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Abundance of minimal surfaces*

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Abstract. This article is concerned with the existence theory of closed minimal hypersurfaces in closed Riemannian manifolds of dimension at least three. These hypersurfaces are critical points for the area functional, and hence their study can be seen as a high-dimensional generalization of the classical theory of closed geodesics (Birkhoff, Morse, Lusternik, Schnirel'mann,...). The best result until very recently, due to Almgren ([2], 1965), Pitts ([37], 1981), and Schoen–Simon ([43], 1981), was the existence of at least one closed minimal hypersurface in every closed Riemannian manifold.

I will discuss the methods I have developed with André Neves, for the past few years, to approach this problem through the variational point of view. These ideas have culminated with the discovery that minimal hypersurfaces in fact abound.

Keywords and phrases: minimal surfaces, min-max theory, Morse index

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1. Introduction

In these notes, prepared for the occasion of the Takagi Lectures of 2018, we will be concerned with the n-dimensional area functional. We will consider a closed Riemannian manifold (M^{n+1}, g) and closed hypersurfaces $\Sigma^n \subset M^{n+1}$. Crit-

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ical points for the area functional are called minimal hypersurfaces. More precisely, we say that Σ is minimal if

$$\frac{d}{dt}\Big|_{t=0} \text{area}(F_t(\Sigma)) = 0$$

for every smooth one-parameter family $\{F_t\}_{t\in(-\varepsilon,\varepsilon)}$ of ambient diffeomorphisms $F_t : M \to M$ satisfying $F_0 = \text{id}$. Equivalently, Σ is minimal in M if its mean curvature vanishes identically.

If $n = 1$, minimal hypersurfaces are just closed geodesics. This is an important special case, because closed geodesics can also be defined as periodic orbits of the geodesic flow. Hence dynamics plays a role when the ambient dimension is two, but is not available in higher dimensions.

The subject started with a question of Poincaré ([38]), who asked whether any Riemannian sphere (S^2, g) contains a nontrivial closed geodesic. If the genus of the surface is at least one, then one can find a nontrivial closed geodesic by minimizing the length functional inside a nontrivial homotopy class of loops. This method fails for the 2-sphere since the sphere is simply-connected.

Birkhoff [4] was the first to realize that in the sphere this is not a problem of minimization. His main insight was that the space of closed loops in S^2 has interesting topology, and that this would force the existence of a nontrivial critical point. Birkhoff introduced the notion of sweepout, a homotopically nontrivial one-parameter family of closed loops in S^2. A sweepout is a family $\{c_s = f \circ \overline{c}_s\}_{s\in[-1,1]}$, where $\overline{c}_s$ is the loop $x_3 = s$ and $f : S^2 \to S^2$ is a degree one map. He proved that the min-max number

$$L = \inf_{\{c_s\}} \sup_{s\in[-1,1]} L(c_s),$$

where $L(c)$ denotes the length of c, is positive and that there exists a closed geodesic $\gamma : S^1 \to (S^2, g)$ such that

$$L(\gamma) = L.$$

Hence

Theorem 1.1 (Birkhoff, [4]). *Every Riemannian 2-sphere contains at least one nontrivial closed geodesic.*

One way of seeing this is by considering the curve shortening flow, defined to be the process of evolving a closed curve α in the direction of the curvature vector:

$$\frac{\partial \alpha}{\partial t} = \overrightarrow{k}_\alpha.$$

The equation is parabolic hence there is always a solution for small positive time $t \in [0, \varepsilon)$. Grayson [14] studied the longtime behavior as t converges to the maximal time of existence $T_{\max}$. He proved that either $T_{\max} < \infty$ and the curve converges to a point or $T_{\max} = \infty$ and there is a sequence $t_k \to \infty$ such that α_{t_k} converges to a nontrivial closed geodesic.

The idea then is to apply curve shortening flow simultaneously to all curves in a sweepout. Because f has degree one it cannot be that every curve in a sweepout converges to a point through curve shortening flow. Hence there must be at least one curve in any sweepout that subsequentially converges to a closed geodesic. Birkhoff did not have the technology of parabolic curve shortening flow available in his time, but he devised a more elementary discrete curve shortening process based on local replacements by length minimizing geodesic segments. Curve shortening flow has the advantage of preserving embeddedness of curves and hence can be used as above to prove the existence of a simple (embedded) closed geodesic.

The work of Birkhoff later inspired Morse [34] and also Lusternik and Schnirel'mann [25] to study more general types of critical points. These theories brought together the fields of topology and calculus of variations.

In the early 1990s, by combining dynamical and variational arguments one obtains:

Theorem 1.2 (Bangert [3], Franks [11], Hingston [19]). *Every Riemannian 2-sphere contains (geometrically distinct) infinitely many closed geodesics.*

Here geometrically distinct means that the closed geodesics as maps from S^1 do not differ by iteration. The possibility of getting the same geodesic iterated many times is a crucial difficulty in the theory of closed geodesics. In higher dimensions this is replaced by the multiplicity problem, as mentioned later in this article.

We would like to consider the higher dimensional case $n \geq 2$, through the variational point of view. Classically one can produce closed minimal surfaces by the parametric approach, in which surfaces are thought as maps from a fixed topological 2-dimensional surface into the manifold. This works only in dimension two (although the codimension can be arbitrary), where one can take advantage of the existence of conformal parametrizations. The area of the image of a conformal map coincides with its Dirichlet energy, which has more coercivity properties than the area functional.

Schoen–Yau [44] and Sacks–Uhlenbeck [42] used this approach to produce incompressible minimal surfaces, while Sacks–Uhlenbeck [41] used it to understand bubbling and produce minimal 2-spheres. Recently, Rivière [40] revisited the subject and proposed a viscosity alternative to the perturbed functionals of Sacks–Uhlenbeck. The minimal surfaces produced by this method might have branch points and self-intersections.

We are going to take the Geometric Measure Theory (GMT) point of view, in which one thinks of submanifolds by themselves and not necessarily as parametrized objects. This can be used to solve the Plateau Problem [8] in great generality, by allowing hypersurfaces of arbitrary topological type. The Plateau Problem asks for the hypersurface of least area with a given boundary. This produces embedded hypersurfaces and is in contrast with the classical/parametric approach of Douglas [7] and Rado [39].

We will consider the space of flat chains mod 2 endowed with the flat topology. Two closed submanifolds Σ_1, Σ_2 of dimension p are close to each other in the flat topology if there is a $(p+1)$-dimensional submanifold Ω with very small $(p+1)$-dimensional area and $\partial\Omega = \Sigma_1 - \Sigma_2$. The space of flat cycles mod 2 of dimension n in $\mathbb{R}^J$ is the completion of the space of boundaryless polyhedral chains mod 2 with respect to the flat topology. The space of flat cycles has the right kind of compactness properties which together with the lower semicontinuity of the area (or mass in GMT language) allow the solution of the Plateau Problem in that class.

We will isometrically embed our closed Riemannian manifold (M^{n+1}, g) into some Euclidean space $\mathbb{R}^J$ by Nash's Theorem [35] and consider the space $\mathcal{Z}_n(M;\mathbb{Z}_2)$ of n-dimensional flat cycles Σ of $\mathbb{R}^J$ with support contained in M. We will take only those cycles that are boundaries in M: $\Sigma = \partial\Omega$, where Ω is an $(n+1)$-dimensional flat chain mod 2 in M.

A one-parameter family $\{\Sigma_t\}_{t\in[0,1]}$ of cycles, with $\Sigma_0 = \Sigma_1 = 0$, is a sweepout of M if one can find a continuous family of $(n+1)$-dimensional chains mod 2 $\{\Omega_t\}$ in M with $\partial\Omega_t = \Sigma_t$, $\Omega_0 = 0$ and $\Omega_1 = M$. An example is given by $\{\Sigma_t = \partial\{f \le t\}\}$, where $f : M \to [0,1]$ is any Morse function.

As in Birkhoff, one can define the width of M as the min-max invariant

$$W = \inf_{\{\Sigma_t\}} \sup_{t\in[0,1]} \text{area}(\Sigma_t).$$

There is always some $t_0 \in [0,1]$ such that $\text{vol}(\Omega_{t_0}) = \text{vol}(M)/2$, and hence $\text{area}(\Sigma_{t_0}) \ge c$ for some fixed constant $c > 0$ by the Isoperimetric Inequality. This implies $W > 0$.

The Min-Max Theorem states:

Theorem 1.3 (Almgren'65 [2], Pitts'81 [37], Schoen–Simon'81 [43]). *Suppose $3 \le (n+1) \le 7$. Then there exist a disjoint collection $\{\Sigma_1, \dots, \Sigma_q\}$ of smooth, closed, embedded minimal hypersurfaces in M and a collection $\{m_1, \dots, m_q\} \subset \mathbb{N}$ such that*

$$W = m_1 \text{area}(\Sigma_1) + \cdots + m_q \text{area}(\Sigma_q).$$

Almgren ([2]) devised a general min-max theory that succeeded in proving the existence of minimal varieties (or stationary integral varifolds) of any dimension, and Pitts ([37]) proved smoothness of the varifold in the codimension one case for $(n+1) \le 6$. Regularity for higher dimensions was proven by Schoen and Simon ([43]).

If $(n+1) \geq 8$, the above theorem is still true but the minimal hypersurface can have a singular set of Hausdorff dimension at most $(n-7)$. This follows from the existence theory of Almgren–Pitts combined with the regularity theory of Schoen–Simon.

If $(n+1) = 2$, the min-max theory with flat cycles does not necessarily produce a closed geodesic. In general the min-max minimal variety could be a stationary geodesic network (with integer multiplicities).

In 1982, Yau [53] conjectured:

Conjecture. Every closed Riemannian three-manifold (M^3, g) contains infinitely many smooth, closed minimal surfaces.

A few years ago, I started working with André Neves on the subject. The starting point is the rich topology of the space of cycles mod 2.

Almgren [1] proved that there is a canonical isomorphism between the homotopy group $\pi_l(\mathscr{Z}_k(M, \mathbb{Z}_2))$ of the space of k-dimensional cycles and the homology group $H_{k+l}(M, \mathbb{Z}_2)$ of M (he did it for integer coefficients but the same proof applies to coefficients in $\mathbb{Z}_2$). In the codimension one case this gives

$$\pi_1(\mathscr{Z}_n(M, \mathbb{Z}_2)) = \mathbb{Z}_2,$$

and

$$\pi_k(\mathscr{Z}_n(M, \mathbb{Z}_2)) = 0$$

for all $k \geq 2$. This is exactly the list of homotopy groups of $\mathbb{RP}^\infty$. This gives us homotopically nontrivial k-parameter families of hypersurfaces to work with, like in the inclusion $\mathbb{RP}^k \subset \mathbb{RP}^\infty$.

There is another way of understanding this $\mathbb{RP}^\infty$ structure (see Sect. 4), through the boundary map

$$\partial : \mathbf{I}_{n+1}(M^{n+1}, \mathbb{Z}_2) \longrightarrow \mathscr{Z}_n(M^{n+1}, \mathbb{Z}_2),$$

where $\mathbf{I}_{n+1}(M^{n+1}, \mathbb{Z}_2)$ is the space of $(n+1)$-dimensional flat chains modulo two. If $T = \partial U$, then also $T = \partial(M - U)$. The chains U and $M - U$ are, by the Constancy Theorem of Geometric Measure Theory ([46]), the only two chains with boundary equal to T. This implies that the boundary map is a two cover. The involution

$$\alpha : \mathbf{I}_{n+1}(M^{n+1}, \mathbb{Z}_2) \longrightarrow \mathbf{I}_{n+1}(M^{n+1}, \mathbb{Z}_2)$$

defined by $\alpha(U) = M - U$ plays the role of the antipodal map and the boundary map is a two-cover just like the standard projection $\mathbb{S}^\infty \to \mathbb{RP}^\infty$.

In 1988, Gromov [15] performed a study of the area functional in which hypersurfaces were thought as zero sets of a real function defined on M. The space of real functions is an infinite dimensional vector space, and since multiplication by a nonzero scalar does not change the zero set we can think of the area as defined in an infinite-dimensional projective space. Gromov was interested in defining the spectrum of such functional, in analogy with the standard spectrum of the Laplacian defined through the Rayleigh functional

$$f \longmapsto \int_M |\nabla f|^2 \Big/ \int_M f^2.$$

We will come back to this later.

In Sect. 2, I will discuss recent existence theorems of closed minimal hypersurfaces. In Sect. 3, I will discuss characterizations of the Morse index of min-max minimal hypersurfaces. In Sect. 4, I will prove that the space of flat hypercycles mod 2 is weakly homotopically equivalent to $\mathbb{RP}^\infty$. In Sect. 5, I will sketch the proof of density of minimal hypersurfaces for generic metrics. In Sect. 6, I will be more technical and describe how the min-max theory for the area functional is formulated.

2. Infinitely many minimal hypersurfaces

We proved:

Theorem 2.1 (—, Neves [31]). *Suppose* $3 \leq (n+1) \leq 7$. *Then for any closed Riemannian manifold* (M^{n+1}, g),

(i) *either there exist infinitely many closed, embedded, minimal hypersurfaces,*
(ii) *or there exists a disjoint collection of* $(n+1)$ *closed, embedded, minimal hypersurfaces* $\{\Sigma_1, \dots, \Sigma_{n+1}\}$.

We say the manifold satisfies the Frankel property if any two closed minimal hypersurfaces have to intersect. Hence we get:

Corollary 2.2 (—, Neves [31]). *Suppose* $3 \leq (n+1) \leq 7$. *Then any closed Riemannian manifold* (M^{n+1}, g) *that satisfies the Frankel property contains infinitely many smooth, closed, embedded minimal hypersurfaces.*

If the Ricci curvature of g is positive, it follows from Frankel's theorem ([10]) that (M, g) satisfies the Frankel property.

This property is also implied when there are no stable minimal hypersurfaces. A minimal hypersurface is said to be stable if the second variation of the area functional is nonnegative for any variation, like in a minimum point. If the initial velocity of the variation is given by $X = fN$, where N is a unit normal to Σ (assuming Σ is two-sided and minimal), the Second Variation Formula states that

$$\frac{d^2}{dt^2}\Big|_{t=0} \text{area}(\Sigma_t) = \int_\Sigma \{|\nabla_\Sigma f|^2 - (|A|^2 + Ric(N,N)) f^2\}\, d\Sigma,$$

where A denotes the second fundamental form of Σ. Hence one immediately sees from the formula, by choosing f to be constant equal to 1, that Σ cannot be stable if $Ric(g) > 0$. If Σ_1 and Σ_2 are two disjoint and homologous minimal hypersurfaces, by minimizing area in the homology class one finds a stable minimal hypersurface between Σ_1 and Σ_2. Hence Σ_1 and Σ_2 must intersect if there are no stable hypersurfaces.

In order to prove Theorem 2.1, we need to use the $\mathbb{RP}^\infty$ structure. The definition of the sweepouts will be given in terms of cohomology classes.

The cohomology ring $H^*(\mathbb{RP}^\infty, \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\overline{\lambda}]$ where $\overline{\lambda}$ is the generator of $H^1(\mathbb{RP}^\infty, \mathbb{Z}_2) = \mathbb{Z}_2$. In particular, $H^k(\mathbb{RP}^\infty, \mathbb{Z}_2) = \{0, \overline{\lambda}^k\}$ where $\overline{\lambda}^k = \overline{\lambda} \cup \cdots \cup \overline{\lambda}$ is the cup product power. We use the same notation $\overline{\lambda}$ to denote the generator of $H^1(\mathscr{Z}_n(M^{n+1}, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2$.

The cohomology class $\overline{\lambda}$ has geometric meaning. Given a continuous loop of cycles $\gamma : S^1 \to \mathscr{Z}_n(M^{n+1}, \mathbb{Z}_2)$, we have that $\overline{\lambda} \cdot (\gamma) = 1$ if and only if γ is a sweepout of M (i.e., if and only if the lift $\tilde{\gamma}$ of γ to $\mathbf{I}_{n+1}(M^{n+1}, \mathbb{Z}_2)$ is open).

Let X be a finite dimensional simplicial complex. We make the following definitions.

Definition. *A continuous map $\Phi : X \to \mathscr{Z}_n(M^{n+1}, \mathbb{Z}_2)$ is called a k-*sweepout *if it detects the cohomology class $\overline{\lambda}^k$ in the sense that*

$$\Phi^*(\overline{\lambda}^k) \neq 0 \in H^k(X, \mathbb{Z}_2).$$

In this case we write $\Phi \in \mathscr{P}_k$.

Definition. *The k-*width *of M is the number*

$$\omega_k(M, g) = \inf_{\Phi \in \mathscr{P}_k} \sup_{x \in \mathrm{dmn}(\Phi)} \mathrm{area}(\Phi(x)),$$

where dmn(Φ) *denotes the domain of Φ, which might depend on Φ.*

Definition. *The* volume spectrum *of M is the sequence of numbers:*

$$\{\omega_1(M, g) \leq \omega_2(M, g) \leq \cdots \leq \omega_k(M, g) \leq \cdots\}.$$

The sequence is ordered because every $(k+1)$-sweepout is also a k-sweepout.

Example. The volume spectrum is nonlinear and hence extremely hard to compute even for the simplest manifolds. For the unit three-sphere endowed with the standard metric $\overline{g}$, we have:

$$\omega_1(S^3, \overline{g}) = \omega_2(S^3, \overline{g}) = \omega_3(S^3, \overline{g}) = \omega_4(S^3, \overline{g}) = 4\pi.$$

The fact that

$$\omega_4(S^3, \overline{g}) \leq 4\pi$$

follows from the existence of the 4-sweepout $\Phi : \mathbb{RP}^4 \to \mathscr{Z}_2(S^3, \mathbb{Z}_2)$ given by:

$$\Phi([a_0 : a_1 : a_2 : a_3 : a_4]) = \{x \in S^3 : a_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0\}.$$

The lower bound

$$4\pi \leq \omega_1(S^3)$$

follows from min-max theory and the fact that any closed minimal surface in S^3 has area at least 4π.

It turns out we know the next nontrivial element:

$$\omega_5(S^3) = 2\pi^2.$$

The proof of this is not easy and requires the solution of the Willmore Conjecture ([28]).

If there are no stable minimal hypersurfaces, for instance if the Ricci curvature is positive, then Almgren–Pitts min-max theory applied to the class of k-sweepouts gives that

$$\omega_k(M, g) = m_k \cdot \text{area}(\Sigma_k),$$

with $m_k \in \mathbb{N}$ and Σ_k a connected minimal hypersurface. In [31], we prove that this contradicts the sublinear bounds:

Theorem 2.3 (Gromov [16], Guth [18]). *There exist constants $c_1, c_2 > 0$ depending on M such that*

$$c_1 k^{\frac{1}{n+1}} \leq \omega_k(M) \leq c_2 k^{\frac{1}{n+1}}.$$

for every $k \in \mathbb{N}$.

To be precise, the fact that

$$\omega_k(M, g) = m_1^{(k)} \text{area}(\Sigma_1^{(k)}) + \cdots + m_{q_k}^{(k)} \text{area}(\Sigma_{q_k}^{(k)})$$

as in Theorem 1.3 does not follow immediately from Almgren–Pitts min-max theory because that theory worked with homotopy (and not cohomology) classes. In any case, the above formula can be proven using the Morse index estimates of the author and Neves [30] and Sharp's Compactness Theorem [45]. See Theorem 3.3 of Sect. 3.

In [16], Gromov conjectured:

Conjecture. The volume spectrum obeys a Weyl law (just like the standard Weyl law for the spectrum of the Laplacian, [49]).

In order to explain the analogy, recall that the p-th eigenvalue λ_p of the Laplace–Beltrami operator Δ_g can be given a min-max characterization:

$$\lambda_p(M) = \inf_{\dim Q = p+1} \sup_{f \in Q, f \neq 0} E(f),$$

where $Q \subset W^{1,2}(M)$ and $E(f) = \frac{\int_M |\nabla f|^2 \, dM}{\int_M f^2 \, dM}$ is the Rayleigh functional. Note that $E(c \cdot f) = E(c)$ for any constant $c \in \mathbb{R} \setminus \{0\}$ and so the Rayleigh functional descends to the projectivization of $W^{1,2}(M)$:

$$E : \mathbb{P}W^{1,2}(M) \longrightarrow \mathbb{R}.$$

A $(p+1)$-plane in $W^{1,2}(M)$ becomes a p-projective space in $\mathbb{P}W^{1,2}(M)$ and one should think of $\mathbb{P}W^{1,2}(M)$ as an $\mathbb{RP}^\infty$. Hence we have the analogous characterization:

$$\lambda_p(M) = \inf_{\mathbb{RP}^p \subset \mathbb{P}W^{1,2}(M)} \sup_{[f] \in \mathbb{RP}^p} E(f).$$

In the late 1980s, Gromov [15] wrote a paper in which he first mentions the analogies explained above and explores applications of the classical Borsuk–Ulam Theorem. This theorem states that for any continuous map $f : S^k \to \mathbb{R}^k$, there is always a point $x \in S^k$ such that $f(x) = f(-x)$. Here is one such application. Take a bounded domain $\Omega \subset \mathbb{R}^{n+1}$, an integer $k \in \mathbb{N}$, a disjoint collection $\Omega_1, \ldots, \Omega_k$ of subdomains of Ω and a vector subspace $E \subset C^\infty(\Omega)$ of $\dim E = k+1$. Then for every $u \in S^k_E$, where $S^k_E \subset E$ is the unit sphere with respect to some norm, consider

$$F(u) = (\mathrm{vol}\,\{u < 0\} \cap \Omega_1, \ldots, \mathrm{vol}\,\{u < 0\} \cap \Omega_k) \in \mathbb{R}^k.$$

The Borsuk–Ulam Theorem implies the existence of a function $u_0 \in E$ such $F(u_0) = F(-u_0)$, which means that the zero set $Z(u_0) = \{x \in \Omega : u(x) = 0\}$ bisects each Ω_i into two regions of equal volume.

From this we can derive an estimate for the area of $Z(u_0)$. Choose a cube $C \subset \Omega$ and let $l = \lfloor k^{\frac{1}{n+1}} \rfloor$. Denote by a the length of the sides of C. Divide C into l^{n+1} subcubes of size a/l. Since $k \geq l^{n+1}$, there exists $u_0 \in E$ such that the zero set $Z(u_0) = \{x \in \Omega : u(x) = 0\}$ bisects each subcube C_i into two regions of equal volume. The relative isoperimetric inequality then implies area $Z(u_0) \cap C_i \geq d(n) a^n l^{-n}$ for every i. Hence

$$\mathrm{area}\, Z(u_0) \geq d(n) a^n l \geq C(n,a) k^{\frac{1}{n+1}}.$$

This is another instance of some kind of similarity with the eigenvalue problem.

In [24], we proved Gromov's conjecture:

Theorem 2.4 (Liokumovich, —, Neves). *There exists a dimensional constant $a(n) > 0$ such that*

$$\lim_{k\to\infty} \omega_k(M, g)k^{-\frac{1}{n+1}} = a(n)\mathrm{vol}(M, g)^{\frac{n}{n+1}}.$$

This theorem holds in any dimension as it has nothing to do with the regularity of minimal hypersurfaces. The constant $a(n)$ can be estimated (for instance, one can take families of zero sets of polynomials on the standard sphere) but it is not known explicitly. The Weyl Law also holds for manifolds with boundary, in which case one does everything with relative cycles (relative to the boundary of M). In fact, the proof of Theorem 2.4 starts by considering the case in which M is the Euclidean unit cube.

The proof of our Weyl law is based on a result inspired by Lusternik–Schnirel'mann theory that we proved in [24]:

Theorem 2.5. *Let $\{\Omega_1, \dots, \Omega_p\}$ be a disjoint collection of subregions of Ω. Then*

$$\omega_k(\Omega, g) \geq \sum_{i=1}^{p} \omega_{k_i}(\Omega_i, g)$$

as long as $\sum_{i=1}^{p} k_i \leq k$.

The proof uses the cup product structure and goes as follows. Let $\Phi : X \to \mathscr{Z}_n(\Omega, \partial\Omega, \mathbb{Z}_2)$ be a k-sweepout of mod 2 relative cycles of Ω. This means $\Phi^*(\overline{\lambda}^k)|_X \neq 0$.

Define $X_i = \{x \in X : M(\Phi(x) \cap \Omega_i) < \omega_{k_i}(\Omega_i, g)\}$. The map $T \mapsto T \cap \Omega_i$ from relative cycles of Ω to relative cycles of Ω_i preserves the fundamental cohomology class. Hence $\Phi^*(\overline{\lambda}^{k_i})|_{X_i} = 0$. A basic property of the cup product implies $\Phi^*(\overline{\lambda}^{k_1+\dots+k_p})|_{X_1\cup\dots\cup X_p} = 0$. Because $\sum_{i=1}^{p} k_i \leq k$, we get $\Phi^*(\overline{\lambda}^k)|_{X_1\cup\dots\cup X_p} = 0$. This implies there must exist $x \in X \setminus (X_1 \cup \dots \cup X_p)$. Hence

$$M(\Phi(x)) \geq M(\Phi(x) \cap \Omega_1) + \dots + M(\Phi(x) \cap \Omega_p) \geq \sum_{i=1}^{p} \omega_{k_i}(\Omega_i, g).$$

Since Φ is an arbitrary k-sweepout of Ω, we are done.

As an application of the Weyl Law for the volume spectrum, we were able to prove Yau's Conjecture for generic metrics by proving a stronger property is true:

Theorem 2.6 (Irie, —, Neves, [21]). *Suppose $3 \leq (n + 1) \leq 7$. For a generic metric on M^{n+1}, the union of all closed, smooth, embedded, minimal hypersurfaces of M is dense in M.*

White ([50], [52]) had proved that a generic metric is bumpy, meaning that every minimal hypersurface is nondegenerate as a critical point of the area functional. Nondegeneracy here means the second variation quadratic form has no kernel, as in traditional Morse theory. Together with Sharp's Compactness Theorem ([45]), this implies the set of minimal hypersurfaces for such metrics is countable.

The basic idea for proving Theorem 2.6 goes as follows (see Sect. 5 for more details). Suppose (M, g) has precisely m minimal hypersurfaces. We can choose a point p outside the union of such hypersurfaces and a small ball B around p that is disjoint from the hypersurfaces. We can consider a one-parameter family of Riemannian metrics $(g(t))_{t\in[0,\varepsilon]}$ near g, with $g(0) = g$, $\mathrm{vol}(M, g(\varepsilon)) > \mathrm{vol}(M, g)$ and such that $g(t)$ coincides with g outside U. This can be achieved easily through a conformal deformation. The fact that $\mathrm{vol}(M, g(\varepsilon)) > \mathrm{vol}(M, g)$ implies together with the Weyl Law that for some $k \in \mathbb{N}$, we must have $\omega_k(M, g(\varepsilon)) > \omega_k(M, g)$. If for every $t \in [0, \varepsilon]$, no new minimal surface is created intersecting U, we would have that $\omega_k(M, g(t))$ is an integer combination of the areas of the m minimal hypersurfaces of g. But the function $t \mapsto \omega_k(M, g(t))$ is continuous (Lipschitz, in fact), which leads to a contradiction. We conclude that for some $t' \in [0, \varepsilon]$, the metric $g(t')$ has the original m minimal hypersurfaces of g and at least one new minimal hypersurface crossing U. We can iterate this argument. A similar argument was used by Irie [20] for the case of (immersed) closed geodesics in dimension two, using a different kind of asymptotic law ([6]).

At this point we do not know anything about the topology or geometry of these hypersurfaces. It should be interesting, for instance, to understand how the Morse index grows. In the above argument we have no control on how large k has to be, depending on the choice of U and $g(t)$.

In [33], we were able to make the argument more quantitative and proved an equidistribution property:

Theorem 2.7 (—, Neves, Song). *Suppose $3 \le (n+1) \le 7$. For a generic metric on M^{n+1}, there is a sequence $\{\Sigma_i\}$ of closed, connected, smooth, embedded, minimal hypersurfaces of M such that*

$$\lim_{p\to\infty} \frac{\sum_{i=1}^{p} \int_{\Sigma_i} f \, d\Sigma_i}{\sum_{i=1}^{p} \mathrm{area}(\Sigma_i)} = \frac{\int_M f \, dM}{\mathrm{vol}(M)}$$

for any continuous function $f : M \to \mathbb{R}$.

In [47], Song settled Yau's Conjecture:

Theorem 2.8. *Suppose $3 \le (n + 1) \le 7$. Any closed Riemannian manifold (M^{n+1}, g) contains infinitely many smooth, closed, embedded minimal hypersurfaces.*

Song was able to localize the methods of [31] and proved that there are infinitely many minimal hypersurfaces inside any domain bounded by stable minimal hypersurfaces. In order to do that, he considers the least volume such domain Ω in M, called the core. The point is that the Frankel property holds for minimal surfaces that are completely contained in the interior of Ω. He defines a cylindrical extension

$$\hat{\Omega} = \Omega \cup (\partial\Omega \times [0, \infty)),$$

and outside Ω he puts the product metric. This is a noncompact Riemannian manifold with a metric $\hat{g}$ that is not smooth. Song defines $\omega_k(\hat{\Omega}, \hat{g})$ as the limit of $\omega_k(\Omega_i, \hat{g})$ over an exhaustion Ω_i of $\hat{\Omega}$. In contrast with the compact case these numbers grow linearly, but Song is able to determine the linear coefficient:

$$k \cdot \text{area}(\Sigma_1) \le \omega_k(\hat{\Omega}, \hat{g}) \le k \cdot \text{area}(\Sigma_1) + Ck^{\frac{1}{n+1}},$$

where Σ_1 is the largest area component of $\partial\Omega$. Song finishes the argument with an arithmetic lemma that generalizes the counting argument of the author and Neves [31]. The minimal hypersurfaces he constructs are limits of free boundary minimal hypersurfaces produced by applying the min-max theory for relative cycles of Li and Zhou [23] to each Ω_i.

3. Morse index of min-max minimal hypersurfaces

For generic metrics one can hope to have a Morse theory. In a series of papers ([26], [29], [30], [36]), the authors proposed a program to obtain a Morse-theoretic description of the set of minimal hypersurfaces in the generic case. The authors conjectured:

Morse Index Conjecture 3.1. For a generic metric g on M^{n+1}, $3 \le (n+1) \le 7$, there exists a sequence $\{\Sigma_k\}$ of smooth, embedded, two-sided, closed minimal hypersurfaces such that:

(1) $\text{index}(\Sigma_k) = k$,
(2) $C^{-1}k^{\frac{1}{n+1}} \le \text{area}(\Sigma_k) \le Ck^{\frac{1}{n+1}}$ for some $C > 0$.

The authors proposed a program to prove this conjecture based on three main components: the use of min-max constructions over multiparameter sweepouts to obtain existence results, the characterization of the Morse index of min-max minimal hypersurfaces under the multiplicity one assumption, and a proof of the Multiplicity One Conjecture:

Multiplicity One Conjecture 3.2. For generic metrics on M^{n+1}, $3 \le (n+1) \le 7$, any component of a closed, minimal hypersurface obtained by min-max methods is two-sided and has multiplicity one.

The first part of the program was done in the Almgren–Pitts setting by the authors ([31], [30]). In [32], we completed the characterization of the Morse index of Almgren–Pitts min-max minimal hypersurfaces under the multiplicity one assumption for bumpy metrics. We used the fact that any nondegenerate minimal hypersurface is a solution of a local min-max problem (White, [51]).

In [30], we proved the upper bound:

Theorem 3.3 (—, Neves). *If Σ is the min-max minimal hypersurface produced by min-max over a k-dimensional homotopy class, then*

$$\text{index}(\text{support}(\Sigma)) \leq k.$$

If $\Sigma = m_1\Sigma_1 + \cdots + m_q\Sigma_q$, then we define

$$\text{index}(\text{support}(\Sigma)) = \text{index}(\Sigma_1) + \cdots + \text{index}(\Sigma_q).$$

Recently, we proved also the lower bound:

Theorem 3.4 (—, Neves, [32]). *Suppose the metric is bumpy. If Σ is the min-max minimal hypersurface produced by min-max over the class of k-sweepouts, and if it is achieved with multiplicity one:*

$$\omega_k(M, g) = 1 \cdot \text{area}(\Sigma),$$

then

$$\text{index}(\Sigma) = k.$$

It remains to prove the Multiplicity One Conjecture in the Almgren–Pitts setting[1]. We proved the Multiplicity One Conjecture for two-sided components in the one-parameter case and we were also able to rule out one-sided components with multiplicity in some settings ([30], [22], see also [27], [54], [55]).

The Morse Index Conjecture would also follow if one can implement the three parts of the program in the alternative Allen–Cahn setting, which can be seen as an ε-regularization of the Almgren–Pitts setting. In the Allen–Cahn setting one considers the Sobolev space $W^{1,2}(M)$ and associates to a function $u \in W^{1,2}(M)$, for fixed $\varepsilon > 0$, the energy:

$$E_\varepsilon(u) = \int_M \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right),$$

where W is a double-well potential like $W(u) = \frac{(1-u^2)^2}{4}$. Critical points of E_ε satisfy the Euler–Lagrange equation:

$$\varepsilon \Delta u = \frac{W'(u)}{\varepsilon},$$

[1] A proof of the Multiplicity One Conjecture has been recently announced by X. Zhou [56], using the prescribed mean curvature min-max theory of Zhou–Zhu [57].

from which one can see that the constant functions -1, 1 and 0 are trivial critical points. The variational problem associated to the Allen–Cahn energy satisfies the Palais–Smale condition for fixed ε. Interesting behavior, like formation of phase transitions, happens in the singular limit, as ε goes to zero. Notice that $E_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, so the constant function 0 will not be a relevant critical point in our discussion.

In [17], Guaraco used the regularity theory of Tonegawa and Wickramasekera [48] to prove the following result:

Theorem 3.5 (Guaraco, [17]). *Suppose $(u_{\varepsilon_i})_i$ is a sequence of critical points, $\varepsilon_i \to 0$, with bounded energy and Morse index. Then there is a subsequence $(u_{\varepsilon_j})_j$ such that the level sets of u_{ε_j} accumulate at a minimal hypersurface possibly with integer multiplicities.*

In the complement of the limiting minimal hypersurface the solutions converge to either $+1$ or -1 smoothly in compact sets. This is the so-called phase transition phenomenon.

In [17], Guaraco performed a mountain-pass argument to construct nontrivial solutions to the Allen–Cahn equations and by passing to the limit as $\varepsilon \to 0$ he obtained a PDE-based proof of the existence of at least one minimal hypersurface. In [12], Gaspar and Guaraco generalized this existence theory to the case of multiparameter sweepouts for each fixed ε. This again explores an $\mathbb{RP}^\infty$ structure coming from the identity $E_\varepsilon(u) = E_\varepsilon(-u)$. The areas of the minimal hypersurfaces produced by Gaspar and Guaraco grow sublinearly with the dimension of the parameter space, as in the Almgren–Pitts setting.

The Multiplicity One Conjecture 3.2, adapted to the Allen–Cahn setting, was recently proven when $(n+1) = 3$ in work by Chodosh and Mantoulidis [5]. They proved that in this dimension, for bumpy metrics, the limiting minimal hypersurfaces of Gaspar and Guaraco are two-sided and occur with multiplicity one. In [5], they also finish the Morse index characterization for multiplicity one Allen–Cahn minimal hypersurfaces (assuming smoothness) in any dimension. Putting this all together, they obtain:

Theorem 3.6 (Chodosh, Mantoulidis, [5]). *The Morse Index Conjecture is true if* $(n+1) = 3$.

Recently, Gaspar and Guaraco [13] used the Allen–Cahn regularization to obtain different proofs of density and equidistribution of minimal hypersurfaces for generic metrics. The proofs follow the lines of [21] and [33] and are based on a Weyl law for the Allen–Cahn volume spectrum. It is not known whether the constants in the Weyl laws of [24] and [13] are the same.

4. Topology of the space of hypercycles

In this section, we will prove the following theorem:

Theorem 4.1. *The space of cycles $\mathscr{Z}_n(M;\mathbb{Z}_2)$ is weakly homotopically equivalent to $\mathbb{RP}^\infty$.*

Proof. Let $f : M \to \mathbb{R}$ be a Morse function, with $f(M) = [0,1]$. We assert that the map $\hat{\Phi} : \mathbb{RP}^\infty \to \mathscr{Z}_n(M;\mathbb{Z}_2)$ given by the formula

$$\hat{\Phi}([a_0 : a_1 : \cdots : a_k : 0 : \cdots]) = \partial\{x \in M : a_0 + a_1 f(x) + \cdots + a_k f(x)^k \leq 0\}$$

is a weak homotopy equivalence, i.e. it induces isomorphisms in every homotopy group. In [31] (Claim 5.5), we proved the map $\hat{\Phi}$ is continuous in the flat topology.

Lifting Property 4.2. Let $\Psi : I^p \to \mathscr{Z}_n(M;\mathbb{Z}_2)$ be a continuous map, $p \in \mathbb{N}$, and $U_0 \in \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ be such that $\partial U_0 = \Psi(0)$. Then there exists a unique continuous map $U : I^p \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ such that $U(0) = U_0$ and $\partial U(x) = \Psi(x)$ for every $x \in I^p$.

We start with uniqueness. Let U, U' be two such maps and consider the difference $V = U - U'$. Then $V : I^p \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ satisfies $V(0) = 0$ and $\partial V(x) = 0$ for every $x \in I^p$. By the Constancy Theorem for mod 2 flat chains, $V(x) \in \{0, M\}$ for every $x \in I^p$. This implies that the set $A = \{x \in I^p : V(x) = 0\}$ is both closed and open. Since $0 \in A$, we have $A = I^p$ and therefore $U = U'$.

Now we prove the existence of the lifting U when $p = 1$. By the Isoperimetric Inequality of Federer–Fleming (see Proposition 1.11 or Corollary 1.14 of [1]), adapted to the setting of mod 2 flat chains, there exist constants $\varepsilon_M > 0$ and $\nu_M > 0$ such that if $T \in \mathscr{Z}_n(M;\mathbb{Z}_2)$ satisfies $\mathscr{F}(T) < \varepsilon_M$, then there exists $W \in \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ with $\partial W = T$ and $\mathbf{M}(W) \leq \nu_M \mathscr{F}(T)$. Here $\mathscr{F}$ and $\mathbf{M}$ denote the flat norm and the mass norm, respectively. We can choose ε_M to be small so that W is unique by the Constancy Theorem.

Since Ψ is continuous, we can find a partition $0 = t_0 < t_1 < \cdots < t_{q-1} < t_q = 1$ such that for every $s, t \in [t_{i-1}, t_i]$, $1 \leq i \leq q$, we have

$$\mathscr{F}(\Psi(s), \Psi(t)) < \varepsilon_M.$$

For $t \in [t_{i-1}, t_i]$, let $W_i(t)$ be the unique element of $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ with $\partial W_i(t) = \Psi(t) - \Psi(t_{i-1})$ and $\mathbf{M}(W_i(t)) \leq \nu_M \mathscr{F}(\Psi(t) - \Psi(t_{i-1}))$. The map $W_i : [t_{i-1}, t_i] \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ is continuous and $W_i(t_{i-1}) = 0$. For $t \in [0, t_1]$, we define $U(t) = U_0 + W_1(t)$. Then $U : [0, t_1] \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ is continuous, with $U(0) = U_0$ and $\partial U(t) = \Psi(t)$ for all $t \in [0, t_1]$. Suppose that we have found a continuous map $U : [0, t_{i-1}] \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ with $U(0) = U_0$ and $\partial U(t) = \Psi(t)$ for all $t \in [0, t_{i-1}]$. Then we extend it to $[0, t_i]$ by putting $U(t) = U(t_{i-1}) + W_i(t)$ for $t \in [t_{i-1}, t_i]$. The existence of the lifting $U : [0,1] \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ follows by induction.

Now suppose $p > 1$. Given $x \in I^p$, we choose a continuous path $\sigma : [0,1] \to I^p$ with $\sigma(0) = 0$ and $\sigma(1) = x$ and define $\Psi_\sigma = \Psi \circ \sigma : [0,1] \to \mathcal{Z}_n(M;\mathbb{Z}_2)$. We know there exists a continuous map $U_\sigma : [0,1] \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ with $U_\sigma(0) = U_0$ and $\partial U_\sigma(t) = \Psi_\sigma(t)$ for all $t \in [0,1]$. Then we put $U(x) = U_\sigma(1)$. Note that $\partial U(x) = \Psi(x)$. Because I^p is simply-connected, a standard argument gives that $U(x)$ does not depend on σ and the obtained extension $U : I^p \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ is a continuous map. This ends the proof of the lifting property.

Claim 4.3. The space $\mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ is contractible.

We define the deformation $H : [0,1] \times \mathbf{I}_{n+1}(M;\mathbb{Z}_2) \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ by putting

$$H(t,U) = U \llcorner \{f \leq t\}.$$

The map H is continuous, $H(1,U) = U$ and $H(0,U) = 0$ for every $U \in \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$. This proves the claim.

The homotopy groups of the infinite-dimensional projective space are given by

$$\pi_1(\mathbb{RP}^\infty, 1) = \mathbb{Z}_2, \quad \text{and}$$
$$\pi_k(\mathbb{RP}^\infty, 1) = 0 \quad \text{for every } k \geq 2.$$

If $k \geq 2$, and $\Psi : I^k \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ is a continuous map with $\Psi(\partial I^k) = \{0\}$, the Lifting Property implies there exists a unique continuous map $U : I^k \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ with $U(0) = 0$ and $\partial U(x) = \Psi(x)$ for every $x \in I^k$. By the uniqueness of the liftings of maps defined on I^{k-1}, we have $U(\partial I^k) = 0$. Claim 4.3 implies U is homotopically constant relative to ∂I^k. Hence Ψ is homotopically constant relative to ∂I^k. This establishes that

$$\pi_k(\mathcal{Z}_n(M;\mathbb{Z}_2), 0) = 0$$

for every $k \geq 2$.

Now let $\sigma : [0,1] \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ be a continuous map with $\sigma(0) = \sigma(1) = 0$. The Lifting Property gives a unique continuous map $U : [0,1] \to \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ with $U(0) = 0$ and $\partial U(t) = \sigma(t)$ for every $t \in [0,1]$. Then $\partial U(1) = \sigma(1) = 0$, hence either $U(1) = 0$ or $U(1) = M$. If $U(1) = 0$, the map σ is homotopically constant relative to $\{0,1\}$ as in the higher-dimensional argument. Conversely, if such a homotopy exists then we can lift it and a standard argument implies $U(1) = 0$. These arguments give that

$$\pi_1(\mathcal{Z}_n(M;\mathbb{Z}_2), 0) = \mathbb{Z}_2,$$

and $\sigma : [0,1] \to \mathcal{Z}_n(M;\mathbb{Z}_2)$ given by $\sigma(t) = \partial\{f \leq t\}$ is a generator. Since $\hat{\Phi}([\cos(\pi t) : \sin(\pi t) : 0 : \cdots]) = \partial\{f \leq -\cot(\pi t)\}$, the map

$$\hat{\Phi}_* : \pi_1(\mathbb{RP}^\infty, 1) \longrightarrow \pi_1(\mathcal{Z}_n(M;\mathbb{Z}_2), 0)$$

is an isomorphism. The higher homotopy groups of both spaces are trivial, hence $\hat{\Phi}$ is a weak homotopy equivalence. □

Note that if we take integer coefficients (flat chains with integer coefficients are called integral currents), and M is oriented, Almgren's formula will give

$$\pi_1(\mathscr{Z}_n(M, \mathbb{Z})) = \mathbb{Z},$$

and

$$\pi_k(\mathscr{Z}_n(M, \mathbb{Z})) = 0$$

for all $k \geq 2$. It follows that $\mathscr{Z}_n(M, \mathbb{Z})$ is weakly homotopically equivalent to the circle S^1, and hence there are no nontrivial multiparameter families of integral currents.

5. Density of minimal hypersurfaces for generic metrics

In what follows we explain in more detail the proof of density of minimal hypersurfaces for generic metrics ([21]).

We denote by $\mathscr{M}$ the space of all smooth Riemannian metrics on M, endowed with the C^∞ topology.

The key proposition is the following:

Proposition 5.1. *Suppose* $3 \leq (n+1) \leq 7$*, and let* $U \subset M$ *be a nonempty open set. Then the set* $\mathscr{M}_U$ *of all smooth Riemannian metrics on* M *such that there exists a nondegenerate, closed, smooth, embedded, minimal hypersurface* Σ *that intersects* U *is open and dense in the* C^∞ *topology.*

Proof (Openness). Let $g \in \mathscr{M}_U$ and Σ be as in the statement of the proposition. Because Σ is nondegenerate, an application of the Inverse Function Theorem implies that for every Riemannian metric g' sufficiently close to g, there exists a unique nondegenerate closed, smooth, embedded minimal hypersurface Σ' close to Σ. In particular, $\Sigma' \cap U \neq \emptyset$ if g' is sufficiently close to g. This implies $\mathscr{M}_U$ is open.

(Density) Let g be an arbitrary smooth Riemannian metric on M and $\mathscr{V}$ be an arbitrary neighborhood of g in the C^∞ topology. By the Bumpy Metrics Theorem of White (Theorem 2.1, [52]), there exists $g' \in \mathscr{V}$ such that every closed, smooth immersed minimal hypersurface with respect to g' is nondegenerate. If one of these minimal hypersurfaces is embedded and intersects U then $g' \in \mathscr{M}_U$, and we are done.

Therefore we can suppose that every closed, smooth, embedded minimal hypersurface with respect to g' is contained in the complement of U. Since g' is bumpy, it follows from Sharp (Theorem 2.3 and Remark 2.4, [45]) that the set of connected, closed, smooth, embedded minimal hypersurfaces in (M, g') with both area and index bounded from above by q is finite for every $q > 0$. This implies that the set

$$\mathscr{C} = \Big\{ \sum_{j=1}^{N} m_j \operatorname{vol}_{g'}(\Sigma_j) : N \in \mathbb{N}, \{m_j\}_{j=1}^N \subset \mathbb{N}, \{\Sigma_j\}_{j=1}^N \text{ disjoint collection}$$
$$\text{of closed, smooth, embedded minimal hypersurfaces in } (M, g') \Big\}$$

is countable.

Now we choose $h : M \to \mathbb{R}$ a smooth nonnegative function such that $\operatorname{supp}(h) \subset U$ and $h(x) > 0$ for some $x \in U$. We define $g'(t) = (1 + th)g'$ for $t \geq 0$, and choose $t_0 > 0$ sufficiently small so that $g'(t) \in \mathscr{V}$ for every $t \in [0, t_0]$. By construction $g'(t) = g'$ outside some compact set $K \subset U$ for every $t > 0$.

We have $\operatorname{vol}(M, g'(t_0)) > \operatorname{vol}(M, g')$. It follows from the Weyl Law for the Volume Spectrum that there exists $k \in \mathbb{N}$ such that $\omega_k(M, g'(t_0)) > \omega_k(M, g')$. Assume by contradiction that for every $t \in [0, t_0]$, every closed, smooth, embedded minimal hypersurface in $(M, g'(t))$ is contained in $M \setminus U$. Since $g'(t) = g'$ outside $K \subset U$ we conclude from that $\omega_k(M, g'(t)) \in \mathscr{C}$ for all $t \in [0, t_0]$. But $\mathscr{C}$ is countable and we know that the function $t \mapsto \omega_k(M, g'(t))$ is continuous. Hence $t \mapsto \omega_k(M, g'(t))$ is constant in the interval $[0, t_0]$. This contradicts the fact that $\omega_k(M, g'(t_0)) > \omega_k(M, g')$.

Therefore we can find $t \in [0, t_0]$ such that there exists a closed, smooth, embedded minimal hypersurface Σ with respect to $g'(t)$ that intersects U. Through a conformal deformation it is possible to perturb the metric slightly to make Σ nondegenerate (and remains minimal). Since $g'(t) \in \mathscr{V}$, we find a Riemannian metric $g'' \in \mathscr{V}$ such that Σ is minimal and nondegenerate with respect to g''. Therefore $g'' \in \mathscr{V} \cap \mathscr{M}_U$ and we have finished the proof of the Proposition. □

Let $\{U_i\}$ be a countable basis of M. Since, by Proposition 5.1, each $\mathscr{M}_{U_i}$ is open and dense in $\mathscr{M}$ the set $\bigcap_i \mathscr{M}_{U_i}$ is C^∞ Baire-generic in $\mathscr{M}$. This finishes the proof of density of minimal hypersurfaces for generic metrics.

6. Min-max Theory for the area functional

In this section we explain in more detail the Min-max Theorem applied to homotopy classes of maps.

Let (M^{n+1}, g) be an $(n+1)$-dimensional closed Riemannian manifold. We assume, for convenience, that (M, g) is isometrically embedded in some Euclidean space $\mathbb{R}^J$.

We consider the following spaces:

- the space $\mathbf{I}_l(M;\mathbb{Z}_2)$ of l-dimensional flat chains in $\mathbb{R}^J$ with coefficients in $\mathbb{Z}_2$ and support contained in M, where $l = n$ or $n+1$;
- the space $\mathcal{Z}_n(M;\mathbb{Z}_2)$ of flat chains $T \in \mathbf{I}_n(M;\mathbb{Z}_2)$ such that there exists $U \in \mathbf{I}_{n+1}(M;\mathbb{Z}_2)$ with $\partial U = T$;
- the closure $\mathcal{V}_n(M)$, in the weak topology, of the space of n-dimensional rectifiable varifolds in $\mathbb{R}^J$ with support contained in M.

We assume implicitly that $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$ for every $T \in \mathbf{I}_l(M;\mathbb{Z}_2)$. We will refer to $\mathcal{Z}_n(M;\mathbb{Z}_2)$ as the *space of cycles*. Flat chains over a finite coefficient group were introduced by Fleming [9].

Given $T \in \mathbf{I}_l(M;\mathbb{Z}_2)$, we denote by $|T|$ and $\|T\|$ the integral varifold and the Radon measure in M associated with $|T|$, respectively; given $V \in \mathcal{V}_n(M)$, $\|V\|$ denotes the Radon measure in M associated with V. The space of n-dimensional integral varifolds with support in M is denoted by $\mathcal{IV}_n(M)$.

The spaces above come with several relevant metrics. The *mass* of $T \in \mathbf{I}_l(M;\mathbb{Z}_2)$ is denoted by $\mathbf{M}(T)$, and the metric $\mathbf{M}(T_1, T_2) = \mathbf{M}(T_1 - T_2)$ defines the mass topology. The *flat metric*

$$\mathcal{F}(T_1, T_2) = \inf\{\mathbf{M}(Q) + \mathbf{M}(R) : T_1 - T_2 = Q + \partial R\}$$

induces the flat topology (we put $\mathcal{F}(T) = \mathcal{F}(T, 0)$). The **F**-*metric* is defined in the book of Pitts [37, page 66] and induces the varifold weak topology on $\mathcal{V}_n(M) \cap \{V : \|V\|(M) \leq a\}$ for any a. It satisfies

$$\|V\|(M) \leq \|W\|(M) + \mathbf{F}(V, W)$$

for all $V, W \in \mathcal{V}_n(M)$. We denote by $\overline{\mathbf{B}}^{\mathbf{F}}_\delta(V)$ and $\mathbf{B}^{\mathbf{F}}_\delta(V)$ the closed and open metric balls, respectively, with radius δ and center $V \in \mathcal{V}_n(M)$. Similarly, we denote by $\overline{\mathbf{B}}^{\mathcal{F}}_\delta(T)$ and $\mathbf{B}^{\mathcal{F}}_\delta(T)$ the corresponding balls with center $T \in \mathcal{Z}_n(M;\mathbb{Z}_2)$ in the flat metric. Finally, the **F**-*metric* on $\mathbf{I}_l(M;\mathbb{Z}_2)$ is defined by

$$\mathbf{F}(S, T) = \mathcal{F}(S - T) + \mathbf{F}(|S|, |T|).$$

We have $\mathbf{F}(|S|, |T|) \leq \mathbf{M}(S, T)$ and hence $\mathbf{F}(S, T) \leq 2\mathbf{M}(S, T)$ for any $S, T \in \mathbf{I}_l(M;\mathbb{Z}_2)$.

We assume that $\mathbf{I}_l(M;\mathbb{Z}_2)$ and $\mathcal{Z}_n(M;\mathbb{Z}_2)$ have the topology induced by the flat metric. When endowed with the topology of the **F**-metric or the mass norm, these spaces will be denoted by $\mathbf{I}_l(M;\mathbf{F};\mathbb{Z}_2)$, $\mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)$, $\mathbf{I}_l(M;\mathbf{M};\mathbb{Z}_2)$, $\mathcal{Z}_n(M;\mathbf{M};\mathbb{Z}_2)$, respectively. The space $\mathcal{V}_n(M)$ is considered with the weak topology of varifolds.

Our parameter space X will be a cubical complex of dimension k, meaning a subcomplex of dimension k of $I(m, j)$ for some m and j.

Definition 6.1. *Let* $\Phi : X \to \mathcal{Z}_n(M^{n+1};\mathbf{F};\mathbb{Z}_2)$ *be a continuous map. The* homotopy class *of* Φ *is the class* Π *of all continuous maps* $\Phi' : X \to \mathcal{Z}_n(M^{n+1};\mathbf{F};\mathbb{Z}_2)$ *such that* Φ *and* Φ' *are homotopic to each other in the flat topology.*

Remark 6.2. Notice that our definition of homotopy class is slightly unusual, as we allow homotopies that are continuous in a weaker topology.

Definition 6.3. *The* width *of* Π *is defined by:*

$$\mathbf{L}(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in X} \{\mathbf{M}(\Phi(x))\}.$$

Definition 6.4. *A sequence* $\{\Phi_i\} \subset \Pi$ *is called a* minimizing sequence *if*

$$\mathbf{L}(\Phi_i) := \sup_{x \in X} \mathbf{M}(\Phi_i(x))$$

satisfies $\mathbf{L}(\{\Phi_i\}) := \limsup_{i\to\infty} \mathbf{L}(\Phi_i) = \mathbf{L}(\Pi)$. *Any sequence* $\{\Phi_{i_j}(x_j)\}$ *with* $\lim_{j\to\infty} \mathbf{M}(\Phi_{i_j}(x_j)) = \mathbf{L}(\Pi)$, *where* $\{i_j\} \subset \{i\}$ *is a subsequence and* $\{x_j\} \subset X$, *is called a* min-max sequence.

Definition 6.5. *The* image set *of* $\{\Phi_i\}$ *is defined by*

$$\begin{aligned}\check{\mathbf{C}}(\{\Phi_i\}) = &\{V \in \mathcal{V}_n(M) : \exists \text{ sequences } \{i_j\} \to \infty, x_j \in X \\ &\text{such that } \lim_{j\to\infty} \mathbf{F}(|\Phi_{i_j}(x_j)|, V) = 0\}.\end{aligned}$$

Definition 6.6. *If* $\{\Phi_i\}$ *is a minimizing sequence in* Π, *with* $L = \mathbf{L}(\{\Phi_i\})$, *the* critical set *of* $\{\Phi_i\}$ *is defined by*

$$\mathbf{C}(\{\Phi_i\}) = \{V \in \check{\mathbf{C}}(\{\Phi_i\}) : \|V\|(M) = L\}.$$

Pull-tight 6.7. *Following Pitts ([37] p.153), we can define for each* $\varepsilon > 0$ *a continuous map*

$$\begin{aligned}H : I \times (\mathcal{Z}_n(M^{n+1}; \mathbf{F}; \mathbb{Z}_2) \cap \{T : \mathbf{M}(T) \leq 2\mathbf{L}(\Pi)\}) \\ \longrightarrow \mathcal{Z}_n(M^{n+1}; \mathbf{F}; \mathbb{Z}_2) \cap \{T : \mathbf{M}(T) \leq 2\mathbf{L}(\Pi)\}\end{aligned}$$

such that:

- $H(0, T) = T$ *for all* T*;*
- $H(t, T) = T$ *for all* $t \in [0, 1]$ *if* $|T|$ *is stationary;*
- $\mathbf{M}(H(1, T)) < \mathbf{M}(T)$ *if* $|T|$ *is not stationary;*
- $\mathbf{F}(T, H(t, T)) \leq \varepsilon$ *for all* t *and* T.

Given a minimizing sequence $\{\Phi_i^*\} \subset \Pi$, *we define* $\Phi_i(x) = H(1, \Phi_i^*(x))$ *for every* $x \in X$. *Then* $\{\Phi_i\} \subset \Pi$ *is also a minimizing sequence. It follows from the construction that* $\mathbf{C}(\{\Phi_i\}) \subset \mathbf{C}(\{\Phi_i^*\})$ *and that every element of* $\mathbf{C}(\{\Phi_i\})$ *is stationary.*

Definition 6.8. *Any minimizing sequence* $\{\Phi_i\} \subset \Pi$ *such that every element of* $\mathbf{C}(\{\Phi_i\})$ *is stationary is called* pulled-tight.

The next result follows from the Almgren–Pitts ([2], [37]) min-max theory together with the regularity theory of Schoen–Simon ([43]). See Sect. 3 of [30] for the formulation presented below.

Definition 6.9 (Min-max Theorem). *Suppose* $\mathbf{L}(\Pi) > 0$*, and let* $\{\Phi_i\}$ *be a minimizing sequence in* Π*. Then there exists a stationary integral varifold* $V \in \mathbf{C}(\{\Phi_i\})$ *(hence* $\|V\|(M) = \mathbf{L}(\Pi)$*), with support a closed minimal hypersurface that is smooth and embedded outside a set of Hausdorff dimension less than or equal to* $(n-7)$*.*

If $3 \le (n+1) \le 7$, we conclude that there is a disjoint collection $\{\Sigma_1, \dots, \Sigma_N\}$ of closed, smooth, embedded, minimal hypersurfaces in M and a set of integers $\{m_1, \dots, m_N\} \subset \mathbb{N}$, such that

$$V = m_1 \cdot |\Sigma_1| + \cdots + m_N \cdot |\Sigma_N|.$$

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