t]\}}, \end{aligned}$$

where $\tau \in [0, t]$. Using (A.3) again leads to that $[\ln\{y_i(t)\}/t]^* \leq 0$ a.s., thus

$$[\ln\{x_i(t)\}/t]^* \leq 0 \quad \text{a.s.} \quad \square$$

Proof of Lemma 4 (I) Suppose that $\lambda > 0$. Denote $g(t) = \int_0^t x(s) ds$ for all $t \geq T$, then $dg/dt = x(t) \geq 0$ for all $t \geq T$. Substituting g into (10) one can derive that

$$\ln \frac{dg}{dt} \leq \lambda t - \lambda_0 g + \sum_{i=1}^n \beta_i B_i(t), \quad t \geq T$$

which is to say

$$\exp\{\lambda_0 g\} \frac{dg}{dt} \leq \exp\left\{ \sum_{i=1}^n \beta_i B_i(t) \right\} \exp\{\lambda t\}, \quad t \geq T.$$

Integrating the above inequality from T to t and then using mean value theorem for integral results in

$$\begin{aligned} \lambda_0^{-1} [\exp\{\lambda_0 g(t)\} - \exp\{\lambda_0 g(T)\}] &\leq \int_T^t \exp\left\{ \sum_{i=1}^n \beta_i B_i(s) \right\} \exp\{\lambda s\} ds \\ &= \lambda^{-1} \exp\left\{ \sum_{i=1}^n \beta_i B_i(\tau) \right\} [\exp\{\lambda t\} - \exp\{\lambda T\}], \end{aligned}$$

where $\tau \in [T, t]$. Rewriting the above inequality, one can see that

$$\begin{aligned} \exp\{\lambda_0 g(t)\} &\leq \exp\{\lambda_0 g(T)\} + \lambda_0 \lambda^{-1} \exp\left\{ \sum_{i=1}^n \beta_i B_i(\tau) \right\} \exp\{\lambda t\} \\ &\quad - \lambda_0 \lambda^{-1} \exp\left\{ \sum_{i=1}^n \beta_i B_i(\tau) \right\} \exp\{\lambda T\}. \end{aligned}$$

Taking logarithm of both sides yields that

$$g \leq \lambda_0^{-1} \ln \left\{ \left[\lambda_0 \lambda^{-1} \exp \left\{ \sum_{i=1}^n \beta_i B_i(\tau) \right\} \exp\{\lambda t\} \right] + \exp\{\lambda_0 g(T)\} - \lambda_0 \lambda^{-1} \exp \left\{ \sum_{i=1}^n \beta_i B_i(\tau) \right\} \exp\{\lambda T\} \right\},$$

which means that

$$\begin{aligned} \langle x(t) \rangle &= t^{-1} \int_0^t x(s) ds \\ &\leq \lambda_0^{-1} \ln \left\{ \exp \left(\sum_{i=1}^n \beta_i B_i(\tau) / t \right) \left[\lambda_0 \lambda^{-1} \exp(\lambda t) + \exp \left[\lambda_0 g(T) - \sum_{i=1}^n \beta_i B_i(\tau) \right] - \lambda_0 \lambda^{-1} \exp(\lambda T) \right]^{\frac{1}{t}} \right\}. \end{aligned}$$

In view of (A.3) and the basic inequality $(a + b + c)^p \leq [3(a \vee b \vee c)]^p$, we can derive that

$$\langle x \rangle^* = \limsup_{t \rightarrow +\infty} \langle x(t) \rangle \leq \lambda_0^{-1} \{ \ln [3 \lambda_0 \lambda^{-1} \exp(\lambda t)] / t \}^* = \lambda / \lambda_0.$$

If $\lambda = 0$, then for $\forall \varepsilon > 0$, we must have

$$\ln x(t) \leq \varepsilon t - \lambda_0 \int_0^t x(s) ds + \sum_{i=1}^n \beta_i B_i(t).$$

Consequently, $\langle x \rangle^* \leq \varepsilon / \lambda_0$. Since ε is arbitrary, we therefore obtain $\langle x \rangle^* \leq 0 = \lambda / \lambda_0$.

(II) The proof of (II) is similar to (I). This completes the proof. □

Proof of Theorem 1 First, let us consider population x_1 .

Case 1: If $\vartheta \leq 0$, then $r_{10} - \alpha_1^2 / 2 < r_{11} \langle C_0 \rangle_*$. Applying Itô's formula (see, e.g. Mao and Yuan 2006 on p. 39) to (4), one then sees that

$$\begin{aligned} \ln(x_1(t) / x_1(0)) / t &= r_{10} - \alpha_1^2 / 2 - r_{11} \langle C_0(t) \rangle \\ &\quad - a_{11} \langle x_1(t) \rangle - a_{12} \langle x_2(t) \rangle + \alpha_1 B_1(t) / t. \end{aligned} \tag{A.5}$$

Making use of (A.3) yields

$$\{ t^{-1} \ln x_1(t) \}^* \leq r_{10} - \alpha_1^2 / 2 - r_{11} \langle C_0 \rangle_* < 0,$$

which results in $\lim_{t \rightarrow +\infty} x_1(t) = 0$ a.s.

Case 2: If $\vartheta > 0$, then $\Delta_1 < \Phi_1\langle C_0 \rangle_*$. Applying Itô's formula to (5) gives

$$\begin{aligned} \ln(x_2(t)/x_2(0))/t &= r_{20} - \alpha_2^2/2 - r_{21}\langle C_0(t) \rangle - a_{21}\langle x_1(t) \rangle \\ &\quad - a_{22}\langle x_2(t) \rangle + \alpha_2 B_2(t)/t. \end{aligned} \tag{A.6}$$

Computing (A.5) $\times a_{22} -$ (A.6) $\times a_{12}$ leads to

$$\begin{aligned} a_{22} \frac{\ln(x_1(t)/x_1(0))}{t} - a_{12} \frac{\ln(x_2(t)/x_2(0))}{t} &= \Delta_1 - \Phi_1\langle C_0(t) \rangle - \Delta\langle x_1(t) \rangle \\ &\quad + \alpha_1 a_{22} \frac{B_1(t)}{t} - \alpha_2 a_{12} \frac{B_2(t)}{t}. \end{aligned} \tag{A.7}$$

It follows from Lemma 3, (A.3) and $\Delta > 0$ that

$$a_{22}\{\ln x_1(t)/t\}^* \leq \Delta_1 - \Phi_1\langle C_0 \rangle_* + \varepsilon < 0,$$

where ε is sufficiently small such that $\Delta_1 - \Phi_1\langle C_0 \rangle_* + \varepsilon < 0$. In other words, we have already shown that $\lim_{t \rightarrow +\infty} x_1(t) = 0$.

Similarly, we can conclude that x_2 will go to extinction provided $\langle C_0 \rangle_* > \nu$. \square

Proof of Theorem 2 First, let us consider species x_1 .

Case 1: If $\vartheta \leq 0$, then $r_{10} - \alpha_1^2/2 = r_{11}\langle C_0 \rangle_*$. For $\forall \varepsilon > 0, \exists T$ such that $\langle C_0(t) \rangle \geq \langle C_0 \rangle_* - \varepsilon$ for all $t > T$. Substituting the above inequality into (A.5) results in

$$t^{-1} \ln[x_1(t)/x_1(0)] \leq r_{10} - \alpha_1^2/2 - r_{11}\langle C_0 \rangle_* + r_{11}\varepsilon - a_{11}\langle x_1(t) \rangle + \alpha_1 B_1(t)/t,$$

that is to say

$$\ln[x_1(t)/x_1(0)] \leq r_{11}\varepsilon t - a_{11} \int_0^t x_1(s) ds + \alpha_1 B_1(t).$$

According to Lemma 4, we must have that $\langle x_1 \rangle^* \leq r_{11}\varepsilon/a_{11}$. Because ε is arbitrarily small, then $\langle x_1 \rangle^* \leq 0$. In other words, $\lim_{t \rightarrow +\infty} \langle x_1(t) \rangle = 0$.

Case 2: If $\delta > 0$, it follows from Lemma 3 that for $\forall \varepsilon > 0, \exists T$ such that $a_{12} \frac{\ln(x_2(t)/x_2(0))}{t} \leq \varepsilon/2$ and $\langle C_0(t) \rangle \geq \langle C_0 \rangle_* - \varepsilon/(2\Phi_1)$ for all $t > T$. Substituting the above inequalities into (A.7) one can derive that

$$a_{22} \frac{\ln(x_1(t)/x_1(0))}{t} \leq \Delta_1 - \Phi_1\langle C_0 \rangle_* - \Delta\langle x_1(t) \rangle + \varepsilon + \alpha_1 a_{22} \frac{B_1(t)}{t} - \alpha_2 a_{12} \frac{B_2(t)}{t},$$

which indicates that

$$\ln(x_1(t)/x_1(0)) \leq \varepsilon t/a_{22} - \Delta \int_0^t x_1(s) ds/a_{22} + \alpha_1 B_1(t) - a_{12}\alpha_2 B_2(t)/a_{22}.$$

Making use of Lemma 4, we obtain

$$\langle x_1 \rangle^* \leq \varepsilon/\Delta.$$

Then it follows from the arbitrariness of ε that $\langle x_1 \rangle^* \leq 0$. Thus, $\lim_{t \rightarrow +\infty} \langle x_1(t) \rangle = 0$. Similarly, we can show that x_2 will be nonpersistent in the mean if $\langle C_0 \rangle_* = \nu$. \square

Proof of Theorem 3 Consider species x_1 ,

Case 1: If $\vartheta < 0$, then

$$r_{11}/r_{21} < (r_{10} - \alpha_1^2/2)/(r_{20} - \alpha_2^2/2) \tag{A.8}$$

and

$$r_{10} - \alpha_1^2/2 > r_{11}\langle C_0 \rangle_*.$$

According to Lemma 3 and (A.5), one can observe that

$$a_{11}\langle x_1 \rangle^* + a_{12}\langle x_2 \rangle^* \geq r_{10} - \alpha_1^2/2 - r_{11}\langle C_0 \rangle_* > 0. \tag{A.9}$$

Therefore, $\langle x_1 \rangle^* > 0$ a.s. In fact, for $\forall \omega \in \{\langle x_1 \rangle^* = 0\}$, we must have that $\langle x_2(\omega) \rangle^* > 0$. Making use of $\Delta_1 > 0$, $\Delta_2 > 0$, and (A.8), we see that

$$\frac{a_{12}}{a_{22}} < \frac{r_{10} - \alpha_1^2/2}{r_{20} - \alpha_2^2/2} < \frac{a_{11}}{a_{21}},$$

which implies that $r_{11}/r_{21} \in (0, a_{11}/a_{21})$.

(a) If $r_{11}/r_{21} \in (0, a_{12}/a_{22}]$, then $\Phi_1 \leq 0$. Using Lemma 3 and (A.7), one can get that

$$-a_{12} \left[\frac{\ln x_2(t, \omega)}{t} \right]^* \geq \Delta_1 - \Phi_1 \langle C_0 \rangle_* > 0.$$

So, $\lim_{t \rightarrow +\infty} x_2(t, \omega) = 0$, which contradicts with $\langle x_2(\omega) \rangle^* > 0$. Thus, $r_{11}/r_{21} \notin (0, a_{12}/a_{22}]$.

(b) If $r_{11}/r_{21} \in (a_{12}/a_{22}, a_{11}/a_{21})$, then $\Phi_1 > 0$. From $r_{11} \times (A.6) - r_{21} \times (A.5)$,

$$\begin{aligned} & -r_{21} \frac{\ln(x_1(t, \omega)/x_1(0))}{t} + r_{11} \frac{\ln(x_2(t, \omega)/x_2(0))}{t} \\ & = \vartheta + \Phi_2 \langle x_1(t, \omega) \rangle - \Phi_1 \langle x_2(t, \omega) \rangle + \alpha_2 r_{11} \frac{B_2(t)}{t} - \alpha_1 r_{21} \frac{B_1(t)}{t} \end{aligned} \tag{A.10}$$

It then follows from $[\frac{\ln(x_1(t, \omega)/x_1(0))}{t}]^* \leq 0$ and $\langle x_1(\omega) \rangle^* = 0$ that

$$r_{11} \left[\frac{\ln x_2(t, \omega)}{t} \right]^* \leq \vartheta - \Phi_1 \langle x_2(\omega) \rangle_* < 0.$$

That is to say, $\lim_{t \rightarrow +\infty} x_2(t, \omega) = 0$, which contradicts with $\langle x_2(\omega) \rangle^* > 0$. So $r_{11}/r_{21} \notin (a_{12}/a_{22}, a_{11}/a_{21})$, in other words, $\langle x_1 \rangle^* > 0$.

Case 2: If $\vartheta = 0$, then

$$r_{11}/r_{21} = (r_{10} - \alpha_1^2/2)/(r_{20} - \alpha_2^2/2).$$

Since

$$r_{10} - \alpha_1^2/2 > r_{11} \langle C_0 \rangle_*,$$

then (A.9) is valid, which demonstrates that $\langle x_1 \rangle^* > 0$ a.s. In fact, for $\forall \omega \in \{\langle x_1 \rangle^* = 0\}$, we can obtain that $\langle x_2(\omega) \rangle^* > 0$. At the same time, $\lim_{t \rightarrow +\infty} \langle x_1(t, \omega) \rangle = 0$ and $[\ln x_1(t, \omega)/t]^* \leq 0$ mean that for $\forall \varepsilon > 0, \exists T$ such that $|\Phi_2 \langle x_1(t, \omega) \rangle| < \varepsilon/2$ and $r_{21} \ln(x_1(t, \omega)/x_1(0))/t < \varepsilon/2$ for all $t > T$. Substituting above inequalities into (A.10) yields that

$$\begin{aligned} \ln[x_2(t, \omega)/x_2(0)] &< r_{11}^{-1}(\varepsilon + \vartheta)t - r_{11}^{-1}\Phi_1 \int_0^t x_2(s, \omega) ds \\ &+ \alpha_2 B_2(t) - r_{11}^{-1}r_{21}\alpha_1 B_1(t). \end{aligned}$$

According to Lemma 4, we obtain that

$$\langle x_2(\omega) \rangle^* \leq (\varepsilon + \vartheta)/\Phi_1. \tag{A.11}$$

Then it follows from the arbitrariness of ε and $\vartheta = 0$ that $\langle x_2(\omega) \rangle^* = 0$, which contradicts with $\langle x_2(\omega) \rangle^* > 0$.

Case 3: Suppose $\vartheta > 0$. We can deduce from $\vartheta > 0$ and $\Delta_1 > 0$ that $\Phi_1 > 0$ and $\mu = \Delta/\Phi_1 > 0$. Applying (A.9) gives that

$$a_{11} \langle x_1 \rangle^* + a_{12} \langle x_2 \rangle^* \geq r_{10} - \frac{\alpha_1^2}{2} - r_{11} \Delta_1/\Phi_1 = a_{12} \vartheta/\Phi_1 > 0. \tag{A.12}$$

For $\forall \omega \in \{\langle x_1 \rangle^* = 0\}$, it follows from (A.12) that

$$\langle x_2(\omega) \rangle^* > \vartheta/\Phi_1, \tag{A.13}$$

then $\varepsilon = (\Phi_1 \langle x_2(\omega) \rangle^* - \vartheta)/2 > 0$. Substituting this ε into (A.11) yields that $\langle x_2(\omega) \rangle^* \leq \vartheta/\Phi_1$, which contradicts with (A.13).

Similarly, if $\langle C_0 \rangle_* > \nu$, x_2 will be weakly persistent in the mean. □

Proof of Theorem 4 Since $[\ln x_2(t)/t]^* \leq 0$, then for $\forall \varepsilon > 0, \exists T$ such that

$$a_{12}t^{-1} \ln(x_2(t)/x_2(0)) < \varepsilon/2$$

for $t > T$. Substituting the above inequality into (A.7) leads to

$$\begin{aligned} a_{22}t^{-1} \ln(x_1(t)/x_1(0)) &< \Delta_1 - \Phi_1 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle + \varepsilon - \Delta \langle x_1(t) \rangle \\ &+ \alpha_1 a_{22} B_1(t)/t - \alpha_2 a_{12} B_2(t)/t. \end{aligned}$$

Note that $\Delta > 0$ and $\Delta_1 - \Phi_1 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle > 0$, then according to Lemma 4 and the arbitrariness of ε , one can see that

$$\langle x_1 \rangle^* \leq \left(\Delta_1 - \Phi_1 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) / \Delta. \tag{A.14}$$

At the same time, computing $a_{11} \times (A.6) - a_{21} \times (A.5)$ gives

$$\begin{aligned} & a_{11} \frac{\ln(x_2(t)/x_2(0))}{t} - a_{21} \frac{\ln(x_1(t)/x_1(0))}{t} \\ &= \Delta_2 - \Phi_2 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle - \Delta \langle x_2(t) \rangle + \varepsilon \\ & \quad + \alpha_2 a_{11} \frac{B_2(t)}{t} - \alpha_1 a_{21} \frac{B_1(t)}{t}. \end{aligned} \tag{A.15}$$

Then similar to (A.14), we get

$$\langle x_2 \rangle^* \leq \left(\Delta_2 - \Phi_2 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) / \Delta. \tag{A.16}$$

Computing (A.7) $\times a_{21} + (A.15) \times a_{22}$ and then taking the superior limit leads to

$$\begin{aligned} [t^{-1} \ln x_2(t)]^* &\geq a_{21} \left(\Delta_1 - \Phi_1 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) / \Delta \\ & \quad + a_{22} \left(\Delta_2 - \Phi_2 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) / \Delta \\ & \quad - a_{21} \langle x_1 \rangle^* - a_{22} \langle x_2 \rangle^*. \end{aligned}$$

Making use of $[t^{-1} \ln x_2(t)]^* \leq 0$ we can demonstrates that

$$\begin{aligned} a_{21} \langle x_1 \rangle^* + a_{22} \langle x_2 \rangle^* &\geq \left[a_{21} \left(\Delta_1 - \Phi_1 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) \right. \\ & \quad \left. + a_{22} \left(\Delta_2 - \Phi_2 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) \right] / \Delta. \end{aligned} \tag{A.17}$$

It then follows from (A.14), (A.16), and (A.17) that

$$\begin{aligned} \langle x_1 \rangle^* &= \left(\Delta_1 - \Phi_1 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) / \Delta > 0, \\ \langle x_2 \rangle^* &= \left(\Delta_2 - \Phi_2 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) / \Delta > 0. \end{aligned}$$

Then for $\forall \varepsilon > 0, \exists T > 0$ such that for all $t > T$,

$$\begin{aligned} a_{11} \langle x_1(t) \rangle &< a_{11} \left(\Delta_1 - \Phi_1 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) / \Delta + \varepsilon/2; \\ a_{12} \langle x_2(t) \rangle &< a_{12} \left(\Delta_2 - \Phi_2 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) / \Delta + \varepsilon/2. \end{aligned}$$

Substituting those inequalities into (A.5) results in

$$\begin{aligned} t^{-1} \ln(x_1(t)/x_1(0)) &> r_{10} - \alpha_1^2/2 - r_{11} \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle - \varepsilon \\ & \quad - a_{11} \left(\Delta_1 - \Phi_1 \lim_{t \rightarrow +\infty} \langle C_0(t) \rangle \right) / \Delta \end{aligned}$$

$$\begin{aligned}
 & -a_{12} \left(\Delta_2 - \Phi_2 \lim_{t \rightarrow +\infty} (C_0(t)) \right) / \Delta + \alpha_1 B_1(t) / t \\
 & = -\varepsilon + \alpha_1 B_1(t) / t,
 \end{aligned}$$

which implies that $[t^{-1} \ln x_1(t)]_* \geq -\varepsilon$. Since ε is arbitrary, from Lemma 3, we can derive that $\lim_{t \rightarrow +\infty} t^{-1} \ln x_1(t) = 0$. Similarly, one can derive that $\lim_{t \rightarrow +\infty} t^{-1} \times \ln x_2(t) = 0$. Taking limits on both sides of (A.7) and (A.15), we get

$$\begin{aligned}
 \lim_{t \rightarrow +\infty} \langle x_2(t) \rangle & = \left(\Delta_2 - \Phi_2 \lim_{t \rightarrow +\infty} (C_0(t)) \right) / \Delta; \\
 \lim_{t \rightarrow +\infty} \langle x_1(t) \rangle & = \left(\Delta_1 - \Phi_1 \lim_{t \rightarrow +\infty} (C_0(t)) \right) / \Delta. \quad \square
 \end{aligned}$$

Proof of Theorem 5 We shall divide the whole proof into two parts.

Firstly, we claim that for arbitrary $\varepsilon > 0$, there exists a constant $\beta > 0$ such that $\mathcal{P}_* \{|x(t)| \geq \beta\} \geq 1 - \varepsilon$. Define

$$V_1(x) = \frac{1}{x_1 + x_2}$$

for $x \in R_+^2$. Then Itô's formula implies that

$$\begin{aligned}
 dV_1(x) & = \left\{ -V_1^2(x) [x_1(r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2) \right. \\
 & \quad \left. + x_2(r_{20} - r_{21}C_0(t) - a_{21}x_1 - a_{22}x_2)] + V_1^3(x) [\alpha_1^2 x_1^2 + \alpha_2^2 x_2^2] \right\} dt \\
 & \quad - V_1^2(x) [\alpha_1 x_1 dB_1(t) + \alpha_2 x_2 dB_2(t)].
 \end{aligned}$$

Since $\min_{i=1,2} \{r_{i0} - r_{i1} \limsup_{t \rightarrow +\infty} C_0(t)\} > \max_{i=1,2} \{\alpha_i^2/2\}$, we can choose a positive constant θ such that it obeys

$$\min_{i=1,2} \left\{ r_{i0} - r_{i1} \limsup_{t \rightarrow +\infty} C_0(t) \right\} - \max_{i=1,2} \{ \alpha_i^2 / 2 \} > \frac{\theta}{2} \max_{i=1,2} \{ \alpha_i^2 \}.$$

Define $V_2(x) = (1 + V_1(x))^\theta$. Making use of Itô's formula again gives

$$\begin{aligned}
 dV_2(x) & = \theta(1 + V_1(x))^{\theta-1} dV_1(x) + \frac{\theta(\theta-1)}{2} (1 + V_1(x))^{\theta-2} d(V_1(x))^2 \\
 & = \theta(1 + V_1(x))^{\theta-2} \left\{ -(1 + V_1(x)) V_1^2(x) \right. \\
 & \quad \times [x_1(r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2) \\
 & \quad \left. + x_2(r_{20} - r_{21}C_0(t) - a_{21}x_1 - a_{22}x_2)] + V_1^3(x) [\alpha_1^2 x_1^2 + \alpha_2^2 x_2^2] \right. \\
 & \quad \left. + \frac{\theta+1}{2} V_1^4(x) [\alpha_1^2 x_1^2 + \alpha_2^2 x_2^2] \right\} dt \\
 & \quad - \theta(1 + V_1(x))^{\theta-1} V_1^2(x) [\alpha_1 x_1 dB_1(t) + \alpha_2 x_2 dB_2(t)]
 \end{aligned}$$

$$\begin{aligned}
 &= \theta(1 + V_1(x))^{\theta-2} F(x) dt \\
 &\quad - \theta(1 + V_1(x))^{\theta-1} V_1^2(x) [\alpha_1 x_1 dB_1(t) + \alpha_2 x_2 dB_2(t)], \tag{A.18}
 \end{aligned}$$

where

$$\begin{aligned}
 F(x) = &-(1 + V_1(x)) V_1^2(x) [x_1(r_{10} - r_{11} C_0(t) - a_{11}x_1 - a_{12}x_2) \\
 &+ x_2(r_{20} - r_{21} C_0(t) - a_{21}x_1 - a_{22}x_2)] \\
 &+ V_1^3(x) [\alpha_1^2 x_1^2 + \alpha_2^2 x_2^2] + \frac{\theta + 1}{2} V_1^4(x) [\alpha_1^2 x_1^2 + \alpha_2^2 x_2^2].
 \end{aligned}$$

Estimate that

$$\begin{aligned}
 F(x) \leq &-V_1^2(x) \left(1 + \frac{1}{x_1 + x_2}\right) \left[x_1 \left(r_{10} - r_{11} \limsup_{t \rightarrow +\infty} C_0(t) - \varepsilon\right) \right. \\
 &+ \left. x_2 \left(r_{20} - r_{21} \limsup_{t \rightarrow +\infty} C_0(t) - \varepsilon\right)\right] \\
 &+ V_1^2(x) \left(1 + \frac{1}{x_1 + x_2}\right) [a_{11}x_1^2 + a_{12}x_1x_2 + a_{21}x_1x_2 + a_{22}x_2^2] \\
 &+ \max_{i=1,2} \{\alpha_i^2\} [x_1^2 + x_2^2] \frac{1}{(x_1 + x_2)^3} \\
 &+ \frac{\theta + 1}{2} \max_{i=1,2} \{\alpha_i^2\} [x_1^2 + x_2^2] \frac{1}{(x_1 + x_2)^4} \\
 \leq &-V_1^2(x) \min_{i=1,2} \left\{r_{i0} - r_{i1} \limsup_{t \rightarrow +\infty} C_0(t) - \varepsilon\right\} + V_1^2(x) \frac{\theta + 1}{2} \max_{i=1,2} \{\alpha_i^2\} \\
 &+ \max_{i,j=1,2} \{a_{ij}\} (1 + V_1(x)) \frac{1}{(x_1 + x_2)^2} (x_1^2 + 2x_1x_2 + x_2^2) + \max_{i=1,2} \{\alpha_i^2\} V_1(x) \\
 \leq &-V_1^2(x) \left[\min_{i=1,2} \left\{r_{i0} - r_{i1} \limsup_{t \rightarrow +\infty} C_0(t) - \varepsilon\right\} - \frac{\theta + 1}{2} \max_{i=1,2} \{\alpha_i^2\}\right] \\
 &+ V_1(x) \left[\max_{i,j=1,2} \{a_{ij}\} + \max_{i=1,2} \{\alpha_i^2\}\right] + \max_{i,j=1,2} \{a_{ij}\}.
 \end{aligned}$$

Substituting the above inequality into (A.18) results in

$$\begin{aligned}
 dV_2(x(t)) \leq &\theta(1 + V_1(x))^{\theta-2} \left\{-V_1^2(x) \left[\min_{i=1,2} \left\{r_{i0} - r_{i1} \limsup_{t \rightarrow +\infty} C_0(t) - \varepsilon\right\} \right. \right. \\
 &\left. \left. - \frac{\theta + 1}{2} \max_{i=1,2} \{\alpha_i^2\}\right] \right. \\
 &+ \left. V_1(x) \left[\max_{i,j=1,2} \{a_{ij}\} + \max_{i=1,2} \{\alpha_i^2\}\right] + \max_{i,j=1,2} \{a_{ij}\}\right\} dt \\
 &- \theta(1 + V_1(x))^{\theta-1} V_1^2(x) [\alpha_1 x_1 dB_1(t) + \alpha_2 x_2 dB_2(t)].
 \end{aligned}$$

Now, choose κ sufficiently small to satisfy

$$0 < \frac{\kappa}{\theta} < \min_{i=1,2} \left\{ r_{i0} - r_{i1} \limsup_{t \rightarrow +\infty} C_0(t) - \varepsilon \right\} - \frac{\theta + 1}{2} \max_{i=1,2} \{ \alpha_i^2 \}.$$

Define $V_3(x(t)) = \exp\{\kappa t\} V_2(x(t))$. Then Itô's formula indicates that

$$\begin{aligned} dV_3(x(t)) &= \kappa \exp\{\kappa t\} V_2(x) dt + \exp\{\kappa t\} dV_2(x) \\ &\leq \exp\{\kappa t\} (1 + V_1(x))^{\theta-2} \left\{ \kappa (1 + V_1(x))^2 \right. \\ &\quad - V_1^2(x) \theta \left[\min_{i=1,2} \left\{ r_{i0} - r_{i1} \limsup_{t \rightarrow +\infty} C_0(t) - \varepsilon \right\} - \frac{\theta + 1}{2} \max_{i=1,2} \{ \alpha_i^2 \} \right] \\ &\quad \left. + V_1(x) \theta \left[\max_{i,j=1,2} \{ a_{ij} \} + \max_{i=1,2} \{ \alpha_i^2 \} \right] + \theta \max_{i,j=1,2} \{ a_{ij} \} \right\} dt \\ &\quad - \theta \exp\{\kappa t\} (1 + V_1(x))^{\theta-1} V_1^2(x) [\alpha_1 x_1 dB_1(t) + \alpha_2 x_2 dB_2(t)] \\ &= \exp\{\kappa t\} \theta (1 + V_1(x))^{\theta-2} \left\{ -V_1^2(x) \left[\min_{i=1,2} \left\{ r_{i0} - r_{i1} \limsup_{t \rightarrow +\infty} C_0(t) - \varepsilon \right\} \right. \right. \\ &\quad \left. \left. - \frac{\theta + 1}{2} \max_{i=1,2} \{ \alpha_i^2 \} - \kappa/\theta \right] \right. \\ &\quad \left. + \left[\max_{i,j=1,2} \{ a_{ij} \} + \max_{i=1,2} \{ \alpha_i^2 \} + 2\kappa/\theta \right] V_1(x) + \max_{i,j=1,2} \{ a_{ij} \} + \kappa/\theta \right\} dt \\ &\quad - \theta \exp\{\kappa t\} (1 + V_1(x))^{\theta-1} V_1^2(x) [\alpha_1 x_1 dB_1(t) + \alpha_2 x_2 dB_2(t)] \\ &=: \exp\{\kappa t\} J(x) dt \\ &\quad - \theta \exp\{\kappa t\} (1 + V_1(x))^{\theta-1} V_1^2(x) [\alpha_1 x_1 dB_1(t) + \alpha_2 x_2 dB_2(t)] \end{aligned}$$

for sufficiently large t , where

$$\begin{aligned} J(x) &= \theta (1 + V_1(x))^{\theta-2} \left\{ -V_1^2(x) \left[\min_{i=1,2} \left\{ r_{i0} - r_{i1} \limsup_{t \rightarrow +\infty} C_0(t) - \varepsilon \right\} \right. \right. \\ &\quad \left. \left. - \frac{\theta + 1}{2} \max_{i=1,2} \{ \alpha_i^2 \} - \kappa/\theta \right] \right. \\ &\quad \left. + \left[\max_{i,j=1,2} \{ a_{ij} \} + \max_{i=1,2} \{ \alpha_i^2 \} + 2\kappa/\theta \right] V_1(x) + \max_{i,j=1,2} \{ a_{ij} \} + \kappa/\theta \right\}. \end{aligned}$$

It then follows from the definition of κ that $J(x)$ is upper bounded in R_+^2 , namely

$$K_1 := \sup_{x \in R_+^2} J(x) < +\infty.$$

Consequently,

$$dV_3(x(t)) \leq K_1 \exp\{\kappa t\} dt - \theta \exp\{\kappa t\} (1 + V_1(x))^{\theta-1} V_1^2(x) \times [\alpha_1 x_1 dB_1(t) + \alpha_2 x_2 dB_2(t)]$$

for sufficiently large t . Integrating both sides of the above inequality and then taking expectations gives

$$E[V_3(x(t))] = E[\exp\{\kappa t\} (1 + V_1(x(t)))^\theta] \leq (1 + V_1(x_0))^\theta + \frac{K_1}{\kappa} \exp\{\kappa t\}.$$

That is to say,

$$\limsup_{t \rightarrow +\infty} E[V_1^\theta(x(t))] \leq \limsup_{t \rightarrow +\infty} E[(1 + V_1(x(t)))^\theta] \leq \frac{K_1}{\kappa}.$$

At the same time, note that for $x(t) \in R_+^2$,

$$(x_1(t) + x_2(t))^\theta \leq \left(2 \max_{i=1,2} x_i(t)\right)^\theta = 2^\theta \left(\max_{i=1,2} x_i^2(t)\right)^{0.5\theta} \leq 2^\theta |x(t)|^\theta.$$

Therefore,

$$\limsup_{t \rightarrow +\infty} E\left[\frac{1}{|x(t)|^\theta}\right] \leq 2^\theta \frac{K_1}{\kappa} =: K.$$

So, for any $\varepsilon > 0$, set $\beta = \varepsilon^{\frac{1}{\theta}} / K^{\frac{1}{\theta}}$, by Chebyshev's inequality, we can derive that

$$\mathcal{P}\{|x(t)| < \beta\} = \mathcal{P}\left\{\frac{1}{|x(t)|^\theta} > \frac{1}{\beta^\theta}\right\} \leq \frac{E\left[\frac{1}{|x(t)|^\theta}\right]}{\frac{1}{\beta^\theta}} = \beta^\theta E\left[\frac{1}{|x(t)|^\theta}\right],$$

that is to say

$$\mathcal{P}^*\{|x(t)| < \beta\} \leq \beta^\theta K = \varepsilon.$$

Consequently,

$$\mathcal{P}_*\{|x(t)| \geq \beta\} \geq 1 - \varepsilon.$$

Next, we prove that for arbitrary $\varepsilon > 0$, there exists $\chi > 0$ such that $\mathcal{P}_*(|x(t)| \leq \chi) \geq 1 - \varepsilon$. The proof is rather standard but for the completeness of the paper we will give it briefly. Define $V(x) = x_1^q + x_2^q$ for $x \in R_+^2$. Then it follows from Itô's formula that

$$dV(x(t)) = qx_1^q \left[r_{10} - r_{11} C_0(t) - a_{11}x_1 - a_{12}x_2 + \frac{q-1}{2} \alpha_1^2 \right] dt + qx_2^q \left[r_{20} - r_{21} C_0(t) - a_{21}x_1 - a_{22}x_2 + \frac{q-1}{2} \alpha_2^2 \right] dt + q\alpha_1 x_1^q dB_1(t) + q\alpha_2 x_2^q dB_2(t)$$

$$\begin{aligned} &\leq qx_1^q \left[r_{10} - a_{11}x_1 + \frac{q-1}{2}\alpha_1^2 \right] dt + qx_2^q \left[r_{20} - a_{22}x_2 + \frac{q-1}{2}\alpha_2^2 \right] dt \\ &\quad + q\alpha_1x_1^q dB_1(t) + q\alpha_2x_2^q dB_2(t). \end{aligned}$$

Let $k_0 > 0$ be sufficiently large such that every component of $x(0)$ is contained within the interval $[1/k_0, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \geq 0 : x_1(t) \notin (1/k, k) \text{ or } x_2(t) \notin (1/k, k)\}.$$

Clearly, $\tau_k \rightarrow \infty$ almost surely as $k \rightarrow \infty$. Applying Itô formula again to $\exp\{t\}V(x)$ and then taking expectations on both sides leads to

$$\begin{aligned} &E[\exp\{t \wedge \tau_k\}x^q(t \wedge \tau_k)] - x^q(0) \\ &\leq qE \int_0^{t \wedge \tau_k} \exp\{s\}x_1^q(s) \left(r_{10} - a_{11}x_1(s) + \frac{q-1}{2}\alpha_1^2 + 1/q \right) ds \\ &\quad + qE \int_0^{t \wedge \tau_k} \exp\{s\}x_2^q(s) \left(r_{20} - a_{22}x_2(s) + \frac{q-1}{2}\alpha_2^2 + 1/q \right) ds \\ &\leq E \int_0^{t \wedge \tau_k} \exp\{s\}(K_2 + K_3) ds \leq (K_2 + K_3)(\exp\{t\} - 1), \end{aligned}$$

where K_2 and K_3 are positive constants. Letting $k \rightarrow \infty$ gives

$$\exp\{t\}E[x^q(t)] \leq x^q(0) + (K_2 + K_3)(\exp\{t\} - 1).$$

That is to say

$$\limsup_{t \rightarrow +\infty} E[x^q(t)] \leq (K_2 + K_3).$$

Then for any $\varepsilon > 0$, set $\chi = (K_2 + K_3)^{1/q} / \varepsilon^{1/q}$, in view of Chebyshev’s inequality, we can obtain that

$$\mathcal{P}\{|x(t)| > \chi\} = \mathcal{P}\{|x(t)|^q > \chi^q\} \leq \frac{E[|x(t)|^q]}{\chi^q}.$$

In other words, we have already shown that

$$\limsup_{t \rightarrow +\infty} \mathcal{P}\{|x(t)| > \chi\} \leq \limsup_{t \rightarrow +\infty} E[|x(t)|^q] / \chi^q \leq \varepsilon,$$

which is the desired assertion. □

Proof of Theorem 6 Applying Itô’s formula to (11) leads to

$$d \ln x = \frac{dx}{x} - \frac{(dx)^2}{2x^2} = [r_0 - r_1C_0(t) - ax - 0.5\sigma^2x^2] dt + \sigma x dB(t).$$

That is to say,

$$\ln x(t) - \ln x(0) = r_0 t - r_1 \int_0^t C_0(s) ds - \int_0^t [ax(s) + 0.5\sigma^2 x^2(s)] ds + M(t), \tag{A.19}$$

where $M(t) = \int_0^t \sigma x(s) dB_1(s)$, whose quadratic variation is

$$\langle M(t), M(t) \rangle = \sigma^2 \int_0^t x^2(s) ds.$$

By virtue of the exponential martingale inequality (see, e.g. Mao and Yuan 2006, on p. 74), for any positive constants T, α , and β , we have

$$\mathcal{P} \left\{ \sup_{0 \leq t \leq T} \left[M(t) - \frac{\alpha}{2} \langle M(t), M(t) \rangle \right] > \beta \right\} \leq \exp^{-\alpha\beta}. \tag{A.20}$$

Choose $T = n, \alpha = 1, \beta = 2 \ln n$. Then it follows that

$$\mathcal{P} \left\{ \sup_{0 \leq t \leq n} \left[M(t) - \frac{1}{2} \langle M(t), M(t) \rangle \right] > 2 \ln n \right\} \leq -1/n^2.$$

An application of the Borel–Cantelli lemma (see, e.g. Mao and Yuan 2006, on p. 10) then yields that for almost all $\omega \in \Omega$, there is a random integer $n_0 = n_0(\omega)$ such that for $n \geq n_0$,

$$\sup_{0 \leq t \leq n} \left[M(t) - \frac{1}{2} \langle M(t), M(t) \rangle \right] \leq 2 \ln n.$$

That is to say

$$M(t) \leq 2 \ln n + \frac{1}{2} \langle M(t), M(t) \rangle = 2 \ln n + 0.5\sigma^2 \int_0^t x(s)^2 ds$$

for all $0 \leq t \leq n, n \geq n_0$ almost surely. Substituting the above inequality into (A.19) leads to

$$\begin{aligned} \ln x(t) - \ln x(0) &\leq r_0 t - r_1 \int_0^t C_0(s) ds - a \int_0^t x(s) ds + 2 \ln n \\ &\leq r_0 t - r_1 \int_0^t C_0(s) ds + 2 \ln n \end{aligned}$$

for all $0 \leq t \leq n, n \geq n_0$ almost surely. In other words, we have already shown that for $0 < n - 1 \leq t \leq n$,

$$\frac{\ln x(t) - \ln x_0}{t} \leq \frac{r_0 t - r_1 \int_0^t C_0(s) ds}{t} + 2 \frac{\ln n}{n - 1},$$

which indicates that

$$\left[\frac{\ln x(t)}{t} \right]^* \leq r_0 - r_1 \langle C_0 \rangle_*.$$

In other words, if $r_0 - r_1 \langle C_0 \rangle_* < 0$, one can see that $\lim_{t \rightarrow +\infty} x(t) = 0$.

Now, we are in the position to prove the second part of Theorem 6. Firstly, we are going to show that

$$\left[\frac{\ln x(t)}{t} \right]^* \leq 0 \quad a.s. \tag{A.21}$$

In fact, making use of Itô's formula to (11) gives

$$\begin{aligned} d(\exp(t) \ln x) &= \exp(t) \ln x \, dt + \exp(t) \, d \ln x \\ &= \exp(t) [\ln x + r_0 - r_1 C_0(t) - ax - 0.5\sigma^2 x^2] \, dt + \exp(t) \sigma x \, dB(t). \end{aligned}$$

In other words, we have already shown that

$$\begin{aligned} \exp(t) \ln x(t) - \ln x_0 &= \int_0^t \exp(s) [\ln x(s) + r_0 - r_1 C_0(s) \\ &\quad - ax(s) - 0.5\sigma^2 x^2(s)] \, ds + N(t), \end{aligned} \tag{A.22}$$

where $N(t) = \sigma \int_0^t \exp(s) x(s) \, dB(s)$ is a local martingale with the quadratic form:

$$\langle N(t), N(t) \rangle = \sigma^2 \int_0^t \exp(2s) x^2(s) \, ds.$$

It then follows from the exponential martingale inequality (A.20) by choosing $T = \gamma k$, $\alpha = \exp(-\gamma k)$, $\beta = \theta \exp(\gamma k) \ln k$ that

$$\mathcal{P} \left\{ \sup_{0 \leq t \leq \gamma k} \left[N(t) - \frac{\exp(-\gamma k)}{2} \langle N(t), N(t) \rangle \right] > \theta \exp(\gamma k) \ln k \right\} \leq k^{-\theta},$$

where $\theta > 1$ and $\gamma > 1$. By virtue of the Borel–Cantelli lemma, for almost $\omega \in \Omega$, there exists $k_0(\omega)$ such that for every $k \geq k_0(\omega)$,

$$N(t) \leq \frac{\exp(-\gamma k)}{2} \langle N(t), N(t) \rangle + \theta \exp(\gamma k) \ln k, \quad 0 \leq t \leq \gamma k.$$

Substituting the above inequality into (A.22) yields that

$$\begin{aligned} \exp(t) \ln x(t) - \ln x_0 &\leq \int_0^t \exp(s) [\ln x(s) + r_0 - r_1 C_0(s) - ax(s) - 0.5\sigma^2 x^2(s)] \, ds \\ &\quad + \frac{\sigma^2 \exp(-\gamma k)}{2} \int_0^t \exp(2s) x^2(s) \, ds + \theta \exp(\gamma k) \ln k \\ &= \int_0^t \exp(s) [\ln x(s) + r_0 - r_1 C_0(s) - ax(s) \\ &\quad - 0.5\sigma^2 x^2(s) [1 - \exp(s - \gamma k)]] \, ds + \theta \exp(\gamma k) \ln k. \end{aligned}$$

It is easy to see that for any $0 \leq s \leq \gamma k$ and $x > 0$, there exists a constant C independent of k such that

$$\ln x(s) + r_0 - r_1 C_0(s) - ax(s) - 0.5\sigma^2 x^2(s)[1 - \exp(s - \gamma k)] \leq C.$$

In other words for any $0 \leq t \leq \gamma k$,

$$\exp(t) \ln x(t) - \ln x_0 \leq C[\exp(t) - 1] + \theta \exp(\gamma k) \ln k.$$

That is to say

$$\ln x(t) \leq \exp(-t) \ln x_0 + C[1 - \exp(-t)] + \theta \exp(-t) \exp(\gamma k) \ln k.$$

If $\gamma(k - 1) \leq t \leq \gamma k$ and $k \geq k_0(\omega)$, we have

$$\ln x(t)/t \leq \exp(-t) \ln x_0/t + C[1 - \exp(-t)]/t + \theta \exp(-\gamma(k - 1)) \exp(\gamma k) \ln k/t,$$

which is the desired assertion (A.21) by letting $k \rightarrow +\infty$.

Now suppose that $r_0 - r_1 \langle C_0 \rangle_* > 0$, we are going to prove that $x^* > 0$ a.s. Otherwise, let E be the set $E = \{x^* = 0\}$ and suppose that $\mathcal{P}(E) > 0$. It follows from (A.19) that

$$\frac{\ln x(t) - \ln x(0)}{t} = r_0 - r_1 \langle C_0(t) \rangle - a \langle x(t) \rangle - 0.5\sigma^2 \langle x^2(t) \rangle + M(t)/t. \tag{A.23}$$

On the other hand, for $\forall \omega \in E$, we have $\lim_{t \rightarrow +\infty} x(t, \omega) = 0$, then the law of large numbers for local martingales (see, e.g. Mao and Yuan 2006, on p. 16) implies that $\lim_{t \rightarrow +\infty} M(t)/t = 0$. Substituting the above inequality into (A.23) gives

$$[\ln x(t, \omega)/t]^* = r_0 - r_1 \langle C_0 \rangle_* > 0.$$

Then $\mathcal{P}([\ln x(t)/t]^*) > 0$, which contradicts with (A.21). This completes the proof. □

Proof of Theorem 7 Without loss of generality, we prove the conclusions for x_1 . Making use of Itô's formula to that (6) leads to

$$\begin{aligned} d \ln x_1 &= \frac{dx_1}{x_1} - \frac{dx_1^2}{2x_1^2} \\ &= [r_{10} - r_{11}C_0(t) - a_{11}x_1 - a_{12}x_2 - 0.5\sigma_{11}^2 x_1^2 - 0.5\sigma_{12}^2 x_2^2] dt \\ &\quad + \sigma_{11}x_1 dB_{11}(t) + \sigma_{12}x_2 dB_{12}(t). \end{aligned} \tag{A.24}$$

In other words,

$$\begin{aligned} \ln x_1(t) - \ln x_1(0) &= \int_0^t [r_{10} - r_{11}C_0(s) - a_{11}x_1(s) - a_{12}x_2(s) - 0.5\sigma_{11}^2 x_1^2(s) \\ &\quad - 0.5\sigma_{12}^2 x_2^2(s)] ds + M_{11}(t) + M_{12}(t), \end{aligned}$$

where $M_{1i}(t) = \int_0^t \sigma_{1i} x_i(s) dB_{1i}(s)$, whose quadratic variation is

$$\langle M_{1i}(t), M_{1i}(t) \rangle = \sigma_{1i}^2 \int_0^t x_i^2(s) ds, \quad i = 1, 2.$$

Then similar to the proof of Theorem 6, by virtue of the Borel–Cantelli lemma and of the exponential martingale inequality for almost $\omega \in \Omega$, there exists $n_0(\omega)$ such that, for every $n \geq n_0(\omega)$,

$$M_{1i}(t) \leq 2 \ln n + 0.5 \sigma_{1i}^2 \int_0^t x_i(s)^2 ds, \quad i = 1, 2,$$

for all $0 \leq t \leq n, n \geq n_0$ almost surely. Substituting the above inequalities into equation (A.24) gives

$$\begin{aligned} \ln x_1(t) - \ln x_1(0) &= \int_0^t [r_{10} - r_{11} C_0(s) - a_{11} x_1(s) - a_{12} x_2(s)] ds + 4 \ln n \\ &\leq r_{10} t - r_{11} \int_0^t C_0(s) ds + 4 \ln n \end{aligned}$$

for all $0 \leq t \leq n, n \geq n_0$. In other words, we have already shown that for $0 < n - 1 \leq t \leq n$,

$$\left[\frac{\ln x_1(t) - \ln x_1(0)}{t} \right] \leq r_{10} - r_{11} \langle C_0(t) \rangle + 4 \frac{\ln n}{n - 1}.$$

Consequently,

$$\left[\frac{\ln x_1(t)}{t} \right]^* \leq r_{10} - r_{11} \langle C_0 \rangle_* < 0,$$

which is the required assertion. □

Proof of Theorem 8 It follows from Theorem 7 that species x_i will go to extinction. Then model (SM₂) becomes a single-species system. Thus, the desired assertion follows from Theorem 6 immediately. □

Proof of Theorem 9 Without loss of generality, we prove the conclusions for x_1 .

- (I) The proof is similar to Lemma 3 (the case $r_{i0} < \alpha_i^2/2$).
- (II) Applying Itô’s formula to (A.1) and setting $i = 1$ yields that

$$\begin{aligned} \ln[y_1(t)/y_1(0)] &= \left(r_{10} - \frac{\alpha_1^2}{2} \right) t - a_{11} \int_0^t y_1(s) ds + \alpha_1 B_1(t) \\ &= -a_{11} \int_0^t y_1(s) ds + \alpha_1 B_1(t). \end{aligned}$$

Then in view of Lemma 4, we have $\langle y_1 \rangle^* \leq 0$, that is to say $\langle x_1 \rangle^* \leq \langle y_1 \rangle^* \leq 0$, thus $\langle x_1 \rangle^* = 0$.

(III) Without loss of generality, we suppose $i = 1, j = 2$. From (I), we know that x_1 will go to extinction, that is to say $\lim_{t \rightarrow +\infty} x_1(t) = 0$. On the other hand, it follows from (A.6) that

$$\ln(x_2(t)/x_2(0))/t \leq r_{20} - \alpha_2^2/2 + \varepsilon - a_{22}\langle x_2(t) \rangle + \alpha_2 B_2(t)/t,$$

$$\ln(x_2(t)/x_2(0))/t \geq r_{20} - \alpha_2^2/2 - \varepsilon - a_{22}\langle x_2(t) \rangle + \alpha_2 B_2(t)/t.$$

Then Lemma 4 means that

$$\langle x_2 \rangle^* \leq (r_{20} - \alpha_2^2/2)/a_{22}, \quad \langle x_2 \rangle_* \geq (r_{20} - \alpha_2^2/2)/a_{22}.$$

This completes the proof. □

Proof of Theorem 10 (I) Without loss of generality, suppose that $\Delta_1 < 0$ and $\Delta_2 > 0$.

(i) Let $\Phi_1 = 0$ in (A.7), one then sees that

$$\begin{aligned} & a_{22} \frac{\ln(x_1(t)/x_1(0))}{t} - a_{12} \frac{\ln(x_2(t)/x_2(0))}{t} \\ &= \Delta_1 - \Delta \langle x_1(t) \rangle + \alpha_1 a_{22} \frac{B_1(t)}{t} - \alpha_2 a_{12} \frac{B_2(t)}{t}, \end{aligned} \tag{A.25}$$

in other words

$$a_{22} \left\{ \frac{\ln x_1(t)}{t} \right\}^* \leq \Delta_1 < 0,$$

which indicates that $\lim_{t \rightarrow +\infty} x_1(t) = 0$.

(ii) The proof of (ii) and (iii) will be presented together. It follows from (A.6) that

$$\ln(x_2(t)/x_2(0))/t \leq r_{20} - \alpha_2^2/2 - a_{22}\langle x_2(t) \rangle + \alpha_2 \frac{B_2(t)}{t}.$$

Using Lemma 4 yields that

$$\langle x_2 \rangle^* \leq (r_{20} - \alpha_2^2/2)/a_{22}.$$

On the other hand, in view of (A.6) and $\lim_{t \rightarrow +\infty} x_1(t) = 0$ we can derive that, for sufficiently large t ,

$$\ln(x_2(t)/x_2(0))/t \geq r_{20} - \alpha_2^2/2 - a_{22}\langle x_2(t) \rangle - \varepsilon + \alpha_2 \frac{B_2(t)}{t}.$$

In view of (II) in Lemma 4, one can observe that

$$\langle x_2 \rangle_* \geq (r_{20} - \alpha_2^2/2)/a_{22}.$$

Consequently,

$$\lim_{t \rightarrow +\infty} \langle x_2(t) \rangle = (r_{20} - \alpha_2^2/2)/a_{22}.$$

The following proof of (iii) is similar to Theorem 4.

(II) (a) Without loss of generality, suppose that $\Delta_1 < 0$, $\Delta_2 > 0$. Making use of (A.15), $\Delta < 0$ and Lemma 3, one then derives

$$-a_{21} \left[\frac{\ln(x_1(t)/x_1(0))}{t} \right]^* \geq \Delta_2 > 0,$$

that is to say

$$\lim_{t \rightarrow +\infty} x_1(t) = 0.$$

At the same time, similar to (A.17) one can show that

$$a_{21} \langle x_1 \rangle^* + a_{22} \langle x_2 \rangle_* \geq (a_{21} \Delta_1 + a_{22} \Delta_2) / \Delta = r_{20} - \alpha_2^2 / 2.$$

Consequently,

$$\langle x_2 \rangle_* \geq (r_{20} - \alpha_2^2 / 2) / a_{22} > 0. \quad (\text{A.26})$$

On the other hand, it follows from (A.6) and $r_{21} = 0$ that

$$\frac{\ln(x_2(t)/x_2(0))}{t} \leq r_{20} - \alpha_2^2 / 2 - a_{22} \langle x_2(t) \rangle + \alpha_2 \frac{B_2(t)}{t}.$$

Then by (I) of Lemma 4 we get

$$\langle x_2 \rangle^* \leq (r_{20} - \alpha_2^2 / 2) / a_{22}.$$

The desired assertion follows from the above inequality and (A.26) immediately.

(b) The fact that x_1 and x_2 will not simultaneously go to local extinction follows from (A.9). The proof of Theorem 10 is complete. \square

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