Japan. J. Math. 14, 231–247 (2019)
DOI: 10.1007/s11537-019-1840-4

JAPANESE JOURNAL OF MATHEMATICS
SINCE 1924

Singularities in mixed characteristic. The perfectoid approach*

Yves André

Received: 25 November 2018 / Revised: 15 February 2019 / Accepted: 25 February 2019
Published online: 30 May 2019

Communicated by: Takeshi Saito

Abstract. The homological conjectures, which date back to Peskine, Szpiro and Hochster in the late 60's, make fundamental predictions about syzygies and intersection problems in commutative algebra. They were settled long ago in the presence of a base field and led to tight closure theory, a powerful tool to investigate singularities in characteristic p.

Recently, perfectoid techniques coming from p-adic Hodge theory have allowed us to get rid of any base field; this solves the direct summand conjecture and establishes the existence and weak functoriality of big Cohen–Macaulay algebras, which solve in turn the homological conjectures in general. This also opens the way to the study of singularities in mixed characteristic.

We sketch a broad outline of this story, taking lastly a glimpse at ongoing work by L. Ma and K. Schwede, which shows how such a study could build a bridge between singularity theory in characteristic p and in characteristic 0.

Keywords and phrases: singularities, homological conjectures, big Cohen–Macaulay algebras, perfectoid spaces

Mathematics Subject Classification (2010): 13D22, 13H05, 14G20

* This article is based on the 22nd Takagi Lectures that the author delivered at The University of Tokyo on November 17–18, 2018.

Y. ANDRÉ
Sorbonne-Université, Institut de Mathématiques de Jussieu, 4 Place Jussieu, 75005 Paris, France
(e-mail: yves.andre@imj-prg.fr)

1. Commutative algebra is not (only) chapter 0 of algebraic geometry

1.1.

Commutative Algebra took its roots in Algebraic Number Theory, Algebraic Geometry and Invariant Theory at the end of the XIXth century, in the wake of Dedekind, Kronecker and Gordan. It soon provided a unified language for the three theories as well as a consolidation of their foundations.

It may be argued that Commutative Algebra started with Hilbert's three fundamental theorems (which still bear their original names: *basis theorem, syzygy theorem, Nullstellensatz*), and was put on firm ground in the 30's with Krull's four cardinal concepts (*locality, dimension, completion and regularity*).

A main source of examples and inspiration is given by affine rings (rings of functions on affine algebraic varieties over some base field). *Regular local rings* (those having the property that the minimal number of generators of their maximal ideal is their Krull dimension) then generalize rings of functions around a *non-singular* point of an algebraic variety. Following this viewpoint, general *local rings* appear as algebraic counterparts of *singularities* (vs. non-singular points).

The theory was so successful that its relation to Algebraic Geometry changed: it was the latter to be founded on the former, in Zariski's and Grothendieck's time. Throughout the canonical text EGA, Commutative Algebra forms an extended, crawling "Chapter 0".

1.2.

However, Commutative Algebra is more than the solfeggio of the music of Algebraic Geometry. In fact, already in the late 50's, a turn happened in the story of Commutative Algebra: *the homological turn*, which was impulsed by Auslander, Buchsbaum and Serre, and originated in Hilbert's syzygy's theorem. From general theory of (Noetherian) commutative rings and their ideals, Commutative Algebra became the homological theory of (Noetherian) modules.

In this new viewpoint, regular local rings are those for which any finite module has a free resolution of finite length.

By the way, let us recall that syzygies generalize the notion of presentation of a module. If a finite module M over a local ring S admits a finite free resolution $0 \to S^{b_s} \to S^{b_{s-1}} \to \cdots \to S^{b_0} \to M \to 0$, which we assume to be minimal, the b_i's are called the *Betti numbers* of M (so that b_0 is the minimal number of generators, b_1 the minimal number of relations between them,...), and $\mathrm{Im}(S^{b_i} \to S^{b_{i-1}})$ is called the *i-th syzygy* of M. Describing syzygies and estimating Betti numbers is a traditional and typical task of Commutative Algebra, not necessarily related to any specific problem from Algebraic Geometry.

1.3.

An important leimotiv in Commutative Algebra is that singularities are always "finite above non-singularities". One version is Noether's theorem: every affine algebra is finite over a polynomial algebra. Another version, in the complete local case, is Cohen's theorem: every complete local domain is finite over a regular complete local ring (in fact a ring of formal power series with coefficients in a field or a discrete valuation ring).

The study of singularities thus focusses on *finite extensions S of a regular (local) ring*. S-modules need not have finite free resolutions; in fact, their homological study is complicated and is the subject of the *homological conjectures* which we will touch upon. But as we will see, it is nevertheless useful to represent the ring S as a finite extension of a regular ring R.

1.4.

We shall report on recent progresses which came from an encounter between two domains:

Commutative Algebra (Hilbert, Krull, ...)	··········	p-adic Hodge Theory (Tate, Fontaine, Faltings, ...)
Noetherian world (finite-dimensional rings, finite modules)		non-Noetherian world (non-archimedean Banach algebras ...)
Homological conjectures (Peskine–Szpiro, Hochster, ...)	*new* ←·········	perfectoid theory (Scholze, ...)

The second domain, p-adic Hodge Theory, which started with Tate's work in the late 60's, studies p-adic representations of p-adic Galois groups, especially those which come from the cohomology of algebraic varieties. Perfectoid theory is a new development (–2010) which has already profoundly transformed p-adic Hodge Theory.

In the "Chapter 0" viewpoint, Commutative Algebra stands as the provider of basic concepts, tools and theorems for Algebraic or Arithmetic Geometry and its advanced developments, such as p-adic Hodge Theory.

In the present story, we will see p-adic Hodge Theory contribute backward to some basic problems of Commutative Algebra.

2. The direct summand theorem

2.1.

Let us tell the first instance of this encounter.

Let R be a Noetherian commutative ring, and $R \subset S$ be a finite extension, i.e., S is a faithful R-algebra, finitely generated as an R-module (as in Noether's or Cohen's theorem). One thus has an exact sequence of finite R-modules:

$$(*) \qquad 0 \longrightarrow R \longrightarrow S \longrightarrow S/R \longrightarrow 0.$$

Question 1: Does this exact sequence splits?

Equivalently: Is R a direct summand in S? Is there an R-linear map $\lambda : S \to R$ with $\lambda(1) = 1$?

This basic question arose in the late 60's (Hochster [15], Raynaud–Gruson [29]), in the framework of the homological conjectures and of descent theory (independently). An algebraic geometer may not find easy to confer it any geometric meaning; this illustrates what we wrote about "Chapter 0". But here is a related question with a more immediate algebro-geometric content:

Question 2: Is there an S-algebra T which is faithfully flat over R?

Equivalently: Is $\operatorname{Spec} S \to \operatorname{Spec} R$ a covering for the fpqc (fidèlement plate quasi-compacte) topology?

(It is not required that T is finitely generated, nor even noetherian).

Actually, a positive answer to Question 2 implies a positive answer to Question 1. Indeed:

a) any faithfully flat map $R \to T$ is *pure*, i.e., universally injective,
b) if a composition $R \to S \to T$ is pure, so is $R \to S$,
c) a finite extension $R \subset S$ is pure if and only if it splits (in the R-module sense).

Example. The sequence (∗) splits if R is a normal $\mathbb{Q}$-algebra (take for λ the trace divided out by the degree). Hence Question 1 has an easy positive answer in this case, but Question 2 is much more difficult.

Counter-example. (∗) does not split for $R = \mathbb{Q}[x, y]/(xy)$ and its normalization S. Besides, (∗) does not always split for normal $\mathbb{F}_p$-algebras. Question 1 and therefore Question 2 have negative answers in these cases.

2.2.

M. Hochster's *direct summand conjecture* [15] (1969) concerns Question 1, namely:

$$\mathrm{DSC}: \textit{The sequence } (*) \textit{ splits if } R \textit{ is } \text{regular.}$$

For instance, this may be the situation of a Noether normalization of an affine ring S (R being a polynomial ring), or a Cohen presentation of a complete local domain S (R being a ring of formal power series).

The state of the art of DSC before 2016 could be summarized as follows:

a) Hochster gave (short) proofs when R contains a field,
b) he reduced the problem to the unramified complete local case with perfect residue field k of characteristic p:

$$R \cong W(k)[[x_2, \cdots, x_d]], \quad (x_1 = p),$$

c) R. Heitmann [13] (2002) gave a proof in dimension $d \leq 3$[1].

Here is one of Hochster's proofs for a regular complete local domain R of characteristic p with perfect residue field $k = R/\mathfrak{m}$. Let F denote the Frobenius endomorphism of R or its finite extension S, and let n be a positive integer. Then $F^n\mathfrak{m}$ is the maximal ideal of F^nR and is contained in $\mathfrak{m}^{p^n}$, so that $\bigcap(R.F^n\mathfrak{m}) = 0$ by Krull's theorem. On the other hand, since k is perfect, R is a finite F^nR-module. Since R is a regular local ring, so is F^nR, and R is a free F^nR-module by Kunz's theorem. Let λ be any non-zero R-linear form on S; rescaling, we may arrange that $\lambda(1) \neq 0$, hence $\lambda(1) \notin R.F^n\mathfrak{m}$ for some n, so that $\lambda(1)$ is part of a basis of the F^nR-module R by Nakayama's lemma. Therefore, there exists a F^nR-linear form μ on R such that $\mu\lambda(1) = 1$ and in particular, $\mu\lambda_{|F^nS}$ provides a splitting of $F^nR \subset F^nS$. By transport of structure via F^n, $(*)$ also splits.

The case of the extension $R \hookrightarrow R^{\frac{1}{p}}$ is already very interesting; local rings R of characteristic p for which its splits are called F-pure[2].

2.3.

As for Question 2, Hochster and C. Huneke have given a positive answer in the geometric case, i.e., in presence of a base field. This is much more difficult than

[1] in dimension ≤ 2, this is easy: up to replacing S by its normalization, it can be assumed to be a reflexive R-module; it is then faithfully flat over the regular ring R of dimension ≤ 2, so that S/R is finite flat, hence projective, which ensures that $(*)$ splits. The first difficult case is dimension 3. Heitmann's proof already has a flavor of "almost algebra", which will play a fundamental role in the general case.

[2] beyond the regular case, they contain characteristic p analogs of log-canonical singularities.

the above short argument, and it turns out that (unlike Question 1) the characteristic 0 case is even more difficult (and is deduced from) the characteristic p case. They proved that if R is a complete regular local domain of characteristic p and S is a finite extension domain, then the absolute integral closure R^+ of R (i.e., the integral closure in an algebraic closure of the fraction field), viewed as an S-algebra, is faithfully flat over R [20].

2.4.

The direct summand conjecture is now a theorem:

2.4.1. Theorem (A. [2] 2016). *Any finite extension of a regular ring splits (as a module).*

In fact, what is stronger, Question 2 also admits a positive answer for regular rings:

2.4.2. Theorem ([2]). *For any finite extension S of a regular ring R, there is an S-algebra T which is faithfully flat over R.*

After Hochster's work, it suffices to deal, in both theorems, with the mixed characteristic case, and more specifically with

$$R = W(k)[[x_2, \cdots, x_d]],$$

where $W(k)$ stands for the Witt ring of a perfect field k of characteristic p. In that case, and if T is p-torsion free and p-adically complete, *faithful flatness over R can be checked by faithful flatness of T/p over R/p.*

The simple argument sketched above for DSC in characteristic p does not extend to the mixed characteristic case: R has a natural Frobenius endomorphism, but it does not extend to the finite extension S in general. This suggests to introduce a mixed characteristic analog of the perfect closure, by introducing p^{th}-power roots of the system of parameters $x_1 = p, x_2, \ldots, x_d$, which brings us into the perfectoid world.

3. Perfectoid notions

3.1.

Let us begin with the notion of perfectoid field, whose origin dates back to J. Tate's studies in Galois cohomology. Let K be a complete non-archimedean field, K^o its valuation ring, and K^{oo} its valuation ideal.

Assume that the valuation is *not discrete* (equivalently: $K^{oo} = (K^{oo})^2$), and that the residue field k is of characteristic $p > 0$.

3.1.1. Proposition (Gabber–Ramero [11]). *The following are equivalent:*

i) $K^o/p \overset{x\mapsto x^p}{\to} K^o/p$ *is surjective,*
ii) for each finite separable L/K, L^o *is almost étale over* K^o, *i.e.,* Ω_{L^o/K^o} *is killed by* K^{oo}.

Such a field K is called *perfectoid* (if one refers to i)) or *deeply ramified* (if one refers to ii)). The residue field k is then a perfect field.

Example. Let $K_0 = W(k)[\frac{1}{p}]$ and K be the completion of $K_0[p^{\frac{1}{p^\infty}}]$. This is the basic perfectoid field in the sequel. Condition i) is easy to checked directly, while condition ii) tells us that K absorbs almost all the ramification of the completion $\hat{\bar{K}}_0$ of the algebraic closure of K_0 (this was used by Tate to compute the Galois cohomology of $\hat{\bar{K}}_0$).

Here and in the proposition, "almost" is used in the sense of Almost Algebra: given a commutative ring $\mathfrak{V}$ and an idempotent ideal $\mathfrak{m}$, one "neglects" all $\mathfrak{V}$-module killed by $\mathfrak{m}$. Almost algebra, introduced by Faltings [9] and developed by O. Gabber and L. Ramero [11], goes much beyond categorical localization, and studies (non-categorical) notions such as almost finite, almost flat, almost étale.

We shall write "$p^{\frac{1}{p^\infty}}$-almost" to specify that the set-up is $(\mathfrak{V}, \mathfrak{m}) = (K^o, K^{oo} = p^{\frac{1}{p^\infty}} K^o)$: "$p^{\frac{1}{p^\infty}}$-almost zero" means "killed by all fractional powers $p^{\frac{1}{p^i}}$".

3.2.

The generalization from fields to algebras, and further to spaces, was initiated by G. Faltings, and fully developed by P. Scholze (and also, to some extent and independently, by K. Kedlaya and R. Liu).

Let A be a Banach K-algebra, and A^o be its sub-K^o-algebra of power-bounded elements.

3.2.1. Definition (Scholze). *A is* perfectoid *if* $A^o/p^{1/p} \overset{x\mapsto x^p}{\to} A^o/p$ *is an isomorphism.*

The norm of such an algebra A is equivalent to the spectral norm, and A^o is the unit ball for the latter.

Example. $A^o = R_\infty := \widehat{\bigcup W(k)[p^{\frac{1}{p^i}}][[x_2^{\frac{1}{p^i}}, \ldots, x_d^{\frac{1}{p^i}}]]}$, $A = R_\infty\left[\frac{1}{p}\right]$.

A fundamental result of perfectoid theory is the so-called "almost purity theorem":

3.2.2. Theorem (Faltings; Scholze [31], Kedlaya–Liu [23]). *Let A be a perfectoid algebra over a perfectoid field, and B be a finite étale A-algebra.*

Then B is perfectoid, and B^o is an $p^{\frac{1}{p^\infty}}$-almost finite étale A^o algebra.

4. Perfectoid Abhyankar lemma

4.1.

Let us come back to the situation of DSC, and keep the above notation. $S \otimes_R R_\infty[\frac{1}{p}]$ may not be étale over $R_\infty[\frac{1}{p}]$, hence one cannot apply Almost Purity: a non-étale finite extension of a perfectoid algebra need not be perfectoid.

To remedy this, we take inspiration from *Abhyankar's classical lemma*, which tells that under appropriate assumptions, one can achieve étaleness by adjoining roots of the discriminant.

We follow this strategy. Let $g \in R = W(k)[[x_2, \ldots, x_d]]$ be a discriminant of $S[\frac{1}{p}]/R[\frac{1}{p}]$. The first step is to note that adjoining p^{th}-power roots of g, in the (non-naive) sense of considering $R_\infty[\frac{1}{p}]\langle g^{\frac{1}{p^\infty}}\rangle^o$, is "harmless", namely:

4.1.1. Theorem (A. [2]). *Let A be a perfectoid $\mathscr{K}$-algebra, and let $g \in A^o$ be a non-zero divisor. Then for any n, $A\langle g^{\frac{1}{p^\infty}}\rangle^o/p^m$ is $p^{\frac{1}{p^\infty}}$-almost faithfully flat over A^o/p^m.*

In particular, $R_{\infty,g} := R_\infty[\frac{1}{p}]\langle g^{\frac{1}{p^\infty}}\rangle^o$ has the property that $R_{\infty,g}/p$ is $p^{\frac{1}{p^\infty}}$-almost faithfully flat over R_∞/p. Note that $R_{\infty,g}$ is in general much bigger (even before completion) than the ring $R_\infty[g^{\frac{1}{p^\infty}}]$: it contains elements such as $p^{-\frac{1}{p}}(g^{\sigma^{-1}} - g^{\frac{1}{p}})$, where $\sigma : W(k)[[x_2^{\frac{1}{p}}, \ldots, x_d^{\frac{1}{p}}]] \to W(k)[[x_2, \ldots, x_d]]$ is the natural $W(k)$-semilinear Frobenius isomorphism.

The proof of the theorem uses a deformation argument: one spreads out the perfectoid space attached to A^o_∞ into a perfectoid space Y by adding one variable x. Let $Y^{<\epsilon}$ be the (perfectoid) ϵ-tubular neighborhood of the locus $x = g$. Then:

- $A\langle g^{\frac{1}{p^\infty}}\rangle^o$ is $p^{\frac{1}{p^\infty}}$-almost equal to $\widehat{\operatorname{colim}_\epsilon}\ \mathscr{O}^+(Y^{<\epsilon})$.
- Using Scholze's (almost) description of $\mathscr{O}^+(Y^{<\epsilon})/p^\epsilon$ in terms of "Puiseux-like" series in the variable $x - g$ with coefficients in $O^+(Y)/p^\epsilon$, one shows that $\mathscr{O}^+(Y^{<\epsilon})/p^\epsilon$ is almost faithfully flat over $\mathscr{O}^+(Y)/p^\epsilon$.

For details, we refer to [2] (and/or to the short account [3])[3].

[3] this theorem and its avatars also play an important role in the latest development of perfectoid theory: prism and prismatic cohomology (Bhatt, Scholze).

4.2.

The next step represents at the same time a perfectoid version of Abhyankar's lemma, and a ramified version of the Almost Purity theorem.

4.2.1. Theorem (A. [1]). *Let A be a perfectoid K-algebra, containing a sequence of p^{th}-power roots of a non-zero divisor $g \in A^o$ (E.g. $A = R_{\infty,g}[\frac{1}{p}]$). Let B' be a finite étale $A[\frac{1}{g}]$-algebra, and B^o be the integral closure of A^o in B' (hence $B^o[\frac{1}{pg}] = B'$).*

Then for any n, B^o/p^n is an $(pg)^{\frac{1}{p^\infty}}$-almost finite étale A^o/p^n algebra.

The proof also uses a deformation argument. Let X be the perfectoid space attached to $A^o = \mathscr{O}^+(X)$, and $X_{>\epsilon}$ be the (perfectoid) complement of ϵ-tubular neighborhood of the discriminant locus $g = 0$. Then:

- $\mathscr{O}^+(X)$ is $(pg)^{\frac{1}{p^\infty}}$-almost equal to $\lim_\epsilon \mathscr{O}^+(X_{>\epsilon})$ (by Scholze's perfectoid Riemann extension theorem).
- By almost purity over $X_{>\epsilon}$, $(B'\mathscr{O}(X_{>\epsilon}))^+$ is almost finite étale over $\mathscr{O}^+(X_{>\epsilon})$.

The bulk of the work is the passage to the limit $\epsilon \to 0$, for which we refer to [1] (and/or again to the short account [3]).

4.3.

Let us go back to DSC, in the mixed characteristic setting:

$$R = W(k)[[x_2, \ldots, x_d]],\ S \text{ finite extension, étale outside } pg = 0.$$

Applying the perfectoid Abhyankar lemma with $A = R_{\infty,g}[\frac{1}{p}]$, $B' = S \otimes_R A[\frac{1}{g}]$, one gets an S-algebra B^o sitting on top of a tower

$$R \xrightarrow{\alpha} R_\infty \xrightarrow{\beta} R_{\infty,g} \xrightarrow{\gamma} B^o,$$

where α is faithfully flat, β is $p^{\frac{1}{p^\infty}}$-almost faithfully flat modulo p, γ is $(pg)^{\frac{1}{p^\infty}}$-almost faithfully flat modulo p. It follows that B^o/p is almost isomorphic to a faithfully flat R/p-algebra. *Thus B^o is "almost" our wanted T* (*cf.* 2.4).

How to get rid of "almost"? For this, we will use some "Cohen–Macaulay notions".

5. Big Cohen–Macaulay algebras

5.1.

Let S be a Noetherian local ring, and T be a (possibly big, i.e., non-Noetherian) extension.

5.1.1. Definition. *T is a (big) Cohen–Macaulay S-algebra if any sequence of parameters $x_1, \dots, x_d$ of S becomes regular in T* (i.e., *x_1 is non-zero divisor in T, x_2 is non-zero divisor in $T/x_1T, \dots$, and $T \neq (x_1, \dots, x_d)T$).*

Hochster conjectured that *such a T always exists*.

Example. If S a complete local domain of characteristic p, its absolute integral closure S^+ is a big Cohen–Macaulay S-algebra (Hochster–Huneke, *cf.* 2.3).

The relevance of Cohen–Macaulay algebras to our purpose comes from the following (classical) lemma:

5.1.2. Lemma. *Assume S is a complete local domain, hence a finite extension of complete regular local ring R.*

Then T is a (big) Cohen–Macaulay S-algebra if and only if T is R-faithfully flat.

Flatness of Cohen–Macaulay algebras T over a regular local ring R can be seen as follows. Any finite R-module M has projective dimension $\leq d = \dim R$, hence $\mathrm{Tor}_i^R(M, T) = 0$ for all $i > d$. One argues by descending induction on i. By dévissage, one may assume that $M = R/\mathfrak{p}$ for a prime $\mathfrak{p}$. If $(x_1, \dots, x_i)$ is a maximal regular sequence contained in $\mathfrak{p}$, M embeds into $N := R/R(x_1, \dots, x_i)$. But $\mathrm{Tor}_i^R(N, T) = 0$ for all $i > 0$ if T is a Cohen–Macaulay R-algebra. The exact sequence $\mathrm{Tor}_{i+1}^R(N/M, T) \to \mathrm{Tor}_i^R(M, T) \to \mathrm{Tor}_i^R(N, T)$ is thus the 0-sequence, hence T is a flat R-module.

5.2.

The problem to pass from an *almost* Cohen–Macaulay S-algebra T_- (such as B^o at the end of the previous section) to a *genuine* Cohen–Macaulay S-algebra T can be settled by Hochster's (classical) technique of algebra modifications [17], [21]. The idea is to force each relation $x_{i+1}t_{i+1} = \sum_1^i x_j t_j$ with $t_j \in T_-, x_i \in S$, to come from a relation $t_{i+1} = \sum_1^i x'_j t'_j$ by introducing step by step new variables: a partial modification of degree n of an S-module M is a homomorphism $M \to M'$ where, given a relation $x_{i+1}m_{i+1} = \sum_1^i x_j m_j$ with coefficients m_j in M, $M' := M[T_1, \dots, T_i]_{\leq n}/(m_{i+1} - \sum_1^i x_j T_j) \cdot$

$M[T_1, \ldots, T_i]_{\leq n-1}$. Starting from T_- and taking the colimit of partial modifications for all such relations yields an S-algebra T for which x_{i+1} is not a zero divisor modulo $x_1, \ldots, x_i$. Condition $T \neq (x_1, \ldots, x_d)T$ comes from the assumption that T_- is almost Cohen–Macaulay.

This provides the last step in the proof of Hochster's conjecture about the existence of big Cohen–Macaulay algebras. Using the above lemma, this implies Theorem 2.4.2 and the Direct Summand Conjecture 2.4.1 (positive answer to Questions 1 and 2 for regular rings).

5.3.

This is the first item of the following:

5.3.1. Theorem (A. [2], [4]).

(1) Any Noetherian local ring S has a big Cohen–Macaulay algebra T.
(2) For any local morphism $S \to S'$ of Noetherian complete local domains, there is a morphism of respective big Cohen–Macaulay algebras $T \to T'$.

The second item is the so-called "weak functoriality of big Cohen–Macaulay algebras", also conjectured by Hochster. Its poof is more subtle and requires a delicate consideration of integral perfectoid algebras (the most difficult case is when S is of mixed characteristic and S' of characteristic p).

In mixed characteristic, one has a more precise result regarding item (1) (which is used in the proof of item (2)):

5.3.2. Theorem (Shimomoto [34], A. [4]). *Any Noetherian complete local domain S of characteristic $(0, p)$ admits an (integral) perfectoid[4] Cohen–Macaulay algebra T.*

K. Shimomoto's idea is to use the tilting equivalence between perfectoid algebras over K and certain perfect algebras over a perfect field $K^\flat \cong \widehat{k((t^{\frac{1}{p^\infty}}))}$ of characteristic p (Scholze): one can apply Hochster modifications in characteristic p, after tilting, and then untilt. An alternative and more recent argument, due to Gabber [10], uses ultraproducts instead of tilting and Hochster modifications.

[4] an integral perfectoid K^o-algebra is a p-adically complete p-torsionfree K^o-algebra T such that the Frobenius map $T/p^{\frac{1}{p}} \to T/p$ is an isomorphism.

5.4. *Kunz' theorem in mixed characteristic*

In the same vein but independently from the above results, B. Bhatt, S. Iyengar and L. Ma extended Kunz's flatness criterion in characteristic p to mixed characteristic. Let us recall that for a Noetherian ring R of characteristic p, Kunz' theorem asserts that *R is regular if and only if $R \stackrel{x\mapsto x^p}{\to} R$ is flat* (which amounts to saying that *there exists a perfect, faithfully flat R-algebra*[5]).

Let R now be any Noetherian p-adically complete ring.

5.4.1. Theorem (Bhatt–Iyengar–Ma [6] 2018). *R is regular if and only if there exists an integral perfectoid*[6]*, faithfully flat, R-algebra.*

6. Applications to the homological conjectures

6.1. *The homological turn*

We have already mentioned the homological turn in commutative algebra in the 60's: from the study of Noetherian rings and their ideals (Krull, Zariski, ...) to the study of the homological properties of their modules (Auslander, Buchsbaum, Serre, ...).

Example. A local ring R is regular if and only if every finite R-module has a finite free resolution.

C. Peskine and L. Szpiro introduced reduction techniques to characteristic p, and the crucial observation that in characteristic p, Frobenius preserves finite free resolutions (which may be seen as an extension of Kunz' theorem).

Example. A local ring S is Cohen–Macaulay if and only if there is an S-module of finite length with a finite free resolution [28], [30].

We refer to [7], [18] for accounts of the homological conjectures (before the introduction of perfectoid techniques). The relevance of Cohen–Macaulay algebras typically comes from the following observation: let

$$(F_*) \quad 0 \longrightarrow F_s \longrightarrow F_{s-1} \longrightarrow \cdots \longrightarrow F_i \stackrel{\phi_i}{\longrightarrow} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0$$

be a complex of finite free S-modules, let r_i be the rank of $\operatorname{Im}\phi_i$, and let $I_{r_i}(\phi_i)$ be the Fitting ideal of S generated the minors of size r_i of ϕ_i.

If (F_*) is acyclic, then $\operatorname{codim} I_{r_i}(\phi_i) \geq i$ (Buchsbaum–Eisenbud). The converse is not true in general if S is not Cohen–Macaulay; however if $\operatorname{codim} I_{r_i}(\phi_i) \geq i$, then $(F_*) \otimes_S T$ is acyclic for any Cohen–Macaulay S-algebra T (*cf.* [7, 9.1.8]).

5 such an algebra is a (big) Cohen–Macaulay R-algebra.

6 in some generalized sense, which does not require a perfectoid base field.

6.2. *Pure subrings of regular rings*

The following consequence of weak functoriality of Cohen–Macaulay algebras generalizes and extends in mixed characteristic the well-known theorem of Hochster–Roberts about Cohen–Macaulayness of rings of invariants of polynomials under reductive groups in characteristic 0.

6.2.1. Theorem (Heitmann–Ma [14]; A. [4]). *Any pure subring of a regular ring (e.g. any subring which is a direct summand as a module) is Cohen–Macaulay.*

Let us recall the (classical) derivation of the theorem from 5.3.1 (2). We denote the embedding by $S \hookrightarrow R'$, where R' is regular. We may also assume that S is a finite extension of a regular ring R, and it suffices to show that S is flat over R. Let $T_S \to T_{R'}$ be a compatible map of Cohen–Macaulay algebras. For any R-module M, one has a commutative square

$$\begin{array}{ccc} \mathrm{Tor}_i^R(M,S) & \xrightarrow{a} & \mathrm{Tor}_i^R(M,R') \\ \downarrow & & \downarrow{\scriptstyle b} \\ \mathrm{Tor}_i^R(M,T_S) & \longrightarrow & \mathrm{Tor}_i^R(M,T_{R'}) \end{array}$$

in which a, b are injective since $S \hookrightarrow R' \hookrightarrow T_{R'}$ are pure morphisms (see e.g. [2, 5.1.1]), and $\mathrm{Tor}_i^R(M,T_S) = 0$ for $i > 0$ since T_S is Cohen–Macaulay over S, hence flat over R; therefore $\mathrm{Tor}_i^R(M,S) = 0$.

In fact, Heitmann and Ma proved just the right amount of weak functoriality needed to get the corollary, but they also obtained more: S has *pseudo-rational singularities*.

6.3. *The syzygy theorem*

Let us turn to the traditional problem of estimating Betti numbers and ranks of syzygies of Noetherian modules.

Let S be a Noetherian local ring, and let M be a finite S-module which admits a finite free resolution.

6.3.1. Theorem. *Let* $0 \to S^{b_s} \to S^{b_{s-1}} \to \cdots \to S^{b_0} \to M \to 0$ *be a minimal free resolution of* M*. Then the* i*-th syzygy has rank* $\geq i$ *for any* $i \leq s-1$.

A fortiori, $b_i \geq 2i+1$ *if* $i < s-1$*, and* $b_{s-1} \geq s$.

These bounds are optimal in general; for instance, if S is the localization of $k[x_1,x_2,x_3]$ at 0 and $M = k$, the minimal free resolution is $0 \to S \to S^3 \to$

$S^3 \to S \to k \to 0$, so that $s = 3$ and the first two syzygies have rank 1 and 2 respectively.

E. Evans and P. Griffiths [8] stated and proved this theorem when S contains a field, using the existence of Cohen–Macaulay algebras in this case. Hochster [16] and T. Ogoma [27] proved that the statement follows from DSC in general. DSC being proven (thanks to perfectoid techniques), these bounds are now *unconditional*.

Similar bounds hold for minimal injective resolutions, S being replaced by the injective hull of the residue field (*cf.* [7, 9.6]).

Actually, most of the homological conjectures which were standing for a while on Hochster's list are now solved, as consequences of theorem 5.3.1 (still, a number of related old problems remain open, such as Serre's positivity conjecture for intersection numbers and several questions posed by Peskine and Szpiro in their thesis).

7. Applications to singularities

7.1.

Starting from theorems 5.3.1 and 5.3.2, Ma and K. Schwede [24], [25] developed an analog of tight closure theory in mixed characteristic, with applications to singularities. In their theory, integral perfectoid Cohen–Macaulay algebras play somehow the role of resolution of singularities in characteristic 0.

Let S be a local domain, essentially of finite type over $\mathbb{C}$, and let $\pi : Y \to \operatorname{Spec} S$ be a resolution of singularities. By Grauert–Riemenschneider, $R^i\Gamma(Y, \omega_Y) = 0$ for $i > 0$, whence $\mathbb{H}^j_{\mathfrak{m}}(R\Gamma(Y, \mathcal{O}_Y)) = 0$ for $j < \dim S$ by local duality. Thus $R\Gamma(Y, \mathcal{O}_Y)$ appears as a "derived avatar" of a Cohen–Macaulay algebra. Since in many questions about singularities, it is not so much resolution itself which matters but the object $R\Gamma(Y, \mathcal{O}_Y) \in D^b(S)$, the idea is to replace, in mixed characteristic or in characteristic p, this object by suitable (big) Cohen–Macaulay S-algebras.

7.2.

Let us just outline one striking application of this idea to *rational singularities*. Recall that, by definition, S (as before) "is" a rational singularity if and only if $R\Gamma(Y, \mathcal{O}_Y) \cong S$. In particular, by Grauert–Riemenschneider and local duality, S is Cohen–Macaulay.

It has been known for a long time (N. Hara [12], K. Smith [35], V. Mehta and V. Srinivas [26]) that S is a rational singularity if and only if, after "spreading out", (S modulo p) is a F-rational singularity (i.e., it is Cohen–Macaulay and in addition, the top local cohomology module is a simple Frobenius module) for $p \gg 0$. While F-rationality is a checkable property (Macaulay2), checking it for all $p \gg 0$ does not give rise to an algorithm.

But remarkably, Ma and Schwede recently proved [25] that it suffices to check that (S *modulo* p) *is a* F*-rational singularity for some* p. This provides an efficient algorithm: in practice, one may choose a very small prime p.

The proof goes through mixed characteristic. More precisely, it uses a perfectoid avatar of the notion of rational singularity in mixed characteristic as link between characteristic p and characteristic 0. In some sense, this is an application of perfectoid theory to complex geometry!

A similar result holds for *log-terminal singularities* (rational singularities for which the multiplier ideal is trivial); in this case, the corresponding property in characteristic p has a simple formulation: *for any non-zero divisor* $s \in S$*, there is a positive integer* n *such that* $S \xrightarrow{\cdot s^{\frac{1}{p^n}}} S^{\frac{1}{p^n}}$ *splits*[7].

These results may be the beginning of a program toward making the connection between singularity theory and birational geometry in characteristic 0 and their counterparts in characteristic p both effective and direct (even though the proof that the algorithms work would be indirect and go through mixed characteristic and perfectoid techniques).

7.3.

Another striking application concerns symbolic powers. For any Noetherian ring S, prime ideal $\mathfrak{p}$, and positive integer n, the symbolic power is defined by

$$\mathfrak{p}^{(n)} := (\mathfrak{p}^n S_{\mathfrak{p}}) \cap \mathfrak{p}.$$

In the affine case, this is just the ideal of functions which vanish at $V(\mathfrak{p})$ at order at least n. Obviously, $\mathfrak{p}^{(n)} \supset \mathfrak{p}^n$, and if $\mathfrak{p}$ is generated by a regular sequence, it is well-known that $\mathfrak{p}^{(n)} = \mathfrak{p}^n$. To compare $\mathfrak{p}^{(n)}$ and $\mathfrak{p}^n$ in general is a classical problem which has far-reaching applications, e.g. in complex analysis or in transcendental number theory (Waldschmidt constants).

7.3.1. Theorem (Ma, Schwede [24]). *Let* S *be excellent regular of dimension* d*. Then for any prime* $\mathfrak{p}$ *and any positive integer* n, $\mathfrak{p}^{(dn)} \subset \mathfrak{p}^n$.

In characteristic 0, this was proven by Ein–Lazarsfeld–Smith using subadditivity of the multiplier ideal. In mixed characteristic, Ma and Schwede follow the same strategy, defining a new notion of multiplier ideal in which the complex $R\Gamma(Y, \mathcal{O}_Y)$ attached to a log-resolution of $V(\mathfrak{p})$ is replaced by an integral perfectoid Cohen–Macaulay algebra for $S_{\mathfrak{p}}$.

[7] for the intermediate notion of log-canonical singularity, the situation is more subtle: for instance, the origin of the affine Fermat surface of degree 3 is a log-canonical singularity, but (S modulo p) is F-pure if and only if $p \equiv 1 \pmod 3$. See also [22, Conjecture 10.3.15] and [36].

References

[1] Y. André, Le lemme d'Abhyankar perfectoïde, Publ. Math. Inst. Hautes Études Sci., **127** (2018), 1–70.

[2] Y. André, La conjecture du facteur direct, Publ. Math. Inst. Hautes Études Sci., **127** (2018), 71–93.

[3] Y. André, Perfectoid spaces and the homological conjectures, In: Proc. Int. Cong. of Math.—2018, Rio de Janeiro, Vol. 1, 2018, pp. 249–260.

[4] Y. André, Weak functoriality of Cohen–Macaulay algebras, preprint, arXiv:1801.10010.

[5] B. Bhatt, Almost direct summands, Nagoya Math. J., **214** (2014), 195–204.

[6] B. Bhatt, S.B. Iyengar and L. Ma, Regular rings and perfect(oid) algebras, preprint, arXiv:1803.03229.

[7] W. Bruns and J. Herzog, Cohen–Macaulay Rings, Cambridge Stud. Adv. Math., **39**, Cambridge Univ. Press, Cambridge, 1993.

[8] E.G. Evans and P. Griffith, The syzygy problem, Ann. of Math. (2), **114** (1981), 323–333.

[9] G. Faltings, Almost étale extensions, In: p-adic Cohomologies and Arithmetic Applications (II), Astérisque, **279**, Soc. Math. France, Paris, 2002, pp. 185–270.

[10] O. Gabber, Some remarks on big Cohen–Macaulay algebras, talk at MSRI, Feb. 2018.

[11] O. Gabber and L. Ramero, Almost Ring Theory, Lecture Notes in Math., **1800**, Springer-Verlag, 2003.

[12] N. Hara, A characterization of rational singularities in terms of injectivity of Frobenius maps, Amer. J. Math., **120** (1998), 981–996.

[13] R.C. Heitmann, The direct summand conjecture in dimension three, Ann. of Math. (2), **156** (2002), 695–712.

[14] R.C. Heitmann and L. Ma, Big Cohen–Macaulay algebras and the vanishing conjecture for maps of Tor in mixed characteristic, Algebra Number Theory, **12** (2018), 1659–1674.

[15] M. Hochster, Contracted ideals from integral extensions of regular rings, Nagoya Math. J., **51** (1973), 25–43.

[16] M. Hochster, Canonical elements in local cohomology modules and the direct summand conjecture, J. Algebra, **84** (1983), 503–553.

[17] M. Hochster, Big Cohen–Macaulay algebras in dimension three via Heitmann's theorem, J. Algebra, **254** (2002), 395–408.

[18] M. Hochster, Homological conjectures, old and new, Illinois J. Math., **51** (2007), 151–169.

[19] M. Hochster and C. Huneke, Tight closure, invariant theory, and the Briançon–Skoda theorem, J. Amer. Math. Soc., **3** (1990), 31–116.

[20] M. Hochster and C. Huneke, Infinite integral extensions and big Cohen–Macaulay algebras, Ann. of Math. (2), **135** (1992), 53–89.

[21] M. Hochster and C. Huneke, Applications of the existence of big Cohen–Macaulay algebras, Adv. Math., **113** (1995), 45–117.

[22] S. Ishii, Introduction to Singularities. 2nd ed., Springer-Verlag, 2018.

[23] K.S. Kedlaya and R. Liu, Relative p-adic Hodge Theory: Foundations, Astérisque, **371**, Soc. Math. France, Paris, 2015.

[24] L. Ma and K. Schwede, Perfectoid multiplier/test ideals in regular rings and bounds on symbolic powers, Invent. Math., **214** (2018), 913–955.

[25] L. Ma and K. Schwede, Singularities in mixed characteristic via perfectoid big Cohen–Macaulay algebras, preprint, arXiv:1806.09567.

[26] V.B. Mehta and V. Srinivas, A characterization of rational singularities, Asian J. Math., **1** (1997), 249–271.

[27] T. Ogoma, A note on the syzygy problem, Comm. Algebra, **17** (1989), 2061–2066.

[28] C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale, Inst. Hautes Études Sci. Publ. Math., **42** (1973), 47–119.

[29] M. Raynaud and L. Gruson, Critères de platitude et de projectivité, Invent. Math., **13** (1971), 1–89.
[30] P. Roberts, Le théorème d'intersection, C. R. Acad. Sci. Paris Sér. I Math., **304** (1987), 177–180.
[31] P. Scholze, Perfectoid spaces, Publ. Math. Inst. Hautes Études Sci., **116** (2012), 245–313.
[32] P. Scholze, On torsion in the cohomology of locally symmetric varieties, Ann. of Math. (2), **182** (2015), 945–1066.
[33] K. Shimomoto, An application of the almost purity theorem to the homological conjectures, J. Pure Appl. Algebra, **220** (2016), 621–632.
[34] K. Shimomoto, Integral perfectoid big Cohen–Macaulay algebras via André's theorem, Math. Ann., **372** (2018), 1167–1188.
[35] K.E. Smith, F-rational rings have rational singularities, Amer. J. Math., **119** (1997), 159–180.
[36] S. Takagi and K.-i. Watanabe, F-singularities: applications of characteristic p methods to singularity theory, Sugaku Expositions, **31** (2018), 1–42.