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An operadic approach to vertex algebra and Poisson vertex algebra cohomology

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Abstract We translate the construction of the chiral operad by Beilinson and Drinfeld to the purely algebraic language of vertex algebras. Consequently, the general construction of a cohomology complex associated to a linear operad produces the vertex algebra cohomology complex. Likewise, the associated graded of the chiral operad leads to the classical operad, which produces a Poisson vertex algebra cohomology complex. The latter is closely related to the variational Poisson cohomology studied by two of the authors.

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1. Introduction

The universal Lie superalgebra associated to a vector superspace V is defined as a $\mathbb{Z}$-graded Lie superalgebra

$$W(V) = \bigoplus_{j \geq -1} W_j(V)\,, \quad \text{with} \quad W_{-1}(V) = V,$$

such that for any $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{j \geq -1} \mathfrak{g}_j$, with $\mathfrak{g}_{-1} = V$, there is a unique grading preserving homomorphism $\mathfrak{g} \to W(V)$ identical on V. It is easy to see that

$$W_j(V) = \mathrm{Hom}(S^{j+1}(V), V),$$

for all $j \geq -1$. The Lie superalgebra bracket on $W(V)$ is given by

$$[X, Y] = X \Box Y - (-1)^{p(X)p(Y)} Y \Box X, \tag{1.1}$$

were p is the parity on $W(V)$, and, for $X \in W_n(V)$, $Y \in W_m(V)$,

$$\begin{aligned}&(X \Box Y)(v_0 \otimes \cdots \otimes v_{m+n}) \\ &= \sum_{\substack{i_0 < \cdots < i_m \\ i_{m+1} < \cdots < i_{m+n}}} \epsilon_v(i_0, \dots, i_{m+n}) X(Y(v_{i_0} \otimes \cdots \otimes v_{i_m}) \otimes v_{i_{m+1}} \otimes \cdots \otimes v_{i_{m+n}}).\end{aligned} \tag{1.2}$$

Here $\epsilon_v(i_0, \dots, i_{m+n})$ is non-zero only if $i_0, \dots, i_{m+n}$ are distinct, and in this case it is equal to $(-1)^N$, where N is the number of interchanges of indices of odd v_i's in the permutation.

Clearly, $W_0(V) = \mathrm{End}\, V$ and $W_1(V) = \mathrm{Hom}(S^2 V, V)$, so that any even element of the vector superspace $W_1(V)$ defines a commutative superalgebra

structure on V, and this correspondence is bijective. On the other hand, any odd element X of the vector superspace $W_1(\Pi V)$ defines a skew-commutative superalgebra structure on V by the formula

$$[a,b] = (-1)^{p(a)} X(a \otimes b), \quad a,b \in V. \tag{1.3}$$

Here and further ΠV stands for the vector superspace V with reversed parity. Moreover, (1.3) defines a Lie superalgebra structure on V if and only if $[X,X] = 0$ in $W(\Pi V)$. Thus, given a Lie superalgebra structure on V, considering the corresponding odd element $X \in W_1(\Pi V)$, we obtain a cohomology complex

$$\Big(C^\bullet = \bigoplus_{j \geq 0} C^j, \operatorname{ad} X\Big), \quad \text{where } C^j = W_{j-1}(\Pi V), \tag{1.4}$$

which coincides with the cohomology complex of the Lie superalgebra V with the bracket defined by X, with coefficients in the adjoint representation. This construction for V purely even goes back to the paper [NR67] on deformations of Lie algebras; for a general superspace V it was explained in [DSK13]. Note also that, more generally, given a module M over the Lie superalgebra V, one considers instead of V the Lie superalgebra $V \ltimes M$ with M an abelian ideal, and by a simple reduction procedure constructs the cohomology complex of the Lie superalgebra V with coefficients in M.

In the paper [DSK13], this point of view on cohomology has been also applied to several other algebraic structures. The most important for the present paper is that of a Lie conformal (super)algebra and the corresponding cohomology complex introduced in [BKV99]; see also [BDAK01], [DSK09]. The complex is constructed in [DSK13] as follows. Assume that the vector superspace V carries an even endomorphism ∂. For each integer $k \geq 0$, denote by $\mathbb{F}_-[\lambda_1, \dots, \lambda_k]$ the space of polynomials in the k variables $\lambda_1, \dots, \lambda_k$ of even parity with coefficients in the field $\mathbb{F}$, endowed with the structure of a left $\mathbb{F}[\partial]^{\otimes k}$-module by letting $P_1(\partial) \otimes \cdots \otimes P_k(\partial)$ act as multiplication by $P_1(-\lambda_1) \cdots P_k(-\lambda_k)$. This space carries also a right $\mathbb{F}[\partial]$-module structure, for which ∂ acts as multiplication by $-\lambda_1 - \cdots - \lambda_k$. Then we let for $k \geq 0$:

$$P_k^\partial(V) = \operatorname{Hom}_{\mathbb{F}[\partial]^{\otimes k}}(V^{\otimes k}, \mathbb{F}_-[\lambda_1, \dots, \lambda_k] \otimes_{\mathbb{F}[\partial]} V). \tag{1.5}$$

The symmetric group S_k acts on the vector superspace $P_k^\partial(V)$ by simultaneous permutation of the factors of the vector superspace $V^{\otimes k}$ and of the λ_i's. We denote by $W_k^\partial(V)$ the subspace of fixed points in $P_{k+1}^\partial(V)$, $k \geq -1$. Then the "conformal" analogue of $W(V)$ is the vector superspace

$$W^\partial(V) = \bigoplus_{j \geq -1} W_j^\partial(V),$$

with a $\mathbb{Z}$-graded Lie superalgebra structure similar to (1.1)–(1.2). Note that we have:

$$W_{-1}^{\partial}(V) = V/\partial V, \quad W_0^{\partial}(V) = \mathrm{End}_{\mathbb{F}[\partial]}\, V.$$

Moreover, the even elements in the vector superspace $W_1^{\partial}(V)$ are identified, letting $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda - \partial$, with maps

$$X : V \otimes V \longrightarrow V[\lambda], \quad a \otimes b \longmapsto X_\lambda(a \otimes b),$$

which satisfy certain sesquilinearity and commutativity conditions.

Proceeding in exactly the same way as in the Lie superalgebra case, consider the Lie superalgebra $W^{\partial}(\Pi V)$. Then we get a bijection between odd elements $X \in W_1^{\partial}(\Pi V)$, such that $[X, X] = 0$, and the *Lie conformal superalgebra* structures on V, i.e., λ-brackets on V satisfying sesquilinearity

$$[\partial a_\lambda b] = -\lambda [a_\lambda b], \quad [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b], \tag{1.6}$$

skew-commutativity

$$[b_\lambda a] = -(-1)^{p(a)p(b)} [a_{-\lambda-\partial} b], \tag{1.7}$$

and Jacobi identity

$$[a_\lambda [b_\mu c]] - (-1)^{p(a)p(b)} [b_\mu [a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c]. \tag{1.8}$$

This bijection is similar to (1.3):

$$[a_\lambda b] = (-1)^{p(a)} X_\lambda(a \otimes b). \tag{1.9}$$

Moreover, similarly to (1.4), we obtain the cohomology complex of the Lie conformal superalgebra V with λ-bracket given by X_λ via (1.9), with coefficients in the adjoint representation. One defines the cohomology of V with coefficients in a V-module M in a similar way as well.

The most relevant to [DSK13] construction is obtained by endowing the $\mathbb{F}[\partial]$-module V with a structure of a (commutative associative) differential superalgebra. In this case one considers the $\mathbb{Z}$-graded subalgebra $W^{\partial,\mathrm{as}}(V) = \bigoplus_{j \geq -1} W_j^{\partial,\mathrm{as}}$ of $W^{\partial}(V)$, where $W_j^{\partial,\mathrm{as}} = W_j^{\partial}$ for $j = -1, 0$, while $W_j^{\partial,\mathrm{as}}$ for $j \geq 1$ consists of the maps from W_j^{∂} satisfying the Leibniz rule. The odd elements $X \in W_1^{\partial,\mathrm{as}}(\Pi V)$, such that $[X, X] = 0$, correspond bijectively to Poisson vertex algebra (PVA) structures on V with the given differential algebra structure, and using this, one constructs the variational Poisson cohomology of the PVA V. Recall that a *Poisson vertex (super)algebra* is a differential (super)algebra endowed with a Lie conformal (super)algebra λ-bracket satisfying the Leibniz rule

$$[a_\lambda bc] = [a_\lambda b]c + (-1)^{p(a)p(b)} b[a_\lambda c]. \tag{1.10}$$

A somewhat different point of view on cohomology complexes of algebraic structures is provided by the theory of linear unital symmetric superoperads, which we call *operads* in this paper for simplicity. (It covers the first two above mentioned examples, but not the third one.) One of its equivalent definitions is that it is a sequence of vector superspaces $\mathscr{P}(n)$, $n \in \mathbb{Z}_{\geq 0}$, endowed with a right action of S_n for $n \geq 1$, and bilinear parity preserving products

$$\circ_i : \mathscr{P}(n) \times \mathscr{P}(m) \longrightarrow \mathscr{P}(n+m-1), \quad i = 1, \dots, n,$$

satisfying the associativity axioms given by formula (3.8) below and the equivariance axioms given by formula (3.9). (There is also a unity $1 \in \mathscr{P}(1)$, satisfying the unity axiom, but this is irrelevant to our paper.) See, e.g. [MSS02], [LV12]. The universal (to the operad $\mathscr{P}$) $\mathbb{Z}$-graded Lie superalgebra $W(\mathscr{P}) = \bigoplus_{j \geq -1} W_j$ is defined by letting $W_n = \mathscr{P}(n+1)^{S_{n+1}}$ with the bracket (1.1), where

$$X \square Y = \sum_{\sigma \in S_{m+1,n}} (X \circ_1 Y)^{\sigma^{-1}}.$$

Here $S_{m,n}$ denotes the set of (m,n)-shuffles in S_{m+n}; see Sect. 3 for details. The earliest reference to this construction that we know of is [Tam02].

The most popular example of an operad is $\mathscr{P} = \mathscr{H}om$, for which

$$\mathscr{H}om(n) = \mathrm{Hom}(V^{\otimes n}, V),$$

for a vector superspace V. The action of S_n on $\mathscr{P}(n)$ is defined via its natural action on $V^{\otimes n}$ (taking into account the parity of V), and the i-th product $X \circ_i Y$ of $X \in \mathscr{H}om(n)$ and $Y \in \mathscr{H}om(m)$ is defined by ($i = 1, \dots, n$)

$$\begin{aligned}&(X \circ_i Y)(v_1, \dots, v_{n+m-1}) \\ &= X(v_1, \dots, v_{i-1}, Y(v_i, \dots, v_{i+m-1}), v_{i+m}, \dots, v_{n+m-1}).\end{aligned}$$

It is easy to see that the Lie superalgebras $W(V)$ and $W(\mathscr{H}om)$ are identical. Likewise, for an $\mathbb{F}[\partial]$-module V, one defines the operad $\mathscr{C}hom$, for which $\mathscr{C}hom(n)$ is the space $P_n^{\partial}(V)$ defined by (1.5), and recovers thereby the associated Lie superalgebra $W^{\partial}(V)$.

Thus, the operads $\mathscr{H}om$ and $\mathscr{C}hom$ "govern" the Lie superalgebras and the Lie conformal superalgebras respectively. In their seminal book [BD04], Beilinson and Drinfeld generalized the notion of a vertex algebra, introduced by Borcherds [Bor86], by defining a chiral algebra in the language of $\mathscr{D}$-modules on any smooth algebraic curve, so that a vertex algebra is a weakly translation covariant chiral algebra on the affine line. They also constructed the corresponding chiral operad and the cohomology theory of chiral algebras, and the associated graded chiral operad.

In the present paper, we translate the construction of the chiral operad from [BD04] to the purely algebraic language of vertex algebras. The resulting operad, which we denote by P^{ch}, not surprisingly turns out to be an extension of the operad $\mathscr{Chom}$ in the same spirit as $\mathscr{Chom}$ is an extension of the operad $\mathscr{Hom}$.

In order to explain the construction of P^{ch} (see Sect. 6.3), let us introduce some notation. For $k \in \mathbb{Z}_{\geq -1}$, let $\mathscr{O}^T_{k+1}$ and $\mathscr{O}^{\star T}_{k+1}$ be respectively the algebras of polynomials and Laurent polynomials in $z_{ij} = z_i - z_j$, where $0 \leq i < j \leq k$, and let $\mathscr{D}^T_{k+1} = \sum_{i=0}^k \mathscr{O}^T_{k+1} \partial_{z_i}$ be the algebra of translation invariant regular differential operators. Let V be an $\mathbb{F}[\partial]$-module. The space $V^{\otimes(k+1)} \otimes \mathscr{O}^{\star T}_{k+1}$ carries the structure of a right $\mathscr{D}^T_{k+1}$-module by letting z_{ij} act by multiplication on $\mathscr{O}^{\star}_{k+1}$, and letting ∂_{z_i} act by

$$(v_0 \otimes \cdots \otimes v_k \otimes f)\partial_{z_i} = v_0 \otimes \cdots \partial v_i \cdots \otimes v_k \otimes f - v_0 \otimes \cdots \otimes v_k \otimes \frac{\partial f}{\partial z_i}.$$

The space $\mathbb{F}_-[\lambda_0, \dots, \lambda_k]$ considered above carries a structure of a $\mathscr{D}^T_{k+1}$-module as well, by letting z_{ij} act as $-\frac{\partial}{\partial \lambda_i} + \frac{\partial}{\partial \lambda_j}$ and ∂_{z_i} act as multiplication by $-\lambda_i$. Then $P^{\mathrm{ch}}(k+1)$ is defined as the space of all right $\mathscr{D}^T_{k+1}$-module homomorphisms

$$V^{\otimes(k+1)} \otimes \mathscr{O}^{\star T}_{k+1} \longrightarrow \mathbb{F}_-[\lambda_0, \dots, \lambda_k] \otimes_{\mathbb{F}[\partial]} V.$$

The right action of the symmetric group S_{k+1} on $P^{\mathrm{ch}}(k+1)$ is defined by simultaneous permutations in $V^{\otimes(k+1)} \otimes \mathscr{O}^{\star T}_{k+1}$ of factors of $V^{\otimes(k+1)}$ and the corresponding variables $z_0, \dots, z_k$ in $\mathscr{O}^{\star T}_{k+1}$. The $\circ_1$ product in P^{ch} is defined by (6.20), and the general composition by (6.25).

We denote by $W^{\mathrm{ch}}(V) = \bigoplus_{j \geq -1} W^{\mathrm{ch}}_j(V)$ the $\mathbb{Z}$-graded Lie superalgebra associated to the operad P^{ch} for the $\mathbb{F}[\partial]$-module V. It is clear that W^{ch}_j for $j = -1, 0$ is the same as for the operad $\mathscr{Chom}$. However, $W^{\mathrm{ch}}_1(\Pi V)$ is identified not with the space of sesquilinear skew-symmetric λ-brackets as for $\mathscr{Chom}$, but with their integrals; see Proposition 6.8. Moreover, the set of odd elements $X \in W^{\mathrm{ch}}_1(\Pi V)$ such that $[X, X] = 0$ is identified with those integrals of λ-brackets satisfying the "integral" Jacobi identity; see Theorem 6.12. Thus, due to the integral of λ-bracket definition of a vertex algebra introduced in [DSK06], such elements X parametrize non-unital vertex algebra structures on the $\mathbb{F}[\partial]$-module V.

As explained above, we thus obtain a cohomology complex for any non-unital vertex algebra V and its module M. The low cohomology is as expected from any Lie-type cohomology. Namely, the 0-th cohomology parametrizes Casimirs (i.e., invariants) of the V-module M, the 1-st cohomology is identified with the quotient of all derivations from V to M by the space of inner derivations, and the 2-nd cohomology parametrizes the $\mathbb{F}[\partial]$-split extensions of

V by M with a trivial structure of a non-unital vertex algebra (see Theorem 7.6). The vertex algebra cohomology studied in [Bor98], [Hua14] and [Lib17] is rather of Harrison type; for example, their 1-st cohomology is identified with the space of all derivations from V to M.

The $\mathbb{Z}$-graded Lie superalgebra associated to the operad P^{ch} and its corresponding differential complex associated to a non-unital vertex algebra structure on V was defined in [Tam02] in the context of chiral algebras as the complex governing deformations of the chiral algebra structure. It was later studied in [Yan16] where the author introduces also a Lie algebra structure on the complex governing deformations of Poisson vertex algebras. Both [Tam02] and [Yan16] rely on the geometric language of [BD04] to construct these Lie algebras. In particular, they associate a deformation complex to any smooth algebraic curve X and any chiral (respectively, coisson) algebra on X. In this article we restrict to the case when X is the affine line and the chiral (respectively, coisson) algebra is translation equivariant, hence associated to a vertex algebra (respectively, Poisson vertex algebra). In this restricted case, we are able to give a more explicit linear algebraic and combinatorial description of these complexes, providing a suitable framework to carry out computations of (Poisson) vertex algebra cohomologies.

The algebras $\mathscr{O}_{k+1}^{\star T}$ carry a natural increasing filtration by the number of poles, which induces a decreasing filtration of the operad P^{ch}. We study the associated graded operad, denoted by P^{cl}. Its explicit description is quite involved and uses the cooperad of graphs (see Theorem 10.6). One can show that the operad P^{ch} studied in our paper is (non-canonically) isomorphic to that in [BD04] in the case of the curve being the affine line.

We also consider a refinement of the above filtration of P^{ch}, associated to an increasing filtration $0 \subset \mathrm{F}^1 V \subset \mathrm{F}^2 V \subset \cdots$ of the $\mathbb{F}[\partial]$-module V, and show that the structures of a filtered vertex algebra on V are in bijective correspondence with odd elements $X \in \mathrm{F}^1 W_1^{\mathrm{ch}}(\Pi V)$ such that $[X, X] = 0$ (see Theorem 8.10). Moreover, one has an injective morphism of complexes

$$(\mathrm{gr}\, W^{\mathrm{ch}}(\Pi V), \mathrm{gr}(\mathrm{ad}\, X)) \hookrightarrow (W^{\mathrm{cl}}(\mathrm{gr}\, \Pi V), \mathrm{ad}(\mathrm{gr}\, X))$$

(see Theorem 10.14), which is an isomorphism under certain conditions (see Remark 10.15).

Next, we show that the structures of a Poisson vertex algebra on the $\mathbb{F}[\partial]$-module V are in bijection with the odd elements $X \in W_1^{\mathrm{cl}}(\Pi V)$ such that $[X, X] = 0$ (see Theorem 10.7). Using this, we relate the cohomology of the corresponding complex, called the PVA cohomology, to the variation Poisson cohomology studied in [DSK13]. In particular, we show that the low vertex algebra cohomology is majorized by the variational Poisson cohomology. Using this and a computation of the variational Poisson cohomology in [DSK12]–

[DSK13], we compute the Casimirs and derivations of the vertex algebra of N bosons.

Throughout the paper, the base field $\mathbb{F}$ is a field of characteristic 0 and, unless otherwise specified, all vector spaces, their tensor products and Hom's are over $\mathbb{F}$.

2. Preliminaries on vector superspaces and the symmetric group

2.1. Vector superspaces, tensor products and linear maps

Recall that a vector superspace is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We denote by $p(v) \in \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ the parity of a homogeneous element $v \in V$. Given two vector superspaces U, V, their tensor product $U \otimes V$ and the space of linear maps $\mathrm{Hom}(U, V)$ are naturally vector superspaces, with $\mathbb{Z}/2\mathbb{Z}$-grading induced by those of U and V, i.e., we have $p(u \otimes v) = p(u) + p(v)$, and $p(f) = p(f(u)) - p(u)$, for $u \in U$, $v \in V$, $f \in \mathrm{Hom}(U, V)$. Let $g_i : U_i \to V_i$, $i = 1, \dots, n$, be linear maps of vector superspaces. One defines their tensor product $g_1 \otimes \cdots \otimes g_n : U_1 \otimes \cdots \otimes U_n \to V_1 \otimes \cdots \otimes V_n$, by

$$(g_1 \otimes \cdots \otimes g_n)(u_1 \otimes \cdots \otimes u_n) = (-1)^{\sum_{i<j} p(g_j)p(u_i)} g_1(u_1) \otimes \cdots \otimes g_n(u_n). \tag{2.1}$$

In other words, we follow the usual Koszul–Quillen rule: every time two odd elements are switched, we change the sign.

2.2. The action of the symmetric group on tensor powers

The symmetric group S_n is, by definition, the group of bijections $\sigma : \{1, \dots, n\} \xrightarrow{\sim} \{1, \dots, n\}$, mapping $i \mapsto \sigma(i)$.

If $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a vector superspace, the symmetric group S_n acts linearly on $V^{\otimes n}$:

$$\sigma(v_1 \otimes \cdots \otimes v_n) := \epsilon_v(\sigma)\, v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}, \tag{2.2}$$

where

$$\epsilon_v(\sigma) = \prod_{i<j \,|\, \sigma(i)>\sigma(j)} (-1)^{p(v_i)p(v_j)}. \tag{2.3}$$

(Again, we follow the Koszul–Quillen rule for the sign factor.) Formula (2.2) defines a left action of S_n on $V^{\otimes n}$, since the signs $\epsilon_v(\sigma)$ satisfy the relation

$$\epsilon_v(\sigma\tau) = \epsilon_{\tau(v)}(\sigma)\epsilon_v(\tau), \tag{2.4}$$

which can be easily checked.

We also have the corresponding *right* action of S_n on the space $\mathrm{Hom}(V^{\otimes n}, V)$ of linear maps $f(v_1 \otimes \cdots \otimes v_n)$, given by

$$f^\sigma(v_1 \otimes \cdots \otimes v_n) = \epsilon_v(\sigma) f(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}) \, (= f(\sigma(v_1 \otimes \cdots \otimes v_n))). \tag{2.5}$$

Note that the same formula (2.2) makes sense when applied to an element $v_1 \otimes \cdots \otimes v_n \in W_1 \otimes \cdots \otimes W_n$, where $W_1, \ldots, W_n$ are different vector superspaces. In this case, $\sigma \in S_n$ defines an (even) linear map

$$\sigma : W_1 \otimes \cdots \otimes W_n \xrightarrow{\sim} W_{\sigma^{-1}(1)} \otimes \cdots \otimes W_{\sigma^{-1}(n)}. \tag{2.6}$$

Lemma 2.1. *Let $g_i : U_i \to V_i$, $i = 1, \ldots, n$, be linear maps of vector superspaces, and let $u_i \in U_i$, $i = 1, \ldots, n$. For every $\sigma \in S_n$, we have*

$$\sigma((g_1 \otimes \cdots \otimes g_n)(u_1 \otimes \cdots \otimes u_n)) = (\sigma(g_1 \otimes \cdots \otimes g_n))(\sigma(u_1 \otimes \cdots \otimes u_n)). \tag{2.7}$$

Proof. Since S_n is generated by transpositions, it suffices to prove that equation (2.7) holds for $\sigma = (s, s+1)$, $s = 1, \ldots, n-1$. In this case it is straightforward. □

We also define the (left) action of the symmetric group S_n on an arbitrary ordered n-tuple of objects $(x_1, \ldots, x_n)$ as follows:

$$\sigma(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}). \tag{2.8}$$

In other words, we put the object x_1 in position $\sigma(1)$, the object x_2 in position $\sigma(2)$, and so on. (Note that, if the objects $x_1, \ldots, x_n$ are the numbers $1, \ldots, n$, this action is obtained by applying not σ to each of the entries of the list, but σ^{-1}.)

2.3. Composition of permutations

Let $n \geq 1$ and $m_1, \ldots, m_n \geq 0$. Given permutations $\sigma \in S_n$, $\tau_1 \in S_{m_1}, \ldots, \tau_n \in S_{m_n}$, we want to define their *composition* $\sigma(\tau_1, \ldots, \tau_n) \in S_{m_1 + \cdots + m_n}$. To describe it, it is easier to say how it acts on the tensor power $V^{\otimes(m_1 + \cdots + m_n)}$ of the vector superspace V, in analogy to (2.2). Let

$$M_i = \sum_{j=1}^{i} m_j, \quad i = 0, \ldots, n. \tag{2.9}$$

To apply $\sigma(\tau_1, \ldots, \tau_n)$ to v, we first apply each $\tau_i \in S_{m_i}$ to the vector $w_i = v_{M_{i-1}+1} \otimes \cdots \otimes v_{M_i} \in V^{\otimes m_i}$ via (2.2), and then we apply $\sigma \in S_n$ to $w = \tau_1(w_1) \otimes \cdots \otimes \tau_n(w_n)$, again by the same formula (2.2), where we view w as an

element of $W_1 \otimes \cdots \otimes W_n$, with $W_i = V^{\otimes m_i}$, and we consider the generalization of (2.2) defined in (2.6). Summarizing this in a formula, we have:

$$(\sigma(\tau_1, \dots, \tau_n))(v) = \sigma(\tau_1(v_1 \otimes \cdots \otimes v_{M_1}) \otimes \cdots \otimes \tau_n(v_{M_{n-1}+1} \otimes \cdots \otimes v_{M_n})). \tag{2.10}$$

Remark 2.2. We can write explicitly how $\sigma(\tau_1, \dots, \tau_n) \in S_{m_1+\cdots+m_n}$ permutes the integers $1, \dots, m_1 + \cdots + m_n$. An integer $k \in \{1, \dots, m_1 + \cdots + m_n\}$ can be uniquely decomposed in the form

$$k = m_1 + \cdots + m_{i-1} + j, \tag{2.11}$$

with $1 \le i \le n$ and $1 \le j \le m_i$. Then, we have

$$(\sigma(\tau_1, \dots, \tau_n))(k) = m_{\sigma^{-1}(1)} + \cdots + m_{\sigma^{-1}(\sigma(i)-1)} + \tau_i(j). \tag{2.12}$$

Proposition 2.3. *The composition of permutations satisfies the following associativity condition: given $\sigma \in S_n$, $\tau_i \in S_{m_i}$ for $i = 1, \dots, n$, and $\rho_j \in S_{\ell_j}$ for $j = 1, \dots, M_n$, we have*

$$\begin{aligned} &(\sigma(\tau_1, \dots, \tau_\ell))(\rho_1, \dots, \rho_{M_n}) \\ &= \sigma(\tau_1(\rho_1, \dots, \rho_{M_1}), \dots, \tau_n(\rho_{M_{n-1}+1}, \dots, \rho_{M_n})) \in S_{\sum_j \ell_j}. \end{aligned} \tag{2.13}$$

Proof. Take a monomial $v_1 \otimes \cdots \otimes v_{L_{M_n}}$, where we define M_i as in (2.9), and let

$$L_j = \sum_{k=1}^{j} \ell_k, \quad j = 0, \dots, M_n. \tag{2.14}$$

By (2.10), when we apply either side of (2.13) to such monomial, we get

$$\begin{aligned} &\sigma(\tau_1(\rho_1(v_1 \times \cdots \times v_{L_1}) \otimes \cdots \otimes \rho_{M_1}(v_{L_{M_1-1}+1} \otimes \cdots \otimes v_{L_{M_1}})) \otimes \cdots \\ &\otimes \tau_n(\rho_{M_{n-1}+1}(v_{L_{M_{n-1}}+1} \otimes \cdots \otimes v_{L_{M_{n-1}+1}}) \otimes \cdots \\ &\otimes \rho_{M_n}(v_{L_{M_n-1}+1} \otimes \cdots \otimes v_{L_{M_n}}))). \end{aligned}$$

The claim follows. □

Proposition 2.4. *The composition of permutations satisfies the following equivariance condition:*

$$(\varphi\sigma)(\psi_1\tau_1, \dots, \psi_n\tau_n) = \varphi(\psi_{\sigma^{-1}(1)}, \dots, \psi_{\sigma^{-1}(n)})\, \sigma(\tau_1, \dots, \tau_n), \tag{2.15}$$

for every $\varphi, \sigma \in S_n$, $\psi_1, \tau_1 \in S_{m_1}$, $\dots$, $\psi_n, \tau_n \in S_{m_n}$.

Proof. It suffices to show that both sides of (2.15) give the same result when applied to a monomial in $V^{\otimes(m_1+\cdots+m_n)}$. When we apply the left-hand side of (2.15) to $v_1 \otimes \cdots \otimes v_{M_n}$, we get

$$\begin{aligned}&((\varphi\sigma)(\psi_1\tau_1,\dots,\psi_n\tau_n))(v_1\otimes\cdots\otimes v_{M_n})\\&=(\varphi\sigma)((\psi_1\tau_1)(v_1\otimes\cdots\otimes v_{M_1})\otimes\cdots\otimes(\psi_n\tau_n)(v_{M_{n-1}+1}\otimes\cdots\otimes v_{M_n})).\end{aligned}$$

On the other hand, if we apply the right-hand side of (2.15) to the same monomial, we get,

$$\begin{aligned}&(\varphi(\psi_{\sigma^{-1}(1)},\dots,\psi_{\sigma^{-1}(n)}))(\sigma(\tau_1,\dots,\tau_n))(v_1\otimes\cdots\otimes v_{M_n})\\&=(\varphi(\psi_{\sigma^{-1}(1)},\dots,\psi_{\sigma^{-1}(n)}))\\&\quad\cdot(\sigma(\tau_1(v_1\otimes\cdots\otimes v_{M_1})\otimes\cdots\otimes\tau_n(v_{M_{n-1}+1}\otimes\cdots\otimes v_{M_n})))\\&=\varphi(\sigma(\psi_1(\tau_1(v_1\otimes\cdots\otimes v_{M_1}))\otimes\cdots\otimes\psi_n(\tau_n(v_{M_{n-1}+1}\otimes\cdots\otimes v_{M_n})))).\end{aligned}$$

For the second equality we used (2.8) and Lemma 2.1. Equation (2.15) follows. □

2.4. $\circ_i$-products of permutations

For $i = 1,\dots,n$, we define the $\circ_i$ product of permutations $\circ_i : S_n \times S_m \to S_{n+m-1}$ as follows ($\beta \in S_n$, $\alpha \in S_m$):

$$\beta \circ_i \alpha := \beta(\overbrace{1,\dots,1}^{i-1},\overset{i}{\alpha},\overbrace{1,\dots,1}^{n-i}). \tag{2.16}$$

In other words, its action on the tensor power $V^{\otimes m+n-1}$ of the vector superspace V, is given by

$$\begin{aligned}(\beta\circ_i\alpha)(v_1\otimes\cdots\otimes v_{n+m-1}) &= \beta(v_1\otimes\cdots\overset{i}{w}\cdots\otimes v_{n+m-1}),\\ w &= \alpha(v_i\otimes\cdots\otimes v_{i+m-1}).\end{aligned} \tag{2.17}$$

As a consequence of Proposition 2.3, the $\circ_i$-products satisfy the following associativity conditions: ($\gamma \in S_n$, $\beta \in S_m$, $\alpha \in S_\ell$, $i = 1,\dots,n$, $j = 1,\dots,n+m-1$):

$$(\gamma\circ_i\beta)\circ_j\alpha = \begin{cases}(\gamma\circ_j\alpha)\circ_{\ell+i-1}\beta & \text{if } 1\le j<i,\\ \gamma\circ_i(\beta\circ_{j-i+1}\alpha) & \text{if } i\le j<i+m,\\ (\gamma\circ_{j-m+1}\alpha)\circ_i\beta & \text{if } i+m\le j<n+m.\end{cases} \tag{2.18}$$

In particular, the $\circ_1$-product is associative. Moreover, as a consequence of Proposition 2.4, the $\circ_i$-products satisfy the following equivariance condition ($\beta,\sigma \in S_n$, $\alpha,\tau \in S_m$, $i = 1,\dots,n$):

$$(\beta\sigma)\circ_i(\alpha\tau) = (\beta\circ_{\sigma(i)}\alpha)(\sigma\circ_i\tau). \tag{2.19}$$

We shall denote the identity element of the symmetric group S_n by 1_n. For every $m, n \geq 1$ and $i = 1, \dots, n$ we have

$$1_n \circ_i 1_m = 1_{n+m-1}. \tag{2.20}$$

By (2.19), we have $(1_n \circ_i \alpha)(1_n \circ_i \tau) = 1_n \circ_i (\alpha\tau)$. In other words, for each $n, m \geq 1$, and each $i = 1, \dots, n$, we have the injective group homomorphism

$$S_m \hookrightarrow S_{n+m-1}, \quad \alpha \longmapsto 1_n \circ_i \alpha. \tag{2.21}$$

For $\alpha \in S_m$ and $i = 1, \dots, n$, we can write explicitly the action of $1_n \circ_i \alpha \in S_{n+m-1}$ on $V^{\otimes(n+m-1)}$:

$$(1_n \circ_i \alpha)(v_1 \otimes \cdots \otimes v_{n+m-1}) = v_1 \otimes \cdots \otimes \alpha(v_i \otimes \cdots \otimes v_{m-1+i}) \otimes \cdots \otimes v_{m+n-1}.$$

In particular, for $\alpha \in S_m$ and $\beta \in S_n$, the actions of

$$1_{n+1} \circ_1 \alpha \text{ and } 1_{m+1} \circ_{m+1} \beta \in S_{m+n} \text{ commute.}$$

In the special case $i = 1$ it is particularly easy to describe the image of the map (2.21) as a permutation of the numbers $\{1, \dots, n+m-1\}$. We have, for $m, n \geq 1$ and $\alpha \in S_m$,

$$(1_n \circ_1 \alpha)(i) = \begin{cases} \alpha(i) & \text{if } 1 \leq i \leq m, \\ i & \text{if } m+1 \leq i \leq m+n-1. \end{cases} \tag{2.22}$$

By (2.19), we also have

$$(\beta \circ_{\sigma(i)} 1_m)(\sigma \circ_i 1_m) = (\beta\sigma) \circ_i 1_m. \tag{2.23}$$

Hence, the injective map $S_n \hookrightarrow S_{n+m-1}$ mapping $\sigma \mapsto \sigma \circ_i 1_m$ is not a group homomorphism. On the other hand, it becomes a group homomorphism when we restrict to the stabilizer $(S_n)_i = \{\sigma \in S_n \,|\, \sigma(i) = i\}$ of i:

$$(S_n)_i \hookrightarrow S_{n+m-1}, \quad \sigma \longmapsto \sigma \circ_i 1_m. \tag{2.24}$$

As special cases of (2.18) (with $\alpha = 1_\ell$, $\gamma = 1_2$, $i = 2$ and $j = 1$), we get the following identity, which we shall need later ($\alpha \in S_m$):

$$1_{\ell+1} \circ_{\ell+1} \alpha = (1_2 \circ_2 \alpha) \circ_1 1_\ell. \tag{2.25}$$

Note that we can write the cyclic permutation $(1, \dots, m+1)$ mapping $1 \mapsto 2 \mapsto \cdots \mapsto m+1 \mapsto 1$ in terms of the $\circ_i$-products as follows:

$$(1, \dots, m+1) = (1, 2) \circ_1 1_m \text{ in } S_{m+1}. \tag{2.26}$$

More generally, if we consider the cyclic permutation $(1, \dots, m+1)$ in the permutation group S_{m+n}, we have the identity

$$(1, \dots, m+1) = 1_n \circ_1 (1, 2) \circ_1 1_m \text{ in } S_{m+n} \tag{2.27}$$

(there is no need to put parentheses in the right-hand side since the $\circ_1$-product is associative). An equivalent way to write equation (2.27) is

$$(1, \dots, m+1) = (1, 2)(1_m, 1_1, 1_{n-1}) \text{ in } S_{m+n}, \tag{2.28}$$

where, in this case, we consider the transposition $(1, 2)$ as an element of S_3.

2.5. Shuffles

A permutation $\sigma \in S_{m+n}$ is called an (m,n)-*shuffle* if

$$\sigma(1) < \cdots < \sigma(m)\,,\ \sigma(m+1) < \cdots < \sigma(m+n). \tag{2.29}$$

In equivalent terms, when acting on the tensor power $V^{\otimes(m+n)}$ of the (purely even) vector space V, a shuffle σ maps the monomial $v = v_1 \otimes \cdots \otimes v_{m+n}$ to a permuted monomial in which the factors $v_1, \dots, v_m$ appear in their order:

$$\sigma(v) = \cdots \otimes \overset{\sigma(1)}{v_1} \otimes \cdots \otimes \overset{\sigma(2)}{v_2} \otimes \cdots \otimes \overset{\sigma(m)}{v_m} \otimes \cdots,$$

and the factors $v_{m+1}, \dots, v_{m+n}$ appear in order. We shall denote by $S_{m,n} \subset S_{m+n}$ the subset (it is not a subgroup) of (m,n)-shuffles. By definition, $S_{n,0} = S_{0,n} = \{1\}$ for every $n \geq 0$ and, by convention, we let $S_{m,n} = \emptyset$ if either m or n is negative.

Similarly, we shall denote by $S_{\ell,m,n} \subset S_{\ell+m+n}$ the subset of (ℓ,m,n)-shuffles, i.e., permutations $\sigma \in S_{\ell+m+n}$ satisfying

$$\begin{aligned} &\sigma(1) < \cdots < \sigma(\ell), \quad \sigma(\ell+1) < \cdots < \sigma(\ell+m), \\ &\sigma(\ell+m+1) < \cdots < \sigma(\ell+m+n), \end{aligned} \tag{2.30}$$

and the same for $(m_1, \dots, m_k)$-shuffles in $S_{m_1+\cdots+m_k}$, for arbitrary $k \geq 2$.

Proposition 2.5. *(a) We have a bijection $S_{m,n} \xrightarrow{\sim} S_{n,m}$ given by $\sigma \mapsto \sigma \cdot (1,2)(1_n, 1_m)$.*
(b) We have a bijection $S_{\ell,m,n} \xrightarrow{\sim} S_{m,\ell,n}$ given by $\sigma \mapsto \sigma \cdot (1,2)(1_m, 1_\ell, 1_n)$.

Proof. The permutation $(1,2)(1_n, 1_m)$ switches the first n factors of $V^{\otimes(m+n)}$ with the last m factors, i.e., it maps

$$\begin{array}{ccccccc} 1 & \cdots & n & 1+n & \cdots & m+n \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ m+1 & \cdots & m+n & 1 & \cdots & m. \end{array}$$

Hence, the product $\sigma \cdot (1,2)(1_n, 1_m)$ maps

$$\begin{array}{ccccccc} 1 & \cdots & n & 1+n & \cdots & m+n \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ \sigma(m+1) & < \cdots < & \sigma(m+n), & \sigma(1) & < \cdots < & \sigma(m), \end{array}$$

so it lies in $S_{n,m}$, provided that $\sigma \in S_{m,n}$. Claim (a) follows. Similarly for claim (b). □

Proposition 2.6. *For $\ell, m, n \geq 1$ we have the following bijections:*

(a) $S_{m,n} \times S_m \times S_n \xrightarrow{\sim} S_{m+n}$, *mapping*

$$\sigma, \alpha, \beta \longmapsto \sigma \cdot (1_{n+1} \circ_1 \alpha) \cdot (1_{m+1} \circ_{m+1} \beta),$$

where $\cdot$ *denotes the product in the symmetric group* S_{m+n}.

(b) $S_{\ell+m,n} \times S_{\ell,m} \xrightarrow{\sim} S_{\ell,m,n}$, *mapping*

$$\sigma, \tau \longmapsto \sigma \cdot (1_{n+1} \circ_1 \tau).$$

(c) $S_{m,n} \times S_{\ell,m+n} \xrightarrow{\sim} S_{\ell,m,n}$, *mapping*

$$\sigma, \tau \longmapsto \tau \cdot (1_{\ell+1} \circ_{\ell+1} \sigma).$$

Proof. First, the two sets $S_{m,n} \times S_m \times S_n$ and S_{m+n} have the same cardinality $(m+n)!$. Hence, to prove that the map (a) is a bijection it suffices to prove that it is injective or surjective. On the other hand, we can see how

$$X = \sigma \cdot (1_{n+1} \circ_1 \alpha) \cdot (1_{m+1} \circ_{m+1} \beta)$$

acts on the tensor power V^{m+n-1}, of a vector space V. First, we permute the factors of $v = v_1 \otimes \cdots \otimes v_m$ by α and the factors of $v_{m+1} \otimes \cdots \otimes v_{m+n}$ by β. Then, we shuffle the resulting monomial, in such a way that the factors of $\alpha(v_1 \otimes \cdots \otimes v_m)$ appear in the same order in $X(v)$, in positions $\sigma(1), \dots, \sigma(m)$, and similarly the factors of $\beta(v_{m+1} \otimes \cdots \otimes v_{m+n})$ appear in the same order in $X(v)$, in positions $\sigma(m+1), \dots, \sigma(m+n)$. Now it is clear that the resulting monomial $X(v)$ is uniquely determined by the choice of $\sigma \in S_{m,n}$, $\alpha \in S_m$ and $\beta \in S_n$. In other words, the map (a) is injective.

Let us now prove that the map (b) is bijective. First note that two sets $S_{\ell+m,n} \times S_{\ell,m}$ and $S_{\ell,m,n}$ have the same cardinality $\frac{(\ell+m+n)!}{\ell!m!n!}$. Next, we need to prove that, for $\sigma \in S_{\ell+m,n}$ and $\tau \in S_{\ell,m}$, the permutation $\sigma(1_{n+1} \circ_1 \tau)$ is an (ℓ, m, n)-shuffle. Indeed, by (2.22),

$$1 \leq (1_{n+1} \circ_1 \tau)(i) = \tau(i) \leq \ell + m \quad \text{for } i = 1, \dots, \ell + m, \tag{2.31}$$

and

$$(1_{\ell+1} \circ_1 \tau)(i) = i \quad \text{for } i = \ell + m + 1, \dots, \ell + m + n. \tag{2.32}$$

On the other hand, since $\tau \in S_{\ell,m}$, we have

$$1 \leq \tau(1) < \cdots < \tau(\ell) \leq \ell + m \text{ and } 1 \leq \tau(\ell+1) < \cdots < \tau(\ell+m) \leq \ell + m, \tag{2.33}$$

and since $\sigma \in S_{\ell+m,n}$, we have

$$\sigma(1) < \cdots < \sigma(\ell + m) \text{ and } \sigma(\ell + m + 1) < \cdots < \sigma(\ell + m + n). \tag{2.34}$$

Combining (2.31)–(2.34), we get

$$\begin{aligned}
&\sigma(1_{n+1} \circ_1 \tau)(1) = \sigma(\tau(1)) < \cdots < \sigma(1_{n+1} \circ_1 \tau)(\ell) = \sigma(\tau(\ell)),\\
&\sigma(1_{n+1} \circ_1 \tau)(\ell+1)\\
&= \sigma(\tau(\ell+1)) < \cdots < \sigma(1_{n+1} \circ_1 \tau)(\ell+m) = \sigma(\tau(\ell+m)),\\
&\sigma(1_{n+1} \circ_1 \tau)(\ell+m+1)\\
&= \sigma(\ell+m+1) < \cdots < \sigma(1_{n+1} \circ_1 \tau)(\ell+m+n) = \sigma(\ell+m+n),
\end{aligned} \tag{2.35}$$

namely, $\sigma(1_{n+1} \circ_1 \tau) \in S_{\ell,m,n}$. To prove that the map (b) is injective, we just observe that, by the third line in (2.35), the values of $\sigma(1_{n+1} \circ_1 \tau)$ on $\ell+m+1, \dots, \ell+m+n$ uniquely determine $\sigma(\ell+m+1), \dots, \sigma(\ell+m+n)$, i.e., uniquely determine the shuffle $\sigma \in S_{\ell+m,n}$. Then, since σ is uniquely determined by $\sigma(1_{n+1} \circ_1 \tau)$, it is clear that τ is uniquely determined as well. A similar proof works for (c). □

Proposition 2.7. *(a) The set of shuffles $S_{m,n}$ decomposes as*

$$S_{m,n} = \{\sigma \in S_{m,n} \,|\, \sigma(1) = 1\} \sqcup \{\sigma \in S_{m,n} \,|\, \sigma(m+1) = 1\}.$$

(b) We have a bijection $\{\sigma \in S_{m+1,n-1} \,|\, \sigma(1) = 1\} \xrightarrow{\sim} \{\sigma \in S_{m,n} \,|\, \sigma(m+1) = 1\}$ given by

$$\sigma \longmapsto \sigma \cdot (1_n \circ_1 (1,2) \circ_1 1_m), \tag{2.36}$$

where $\cdot$ is the product in the symmetric group S_{m+n}.

(c) We have a bijection $S_{m-1,n} \xrightarrow{\sim} \{\sigma \in S_{m,n} \,|\, \sigma(1) = 1\}$ given by

$$\sigma \longmapsto 1_2 \circ_2 \sigma. \tag{2.37}$$

Proof. Claim (a) is obvious since, if σ is an (m,n)-shuffle, then either $1 = \sigma(1)$ or $1 = \sigma(m+1)$. For (b), recall that, by (2.27), $(1_n \circ_1 (1,2) \circ_1 1_m)$ is the cyclic permutation $1 \mapsto 2 \mapsto \cdots \mapsto m+1 \mapsto 1$. Hence, the product $\sigma \cdot (1_n \circ_1 (1,2) \circ_1 1_m)$ maps

$$\begin{array}{ccccccccc}
1 & \cdots & m & & m+1 & & m+2 & \cdots & m+n\\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow\\
\sigma(2) & < \cdots < & \sigma(m+1), & & \sigma(1) = 1 & < & \sigma(m+2) & < \cdots < & \sigma(m+n).
\end{array}$$

It follows that $\sigma \cdot (1_n \circ_1 (1,2) \circ_1 1_m)$ lies in $S_{m,n}$, provided that $\sigma \in S_{m+1,n-1}$, and that it maps $m+1 \mapsto 1$. On the other hand, the map (2.36) is clearly injective, hence it is bijective since the two sets $\{\sigma \in S_{m+1,n-1} \,|\, \sigma(1) = 1\}$ and $\{\sigma \in S_{m,n} \,|\, \sigma(m+1) = 1\}$ have the same cardinality $\frac{(m+n-1)!}{m!(n-1)!}$. This proves (b).

Next, let us prove claim (c). By the definition of the $\circ_i$-products, we have

$$(1_2 \circ_2 \sigma)(1) = 1, \text{ and}$$

$$(1_2 \circ_2 \sigma)(1+i) = 1 + \sigma(i) \quad \text{for} \quad i = 1, \dots, m+n-1.$$

In particular, $(1_2 \circ_2 \sigma) \in S_{m,n}$ provided that $\sigma \in S_{m-1,n}$. On the other hand, the map (2.37) is clearly injective, hence it is bijective since the two sets $S_{m-1,n}$ and $\{\sigma \in S_{m,n} \,|\, \sigma(1) = 1\}$ have the same cardinality $\frac{(m+n-1)!}{(m-1)!n!}$. This proves (c). Finally, claim (d) follows from (b) and (c). □

3. Superoperads and the associated $\mathbb{Z}$-graded Lie superalgebras

In this section, we review the definition and some basic properties of superoperads, which will be needed throughout the rest of the paper. For extended reviews on the theory of operads, see e.g. [LV12], [MSS02].

3.1. Definition of a superoperad

Recall that a (linear, unital, symmetric) *superoperad* $\mathscr{P}$ is a collection of vector superspaces $\mathscr{P}(n)$, $n \geq 0$, with parity p, endowed, for every $f \in \mathscr{P}(n)$ and $m_1, \dots, m_n \geq 0$, with the *composition* parity preserving linear map,

$$\begin{aligned} \mathscr{P}(n) \otimes \mathscr{P}(m_1) \otimes \cdots \otimes \mathscr{P}(m_n) &\longrightarrow \mathscr{P}(M_n), \\ f \otimes g_1 \otimes \cdots \otimes g_n &\longmapsto f(g_1 \otimes \cdots \otimes g_n), \end{aligned} \tag{3.1}$$

where $M_n := m_1 + \cdots + m_n$ (*cf.* (2.9)), satisfying the following associativity axiom:

$$\begin{aligned} &f((g_1 \otimes \cdots \otimes g_n)(h_1 \otimes \cdots \otimes h_{M_n})) \\ &= (f(g_1 \otimes \cdots \otimes g_n))(h_1 \otimes \cdots \otimes h_{M_n}) \in \mathscr{P}\Big(\sum_{j=1}^{M_n} \ell_j\Big), \end{aligned} \tag{3.2}$$

for every $f \in \mathscr{P}(n)$, $g_i \in \mathscr{P}(m_i)$ for $i = 1, \dots, n$, and $h_j \in \mathscr{P}(\ell_j)$ for $j = 1, \dots, M_n$. In the left-hand side of (3.2) the linear map

$$\bigotimes_{i=1}^{n} g_i : \bigotimes_{j=1}^{M_n} \mathscr{P}(\ell_j) \longrightarrow \bigotimes_{i=1}^{n} \mathscr{P}\Big(\sum_{j=M_{i-1}+1}^{M_i} \ell_j\Big)$$

is the tensor product of composition maps, defined by (2.1), applied to

$$\begin{aligned} &h_1 \otimes \cdots \otimes h_{M_n} \\ &= (h_1 \otimes \cdots \otimes h_{M_1}) \otimes (h_{M_1+1} \otimes \cdots \otimes h_{M_2}) \otimes \cdots \otimes (h_{M_{n-1}+1} \otimes \cdots \otimes h_{M_n}). \end{aligned}$$

We assume that $\mathscr{P}$ is endowed with a *unit* element $1 \in \mathscr{P}(1)$ satisfying the following unity axioms:

$$f(1 \otimes \cdots \otimes 1) = 1(f) = f\,, \quad \text{for every} \quad f \in \mathscr{P}(n). \tag{3.3}$$

Furthermore, we assume that, for each $n \geq 1$, $\mathscr{P}(n)$ has a right action of the symmetric group S_n, denoted f^σ, for $f \in \mathscr{P}(n)$ and $\sigma \in S_n$, satisfying the following equivariance axiom ($f \in \mathscr{P}(n)$, $g_1 \in \mathscr{P}(m_1), \dots, g_n \in \mathscr{P}(m_n)$, $\sigma \in S_n$, $\tau_1 \in S_{m_1}, \dots, \tau_n \in S_{m_n}$):

$$f^\sigma(g_1^{\tau_1} \otimes \cdots \otimes g_n^{\tau_n}) = (f(\sigma(g_1 \otimes \cdots \otimes g_n)))^{\sigma(\tau_1,\dots,\tau_n)}, \tag{3.4}$$

where the composition $\sigma(\tau_1, \dots, \tau_n) \in S_{m_1+\cdots+m_n}$ of permutation was defined in Sect. 2.3, and the left action of $\sigma \in S_n$ on the tensor product of vector superspaces was defined in (2.6).

For simplicity, from now on, we will use the term operad in place of superoperad.

Example 3.1. The *symmetric group* operad $\mathscr{S}$ is defined as the collection of purely even superspaces $\mathscr{S}(n) = \mathbb{F}[S_n]$ for $n \geq 1$ and $\mathscr{S}(0) = \mathbb{F}$, with the composition maps obtained by extending linearly (2.10) to the group algebras, the unity $1 \in S_1$, and the action of right multiplication of S_n on $\mathbb{F}[S_n]$. This is an operad, indeed the associativity axiom (3.2) follows from Proposition 2.3, and the equivariance axiom (3.4) follows from Proposition 2.4.

Example 3.2. Given a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, the operad $\mathscr{H}om$ is defined as the collection of superspaces $\mathscr{H}om(n) = \mathrm{Hom}(V^{\otimes n}, V)$, $n \geq 0$, endowed with the composition maps ($f \in \mathscr{P}(n)$, $g_i \in \mathscr{P}(m_i)$ for $i = 1, \dots, n$, $v_j \in V$ for $j = 1, \dots, M_n := m_1 + \cdots + m_n$):

$$(f(g_1 \otimes \cdots \otimes g_n))(v_1 \otimes \cdots \otimes v_{M_n}) := f((g_1 \otimes \cdots \otimes g_n)(v_1 \otimes \cdots \otimes v_{M_n})), \tag{3.5}$$

the unity $1 = \mathbb{1}_V \in \mathrm{End}\, V$, and the right action of S_n on $\mathrm{Hom}(V^{\otimes n}, V)$ given by (2.5). The associativity and unit axioms, for this example, are obvious. Let us prove the equivariance axiom (3.4). When applied to a monomial $v_1 \otimes \cdots \otimes v_{M_n}$, the left-hand side of (3.4) gives (we use the notation (2.9))

$$\begin{aligned}
&(f^\sigma(g_1^{\tau_1} \otimes \cdots \otimes g_n^{\tau_n}))(v_1 \otimes \cdots \otimes v_{M_n}) \\
&= f^\sigma((g_1^{\tau_1} \otimes \cdots \otimes g_n^{\tau_n})(v_1 \otimes \cdots \otimes v_{M_n})) \\
&= f^\sigma((g_1 \otimes \cdots \otimes g_n)(\tau_1(v_1 \otimes \cdots \otimes v_{M_1}) \otimes \cdots \otimes \tau_n(v_{M_{n-1}+1} \otimes \cdots \otimes v_{M_n}))) \\
&= f((\sigma(g_1 \otimes \cdots \otimes g_n))(\sigma(\tau_1, \dots, \tau_n)(v_1 \otimes \cdots \otimes v_{M_n}))) \\
&= (f(\sigma(g_1 \otimes \cdots \otimes g_n)))^{\sigma(\tau_1,\dots,\tau_n)}(v_1 \otimes \cdots \otimes v_{M_n}).
\end{aligned}$$

For the third equality, we used Lemma 2.1 and the definition (2.10) of $\sigma(\tau_1, \dots, \tau_n)$.

Given an operad $\mathscr{P}$, one defines, for each $i = 1, \dots, n$, the $\circ_i$-product $\circ_i : \mathscr{P}(n) \times \mathscr{P}(m) \to \mathscr{P}(n + m - 1)$ by insertion in position i, i.e.,

$$f \circ_i g = f(\overbrace{1 \otimes \cdots \otimes 1}^{i-1} \otimes \overset{i}{g} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-i}). \tag{3.6}$$

Of course, knowing all the $\circ_i$-products allows to reconstruct, thanks to the associativity axiom (3.2), the whole operad structure, by

$$f(g_1,\dots,g_n) = (\cdots((f\circ_1 g_1)\circ_{m_1+1} g_2)\cdots)\circ_{m_1+\cdots+m_{n-1}+1} g_n. \tag{3.7}$$

Then, the unity axiom (i) becomes $1\circ_1 f = f\circ_i 1 = f$ for every $i = 1,\dots,n$, and the associativity axiom (ii) is equivalent to the following identities for the $\circ_i$-products, *cf.* (2.18) ($f\in\mathscr{P}(n)$, $g\in\mathscr{P}(m)$, $h\in\mathscr{P}(\ell)$):

$$(f\circ_i g)\circ_j h = \begin{cases} (-1)^{p(g)p(h)}(f\circ_j h)\circ_{\ell+i-1} g & \text{if } 1\le j<i,\\ f\circ_i (g\circ_{j-i+1} h) & \text{if } i\le j<i+m,\\ (-1)^{p(g)p(h)}(f\circ_{j-m+1} h)\circ_i g & \text{if } i+m\le j<n+m. \end{cases} \tag{3.8}$$

In particular, the $\circ_1$-product is associative. Note that the third identity in (3.8) is equivalent to the first one by flipping the equality. Furthermore, the equivariance condition (3.4) and the supersymmetric equivariance condition (3.4) both become, in terms of the $\circ_i$-products, *cf.* (2.19)

$$f^\sigma\circ_i g^\tau = (f\circ_{\sigma(i)} g)^{\sigma\circ_i\tau}, \tag{3.9}$$

for $f\in\mathscr{P}(n)$, $g\in\mathscr{P}(m)$, $\sigma\in S_n$, $\tau\in S_m$, where $\sigma\circ_i\tau\in S_{n+m-1}$ is defined by (2.16).

By definition, an operad $\mathscr{P}$ is *filtered* if each vector superspace $\mathscr{P}(n)$ is endowed with a filtration $\mathrm{F}^r\,\mathscr{P}(n)$, which is preserved by the action of the symmetric group and is preserved by the composition maps, i.e.,

$$f^\sigma\in\mathrm{F}^r\,\mathscr{P}(n),\quad f(g_1\otimes\cdots\otimes g_n)\in\mathrm{F}^{r+s_1+\cdots+s_n}\,\mathscr{P}(m_1+\cdots+m_n) \tag{3.10}$$

for all $f\in\mathrm{F}^r\,\mathscr{P}(n)$, $\sigma\in S_n$ and $g_i\in\mathrm{F}^{s_i}\,\mathscr{P}(m_i)$. In the case of a decreasing filtration of $\mathscr{P}$, the corresponding *associated graded* operad is

$$\mathrm{gr}^r\,\mathscr{P}(n) = \mathrm{F}^r\,\mathscr{P}(n)/\,\mathrm{F}^{r+1}\,\mathscr{P}(n), \tag{3.11}$$

with the induced action of the symmetric groups and composition maps. This is a *graded operad*, i.e., each superspace $\mathscr{P}(n)$ is graded and the analog of (3.10) holds degreewise.

A *morphism* φ from an operad $\mathscr{P}$ to an operad $\mathscr{Q}$ is a collection of linear maps $\varphi_n\colon\mathscr{P}(n)\to\mathscr{Q}(n)$ commuting with the action of the symmetric groups and compatible with the composition maps. When $\mathscr{P}$ and $\mathscr{Q}$ are filtered, the map φ_n is required to send $\mathrm{F}^r\,\mathscr{P}(n)$ to $\mathrm{F}^r\,\mathscr{Q}(n)$, and similarly for graded operads.

3.2. Universal Lie superalgebra associated to an operad

Let $\mathscr{P}$ be an operad. We let $W = W(\mathscr{P})$ be the $\mathbb{Z}$-graded vector superspace $W = \bigoplus_{n\geq -1} W_n$, where

$$W_n = \mathscr{P}(n+1)^{S_{n+1}} = \{f \in \mathscr{P}(n+1) \mid f^\sigma = f,\ \forall \sigma \in S_{n+1}\}. \tag{3.12}$$

We define the $\square$-product of $f \in W_n$ and $g \in W_m$ as follows:

$$f \square g = \sum_{\sigma \in S_{m+1,n}} (f \circ_1 g)^{\sigma^{-1}}. \tag{3.13}$$

Note that $S_{m+1,-1} = \emptyset$, hence $f \square g = 0$ if $f \in W_{-1}$.

Example 3.3. For $f, g \in W_1$, we have

$$\begin{aligned} f \square g &= f \circ_1 g + (f \circ_1 g)^{(23)} + (f \circ_1 g)^{(132)} \\ &= f \circ_1 g + f \circ_2 g + (f \circ_2 g)^{(12)}. \end{aligned} \tag{3.14}$$

The second equality follows from (3.9) and the fact that f and g are symmetric, using $(23) = (132)(12)$ and $(132) = (12) \circ_2 (1)$.

Theorem 3.4. *(a) For $f \in W_n$ and $g \in W_m$, we have $f \square g \in W_{n+m}$.*
(b) The associator of the $\square$-product is right supersymmetric, i.e.,

$$(f \square g) \square h - f \square (g \square h) = (-1)^{p(g)p(h)}((f \square h) \square g - f \square (h \square g)). \tag{3.15}$$

(c) Consequently, W is a $\mathbb{Z}$-graded Lie superalgebra with Lie bracket given by ($f \in W_n$, $g \in W_m$)

$$[f, g] = f \square g - (-1)^{p(f)p(g)} g \square f. \tag{3.16}$$

Proof. For claim (a), we need to prove that $f \square g$ is fixed by the action of the symmetric group S_{m+n+1}. We have

$$\begin{aligned} \sum_{\sigma \in S_{m+n+1}} (f \circ_1 g)^{\sigma^{-1}} &= \sum_{\substack{\beta \in S_{m+1}, \alpha \in S_n, \\ \sigma \in S_{m+1,n}}} (f \circ_1 g)^{(\sigma \cdot (1_{n+1} \circ_1 \beta) \cdot (1_{m+2} \circ_{m+2} \alpha))^{-1}} \\ &= \sum_{\substack{\beta \in S_{m+1}, \alpha \in S_n, \\ \sigma \in S_{m+1,n}}} (f \circ_1 g)^{(1_{m+2} \circ_{m+2} \alpha^{-1}) \cdot (1_{n+1} \circ_1 \beta^{-1}) \cdot \sigma^{-1}} \\ &= \sum_{\substack{\beta \in S_{m+1}, \alpha \in S_n, \\ \sigma \in S_{m+1,n}}} (f \circ_1 g)^{((1_2 \circ_2 \alpha^{-1}) \circ_1 1_{m+1}) \cdot (1_{n+1} \circ_1 \beta^{-1}) \cdot \sigma^{-1}} \\ &= \sum_{\substack{\beta \in S_{m+1}, \alpha \in S_n, \\ \sigma \in S_{m+1,n}}} (f^{(1_2 \circ_2 \alpha^{-1})} \circ_1 g^{\beta^{-1}})^{\sigma^{-1}} \end{aligned}$$

$$= (m+1)!n! \sum_{\sigma \in S_{m+1,n}} (f \circ_1 g)^{\sigma^{-1}} = (m+1)!n!\ f \square g, \tag{3.17}$$

where we used Proposition 2.6 (a) for the first equality, the homomorphism property of the maps (2.21) for the second equality, identity (2.25) for the third equality, the equivariance conditions (3.9) and the obvious identities $(1_2 \circ_2 \alpha^{-1})(1) = 1 = 1_{n+1}(1)$ for the fourth equality, and the assumptions that $f \in W_n$ and $g \in W_m$ for the fifth equality. Since the left-hand side of (3.17) is manifestly invariant with respect to the action of S_{m+n+1}, we conclude that $f \square g$ is invariant as well, proving (a).

Next, let us prove claim (b). We have

$$\begin{aligned} f \square (g \square h) &= \sum_{\sigma \in S_{\ell+m+1,n}} \sum_{\tau \in S_{\ell+1,m}} (f \circ_1 (g \circ_1 h)^{\tau^{-1}})^{\sigma^{-1}} \\ &= \sum_{\sigma \in S_{\ell+m+1,n}} \sum_{\tau \in S_{\ell+1,m}} (f \circ_1 (g \circ_1 h))^{(1_{n+1} \circ_1 \tau^{-1}) \cdot \sigma^{-1}} \\ &= \sum_{\sigma \in S_{\ell+m+1,n}} \sum_{\tau \in S_{\ell+1,m}} (f \circ_1 (g \circ_1 h))^{(\sigma \cdot (1_{n+1} \circ_1 \tau))^{-1}} \\ &= \sum_{\sigma \in S_{\ell+1,m,n}} (f \circ_1 (g \circ_1 h))^{\sigma^{-1}}. \end{aligned} \tag{3.18}$$

In the second equality we used the equivariance condition (3.9), in the third equality we used the fact that the map (2.21) is a group homomorphism, and in the fourth equality we used Proposition 2.6 (b). On the other hand, we have

$$(f \square g) \square h = \sum_{\sigma \in S_{m+1,n}} \sum_{\tau \in S_{\ell+1,m+n}} ((f \circ_1 g)^{\sigma^{-1}} \circ_1 h)^{\tau^{-1}}. \tag{3.19}$$

By Proposition 2.7 (a), the sum in the right-hand side of (3.19) split as sum of two terms, in the first one we sum over the shuffles $\sigma \in S_{m+1,n}$ such that $\sigma(1) = 1$, and in the second one we sum over the shuffles $\sigma \in S_{m+1,n}$ such that $\sigma(m+2) = 1$. The first term is

$$\begin{aligned} &\sum_{\substack{\sigma \in S_{m+1,n} \\ \text{s.t. } \sigma(1)=1}} \sum_{\tau \in S_{\ell+1,m+n}} ((f \circ_1 g)^{\sigma^{-1}} \circ_1 h)^{\tau^{-1}} \\ &= \sum_{\substack{\sigma \in S_{m+1,n} \\ \text{s.t. } \sigma(1)=1}} \sum_{\tau \in S_{\ell+1,m+n}} ((f \circ_1 g) \circ_1 h)^{(\sigma^{-1} \circ_1 1_{\ell+1}) \cdot \tau^{-1}} \\ &= \sum_{\substack{\sigma \in S_{m+1,n} \\ \text{s.t. } \sigma(1)=1}} \sum_{\tau \in S_{\ell+1,m+n}} ((f \circ_1 g) \circ_1 h)^{(\tau \cdot (\sigma \circ_1 1_{\ell+1}))^{-1}} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\sigma\in S_{m,n}} \sum_{\tau\in S_{\ell+1,m+n}} ((f\circ_1 g)\circ_1 h)^{(\tau\cdot((1_2\circ_2\sigma)\circ_1 1_{\ell+1}))^{-1}} \\
&= \sum_{\sigma\in S_{m,n}} \sum_{\tau\in S_{\ell+1,m+n}} ((f\circ_1 g)\circ_1 h)^{(\tau\cdot(1_{\ell+2}\circ_{\ell+2}\sigma))^{-1}} \\
&= \sum_{\sigma\in S_{\ell+1,m,n}} ((f\circ_1 g)\circ_1 h)^{\sigma^{-1}},
\end{aligned}
\tag{3.20}
$$

where we used the equivariance relation (3.9) for the first equality, the fact that the map (2.24) is a homomorphism for the second equality, Proposition 2.7 (c) for the third equality, equation (2.25) for the fourth equality, and Proposition 2.6 (c) for the fifth equality. Note that the right-hand side of (3.20) coincides with the right-hand side of (3.18) since the $\circ_1$-product is associative. The second term, where we take the sum over the shuffles $\sigma\in S_{m+1,n}$ such that $\sigma(m+2)=1$ in (3.19), is

$$
\begin{aligned}
&\sum_{\substack{\sigma\in S_{m+1,n}\\ \text{s.t. }\sigma(m+2)=1}} \sum_{\tau\in S_{\ell+1,m+n}} ((f\circ_1 g)^{\sigma^{-1}}\circ_1 h)^{\tau^{-1}} \\
&= \sum_{\substack{\sigma\in S_{m+1,n}\\ \text{s.t. }\sigma(m+2)=1}} \sum_{\tau\in S_{\ell+1,m+n}} ((f\circ_1 g)\circ_{\sigma^{-1}(1)} h)^{(\sigma^{-1}\circ_1 1_{\ell+1})\cdot\tau^{-1}} \\
&= \sum_{\substack{\sigma\in S_{m+1,n}\\ \text{s.t. }\sigma(m+2)=1}} \sum_{\tau\in S_{\ell+1,m+n}} ((f\circ_1 g)\circ_{m+2} h)^{(\sigma\circ_{m+2}1_{\ell+1})^{-1}\tau^{-1}} \\
&= \sum_{\substack{\sigma\in S_{m+1,n}\\ \text{s.t. }\sigma(m+2)=1}} \sum_{\tau\in S_{\ell+1,m+n}} (f(g\otimes h\otimes \overbrace{1\otimes\cdots\otimes 1}^{n-1}))^{(\tau\cdot(\sigma\circ_{m+2}1_{\ell+1}))^{-1}} \\
&= \sum_{\sigma\in S_{m+1,n-1}} \sum_{\tau\in S_{\ell+1,m+n}} (f(g\otimes h\otimes \overbrace{1\otimes\cdots\otimes 1}^{n-1}))^{(\tau\cdot(((1_2\circ_2\sigma)\cdot(1_n\circ_1(1,2)\circ_1 1_{m+1}))\circ_{m+2}1_{\ell+1}))^{-1}},
\end{aligned}
\tag{3.21}
$$

where we used the equivariance relation (3.9) for the first equality, equation (2.23) for the second equality, the definition (3.6) of the $\circ_i$-products for the third equality, and Proposition 2.7 (d) for the fourth equality. Equation (2.19), with $(1_2\circ_2\sigma)$ in place of β, $(1_n\circ_1(1,2)\circ_1 1_{m+1})$ in place of σ, $\alpha=\tau=1_{\ell+1}$ and $i=m+2$, gives

$$
\begin{aligned}
&((1_2 \circ_2 \sigma) \cdot (1_n \circ_1 (1,2) \circ_1 1_{m+1})) \circ_{m+2} 1_{\ell+1} \\
&= ((1_2 \circ_2 \sigma) \circ_1 1_{\ell+1}) \cdot ((1_n \circ_1 (1,2) \circ_1 1_{m+1}) \circ_{m+2} 1_{\ell+1}) \\
&= (1_{\ell+2} \circ_{\ell+2} \sigma) \cdot (1,2)(1_{m+1}, 1_{\ell+1}, 1_{n-1}),
\end{aligned}
$$

where, for the second equality, we used equations (2.25) and (2.28). Hence, the right-hand side of (3.21) becomes

$$
\begin{aligned}
&\sum_{\sigma \in S_{m+1,n-1}} \sum_{\tau \in S_{\ell+1,m+n}} (f(g \otimes h \otimes 1 \otimes \cdots \otimes 1))^{(\tau \cdot (1_{\ell+2} \circ_{\ell+2} \sigma) \cdot (1,2)(1_{m+1}, 1_{\ell+1}, 1_{n-1}))^{-1}} \\
&= \sum_{\sigma \in S_{\ell+1,m+1,n-1}} (f(g \otimes h \otimes 1 \otimes \cdots \otimes 1))^{(\sigma \cdot (1,2)(1_{m+1}, 1_{\ell+1}, 1_{n-1}))^{-1}} \\
&= \sum_{\sigma \in S_{m+1,\ell+1,n-1}} (f(g \otimes h \otimes 1 \otimes \cdots \otimes 1))^{\sigma^{-1}},
\end{aligned}
\tag{3.22}
$$

where we used Proposition 2.6 (c) in the first equality and Proposition 2.5 (b) for the second equality. Combining (3.18), (3.19), (3.20), (3.21) and (3.22), we get

$$
(f \square g) \square h - f \square (g \square h) = \sum_{\sigma \in S_{m+1,\ell+1,n-1}} (f(g \otimes h \otimes 1 \otimes \cdots \otimes 1))^{\sigma^{-1}}. \tag{3.23}
$$

To conclude the proof of (b), we observe that the right-hand side of (3.23) is manifestly supersymmetric with respect to the exchange of g and h. Indeed, since $f \in W_n$, we have $f = f^{(1,2)}$. Hence, by the equivariance condition (3.4), we have

$$
\begin{aligned}
&\sum_{\sigma \in S_{m+1,\ell+1,n-1}} (f(g \otimes h \otimes 1 \otimes \cdots \otimes 1))^{\sigma^{-1}} \\
&= \sum_{\sigma \in S_{m+1,\ell+1,n-1}} (f^{(1,2)}(g^{1_{m+1}} \otimes h^{1_{\ell+1}} \otimes 1 \otimes \cdots \otimes 1))^{\sigma^{-1}} \\
&= (-1)^{p(g)p(h)} \sum_{\sigma \in S_{m+1,\ell+1,n-1}} (f(h \otimes g \otimes 1 \otimes \cdots \otimes 1))^{(1,2)(1_{m+1}, 1_{\ell+1}, 1_{n-1}) \cdot \sigma^{-1}} \\
&= (-1)^{p(g)p(h)} \sum_{\sigma \in S_{m+1,\ell+1,n-1}} (f(h \otimes g \otimes 1 \otimes \cdots \otimes 1))^{(\sigma \cdot (1,2)(1_{\ell+1}, 1_{m+1}, 1_{n-1}))^{-1}} \\
&= (-1)^{p(g)p(h)} \sum_{\sigma \in S_{\ell+1,m+1,n-1}} (f(h \otimes g \otimes 1 \otimes \cdots \otimes 1))^{\sigma^{-1}},
\end{aligned}
$$

again by Proposition 2.5 (b). Claim (c) is an obvious consequence of (b). □

Remark 3.5. For an arbitrary non-symmetric operad $\mathscr{P}$ (i.e., for which we do not require the action of the symmetric groups and the equivariance axiom (3.4)), we can also construct a $\mathbb{Z}$-graded Lie superalgebra

$$\mathscr{G} = \bigoplus_{n \geq -1} \mathscr{G}_n, \quad \mathscr{G}_{n-1} = \mathscr{P}(n), \tag{3.24}$$

with Lie bracket ($f \in \mathscr{P}(n)$, $g \in \mathscr{P}(m)$)

$$[f, g] = \sum_{i=1}^{n} f \circ_i g - (-1)^{p(f)p(g)} \sum_{i=1}^{m} g \circ_i f. \tag{3.25}$$

Indeed, letting $f \circ g = \sum_{i=1}^{n} f \circ_i g$, we have, by the associativity condition (3.8),

$$(f \circ g) \circ h - f \circ (g \circ h) = \sum_{i=1}^{n} \sum_{j=1}^{m+n-1} (f \circ_i g) \circ_j h - \sum_{i=1}^{n} \sum_{j=1}^{m} f \circ_i (g \circ_j h)$$
$$= (-1)^{p(g)p(h)} \sum_{1 \leq j < i \leq n} (f \circ_j h) \circ_{\ell+i-1} g + \sum_{1 \leq i < j \leq n} (f \circ_i g) \circ_{m+j-1} h.$$

Since the expression in the right-hand side is supersymmetric with respect to the exchange of g and h, it follows that (3.25) is a Lie superalgebra bracket. In the special case when each $\mathscr{P}(n)$ has the same parity as $(n+1)$, the resulting bracket (3.25) is known as the Gerstenhaber bracket [Ger63].

4. The operad governing Lie superalgebras

Given the vector superspace V, with parity p, we denote by ΠV the same vector space with reversed parity $\bar{p} = 1 - p$, and we consider the corresponding operad $\mathscr{H}om(\Pi V)$ from Example 3.2, and the associated $\mathbb{Z}$-graded Lie superalgebra $W(\Pi V) := W(\mathscr{H}om(\Pi V))$ given by Theorem 3.4.

Proposition 4.1 ([NR67], [DSK13]). *We have a bijective correspondence between the odd elements $X \in W_1(\Pi V)$ such that $X \Box X = 0$ and the Lie superalgebra brackets $[\cdot\,,\cdot] \colon V \times V \to V$ on V, given by*

$$[a, b] = (-1)^{p(a)} X(a \otimes b). \tag{4.1}$$

Proof. By definition, $X \in (\mathscr{H}om(\Pi V))(2)_{\bar{1}}$ is an odd linear map $X \colon (\Pi V)^{\otimes 2} \to \Pi V$, and it corresponds, via (4.1), to a parity preserving bilinear map $[\cdot\,,\cdot] : V \times V \to V$. Moreover, to say that X lies in $W_1(\Pi V) = (\mathscr{H}om(\Pi V))(2)^{S_2}$ is

equivalent to say that the corresponding bracket $[\cdot\,,\cdot]$ satisfies skew-symmetry. Finally, by the definition (3.13) of the $\square$-product, we have $(a,b,c\in V)$

$$(X\square X)(a\otimes b\otimes c)=\sum_{\sigma\in S_{2,1}}(X\circ_1 X)^{\sigma^{-1}}(a\otimes b\otimes c)$$
$$=(-1)^{p(b)}([[a,b],c]-[a,[b,c]]+(-1)^{p(a)p(b)}[b,[a,c]]).$$

Hence, $X\square X=0$ if and only if the Jacobi identity holds. □

Note that, if $X\in W_1(\Pi V)_{\bar{1}}$ satisfies $X\square X=0$, then it follows by the Jacobi identity for the Lie superalgebra $W(\Pi V)$ that $(\operatorname{ad}X)^2=0$, i.e., we have a cohomology complex $(W(\Pi V),\operatorname{ad}X)$.

Definition 4.2. *Let V be a Lie superalgebra. The corresponding* Lie superalgebra cohomology complex *is defined as*

$$(W(\Pi V),\operatorname{ad}X),$$

where $X\in W(\Pi V)_{\bar{1}}$ is given by (4.1).

Obviously, the kernel of $\operatorname{ad}X$ is a subalgebra of $W(\Pi V)$ and the image of $\operatorname{ad}X$ is its ideal. Hence, the cohomology $H(W(\Pi V),\operatorname{ad}X)$ has the structure of a Lie superalgebra.

Remark 4.3. One can define the $\mathscr{Lie}$ operad as follows: $\mathscr{Lie}(1)=\mathbb{F}1$; $\mathscr{Lie}(2)$ is the non-trivial 1-dimensional representation of S_2, with basis element denoted by $[\cdot\,,\cdot]$; for every $n\geq 2$, all the elements of $\mathscr{Lie}(n)$ are obtained by composition of $[\cdot\,,\cdot]\in\mathscr{Lie}(2)$, and they are subject to the relation in $\mathscr{Lie}(3)$ corresponding to the Jacobi identity:

$$[\cdot\,,\cdot]\circ_2[\cdot\,,\cdot]-\sigma_{12}([\cdot\,,\cdot]\circ_2[\cdot\,,\cdot])=[\cdot\,,\cdot]\circ_1[\cdot\,,\cdot].$$

Then, a Lie superalgebra structure on a vector superspace V is the same as a morphism of (symmetric) operads $\mathscr{Lie}\to\mathscr{Hom}(V)$. Proposition 4.1 gives such a morphism by sending $[\cdot\,,\cdot]$ to X.

In the following sections, we will repeat the same line of reasoning as the one used in the present section for the cohomology theories of Lie conformal algebras, Poisson algebras, Poisson vertex algebras and vertex algebras: after reviewing their definition, we will construct, for each of them, an operad $\mathscr{P}$, and we will describe their algebraic structures as an element $X\in W_1\subset W(\mathscr{P})$ such that $X\square X=0$. In this way, we automatically get, for each algebraic structure of interest, the corresponding cohomology complex $(W(\mathscr{P}),\operatorname{ad}X)$.

5. The operad governing Lie conformal superalgebras

5.1. Lie conformal superalgebras

Recall that a Lie conformal superalgebra is a vector superspace V, endowed with an even endomorphism $\partial \in \mathrm{End}(V)$ and a bilinear (over $\mathbb{F}$) λ-bracket $[\cdot\,{}_\lambda\,\cdot]\colon V \times V \to V[\lambda]$ satisfying sesquilinearity ($a, b \in V$):

$$[\partial a\,{}_\lambda\, b] = -\lambda [a\,{}_\lambda\, b], \quad [a\,{}_\lambda\, \partial b] = (\lambda + \partial)[a\,{}_\lambda\, b], \tag{5.1}$$

skew-symmetry ($a, b \in V$):

$$[a\,{}_\lambda\, b] = -(-1)^{p(a)p(b)} [b_{-\lambda-\partial} a], \tag{5.2}$$

and the Jacobi identity ($a, b, c \in V$):

$$[a\,{}_\lambda\, [b\,{}_\mu\, c]] - (-1)^{p(a)p(b)} [b\,{}_\mu\, [a\,{}_\lambda\, b]] = [[a\,{}_\lambda\, b]\,{}_{\lambda+\mu}\, c]. \tag{5.3}$$

5.2. The $\mathscr{Chom}$ operad

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace endowed with an even endomorphism $\partial \in \mathrm{End}\, V$. The operad $\mathscr{Chom}$ is defined as the collection of superspaces

$$\mathscr{Chom}(n) = \mathrm{Hom}_{\mathbb{F}[\partial]^{\otimes n}}(V^{\otimes n}, \mathbb{F}_-[\lambda_1, \dots, \lambda_n] \otimes_{\mathbb{F}[\partial]} V), \quad n \geq 0. \tag{5.4}$$

Here and further $\lambda_1, \dots, \lambda_k$ are commuting indeterminates of even parity and $\mathbb{F}_-[\lambda_1, \dots, \lambda_k]$ denotes the space of polynomials in the variables $\lambda_1, \dots, \lambda_n$. This space is endowed with a structure of a left $\mathbb{F}[\partial]^{\otimes n}$-module by letting $P_1(\partial) \otimes \cdots \otimes P_n(\partial)$ act as multiplication by $P_1(-\lambda_1) \cdots P_n(-\lambda_n)$, and a structure of a right $\mathbb{F}[\partial]$-module by letting ∂ act as multiplication by $-\lambda_1 - \cdots - \lambda_n$.

Note that $\mathscr{Chom}(0) = V/\partial V$. Obviously, we can identify $\mathbb{F}_-[\lambda] \otimes_{\mathbb{F}[\partial]} V \simeq V$, so that $\mathscr{Chom}(1) = \mathrm{End}_{\mathbb{F}[\partial]}(V)$. For arbitrary $n \geq 1$, $\mathscr{Chom}(n)$ consists of all linear maps

$$\begin{aligned} f\colon V^{\otimes n} &\longrightarrow \mathbb{F}_-[\lambda_1, \dots, \lambda_n] \otimes_{\mathbb{F}[\partial]} V, \\ v_1 \otimes \cdots \otimes v_n &\longmapsto f_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n), \end{aligned}$$

satisfying the sesquilinearity conditions:

$$\begin{aligned} &f_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \partial v_i \cdots \otimes v_n) \\ &= -\lambda_i f_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n) \quad \text{for all } i = 1, \dots, n. \end{aligned} \tag{5.5}$$

In particular, $\mathscr{Chom}(2)$ is identified with the space of all λ-brackets on V, satisfying (5.1).

The $\mathbb{Z}/2\mathbb{Z}$-structure of $\mathscr{Chom}(n)$ is induced by that of V. The composition of $f \in \mathscr{Chom}(n)$ and $g_1 \in \mathscr{Chom}(m_1), \dots, g_n \in \mathscr{Chom}(m_n)$ is defined as follows:

$$\begin{aligned}&(f(g_1 \otimes \cdots \otimes g_n))_{\lambda_1,\dots,\lambda_{M_n}}(v_1 \otimes \cdots \otimes v_{M_n}) \\ &:= f_{\Lambda_1,\dots,\Lambda_n}(((g_1)_{\lambda_1,\dots,\lambda_{M_1}} \otimes \cdots \otimes (g_n)_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}})(v_1 \otimes \cdots \otimes v_{M_n})),\end{aligned} \tag{5.6}$$

where we let (*cf.* (2.9))

$$M_i = \sum_{j=1}^{i} m_j, \ i = 0, \dots, n, \quad \text{and} \quad \Lambda_i = \sum_{j=M_{i-1}+1}^{M_i} \lambda_j, \ i = 1, \dots, n, \tag{5.7}$$

and, recalling (2.1), we have

$$\begin{aligned}&((g_1)_{\lambda_1,\dots,\lambda_{M_1}} \otimes \cdots \otimes (g_n)_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}})(v_1 \otimes \cdots \otimes v_{M_n}) \\ &= \pm (g_1)_{\lambda_1,\dots,\lambda_{M_1}}(v_1 \otimes \cdots \otimes v_{M_1}) \otimes \cdots \\ &\quad \otimes (g_n)_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}}(v_{M_{n-1}+1} \otimes \cdots \otimes v_{M_n}),\end{aligned} \tag{5.8}$$

where

$$\pm = (-1)^{\sum_{i<j} p(g_j)(p(v_{M_{i-1}+1})+\cdots+p(v_{M_i}))}. \tag{5.9}$$

The unity in the $\mathscr{Chom}$-operad is $1 = \mathbb{1}_V \in \mathscr{Chom}(1) = \mathrm{End}_{\mathbb{F}[\partial]} V$, and the right action of S_n on $\mathscr{Chom}(n)$ is given by (*cf.* (2.3), (2.5) and (2.8)):

$$\begin{aligned}&(f^\sigma)_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n) \\ &= f_{\sigma(\lambda_1,\dots,\lambda_n)}(\sigma(v_1 \otimes \cdots \otimes v_n)) \\ &= \epsilon_v(\sigma) f_{\lambda_{\sigma^{-1}(1)},\dots,\lambda_{\sigma^{-1}(n)}}(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}),\end{aligned} \tag{5.10}$$

for every $\sigma \in S_n$, where $\epsilon_v(\sigma)$ is given by (2.3).

Let us first check the associativity axiom for the operad $\mathscr{Chom}$, which reads

$$\begin{aligned}&((f(g_1 \otimes \cdots \otimes g_n))(h_1 \otimes \cdots \otimes h_{M_n}))_{\lambda_1,\dots,\lambda_{L_{M_n}}}(v_1 \otimes \cdots \otimes v_{L_{M_n}}) \\ &= (f((g_1 \otimes \cdots \otimes g_n)(h_1 \otimes \cdots \otimes h_{M_n})))_{\lambda_1,\dots,\lambda_{L_{M_n}}}(v_1 \otimes \cdots \otimes v_{L_{M_n}}),\end{aligned} \tag{5.11}$$

for every $f \in \mathscr{Chom}(n)$, $g_i \in \mathscr{Chom}(m_i)$, $i = 1, \dots, n$, $h_j \in \mathscr{Chom}(\ell_j)$, $j = 1, \dots, m_1 + \cdots + m_n =: M_n$, and $v_k \in V$, $k = 1, \dots, \ell_1 + \cdots + \ell_{M_n} =: L_{M_n}$. Let us denote, in accordance to (5.7),

$$M_i = \sum_{j=1}^{i} m_j, \ i = 0, \dots, n, \quad \text{and} \quad L_j = \sum_{k=1}^{j} \ell_k. \tag{5.12}$$

Then, using the definition (5.6) of the composition map in the operad $\mathscr{Chom}$, one easily checks that both sides of (5.11) are equal to

$$
\begin{aligned}
&f_{\sum_{i=1}^{L_{M_1}}\lambda_i,\dots,\sum_{i=L_{M_{n-1}}+1}^{L_{M_n}}\lambda_i}\\
&\cdot\big(((g_1)_{\sum_{j=1}^{L_1}\lambda_j,\dots,\sum_{j=L_{M_1-1}+1}^{L_{M_1}}\lambda_j}\otimes\cdots\otimes(g_n)_{\sum_{j=L_{M_{n-1}}+1}^{L_{M_{n-1}}+2}\lambda_j,\dots,\sum_{j=L_{M_{n-1}}+1}^{L_{M_n}}\lambda_j}\big)\\
&\cdot\big(((h_1)_{\lambda_1,\dots,\lambda_{L_1}}\otimes\cdots\otimes(h_{M_n})_{\lambda_{L_{M_n-1}+1},\dots,\lambda_{L_{M_n}}})(v_1\otimes\cdots\otimes v_{L_{M_n}})\big)\big),
\end{aligned}
$$

proving associativity. The unity axiom is immediate to check. Next, we prove the equivariance axiom (3.4). If we apply the left-hand side of (3.4) to a monomial $v_1\otimes\cdots\otimes v_{M_n}$ and we evaluate it on the variables $\lambda_1,\dots,\lambda_{M_n}$, we get

$$
\begin{aligned}
&(f^\sigma(g_1^{\tau_1}\otimes\cdots\otimes g_n^{\tau_n}))_{\lambda_1,\dots,\lambda_{M_n}}(v_1\otimes\cdots\otimes v_{M_n})\\
&=(f^\sigma)_{\Lambda_1,\dots,\Lambda_n}(((g_1^{\tau_1})_{\lambda_1,\dots,\lambda_{M_1}}\otimes\cdots\otimes(g_n^{\tau_n})_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}})\\
&\quad\cdot(v_1\otimes\cdots\otimes v_{M_n}))\\
&=f_{\sigma(\Lambda_1,\dots,\Lambda_n)}((((g_1)_{\tau_1(\lambda_1,\dots,\lambda_{M_1})}\otimes\cdots\otimes(g_n)_{\tau_n(\lambda_{M_{n-1}+1},\dots,\lambda_{M_n})})\\
&\quad\cdot(\tau_1(v_1\otimes\cdots\otimes v_{M_1})\otimes\cdots\otimes\tau_n(v_{M_{n-1}+1}\otimes\cdots\otimes v_{M_n}))))\\
&=f_{\sigma(\Lambda_1,\dots,\Lambda_n)}((\sigma((g_1)_{\tau_1(\lambda_1,\dots,\lambda_{M_1})}\otimes\cdots\otimes(g_n)_{\tau_n(\lambda_{M_{n-1}+1},\dots,\lambda_{M_n})}))\\
&\quad\cdot(\sigma(\tau_1,\dots,\tau_n)(v_1\otimes\cdots\otimes v_{M_n})))\\
&=(f(\sigma(g_1\otimes\cdots\otimes g_n)))_{(\sigma(\tau_1,\dots,\tau_n))(\lambda_1,\dots,\lambda_{M_n})}(\sigma(\tau_1,\dots,\tau_n)\\
&\quad\cdot(v_1\otimes\cdots\otimes v_{M_n}))\\
&=((f(\sigma(g_1\otimes\cdots\otimes g_n)))^{\sigma(\tau_1,\dots,\tau_n)})_{\lambda_1,\dots,\lambda_{M_n}}(v_1\otimes\cdots\otimes v_{M_n}).
\end{aligned}
$$

For the first equality we used the definition (5.6) of the composition maps in $\mathscr{Chom}$, for the second equality we used the definition (5.10) of the action of the symmetric group on $\mathscr{Chom}$, for the third equality we used Lemma 2.1 and the definition (2.10) of the composition map of permutations, for the fourth equality we used again (2.10) and (5.6), and for the last equality we used again (5.10). This proves the equivariance condition (3.4).

5.3. Lie conformal superalgebras and the operad $\mathscr{Chom}$

Given the vector superspace V, with parity p, and the even endomorphism $\partial\in\mathrm{End}(V)$, we denote by ΠV the same vector space with reversed parity $\bar{p}=1-p$. Obviously, ∂ is also an even endomorphism of ΠV. We consider the corresponding operad $\mathscr{Chom}(\Pi V)$ from Sect. 5.2 and the associated $\mathbb{Z}$-graded Lie superalgebra $W^\partial(\Pi V):=W(\mathscr{Chom}(\Pi V))$ given by Theorem 3.4.

Proposition 5.1 ([DSK13]). *We have a bijective correspondence between the odd elements $X \in W_1^\partial(\Pi V)$ such that $X \square X = 0$ and the Lie conformal superalgebra λ-brackets $[\cdot\,_\lambda\,\cdot]\colon V \times V \to V[\lambda]$ on V, given by*

$$[a_\lambda b] = (-1)^{p(a)} X_{\lambda,-\lambda-\partial}(a \otimes b). \tag{5.13}$$

Proof. First, $X \in (\mathscr{C}\!\mathit{hom}(\Pi V))(2)$ is, by definition, an odd $\mathbb{F}[\partial]^{\otimes 2}$-module homomorphism $X_{\lambda,\mu}\colon (\Pi V)^{\otimes 2} \to \mathbb{F}_-[\lambda,\mu] \otimes_{\mathbb{F}[\partial]} \Pi V \simeq V[\lambda]$ (the last isomorphism being obtained by letting $\mu = -\lambda - \partial$), and it corresponds, via (4.1), to λ-bracket $[\cdot\,,\,\cdot]\colon V \times V \to V[\lambda]$ satisfying the sesquilinearity conditions (5.1). The condition that $X \in (\mathscr{C}\!\mathit{hom}(\Pi V))(2)$ is odd (with respect to the parity $\bar{p}$ induced by that ΠV), translates into saying that the corresponding λ-bracket $[\cdot\,,\,\cdot]$ is parity preserving. Moreover, the condition that X is fixed by the action (5.10) of the symmetric group S_2 translates into saying that the corresponding λ-bracket $[\cdot\,_\lambda\,\cdot]$ satisfies the skew-symmetry axiom (5.2). To complete the proof, we need to check that the equation $X \square X = 0$ translates to the Jacobi identity for the λ-bracket $[\cdot\,_\lambda\,\cdot]$. By equation (3.14), we have

$$\begin{aligned}
(X \square X)_{\lambda,\mu,\nu}(a \otimes b \otimes c) &= \sum_{\sigma \in S_{2,1}} ((X \circ_1 X)^{\sigma^{-1}})_{\lambda,\mu,\nu}(a \otimes b \otimes c) \\
&= X_{\lambda+\mu,\nu}(X_{\lambda,\mu}(a \otimes b) \otimes c) + (-1)^{\bar{p}(b)\bar{p}(c)} X_{\lambda+\nu,\mu}(X_{\lambda,\nu}(a \otimes c) \otimes b) \\
&\quad + (-1)^{\bar{p}(a)(\bar{p}(b)+\bar{p}(c))} X_{\mu+\nu,\lambda}(X_{\mu,\nu}(b \otimes c) \otimes a) \\
&= X_{\lambda+\mu,\nu}(X_{\lambda,\mu}(a \otimes b) \otimes c) + (-1)^{\bar{p}(b)(1+\bar{p}(a))} X_{\mu,\lambda+\nu}(b \otimes X_{\lambda,\nu}(a \otimes c)) \\
&\quad + (-1)^{\bar{p}(a)} X_{\lambda,\mu+\nu}(a \otimes X_{\mu,\nu}(b \otimes c)) \\
&= (-1)^{p(b)}([[a_\lambda b]_{\lambda+\mu} c] - [a_\lambda [b_\mu c]] + (-1)^{p(a)p(b)} [b_\mu [a_\lambda c]]).
\end{aligned}$$

Hence, $X \square X = 0$ if and only if the Jacobi identity (5.3) holds. □

Definition 5.2 ([BKV99], [DSK09]). *Let V be a Lie conformal superalgebra. The corresponding* Lie conformal superalgebra cohomology complex *is defined as*

$$(W^\partial(\Pi V), \operatorname{ad} X),$$

where $X \in W^\partial(\Pi V)_{\bar{1}}$ is given by (5.13).

Remark 5.3. One can introduce a conformal version of the operad $\mathscr{C}\!\mathit{hom}$, which is associated to the *basic* Lie conformal algebra complex (see [DSK13]). This leads to the notion of a *conformal operad*, which will be developed in a forthcoming publication. In the geometric context of chiral algebras, the corresponding object was constructed by Tamarkin in [Tam02].

6. The chiral operad

6.1. Vertex algebras

In this subsection, we recall the "fifth definition" of a vertex algebra, given in [DSK06]. In a nutshell, this definition says that a vertex algebra is a Lie conformal algebra in which the λ-bracket can be "integrated". More precisely we have:

Definition 6.1. *A vertex algebra is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{F}[\partial]$-module V, endowed with an even element $|0\rangle \in V_{\bar{0}}$ and an* integral *of* λ-bracket, *namely a linear map $V \otimes V \to \mathbb{F}[\lambda] \otimes V$, denoted by*

$$\int^{\lambda} d\sigma\, [u_\sigma v] = :uv: + \int_0^{\lambda} d\sigma\, [u_\sigma v],$$

such that the following axioms hold:

$$\textit{(i)}\ \int^{\lambda} d\sigma\, [|0\rangle_\sigma v] = \int^{\lambda} d\sigma\, [v_\sigma |0\rangle] = v,$$

$$\textit{(ii)}\ \int^{\lambda} d\sigma\, [\partial u_\sigma v] = -\int^{\lambda} d\sigma\, \sigma\, [u_\sigma v],\ \int^{\lambda} d\sigma\, [u_\sigma \partial v] = \int^{\lambda} d\sigma\, (\partial + \sigma)[u_\sigma v],$$

$$\textit{(iii)}\ \int^{\lambda} d\sigma\, [v_\sigma u] = (-1)^{p(u)p(v)} \int^{-\lambda-\partial} d\sigma\, [u_\sigma v],$$

$$\textit{(iv)}\ \int^{\lambda} d\sigma \int^{\mu} d\tau\, ([u_\sigma [v_\tau w]] - (-1)^{p(u)p(v)} [v_\tau [u_\sigma w]] - [[u_\sigma v]_{\sigma+\tau} w]) = 0.$$

If we do not assume the existence of the unit element $|0\rangle \in V$ and we drop axiom (i), we call V a non-unital vertex algebra.

Paper [DSK06] contains a detailed discussion of this definition and the proof of its equivalence to other definitions of vertex algebras. We shall call $[u_\lambda v]$, defined as the derivative by λ of the polynomial $\int^{\lambda} d\sigma\, [u_\sigma v]$, the *$\lambda$-bracket* of u and v. Their *normally ordered product* $:uv:$ is defined as the constant term of the polynomial $\int^{\lambda} d\sigma\, [u_\sigma v]$. The polynomial $\int^{\lambda} d\sigma\, [u_\sigma v]$ will be called the *integral of the λ-bracket* of u and v.

Axioms (i)–(iv) are a concise way to write more complicated relations involving the normally ordered products $:uv:$ and the λ-brackets $[u_\lambda v]$. To explain this, let us describe the meaning of axiom (ii). Taking the derivative with respect to λ of both equations we get the sesquilinearity conditions of the λ-bracket: $[\partial u_\lambda v] = -\lambda [u_\lambda v]$ and $[u_\lambda \partial v] = (\partial + \lambda)[u_\lambda v]$, while putting $\lambda = 0$ in axiom (ii) we get that ∂ is a derivation of the normally ordered product, plus a new piece of notation:

$$\int^{0} d\sigma\, \sigma [u_\sigma v] = -:(\partial u) v:. \tag{6.1}$$

Similarly, axiom (iii) gives two conditions: taking the derivative of both sides with respect to λ we get the skew-symmetry of the λ-bracket:

$$[v_\lambda u] = -(-1)^{p(u)p(v)}[u_{-\lambda-\partial}v],$$

while taking the constant term in λ we get the quasi-commutativity of the normally ordered product:

$$:uv: - (-1)^{p(u)p(v)}:vu: \,=\, \int_{-\partial}^{0} d\lambda\, [u_\lambda v].$$

Finally, to explain of axiom (iv) we expand all three summand in terms of normally ordered product and λ-brackets. The first term is immediate to understand:

$$\int^\lambda d\sigma \int^\mu d\tau\, [u_\sigma [v_\tau w]] = :u(:vw:): + \int_0^\mu d\tau\, :u[v_\tau w]: \\ + \int_0^\lambda d\sigma\, [u_\sigma :vw:] + \int_0^\lambda d\sigma \int_0^\mu d\tau\, [u_\sigma [v_\tau w]].$$

Similarly for the second term. To correctly expand the third term, we first perform the change of variable $\sigma + \tau \mapsto \tau$, we exchange the order of integration in $d\sigma$ and $d\tau$, and we use the notation (6.1). As a result, we get

$$\int^\lambda d\sigma \int^\mu d\tau\, [[u_\sigma v]_{\sigma+\tau} w] = \int^{\lambda+\mu} d\tau \Big[\Big(\int_{-\mu-\partial}^{\lambda} d\sigma\, [u_\sigma v] \Big)_\tau w \Big] \\ = :\Big(\int_{-\mu-\partial}^{\lambda} d\sigma\, [u_\sigma v] \Big) w: + \int_0^{\lambda+\mu} d\tau \Big[\Big(\int_{-\mu-\partial}^{\lambda} d\sigma\, [u_\sigma v] \Big)_\tau w \Big].$$

Hence, by taking constant term or derivative with respect to λ and/or μ, axiom (iv) produces four different axioms on the normally ordered product and the λ-bracket.

Axiom (iv), i.e., Jacobi identity under integration, could be written in the seemingly equivalent form:

$$\text{(iv')} \quad \int^\lambda d\sigma \int^\mu d\tau\, ([u_\sigma [v_{\tau-\sigma} w]] - (-1)^{p(u)p(v)}[v_{\tau-\sigma}[u_\sigma w]] - [[u_\sigma v]_\tau w]) = 0.$$

Not surprisingly, (iv) and (iv') are equivalent, as a consequence of the following:

Lemma 6.2. *Let $\int^\lambda d\sigma\, [\cdot_\sigma \cdot] \colon V \otimes V \to V[\lambda]$ be a linear map satisfying the sesquilinearity and skew-symmetry conditions under integration, i.e., axioms (ii) and (iii) of Definition 6.1. Let*

$$J_{\lambda,\mu}(u,v,w) \\ = \int^\lambda d\sigma \int^\mu d\tau\, ([u_\sigma [v_\tau w]] - (-1)^{p(u)p(v)}[v_\tau [u_\sigma w]] - [[u_\sigma v]_{\sigma+\tau} w]), \tag{6.2}$$

and let

$$\widetilde{J}_{\lambda,\mu}(u,v,w) = \int^{\lambda} d\sigma \int^{\mu} d\tau\, ([u_\sigma [v_{\tau-\sigma} w]] - (-1)^{p(u)p(v)} [v_{\tau-\sigma}[u_\sigma w]] - [[u_\sigma v]_\tau w]). \tag{6.3}$$

Then, we have

$$\widetilde{J}_{\lambda,\mu}(u,v,w) = (-1)^{p(v)p(w)} J_{\lambda,-\mu-\partial}(u,w,v). \tag{6.4}$$

Proof. Applying the skew-symmetry condition under the sign of integral, we have

$$\begin{aligned}
&\widetilde{J}_{\lambda,\mu}(u,v,w) \\
&= \int^{\lambda} d\sigma \int^{\mu} d\tau\, ([u_\sigma [v_{\tau-\sigma} w]] - (-1)^{p(u)p(v)} [v_{\tau-\sigma}[u_\sigma w]] - [[u_\sigma v]_\tau w]) \\
&= \int^{\lambda} d\sigma \int^{\mu} d\tau\, (-(-1)^{p(v)p(w)} [u_\sigma [w_{-\tau+\sigma-\partial} v]] + (-1)^{p(v)p(w)} [[u_\sigma w]_{-\tau+\sigma-\partial} v] \\
&\quad + (-1)^{(p(u)+p(v))p(w)} [w_{-\tau-\partial}[u_\sigma v]]).
\end{aligned}$$

Note that, by the sesquilinearity condition, σ in the first term of the right-hand side is the same as $-\partial$ acting on the first factor u. If we then perform the change of variable $-\tau - \partial = \rho$, the right-hand side becomes

$$\begin{aligned}
&(-1)^{p(v)p(w)} \int^{\lambda} d\sigma \int^{-\mu-\partial} d\rho\, ([u_\sigma [w_\rho v]] - [[u_\sigma w]_{\rho+\sigma} v] - (-1)^{p(u)p(w)} [w_\rho [u_\sigma v]]) \\
&= (-1)^{p(v)p(w)} J_{\lambda,-\mu-\partial}(u,w,v).
\end{aligned}$$

This completes the proof. □

Definition 6.3. *A left* module M *over a non-unital vertex algebra* V *is a* $\mathbb{Z}/2\mathbb{Z}$*-graded* $\mathbb{F}[\partial]$*-module endowed with an* integral λ-action*,* $V \otimes M \to \mathbb{F}[\lambda] \otimes M$*, denoted* $v \otimes m \mapsto \int^{\lambda} d\sigma\, (v_\sigma m)$*, preserving the* $\mathbb{Z}/2\mathbb{Z}$*-grading, such that the following axioms hold:*

$$\textit{(i)} \int^{\lambda} d\sigma\, (\partial v_\sigma m) = -\int^{\lambda} d\sigma\, \sigma (v_\sigma m),\ \int^{\lambda} d\sigma\, (v_\sigma \partial m) = \int^{\lambda} d\sigma\, (\partial + \sigma)(v_\sigma m),$$

$$\textit{(ii)} \int^{\lambda} d\sigma \int^{\mu} d\tau\, (u_\sigma (v_\tau m) - (-1)^{p(u)p(v)} v_\tau (u_\sigma m) - [u_\sigma v]_{\sigma+\tau} m) = 0.$$

This is equivalent to say that the $\mathbb{F}[\partial]$*-module* $V \oplus M$ *has a vertex algebra structure* $\int^{\lambda} d\sigma\, [\cdot_\sigma \cdot]^{\sim}$ *such that* $\int^{\lambda} d\sigma\, [u_\sigma v]^{\sim} \in V[\lambda]$ *for all* $u, v \in V$*, making* V *a vertex subalgebra, and such that* $\int^{\lambda} d\sigma\, [v_\sigma m]^{\sim} \in M[\lambda]$ *for all* $v \in V$ *and* $m \in M$*, and* $\int^{\lambda} d\sigma\, [m_\sigma n]^{\sim} = 0$ *for all* $m, n \in M$*, making* M *an abelian ideal.*

The left V-module structure on M induces a right V-module structure on M given by

$$\int^{\lambda} d\sigma\, m_\sigma v = (-1)^{p(v)p(m)} \int^{-\lambda-\partial} d\sigma\, v_\sigma m. \tag{6.5}$$

6.2. The spaces $\mathscr{O}^{\star T}_{k+1}$

In the following subsection, we will introduce the chiral operad, which governs vertex algebras. In order to do so, we need to define certain spaces of rational functions. For $k \geq -1$, let $\mathscr{O}^{\star}_{k+1}$ be the algebra of polynomials in the variables $z_0, \dots, z_k$ localized on the diagonals $z_i = z_j$ for $i \neq j$. In other words, $\mathscr{O}^{\star}_0 = \mathbb{F}$ and

$$\mathscr{O}^{\star}_{k+1} = \mathbb{F}[z_0, \dots, z_k][z_{ij}^{-1}]_{0\leq i<j\leq k}, \quad \text{where} \quad z_{ij} = z_i - z_j, \quad k \geq 0.$$

We will denote by $\mathscr{O}^{\star T}_{k+1}$ the subalgebra of translation invariant elements, i.e.,

$$\mathscr{O}^{\star T}_{k+1} = \operatorname{Ker}\Big(\sum_{i=0}^{k} \partial_{z_i}\Big) = \mathbb{F}[z_{ij}^{\pm 1}]_{0\leq i<j\leq k}.$$

Note that $\mathscr{O}^{\star T}_0 = \mathscr{O}^{\star T}_1 = \mathbb{F}$.

Let $\mathscr{D}_{k+1}$ be the algebra of regular differential operators in $z_0, \dots, z_k$, i.e., $\mathscr{D}_0 = \mathbb{F}$ and

$$\mathscr{D}_{k+1} = \mathbb{F}[z_0, \dots, z_k][\partial_{z_0}, \dots, \partial_{z_k}], \quad k \geq 0.$$

Let $\mathscr{D}^{T}_{k+1}$ be the subalgebra of translation invariant elements: $\mathscr{D}^{T}_0 = \mathbb{F}$ and

$$\mathscr{D}^{T}_{k+1} = \operatorname{Ker} \operatorname{ad}\Big(\sum_{i=0}^{k} \partial_{z_i}\Big) = (\mathbb{F}[z_{ij}]_{0\leq i<j\leq k})[\partial_{z_0}, \dots, \partial_{z_k}], \quad k \geq 0.$$

Lemma 6.4. *The function $f(z_0, \dots, z_k) = \prod_{0\leq i<j\leq k} z_{ij}^{-1}$ is a cyclic element of the $\mathscr{D}_{k+1}$-module $\mathscr{O}^{\star}_{k+1}$. Consequently, f is a cyclic element of the $\mathscr{D}^{T}_{k+1}$-module $\mathscr{O}^{\star T}_{k+1}$.*

Proof (P. Etingof). Consider the Bernstein–Sato polynomial $b(s)$ associated to f^{-1}, which admits a differential operator $L(s)$ (regular in z_i and s) such that $L(s) f^{-s-1} = b(s) f^{-s}$. It is known that the roots of the function $b(s)$ are negative, and by [BW15, Corollary 1.3] we have that $b(s) = \pm b(-s-2)$. So $-2, -3, \dots$ are not roots of $b(s)$. Hence, $f^2 = \frac{1}{b(-2)} L(-2) f$, $f^3 = \frac{1}{b(-3)} L(-3) f^2, \dots$, all lie in the $\mathscr{D}_{k+1}$-submodule of $\mathscr{O}^{\star}_{k+1}$ generated by f. The claim follows. □

6.3. The operad P^{ch}

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a vector superspace endowed with an even endomorphism ∂. For $k \geq 0$, the space $V^{\otimes(k+1)} \otimes \mathscr{O}^{\star}_{k+1}$ carries the structure of a right $\mathscr{D}_{k+1}$-module defined by letting z_i act by multiplication on $\mathscr{O}^{\star}_{k+1}$, and letting ∂_{z_i} act by

$$\begin{aligned}&(v_0 \otimes \cdots \otimes v_k \otimes f(z_0, \dots, z_k)) \cdot \partial_{z_i} \\ &= v_0 \otimes \cdots \otimes (\partial v_i) \otimes \cdots \otimes v_k \otimes f(z_0, \dots, z_k) \\ &\quad - v_0 \otimes \cdots \otimes v_k \otimes \frac{\partial f}{\partial z_i}(z_0, \dots, z_k).\end{aligned} \tag{6.6}$$

By restriction, $V^{\otimes(k+1)} \otimes \mathscr{O}^{\star T}_{k+1}$ is a right $\mathscr{D}^T_{k+1}$-module, where $z_{ij} \in \mathscr{D}^T_{k+1}$ act by multiplication on $\mathscr{O}^{\star T}_{k+1}$. For $k = -1$, $V^{\otimes 0} \otimes \mathscr{O}^{\star}_0 \cong \mathbb{F}$ is also a module over $\mathscr{D}_0 = \mathscr{D}^T_0 = \mathbb{F}$.

Consider the space

$$V[\lambda_0, \dots, \lambda_k]/\langle \partial + \lambda_0 + \cdots + \lambda_k \rangle. \tag{6.7}$$

Here and further, $\langle \Phi \rangle$ denotes the image of an endomorphim Φ. The space (6.7) carries the structure of a right $\mathscr{D}_{k+1}$-module defined by letting z_i act by

$$A(\lambda_0, \dots, \lambda_k) \cdot z_i = -\frac{\partial}{\partial \lambda_i} A(\lambda_0, \dots, \lambda_k), \tag{6.8}$$

and ∂_{z_i} act by

$$A(\lambda_0, \dots, \lambda_k) \cdot \partial_{z_i} = -\lambda_i A(\lambda_0, \dots, \lambda_k). \tag{6.9}$$

Indeed, it is straightforward to check that both the actions (6.6) and (6.8)–(6.9) satisfy the defining relations $A \cdot \partial_{z_i} \cdot z_j = A \cdot z_j \cdot \partial_{z_i} + \delta_{ij} A$, and that the actions (6.8)–(6.9) commute with the operator $\partial + \lambda_0 + \cdots + \lambda_k$. Note that formula (6.8) can be generalized to an arbitrary polynomial $P(z_0, \dots, z_k)$ as follows:

$$A(\lambda_0, \dots, \lambda_k) \cdot P(z_0, \dots, z_k) = P\Big(-\frac{\partial}{\partial \lambda_0}, \dots, -\frac{\partial}{\partial \lambda_k} \Big) A(\lambda_0, \dots, \lambda_k). \tag{6.10}$$

A right $\mathscr{D}^T_{k+1}$-module homomorphism from $V^{\otimes(k+1)} \otimes \mathscr{O}^{\star T}_{k+1}$ to $V[\lambda_0, \dots, \lambda_k]/\langle \partial + \lambda_0 + \cdots + \lambda_k \rangle$ is then a linear map

$$\begin{aligned}&X \colon V^{\otimes(k+1)} \otimes \mathscr{O}^{\star T}_{k+1} \longrightarrow V[\lambda_0, \dots, \lambda_k]/\langle \partial + \lambda_0 + \cdots + \lambda_k \rangle, \\ &v_0 \otimes \cdots \otimes v_k \otimes f(z_0, \dots, z_k) \longmapsto X_{\lambda_0, \dots, \lambda_k}(v_0, \dots, v_k; f),\end{aligned} \tag{6.11}$$

satisfying the following two *sesquilinearity* conditions:

$$\begin{aligned} &X_{\lambda_0,\dots,\lambda_k}(v_0,\dots,(\partial+\lambda_i)v_i,\dots,v_k;f) = X_{\lambda_0,\dots,\lambda_k}\Big(v_0,\dots,v_k;\frac{\partial f}{\partial z_i}\Big), \\ &X_{\lambda_0,\dots,\lambda_k}(v_0,\dots,v_k;z_{ij}f) = \Big(\frac{\partial}{\partial\lambda_j}-\frac{\partial}{\partial\lambda_i}\Big)X_{\lambda_0,\dots,\lambda_k}(v_0,\dots,v_k;f). \end{aligned} \tag{6.12}$$

Remark 6.5. Consider the usual action of ∂ on $V^{\otimes(k+1)}$ as $\sum_{i=0}^k \partial_i$, where ∂_i denotes the action of ∂ on the i-th factor. Then since $\sum_{i=0}^k \frac{\partial f}{\partial z_i} = 0$ for every $f \in \mathscr{O}_{k+1}^{\star T}$, the first sesquilinearity implies

$$X(\partial v\otimes f) = -\sum_{i=0}^k \lambda_i X(v\otimes f) = \partial(X(v\otimes f)), \quad v\in V^{\otimes(k+1)}. \tag{6.13}$$

We let $P^{\mathrm{ch}}(k+1)$ be the space of all right $\mathscr{D}_{k+1}^T$-homomorphisms (6.11), i.e., all linear maps (6.11) satisfying the sesquilinearity conditions (6.12). Sometimes, in order to specify the variables of the function $f \in \mathscr{O}_{k+1}^{\star T}$, we will denote the image of the map X as

$$X_{\lambda_0,\dots,\lambda_k}^{z_0,\dots,z_k}(v_0,\dots,v_k;f(z_0,\dots,z_k)). \tag{6.14}$$

Note that, by definition,

$$P^{\mathrm{ch}}(0) = \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}, V/\langle\partial\rangle) \cong V/\partial V \tag{6.15}$$

and

$$P^{\mathrm{ch}}(1) = \mathrm{Hom}_{\mathbb{F}[\partial]}(V, V[\lambda_0]/\langle\partial+\lambda_0\rangle) \cong \mathrm{End}_{\mathbb{F}[\partial]}(V). \tag{6.16}$$

We will denote by $1 \in P^{\mathrm{ch}}(1)$ the identity endomorphism, so that

$$1_{\lambda_0}(v_0;f) = fv_0 + \langle\partial+\lambda_0\rangle, \quad v_0\in V, \quad f\in\mathscr{O}_1^{\star T} = \mathbb{F}. \tag{6.17}$$

The symmetric group S_{k+1} has a right action on $P^{\mathrm{ch}}(k+1)$ by permuting simultaneously the inputs $v_0,\dots,v_k$ of X and the corresponding variables $z_0,\dots,z_k$ in f. Explicitly, for $X\in P^{\mathrm{ch}}(k+1)$ and $\sigma\in S_{k+1}$, we have

$$\begin{aligned} &(X^\sigma)_{\lambda_0,\dots,\lambda_k}^{z_0,\dots,z_k}(v_0,\dots,v_k;f(z_0,\dots,z_k)) \\ &= \epsilon_v(\sigma)X_{\lambda_{i_0},\dots,\lambda_{i_k}}^{z_0,\dots,z_k}(v_{i_0},\dots,v_{i_k};f(z_{i_0},\dots,z_{i_k})), \end{aligned} \tag{6.18}$$

where $i_s = \sigma^{-1}(s)$ and $\epsilon_v(\sigma)$ is given by (2.3).

To define the structure of an operad, we need to specify how the maps from P^{ch} are composed. Let $X\in P^{\mathrm{ch}}(k+1)$, $Y\in P^{\mathrm{ch}}(m+1)$, and $h\in\mathscr{O}_{k+m+1}^{\star T}$. We can write h in the form

$$h(z_0,\dots,z_{k+m}) = f(z_0,\dots,z_k)g(z_0,\dots,z_{k+m}), \tag{6.19}$$

where $f \in \mathscr{O}^{\star T}_{k+1}$, $g \in \mathscr{O}^{\star T}_{k+m+1}$, and g has no poles at $z_i = z_j$ for $0 \le i < j \le k$. Then we define

$$\begin{aligned}&(Y \circ_1 X)^{z_0,z_1,\dots,z_{k+m}}_{\lambda_0,\lambda_1,\dots,\lambda_{k+m}}(v_0, v_1, \dots, v_{k+m}; h(z_0, \dots, z_{k+m}))\\ &= Y^{z_0,z_{k+1},\dots,z_{k+m}}_{\lambda_0',\lambda_{k+1},\dots,\lambda_{k+m}}(X^{z_0,\dots,z_k}_{\lambda_0-\partial_{z_0},\dots,\lambda_k-\partial_{z_k}}(v_0, \dots, v_k; f(z_0, \dots, z_k))_{\to},\\ &\quad v_{k+1}, \dots, v_{k+m}; g(z_0, \dots, z_{k+m})|_{z_1=\dots=z_k=z_0}),\end{aligned} \tag{6.20}$$

where $\lambda_0' = \lambda_0 + \lambda_1 + \dots + \lambda_k$ and the arrow $\to$ means that we apply the derivatives ∂_{z_i} to g before setting $z_i = z_0$ $(1 \le i \le k)$.

Lemma 6.6. *The product* (6.20) *is a well defined map from* $P^{\mathrm{ch}}(k+1) \times P^{\mathrm{ch}}(m+1)$ *to* $P^{\mathrm{ch}}(k+m+1)$.

Proof. First, we will check that $Y \circ_1 X$ is independent of the choice of factorization (6.19). Let us denote the right-hand side of (6.20) by $R(f, g)$, and consider $R(f, z_{ij}g)$ where $0 \le i < j \le k$. Notice that for any polynomial P and $0 \le i \le k$, we have

$$\begin{aligned}&P(\lambda_0 - \partial_{z_0}, \dots, \lambda_k - \partial_{z_k})(z_i g)|_{z_1=\dots=z_k=z_0}\\ &= (z_0 - \partial_{\lambda_i})P(\lambda_0 - \partial_{z_0}, \dots, \lambda_k - \partial_{z_k})g|_{z_1=\dots=z_k=z_0}.\end{aligned} \tag{6.21}$$

In particular,

$$\begin{aligned}&P(\lambda_0 - \partial_{z_0}, \dots, \lambda_k - \partial_{z_k})(z_{ij} g)|_{z_1=\dots=z_k=z_0}\\ &= (\partial_{\lambda_j} - \partial_{\lambda_i})P(\lambda_0 - \partial_{z_0}, \dots, \lambda_k - \partial_{z_k})g|_{z_1=\dots=z_k=z_0}.\end{aligned} \tag{6.22}$$

Hence, the sesquilinearity of X implies that $R(z_{ij} f, g) = R(f, z_{ij} g)$. This proves that $Y \circ_1 X$ is well defined.

We will show that $Y \circ_1 X$ satisfies the second sesquilinearity in (6.12). First, if again $0 \le i < j \le k$, then $(\partial_{\lambda_j} - \partial_{\lambda_i})\lambda_0' = 0$, and (6.22) implies $R(f, z_{ij} g) = (\partial_{\lambda_j} - \partial_{\lambda_i})R(f, g)$ as desired. On the other hand, if $k+1 \le j \le k+m$, then using (6.21) and the sesquilinearity of Y, we obtain:

$$\begin{aligned}&R(f, z_{0j} g)\\ &= Y^{z_0,z_{k+1},\dots,z_{k+m}}_{\lambda_0',\lambda_{k+1},\dots,\lambda_{k+m}}((z_{0j} - \partial_{\lambda_0})X^{z_0,\dots,z_k}_{\lambda_0-\partial_{z_0},\dots,\lambda_k-\partial_{z_k}}(v_0, \dots, v_k; f)_{\to},\\ &\quad v_{k+1}, \dots, v_{k+m}; g|_{z_1=\dots=z_k=z_0})\\ &= (\partial_{\lambda_j} - \partial_{\lambda_0'})Y^{z_0,z_{k+1},\dots,z_{k+m}}_{\lambda_0',\lambda_{k+1},\dots,\lambda_{k+m}}(X^{z_0,\dots,z_k}_{\lambda_0-\partial_{z_0},\dots,\lambda_k-\partial_{z_k}}(v_0, \dots, v_k; f)_{\to},\\ &\qquad v_{k+1}, \dots, v_{k+m}; g|_{z_1=\dots=z_k=z_0})\\ &\quad - Y^{z_0,z_{k+1},\dots,z_{k+m}}_{\lambda_0',\lambda_{k+1},\dots,\lambda_{k+m}}(\partial_{\lambda_0} X^{z_0,\dots,z_k}_{\lambda_0-\partial_{z_0},\dots,\lambda_k-\partial_{z_k}}(v_0, \dots, v_k; f)_{\to},\\ &\qquad v_{k+1}, \dots, v_{k+m}; g|_{z_1=\dots=z_k=z_0})\\ &= (\partial_{\lambda_j} - \partial_{\lambda_0})R(f, g).\end{aligned}$$

All other cases for i, j can be obtained from the above, by using the identity $z_{lm} = z_{lp} + z_{pm}$. This proves that $Y \circ_1 X$ satisfies the second sesquilinearity in (6.12).

To prove the first sesquilinearity, consider $\partial h/\partial z_i$ instead of h. Then from (6.19), we get

$$\frac{\partial h}{\partial z_i}(z_0, \dots, z_{k+m}) = \frac{\partial f}{\partial z_i}(z_0, \dots, z_k) g(z_0, \dots, z_{k+m}) + f(z_0, \dots, z_k) \frac{\partial g}{\partial z_i}(z_0, \dots, z_{k+m}).$$

In the right-hand side, each of the two summands is factored as in (6.19). Thus,

$$(Y \circ_1 X)^{z_0, z_1, \dots, z_{k+m}}_{\lambda_0, \lambda_1, \dots, \lambda_{k+m}} \Big(v_0, v_1, \dots, v_{k+m}; \frac{\partial h}{\partial z_i}\Big) = R\Big(\frac{\partial f}{\partial z_i}, g\Big) + R\Big(f, \frac{\partial g}{\partial z_i}\Big).$$

We consider two cases: $0 \le i \le k$ and $k+1 \le i \le k+m$. In the first case, by the sesquilinearity of X,

$$\begin{aligned} R\Big(\frac{\partial f}{\partial z_i}, g\Big) = {} & Y^{z_0, z_{k+1}, \dots, z_{k+m}}_{\lambda_0', \lambda_{k+1}, \dots, \lambda_{k+m}} (X^{z_0, \dots, z_k}_{\lambda_0 - \partial_{z_0}, \dots, \lambda_k - \partial_{z_k}} (v_0, \dots, \partial v_i, \dots, v_k; f)_{\to}, \\ & v_{k+1}, \dots, v_{k+m}; g|_{z_1 = \dots = z_k = z_0}) \\ & + Y^{z_0, z_{k+1}, \dots, z_{k+m}}_{\lambda_0', \lambda_{k+1}, \dots, \lambda_{k+m}} (X^{z_0, \dots, z_k}_{\lambda_0 - \partial_{z_0}, \dots, \lambda_k - \partial_{z_k}} (v_0, \dots, v_k; f)_{\to}, \\ & v_{k+1}, \dots, v_{k+m}; ((\lambda_i - \partial_{z_i}) g)|_{z_1 = \dots = z_k = z_0}). \end{aligned}$$

Combined with $R(f, \partial g/\partial z_i)$, this gives exactly the first sesquilinearity (6.12) for $Y \circ_1 X$. In the case $k+1 \le i \le k+m$, we have $\partial f/\partial z_i = 0$ and $R(f, \partial g/\partial z_i)$ gives the sesquilinearity of $Y \circ_1 X$ after applying the sesquilinearity of Y. □

We will extend the definition of the $\circ_1$ product as follows. Fix $n \ge 1$ and $m_1, \dots, m_n \ge 0$, and again use the notation (5.7). Consider $Y \in P^{\mathrm{ch}}(n)$, $X_i \in P^{\mathrm{ch}}(m_i)$, and $v_k \in V$ for $1 \le i \le n$, $1 \le k \le M = M_n$. Let

$$w_i = v_{M_{i-1}+1} \otimes \cdots \otimes v_{M_i} \in V^{\otimes m_i}, \quad i = 1, \dots, n, \tag{6.23}$$

where $M_0 = 0$. For $h \in \mathscr{O}_M^{\star T}$, we can write

$$h(z_1, \dots, z_M) = g(z_1, \dots, z_M) \prod_{i=1}^{n} f_i(z_{M_{i-1}+1}, \dots, z_{M_i}), \tag{6.24}$$

so that $f_i \in \mathscr{O}_{m_i}^{\star T}$ and $g \in \mathscr{O}_M^{\star T}$ has no poles at $z_k = z_l$ for $M_{i-1} + 1 \le k < l \le M_i$ $(1 \le i \le n)$. Then the *composition* $Y(X_1 \otimes \cdots \otimes X_n) \in P^{\mathrm{ch}}(M)$ is

defined as follows:

$$\begin{aligned}&(Y(X_1\otimes\cdots\otimes X_n))^{z_1,\dots,z_M}_{\lambda_1,\dots,\lambda_M}(v_1\otimes\cdots\otimes v_M;h(z_1,\dots,z_M))\\&=\pm Y^{z_{M_1},\dots,z_{M_n}}_{\Lambda_1,\dots,\Lambda_n}\Big(\bigotimes_{i=1}^{n}(X_i)^{z_{M_{i-1}+1},\dots,z_{M_i}}_{\lambda_{M_{i-1}+1}-\partial_{z_{M_{i-1}+1}},\dots,\lambda_{M_i}-\partial_{z_{M_i}}}(w_i;f_i)_{\to};\\&\qquad g(z_1,\dots,z_M)|_{z_k=z_{M_i}\ (M_{i-1}+1\le k\le M_i,\,1\le i\le n)}\Big),\end{aligned}\tag{6.25}$$

where Λ_i are given by (5.7) and the sign $\pm$ is

$$\pm=(-1)^{\sum_{i<j}p(X_j)(p(v_{M_{i-1}+1})+\cdots+p(v_{M_i}))}\tag{6.26}$$

(*cf.* (5.9)). As in (6.20), we first take the partial derivatives of g indicated by the arrows $\to$ and then we make the substitutions $z_k=z_{M_i}$.

It is clear that the $\circ_1$-product is a special case of the composition (6.25), namely $Y\circ_1 X=Y(X\otimes 1\otimes\cdots\otimes 1)$, where $1\in P^{\mathrm{ch}}(1)$ is the identity operator (6.17).

Proposition 6.7. *The collection of vector superspaces $P^{\mathrm{ch}}(n)$ ($n\ge 0$), with the action of S_n described above, the compositions* (6.25)*, and unit $1\in P^{\mathrm{ch}}(1)$, is an operad.*

Proof. First, it is straightforward to generalize Lemma 6.6 for the composition (6.25): the left-hand side of (6.25) is independent of the choice of factorization (6.24), and it satisfies the sesquilinearity (6.12). Hence, (6.25) are well-defined compositions in P^{ch}.

The properties (3.3) of the unit $1\in P^{\mathrm{ch}}(1)$ are obvious. The equivariance of the compositions (6.25) under the action of the symmetric group is also easy to see. Its proof is identical to the proof for the $\mathscr{Chom}$ operad from Sect. 5.2.

Finally, the associativity of the compositions is also similar to the case of $\mathscr{Chom}$. The only additional ingredient is that we have to take derivatives of functions and make substitutions in them. We use that, by the chain rule from calculus,

$$\partial_{z_0}(f(z_0,\dots,z_k)|_{z_1=\cdots=z_k=z_0})=\sum_{i=0}^{k}\partial_{z_i}f(z_0,\dots,z_k)|_{z_1=\cdots=z_k=z_0}.\tag{6.27}$$

Then in both sides of the associativity axiom

$$(Y(X_1\otimes\cdots\otimes X_n))(Z_1\otimes\cdots\otimes Z_{M_n})=Y((X_1\otimes\cdots\otimes X_n)(Z_1\otimes\cdots\otimes Z_{M_n}))$$

the derivatives get spread in the same way over the different variables. □

6.4. Vertex (super) algebra structures

As before, let V be an $\mathbb{F}[\partial]$-module with parity p, and ΠV be the same $\mathbb{F}[\partial]$-module with reversed parity $\bar{p} = 1-p$. Consider the $\mathbb{Z}$-graded Lie superalgebra

$$W^{\mathrm{ch}}(\Pi V) = \bigoplus_{k=-1}^{\infty} W_k^{\mathrm{ch}}(\Pi V) = W(P^{\mathrm{ch}}(\Pi V)),$$

defined in Sect. 3.2 for an arbitrary operad. Note that W_{-1} and W_0 are given by (6.15) and (6.16) with V replaced by ΠV. Hence, they are the same as for the $\mathscr{C}hom$ operad.

By definition, an odd element $X \in W_1^{\mathrm{ch}}(\Pi V)$ is an odd $\mathscr{D}_2^T$-module homomorphism:

$$X_{\lambda_0,\lambda_1} : \Pi V \otimes \Pi V \otimes \mathscr{O}_2^{\star T} \longrightarrow \Pi V[\lambda_0,\lambda_1]/\langle \partial + \lambda_0 + \lambda_1 \rangle, \tag{6.28}$$

satisfying the sesquilinearity axioms (6.12) and the symmetry condition (6.18). Since $\mathscr{O}_2^{\star T} = \mathbb{F}[z_{01}^{\pm 1}]$, the $\mathscr{D}_2^T$-module homomorphism (6.28) is uniquely determined, via the sesquilinearity axioms (6.12), by its values on $(\Pi V)^{\otimes 2} \otimes z_{01}^{-1}$. We have

$$\Pi V[\lambda_0,\lambda_1]/\langle \partial + \lambda_0 + \lambda_1 \rangle \simeq \Pi V[\lambda]$$

by equating $\lambda_0 = \lambda$, $\lambda_1 = -\lambda - \partial$. Hence, an odd $X \in W_1^{\mathrm{ch}}(\Pi V)$ corresponds bijectively to an even linear map $V \otimes V \to V[\lambda]$, which we shall denote as follows

$$u \otimes v \longmapsto \int^{\lambda} d\sigma\, [u_\sigma v] = :uv: + \int_0^{\lambda} d\sigma\, [u_\sigma v]. \tag{6.29}$$

Here and further, when passing from the "X"-notation to the "$\int^{\lambda} d\sigma\, [\cdot_\sigma \cdot]$"-notation, we identify the vector spaces ΠV and V. The correspondence between $X \in W^{\mathrm{ch}}(\Pi V)$ and the map (6.29) is as follows: the corresponding to X integral of λ-bracket is

$$\int^{\lambda} d\sigma\, [u_\sigma v] = (-1)^{p(u)} X^{z_0,z_1}_{\lambda,-\lambda-\partial}(u,v;z_{10}^{-1}). \tag{6.30}$$

Conversely, given the integral of λ-bracket (6.29), we associate to it the map X as in (6.28) by letting

$$X^{z_0,z_1}_{\lambda_0,\lambda_1}(v_0,v_1;z_{10}^{-1}) = (-1)^{1+\bar{p}(v_0)} \int^{\lambda_0} d\sigma\, [v_{0\sigma} v_1], \tag{6.31}$$

and extending it to $(\Pi V)^{\otimes 2} \otimes \mathscr{O}_2^{\star T}$ via the sesquilinearity axioms (6.12). In particular, by sesquilinearity, we have

$$X^{z_0,z_1}_{\lambda_0,\lambda_1}(v_0,v_1;1) = (-1)^{p(v_0)} [v_{0\lambda_0} v_1]. \tag{6.32}$$

We can translate the sesquilinearity and symmetry conditions for X to axioms on the corresponding integral of λ-bracket (6.29). All the sesquilinearity conditions (6.12) translate to

$$\int^\lambda d\sigma\,[\partial u_\sigma v] = -\int^\lambda d\sigma\,\sigma\,[u_\sigma v], \quad \int^\lambda d\sigma\,[u_\sigma \partial v] = \int^\lambda d\sigma\,(\partial+\sigma)[u_\sigma v]. \tag{6.33}$$

while the symmetry conditions (6.18) on X translate, in the notation (6.29), to

$$\int^\lambda d\sigma\,[u_\sigma v] = (-1)^{p(u)p(v)} \int^{-\lambda-\partial} d\sigma\,[v_\sigma u]. \tag{6.34}$$

As a result, we get the following:

Proposition 6.8. *The space $W_1^{\mathrm{ch}}(\Pi V)$ is identified via (6.30) with the space of integrals of λ-brackets*

$$\int^\lambda d\sigma\,[\cdot_\sigma \cdot] : V \otimes V \longrightarrow V[\lambda],$$

satisfying axioms (ii) and (iii) in the Definition 6.1 of a vertex algebra.

Next, let $X, Y \in W_1^{\mathrm{ch}}(\Pi V)_{\bar{1}}$. We can rewrite their box-product (3.13) in terms of the notation (6.30) with the integrals of λ-brackets corresponding to X and Y. By Lemma 6.4, the ring $\mathscr{O}_3^{\star T} = \mathbb{F}[z_{21}^{\pm 1}, z_{20}^{\pm 1}, z_{10}^{\pm 1}]$ is generated as a $\mathscr{D}_3^T$-module by the cyclic element $f = z_{21}^{-1} z_{20}^{-1} z_{10}^{-1}$. Hence, to determine $X \square Y$ (and to prove, for example, that $X \square Y = 0$) it suffices to compute it for this function.

In the following three lemmas, we will compute the three summands contributing to $X \square Y$ from (3.14). We will express them in terms of the notation (6.30), with the above choice of f.

Lemma 6.9. *For $X, Y \in W_1^{\mathrm{ch}}(\Pi V)_{\bar{1}}$, we have:*

$$\begin{aligned}
&(X \circ_1 Y)^{z_0,z_1,z_2}_{\lambda_0,\lambda_1,\lambda_2}\Big(v_0, v_1, v_2; \frac{1}{z_{21}z_{20}z_{10}}\Big) \\
&= (-1)^{p(v_1)} \int^{\lambda_0} d\sigma_0 \int_{\lambda_1}^{\lambda_0+\lambda_1-\sigma_0} d\sigma_1\,(\lambda_0+\lambda_1-\sigma_0-\sigma_1)[[v_{0\sigma_0}v_1]^Y{}_{\sigma_0+\sigma_1}v_2]^X \\
&\quad + (-1)^{p(v_1)} \int^{\lambda_0} d\sigma_0 \int^{\lambda_1} d\sigma_1\,(\lambda_0-\sigma_0)[[v_{0\sigma_0}v_1]^Y{}_{\sigma_0+\sigma_1}v_2]^X.
\end{aligned}$$

Proof. We have by (6.20):

$$\begin{aligned}
&(X \circ_1 Y)^{z_0,z_1,z_2}_{\lambda_0,\lambda_1,\lambda_2}\Big(v_0, v_1, v_2; \frac{1}{z_{21}z_{20}z_{10}}\Big) \\
&= X^{z_0,z_2}_{\lambda_0+\lambda_1,\lambda_2}\Big(Y^{z_0,z_1}_{\lambda_0-\partial_{z_0},\lambda_1-\partial_{z_1}}\Big(v_0, v_1; \frac{1}{z_{10}}\Big)_{\to}, v_2; \frac{1}{z_{21}z_{20}}\Big|_{z_1=z_0}\Big)
\end{aligned}$$

$$= (-1)^{p(v_0)} X^{z_0,z_2}_{\lambda_0+\lambda_1,\lambda_2}\Big(\int^{\lambda_0-\partial_{z_0}} d\sigma\, [v_{0\sigma}v_1]^Y, v_2; \frac{1}{z_{21}z_{20}}\Big|_{z_1=z_0}\Big).$$

Now we use Taylor's formula for a polynomial F:

$$F(\lambda_0-\partial_{z_0})\frac{1}{z_{21}z_{20}}\Big|_{z_1=z_0} = e^{-\partial_{z_0}\partial_{\lambda_0}}\frac{F(\lambda_0)}{z_{21}z_{20}}\Big|_{z_1=z_0} = \frac{1}{z_{20}(z_{20}+\partial_{\lambda_0})}F(\lambda_0)$$
$$= \frac{1-e^{-\partial_{z_0}\partial_{\lambda_0}}}{\partial_{\lambda_0}}\frac{F(\lambda_0)}{z_{20}} = \int_0^{\partial_{z_0}} d\tau\, e^{-\tau\partial_{\lambda_0}}\frac{F(\lambda_0)}{z_{20}}$$
$$= \int_0^{\partial_{z_0}} d\tau F(\lambda_0-\tau)\frac{1}{z_{20}}.$$

Applying this to the previous expression, we obtain:

$$(X\circ_1 Y)^{z_0,z_1,z_2}_{\lambda_0,\lambda_1,\lambda_2}\Big(v_0,v_1,v_2;\frac{1}{z_{21}z_{20}z_{10}}\Big)$$
$$= (-1)^{p(v_0)} X^{z_0,z_2}_{\lambda_0+\lambda_1,\lambda_2}\Big(\int_0^{\partial_{z_0}} d\tau \int^{\lambda_0-\tau} d\sigma\, [v_{0\sigma}v_1]^Y, v_2; \frac{1}{z_{20}}\Big)$$
$$= (-1)^{p(v_0)} X^{z_0,z_2}_{\lambda_0+\lambda_1,\lambda_2}\Big(\int_0^{\partial+\lambda_0+\lambda_1} d\tau \int^{\lambda_0-\tau} d\sigma\, [v_{0\sigma}v_1]^Y, v_2; \frac{1}{z_{20}}\Big),$$

where for the second equality we used the sesquilinearity (6.12). After that, we can write the result in terms of notation (6.30):

$$(-1)^{p(v_1)}\int^{\lambda_0+\lambda_1} d\sigma_1\Big[\Big(\int_0^{\partial+\lambda_0+\lambda_1} d\tau \int^{\lambda_0-\tau} d\sigma\, [v_{0\sigma}v_1]^Y\Big)_{\sigma_1} v_2\Big]^X.$$

Then using sesquilinearity (ii) from Definition 6.1, we replace ∂ by $-\sigma_1$ in the second integral. Now we change the order of integration with respect to τ and σ:

$$\int_0^{\lambda_0+\lambda_1-\sigma_1} d\tau \int^{\lambda_0-\tau} d\sigma\, F(\sigma,\sigma_1)$$
$$= \int^{\sigma_1-\lambda_1} d\sigma \int_0^{\lambda_0+\lambda_1-\sigma_1} d\tau\, F(\sigma,\sigma_1) + \int_{\sigma_1-\lambda_1}^{\lambda_0} d\sigma \int_0^{\lambda_0-\sigma} d\tau\, F(\sigma,\sigma_1)$$
$$= \int^{\sigma_1-\lambda_1} d\sigma\,(\lambda_0+\lambda_1-\sigma_1)F(\sigma,\sigma_1) + \int_{\sigma_1-\lambda_1}^{\lambda_0} d\sigma\,(\lambda_0-\sigma)F(\sigma,\sigma_1).$$

After that, we change the order of integration with respect to σ_1 and $\sigma_0 = \sigma$ and make the change of variables $\sigma_1 \mapsto \sigma_0+\sigma_1$:

$$\int^{\lambda_0+\lambda_1} d\sigma_1 \int^{\sigma_1-\lambda_1} d\sigma\,(\lambda_0+\lambda_1-\sigma_1)F(\sigma,\sigma_1)$$

$$
\begin{aligned}
&+\int^{\lambda_0+\lambda_1} d\sigma_1 \int_{\sigma_1-\lambda_1}^{\lambda_0} d\sigma\,(\lambda_0-\sigma)F(\sigma,\sigma_1)\\
&=\int^{\lambda_0} d\sigma_0 \int_{\lambda_1}^{\lambda_0+\lambda_1-\sigma_0} d\sigma_1\,(\lambda_0+\lambda_1-\sigma_0-\sigma_1)F(\sigma_0,\sigma_0+\sigma_1)\\
&\quad+\int^{\lambda_0} d\sigma_0 \int^{\lambda_1} d\sigma_1\,(\lambda_0-\sigma_0)F(\sigma_0,\sigma_0+\sigma_1).
\end{aligned}
$$

Combining these equations completes the proof of Lemma 6.9. □

Lemma 6.10. *For $X, Y \in W_1^{\mathrm{ch}}(\Pi V)_{\bar{1}}$, we have:*

$$
\begin{aligned}
&(X\circ_2 Y)^{z_0,z_1,z_2}_{\lambda_0,\lambda_1,\lambda_2}\Big(v_0,v_1,v_2;\frac{1}{z_{21}z_{20}z_{10}}\Big)\\
&=(-1)^{1+p(v_1)}\int^{\lambda_0} d\sigma_0 \int^{\lambda_1} d\sigma_1\,(\lambda_0-\sigma_0)[v_{0\sigma_0}[v_{1\sigma_1}v_2]^Y]^X\\
&\quad+(-1)^{1+p(v_1)}\\
&\quad\cdot\int^{\lambda_0} d\sigma_0 \int_{\lambda_1}^{\lambda_0+\lambda_1-\sigma_0} d\sigma_1\,(\lambda_0+\lambda_1-\sigma_0-\sigma_1)[v_{0\sigma_0}[v_{1\sigma_1}v_2]^Y]^X.
\end{aligned}
$$

Proof. Since $X\circ_2 Y = X(1\otimes Y)$, we have by (6.25):

$$
\begin{aligned}
&(X\circ_2 Y)^{z_0,z_1,z_2}_{\lambda_0,\lambda_1,\lambda_2}\Big(v_0,v_1,v_2;\frac{1}{z_{21}z_{20}z_{10}}\Big)\\
&=(-1)^{\bar{p}(v_0)}X^{z_0,z_2}_{\lambda_0,\lambda_1+\lambda_2}\Big(v_0,Y^{z_1,z_2}_{\lambda_1-\partial_{z_1},\lambda_2-\partial_{z_2}}\Big(v_1,v_2;\frac{1}{z_{21}}\Big)_{\to};\frac{1}{z_{20}z_{10}}\Big|_{z_1=z_2}\Big).
\end{aligned}
$$

The rest of the proof is similar to that of Lemma 6.9. □

Lemma 6.11. *For $X, Y \in W_1^{\mathrm{ch}}(\Pi V)_{\bar{1}}$, we have:*

$$
\begin{aligned}
&((X\circ_2 Y)^{(12)})^{z_0,z_1,z_2}_{\lambda_0,\lambda_1,\lambda_2}\Big(v_0,v_1,v_2;\frac{1}{z_{21}z_{20}z_{10}}\Big)\\
&=(-1)^{p(v_1)+p(v_0)p(v_1)}\int^{\lambda_0} d\sigma_0 \int^{\lambda_1} d\sigma_1\,(\lambda_0-\sigma_0)[v_{1\sigma_1}[v_{0\sigma_0}v_2]^Y]^X\\
&\quad+(-1)^{p(v_1)+p(v_0)p(v_1)}\\
&\quad\cdot\int^{\lambda_0} d\sigma_0 \int_{\lambda_1}^{\lambda_0+\lambda_1-\sigma_0} d\sigma_1\,(\lambda_0+\lambda_1-\sigma_0-\sigma_1)[v_{1\sigma_1}[v_{0\sigma_0}v_2]^Y]^X.
\end{aligned}
$$

Proof. Recall that $(X\circ_2 Y)^{(12)}$ is obtained from $X\circ_2 Y$ by switching the roles of z_0 and z_1, v_0 and v_1, λ_0 and λ_1, and σ_0 and σ_1. Then we perform a change of order of integration. Note that there is a double sign change: one is coming from the change of sign of the function $f = z_{21}^{-1}z_{20}^{-1}z_{10}^{-1}$ when we exchange z_0 and z_1, and the other sign change pops out when we change the order of integration in $d\sigma_0$ and $d\sigma_1$. □

As a result of the above three lemmas, the box-product $X\square Y$ can be written as follows

$$\begin{aligned}&(-1)^{p(v_1)+1}(X\square Y)^{z_0,z_1,z_2}_{\lambda_0,\lambda_1,\lambda_2}\Big(v_0,v_1,v_2;\frac{1}{z_{21}z_{20}z_{10}}\Big)\\&=\int^{\lambda_0}d\sigma_0\int^{\lambda_1}d\sigma_1\,(\lambda_0-\sigma_0)j^{X,Y}_{\sigma_0,\sigma_1}(v_0,v_1,v_2)\\&\quad+\int^{\lambda_0}d\sigma_0\int_{\lambda_1}^{\lambda_0+\lambda_1-\sigma_0}d\sigma_1\,(\lambda_0+\lambda_1-\sigma_0-\sigma_1)j^{X,Y}_{\sigma_0,\sigma_1}(v_0,v_1,v_2),\end{aligned}\tag{6.35}$$

where

$$\begin{aligned}j^{X,Y}_{\sigma_0,\sigma_1}(v_0,v_1,v_2)=[v_{0\sigma_0}[v_{1\sigma_1}v_2]^Y]^X-(-1)^{p(v_0)p(v_1)}[v_{1\sigma_1}[v_{0\sigma_0}v_2]^Y]^X\\-[[v_{0\sigma_0}v_1]^Y{}_{\sigma_0+\sigma_1}v_2]^X.\end{aligned}\tag{6.36}$$

From this we can derive the main result of the present section.

Theorem 6.12. *An odd element $X\in W_1^{\mathrm{ch}}(\Pi V)$ satisfies $X\square X=0$ if and only if the corresponding integral of λ-bracket (6.30) satisfies axiom (iv) of Definition 6.1. Consequently, such elements X are in bijective correspondence with the structures of non-unital vertex algebra on the $\mathbb{F}[\partial]$-module V.*

Proof. The symmetry condition $X=X^{(12)}$ on the element $X\in W_1^{\mathrm{ch}}(\Pi V)$ translates to the symmetry axiom (iii) in Definition 6.1. In the notation (6.2), axiom (iv) of the Definition 6.1 of a vertex algebra reads

$$J_{\lambda_0,\lambda_1}(v_0,v_1,v_2)=0,\tag{6.37}$$

while, by (6.35), the condition that $X\square X=0$ can be written as follows:

$$\begin{aligned}&\int^{\lambda_0}d\sigma_0\int^{\lambda_1}d\sigma_1\,(\lambda_0-\sigma_0)j_{\sigma_0,\sigma_1}(v_0,v_1,v_2)\\&+\int^{\lambda_0}d\sigma_0\int_{\lambda_1}^{\lambda_0+\lambda_1-\sigma_0}d\sigma_1\,(\lambda_0+\lambda_1-\sigma_0-\sigma_1)j_{\sigma_0,\sigma_1}(v_0,v_1,v_2)=0,\end{aligned}\tag{6.38}$$

where

$$\begin{aligned}&j_{\lambda_0,\lambda_1}(v_0,v_1,v_2)=\frac{\partial^2}{\partial\lambda_0\partial\lambda_1}J_{\lambda_0,\lambda_1}(v_0,v_1,v_2)\\&=[v_{0\lambda_0}[v_{1\lambda_1}v_2]]-(-1)^{p(v_0)p(v_1)}[v_{1\lambda_1}[v_{0\lambda_0}v_2]]-[[v_{0\lambda_0}v_1]_{\lambda_0+\lambda_1}v_2],\end{aligned}$$

which is the same as (6.36) for $Y=X$.

By the sesquilinearity axiom (ii) and the change of variable of integration $\tilde{\sigma}_1 = \sigma_0 + \sigma_1$, we can rewrite the left-hand side of (6.38), using the notation (6.3), as

$$\widetilde{J}_{\lambda_0,\lambda_0+\lambda_1}((\lambda_0+\partial)v_0,v_1,v_2)+\widetilde{J}_{\lambda_0,\lambda_0+\lambda_1}(v_0,(\lambda_1+\partial)v_1,v_2)\\ -J_{\lambda_0,\lambda_1}(v_0,(\lambda_1+\partial)v_1,v_2).$$

Hence, due to Lemma 6.2, we get that (6.37) implies (6.38). Conversely, if we take the derivative with respect to λ_0 of the left-hand side of (6.38), we get

$$\widetilde{J}_{\lambda_0,\lambda_0+\lambda_1}(v_0,v_1,v_2).$$

Hence, (6.38) implies (6.37), again due to Lemma 6.2. □

7. Vertex algebra modules and cohomology complexes

7.1. Cohomology of vertex algebras

As a consequence of Theorems 3.4 and 6.12, we obtain a cohomology complex associated to a vertex algebra V.

Definition 7.1. *Let V be a (non-unital) vertex algebra. The corresponding* vertex algebra cohomology complex *is defined as*

$$(W^{\mathrm{ch}}(\Pi V),\operatorname{ad} X),$$

where $X \in W^{\mathrm{ch}}(\Pi V)_{\bar{1}}$ is associated to the vertex algebra structure of V via (6.31).

As in Sect. 4, the cohomology

$$H^{\mathrm{ch}}(V)=\operatorname{Ker}(\operatorname{ad} X)/\operatorname{Im}(\operatorname{ad} X)$$

is a $\mathbb{Z}$-graded Lie superalgebra. However, in order to stick to the tradition, we shift the index by 1, namely for $k \geq 0$ we let

$$H^k(V)=\operatorname{Ker}(\operatorname{ad} X|_{W^{\mathrm{ch}}_{k+1}(\Pi V)})/[X,W^{\mathrm{ch}}_k(\Pi V)].$$

We will generalize the above cohomological construction for an arbitrary module M over a vertex algebra V. To this end, we first need to generalize the construction of the Lie superalgebra $W^{\mathrm{ch}}(\Pi V)$.

7.2. The space $W^{\mathrm{ch}}(\Pi V, \Pi M)$

Let V and M be vector superspaces with parity p, endowed with a structure of $\mathbb{F}[\partial]$-modules. As usual we denote by ΠV and ΠM the same spaces with reversed parity $\bar{p} = 1 - p$. We define the $\mathbb{Z}$-graded vector superspace (with parity still denoted by $\bar{p}$)

$$W^{\mathrm{ch}}(\Pi V, \Pi M) = \bigoplus_{k \geq -1} W_k^{\mathrm{ch}}(\Pi V, \Pi M), \tag{7.1}$$

where $W_k^{\mathrm{ch}}(\Pi V, \Pi M)$ is the space of linear maps

$$(\Pi V)^{\otimes(k+1)} \otimes \mathscr{O}_{k+1}^{\star T} \longrightarrow \Pi M[\lambda_0, \dots, \lambda_k]/\langle \partial + \lambda_0 + \cdots + \lambda_k \rangle$$

satisfying the sesquilinearity conditions (6.12), invariant with respect to the action of the symmetric group (6.18), i.e.,

$$W_k^{\mathrm{ch}}(\Pi V, \Pi M) = \mathrm{Hom}_{\mathscr{D}_{k+1}^T}((\Pi V)^{\otimes(k+1)} \otimes \mathscr{O}_{k+1}^{\star T}, \Pi M[\lambda_0, \dots, \lambda_k]/\langle \partial + \lambda_0 + \cdots + \lambda_k \rangle)^{S_{k+1}}.$$

Of course, the Lie superalgebra $W^{\mathrm{ch}}(\Pi V)$ is a special case of (7.1) for $M = V$.

The space $W^{\mathrm{ch}}(\Pi V, \Pi M)$ is obtained as a subquotient of the universal Lie superalgebra $W^{\mathrm{ch}}(\Pi V \oplus \Pi M)$, via the canonical isomorphism of superspaces

$$\mathscr{U}/\mathscr{K} \xrightarrow{\sim} W^{\mathrm{ch}}(\Pi V, \Pi M), \tag{7.2}$$

where $\mathscr{U} = \bigoplus_{k \geq -1} \mathscr{U}_k$ and $\mathscr{K} = \bigoplus_{k \geq -1} \mathscr{K}_k$, and $\mathscr{U}_k$, $\mathscr{K}_k$ are the following subspaces of $W_k^{\mathrm{ch}}(\Pi V \oplus \Pi M)$:

$$\begin{aligned} \mathscr{U}_k = \mathrm{Hom}_{\mathscr{D}_{k+1}^T}&((\Pi V \oplus \Pi M)^{\otimes(k+1)} \\ &\otimes \mathscr{O}_{k+1}^{\star T}, \Pi M[\lambda_0, \dots, \lambda_k]/\langle \partial + \lambda_0 + \cdots + \lambda_k \rangle)^{S_{k+1}}, \\ \mathscr{K}_k = \{Y \in \mathscr{U}_k \mid &Y((\Pi V)^{\otimes(k+1)} \otimes \mathscr{O}_{k+1}^{\star T}) = 0\}, \end{aligned}$$

and (7.2) is the restriction map. For example, we have the canonical isomorphisms

$$W_{-1}^{\mathrm{ch}}(\Pi V, \Pi M) \simeq \Pi M/\partial \Pi M, \quad W_0^{\mathrm{ch}}(\Pi V, \Pi M) \simeq \mathrm{Hom}_{\mathbb{F}[\partial]}(\Pi V, \Pi M). \tag{7.3}$$

The proof of the following proposition is obvious.

Proposition 7.2. *Let $X \in W_h^{\mathrm{ch}}(\Pi V \oplus \Pi M)$. Then the adjoint action of X on $W^{\mathrm{ch}}(\Pi V \oplus \Pi M)$ leaves the subspaces $\mathscr{U}$ and $\mathscr{K}$ invariant provided that the following two conditions hold:*

(i) $X^{z_0,\dots,z_h}_{\lambda_0,\dots,\lambda_h}(v_0,\dots,v_h;f) \in \Pi M[\lambda_0,\dots,\lambda_h]/\langle\partial+\lambda_0+\dots+\lambda_h\rangle$ *if one of the arguments* v_i *lies in* ΠM,
(ii) $X^{z_0,\dots,z_h}_{\lambda_0,\dots,\lambda_h}(v_0,\dots,v_h;f) \in \Pi V[\lambda_0,\dots,\lambda_h]/\langle\partial+\lambda_0+\dots+\lambda_h\rangle$ *if all the arguments* v_i *lie in* ΠV.

In this case, ad X *induces a well-defined linear map on the space* $W^{\mathrm{ch}}(\Pi V,\Pi M)$, *via the isomorphism* (7.2).

7.3. Cohomology of a vertex algebra with coefficients in a module

As before, let V and M be vector superspaces with parity p, endowed with $\mathbb{F}[\partial]$-module structures. Consider the reduced superspace $W^{\mathrm{ch}}(\Pi V,\Pi M)$ introduced in Sect. 7.2, with parity denoted by $\bar{p}$.

According to Definition 6.3, to say that V is a non-unital vertex algebra and M is a V-module is equivalent to say that we have a vertex algebra structure $\int^{\lambda}[\cdot\,_\lambda\,\cdot\widetilde{]}$ on the $\mathbb{F}[\partial]$-module $V\oplus M$ extending that of V, such that the integral of the λ-bracket restricted to $V\otimes M$ is given by the vertex algebra action of V on M, and restricted to $M\otimes M$ vanishes. Hence, such a structure corresponds, bijectively, to an element X of the following set:

$$\begin{aligned}\big\{X\in W_1^{\mathrm{ch}}(\Pi V\oplus\Pi M)_{\bar{1}}\;\big|\;[X,X]=0,\;\; & X_{\lambda_0,\lambda_1}(M\otimes M\otimes\mathscr{O}_2^{\star T})=0,\\ & X_{\lambda_0,\lambda_1}(V\otimes V\otimes\mathscr{O}_2^{\star T})\subset V[\lambda_0,\lambda_1]/\langle\partial+\lambda_0+\lambda_1\rangle,\\ & X_{\lambda_0,\lambda_1}\,(V\otimes M\otimes\mathscr{O}_2^{\star T})\subset M[\lambda_0,\lambda_1]/\langle\partial+\lambda_0+\lambda_1\rangle\big\}.\end{aligned} \tag{7.4}$$

Explicitly, to X in (7.4) we associate the corresponding integral λ-bracket on V given by (6.30) and the corresponding integral of λ-action of V on M given by

$$\int^{\lambda} d\sigma\, v_\sigma m = (-1)^{p(v)} X^{z_0,z_1}_{\lambda,-\lambda-\partial}\Big(v,m;\frac{1}{z_{10}}\Big),\quad v\in V,\ m\in M. \tag{7.5}$$

Note that every element X in the set (7.4) satisfies conditions (i) and (ii) in Proposition 7.2. Hence, ad X induces a well-defined endomorphism d_X of $W^{\mathrm{ch}}(\Pi V,\Pi M)$ such that $d_X^2=0$, thus making $(W^{\mathrm{ch}}(\Pi V,\Pi M),d_X)$ a cohomology complex.

Definition 7.3. *Let* V *be a (non-unital) vertex algebra and* M *be a* V*-module. The corresponding* cohomology complex *of* V *with coefficients in* M *is defined as*

$$(W^{\mathrm{ch}}(\Pi V,\Pi M),d_X),$$

where X *is the element of the set* (7.4) *associated to the integral* λ*-bracket of* V *by* (6.31) *and to the* V*-module structure of* M *by* (7.5). *We denote by* $H^{\mathrm{ch}}(V,M)=\bigoplus_{k\in\mathbb{Z}_+}H^k(V,M)$ *the corresponding vertex algebra cohomology. In the special case of the adjoint representation* $M=V$ *we recover the vertex algebra cohomology* $H^{\mathrm{ch}}(V)$ *from Definition 7.1.*

The explicit formula for the differential d_X can be obtained from (6.20). In order to write such a formula, we need to split $h \in \mathscr{O}_{k+1}^{\star T}$ as in (6.19). For every $i = 0, \dots, k$, we let

$$h(z_0, \dots, z_k) = f_i(z_0, \dots, \overset{i}{\check{\cdot}}, \dots, z_k) g_i(z_0, \dots, z_k),$$

where g_i has no poles at $z_j = z_\ell$ for $j, \ell \neq i$, and for every $0 \leq i < j \leq k$, we let

$$h(z_0, \dots, z_k) = f_{ij}(z_i, z_j) g_{ij}(z_0, \dots, z_k),$$

where g_{ij} has no poles at $z_i = z_j$. As a result, for $Y \in W_{k-1}^{\mathrm{ch}}(\Pi V, \Pi M)$, we have

$$\begin{aligned}
&(d_X Y)_{\lambda_0, \dots, \lambda_k}^{z_0, \dots, z_k}(v_0, \dots, v_k; h(z_0, \dots, z_k)) \\
&= \sum_{i=0}^{k} (-1)^{(1+p(v_i))(s_{i+1,k}+k-i)} \\
&\quad \times X^{w, z_i}_{\lambda_0 + \overset{i}{\check{\cdots}} + \lambda_k, \lambda_i} \big(Y^{z_0, \overset{i}{\check{\cdots}}, z_k}_{\lambda_0 - \partial_{z_0}, \overset{i}{\check{\cdots}}, \lambda_k - \partial_{z_k}} (v_0, \overset{i}{\check{\cdots}}, v_k; f_i)_{\to}, v_i; g_i|_{z_0 = \overset{i}{\check{\cdots}} = z_k = w} \big) \\
&\quad - (-1)^{\bar{p}(Y)} \sum_{0 \leq i < j \leq k} (-1)^{(1+p(v_i))(s_{0,i-1}+i) + (1+p(v_j))(s_{0,i-1}+s_{i+1,j-1}+j-1)} \\
&\quad \times Y^{w, z_0, \overset{i}{\check{\cdots}}, \overset{j}{\check{\cdots}}, z_k}_{\lambda_i + \lambda_j, \lambda_0, \overset{i}{\check{\cdots}}, \overset{j}{\check{\cdots}}, \lambda_k} \big(X^{z_i, z_j}_{\lambda_i - \partial_{z_i}, \lambda_j - \partial_{z_j}} (v_i, v_j; f_{ij})_{\to}, v_0, \overset{i}{\check{\cdots}}, \overset{j}{\check{\cdots}}, v_k; g_{ij}|_{z_i = z_j = w} \big),
\end{aligned} \tag{7.6}$$

where $s_{i,j}$ is given by

$$s_{ij} = p(v_i) + \cdots + p(v_j) \text{ if } i \leq j \text{ and } s_{ij} = 0 \text{ if } i > j. \tag{7.7}$$

As in (7.3), we have the isomorphism $W_{-1}^{\mathrm{ch}}(\Pi V, \Pi M) \simeq M/\partial M$ obtained by identifying the map $Y : \mathbb{F} \to M/\langle \partial \rangle$ with the element

$$Y = Y(1) \in M/\partial M \ , \quad \text{of parity } \ p(Y) = 1 + \bar{p}(Y). \tag{7.8}$$

We have the isomorphism $W_0^{\mathrm{ch}}(\Pi V, \Pi M) \simeq \mathrm{Hom}_{\mathbb{F}[\partial]}(V, M)$, obtained by identifying the map $Y_\lambda^z(v; f(z)) : V \simeq V \otimes \mathscr{O}_1^{\star T} \to M[\lambda_0]/\langle \partial + \lambda_0 \rangle$ with the $\mathbb{F}[\partial]$-module homomorphism $Y : V \to M$ given by

$$Y(v) = Y_\lambda^z(v; 1), \quad \text{of parity } \ \bar{p}(Y). \tag{7.9}$$

Finally, we identify an element

$$Y_{\lambda_0, \lambda_1}^{z_0, z_1}(v_0, v_1; f(z_0, z_1)) : V \otimes V \otimes \mathscr{O}_2^{\star T} \longrightarrow M[\lambda_0, \lambda_1]/\langle \partial + \lambda_0 + \lambda_1 \rangle$$

of $W_1^{\mathrm{ch}}(\Pi V, \Pi M)$ with the integral λ-bracket $\int^\lambda d\sigma\, [\cdot_\sigma \cdot]^Y : V \otimes V \to M[\lambda]$, given by (*cf.* (6.30))

$$\int^\lambda d\sigma\, [u_\sigma v]^Y = (-1)^{p(u)} Y^{z_0,z_1}_{\lambda,-\lambda-\partial}\Big(u, v; \frac{1}{z_{10}}\Big), \quad \text{of parity } 1+\bar{p}(Y). \tag{7.10}$$

As in Proposition 6.8, the sesquilinearity and symmetry conditions for Y translate to the corresponding sesquilinearity and symmetry conditions for the integral λ-bracket, as in axioms (ii) and (iii) of Definition 6.1.

We next write an explicit formula for the differential $d_X : W_{k-1}^{\mathrm{ch}}(\Pi V, \Pi M) \to W_k^{\mathrm{ch}}(\Pi V, \Pi M)$ in the special cases $k = 0, 1$ and 2, under the above identifications. For $k = 0$ we have $Y \in M/\partial M \simeq W_{-1}^{\mathrm{ch}}(\Pi V, \Pi M)$ and, by equation (7.6), $d_X Y$ corresponds to the following $\mathbb{F}[\partial]$-module homomorphism from V to M:

$$(d_X Y)(v) = X^{w,z_0}_{0,\lambda_0}(Y, v; 1) = -(-1)^{(1+p(v))p(Y)} v_{-\partial} Y. \tag{7.11}$$

Next, for $k = 1$, let $Y \in \mathrm{Hom}_{\mathbb{F}[\partial]}(V, M) \simeq W_0^{\mathrm{ch}}(\Pi V, \Pi M)$. Then $d_X Y$, given by equation (7.6), corresponds to the following integral λ-bracket of $u, v \in V$:

$$\begin{aligned}&(-1)^{\bar{p}(Y)} \int^\lambda d\sigma\, [u_\sigma v]^{d_X Y} \\ &= \int^\lambda d\sigma\, [Y(u)_\sigma v] + (-1)^{\bar{p}(Y)p(u)} \int^\lambda d\sigma\, [u_\sigma Y(v)] - Y\Big(\int^\lambda d\sigma\, [u_\sigma v]\Big).\end{aligned} \tag{7.12}$$

Finally, for $k = 2$ we have $X \in W_1^{\mathrm{ch}}(\Pi V)_{\bar{1}}$ and $Y \in W_1^{\mathrm{ch}}(\Pi V, \Pi M)$. In this case $d_X(Y) = X \square Y - (-1)^{\bar{p}(Y)} Y \square X \in W_2^{\mathrm{ch}}(\Pi V, \Pi M)$, where $X \square Y$ is given by the same formula as in (6.35).

7.4. Casimirs, derivations and extensions

Let V be a non-unital vertex algebra and let M be a V-module.

Definition 7.4. *A* Casimir element *is an element* $\int m \in M/\partial M$ *such that* $V_{-\partial} m = 0$. *Denote by* $\mathrm{Cas}(V, M)$ *the space of Casimirs. Note that, due to skew-symmetry of the* λ*-bracket,* $\mathrm{Cas}(V) := \mathrm{Cas}(V, V) = \{\int v \in V/\partial V \mid [v_\lambda V]|_{\lambda=0} = 0\}$.

Definition 7.5. *A* derivation *from* V *to* M *is an* $\mathbb{F}[\partial]$*-module homomorphism* $D : V \to M$ *such that*

$$D\Big(\int^\lambda d\sigma\, [u_\sigma v]\Big) = \int^\lambda d\sigma\, (D(u)_\sigma v) + (-1)^{p(D)p(u)} \int^\lambda d\sigma\, (u_\sigma D(v)). \tag{7.13}$$

We say that a derivation is inner *if it has the following form:*

$$D_Y(v) = Y_\lambda v\,|_{\lambda=0} \quad \text{for some} \quad Y \in M/\partial M. \tag{7.14}$$

In the special case when $V = M$ we have the usual definition of a derivation of the vertex algebra V. Denote by $\mathrm{Der}(V, M)$ *the space of derivations from V to M, and by* $\mathrm{Inder}(V, M)$ *the subspace of inner derivations. We also denote* $\mathrm{Der}(V) = \mathrm{Der}(V, V)$ *and* $\mathrm{Inder}(V) = \mathrm{Inder}(V, V)$.

We can now describe more explicitly the low degree cohomology.

Theorem 7.6. *Let V be a (non-unital) vertex algebra and let M be a V-module. Then:*

(a) $H^0(V, M) = \mathrm{Cas}(V, M)$. *In particular,* $H^0(V) = \mathrm{Cas}(V)$.
(b) $H^1(V, M) = \mathrm{Der}(V, M)/\mathrm{Inder}(V, M)$. *In particular, $H^1(V)$ equals the factor of the Lie algebra* $\mathrm{Der}(V)$ *of all derivations of V by the ideal of all inner derivations.*
(c) $H^2(V, M)$ *is the space of isomorphism classes of $\mathbb{F}[\partial]$-split extensions of the vertex algebra V by the V-module M, viewed as a (non-unital) vertex algebra with trivial integral λ-bracket.*

Proof. Straightforward, using the explicit formulas (7.11), (7.12) and (6.35) for the differential. (*cf.* [BKV99], [DSK09] for a proof in the case of Lie conformal algebras.) □

8. The associated graded of the chiral operad

8.1. Filtration on P^{ch}

We introduce an increasing filtration on the space of translation invariant rational functions $\mathscr{O}^{\star T}_{k+1} = \mathbb{F}[z_{ij}^{\pm 1}]_{0\le i<j\le k}$, given by the number of divisors:

$$\begin{aligned}\mathrm{F}^{-1}\,\mathscr{O}^{\star T}_{k+1} = \{0\} \subset \mathrm{F}^0\,\mathscr{O}^{\star T}_{k+1} = \mathscr{O}^{T}_{k+1} = \mathbb{F}[z_{ij}] \subset \mathrm{F}^1\,\mathscr{O}^{\star T}_{k+1} = \sum_{i<j} \mathscr{O}^{T}_{k+1}[z_{ij}^{-1}] \\ \subset \cdots \subset \mathrm{F}^r\,\mathscr{O}^{\star T}_{k+1} = \sum \mathscr{O}^{T}_{k+1}[z_{i_1,j_1}^{-1}\cdots z_{i_r,j_r}^{-1}] \subset \cdots \subset \mathscr{O}^{\star T}_{k+1}.\end{aligned} \tag{8.1}$$

In other words, the elements of $\mathrm{F}^r\,\mathscr{O}^{\star T}_{k+1}$ are sums of rational functions with at most r divisors each (not counting multiplicities). For example,

$$\frac{1}{z_{01}z_{12}z_{02}} = \frac{1}{z_{01}z_{02}^2} + \frac{1}{z_{12}z_{02}^2} \in \mathrm{F}^2\,\mathscr{O}^{\star T}_3 \tag{8.2}$$

has three divisors, but it lies in $\mathrm{F}^2\,\mathscr{O}_3^{\star T}$. In fact, by using relations similar to (8.2), it is not hard to prove (*cf.* Lemma 8.4 below) that the filtration (8.1) stabilizes:

$$\mathrm{F}^k\,\mathscr{O}_{k+1}^{\star T} = \mathscr{O}_{k+1}^{\star T}.$$

This filtration is invariant under the actions of $\mathscr{D}_{k+1}^T$ and of the symmetric group S_{k+1}. It is compatible with the multiplication:

$$(\mathrm{F}^r\,\mathscr{O}_{k+1}^{\star T})\cdot(\mathrm{F}^s\,\mathscr{O}_{k+1}^{\star T}) \subset \mathrm{F}^{r+s}\,\mathscr{O}_{k+1}^{\star T}. \tag{8.3}$$

Furthermore, if $g \in \mathrm{F}^s\,\mathscr{O}_{k+1}^{\star T}$ has no pole at $z_i = z_j$, then $g|_{z_i=z_j} \in \mathrm{F}^s\,\mathscr{O}_k^{\star T}$.

Now we define a decreasing filtration of $P^{\mathrm{ch}}(k+1)$ by

$$\mathrm{F}^r\,P^{\mathrm{ch}}(k+1) = \{X \in P^{\mathrm{ch}}(k+1) \mid X(V^{\otimes(k+1)} \otimes \mathrm{F}^{r-1}\,\mathscr{O}_{k+1}^{\star T}) = 0\}. \tag{8.4}$$

We have: $\mathrm{F}^0\,P^{\mathrm{ch}}(k+1) = P^{\mathrm{ch}}(k+1)$ and $\mathrm{F}^{k+1}\,P^{\mathrm{ch}}(k+1) = \{0\}$.

Proposition 8.1. *With the above filtration, P^{ch} is a filtered operad (cf. (3.10)).*

Proof. The filtration (8.4) is invariant under the action of the symmetric group because the filtration (8.1) is. In any operad, the compositions can be obtained from the $\circ_1$-product and the action of the symmetric group (see (3.6), (3.7)). Thus, it is enough to prove that

$$Y \circ_1 X \in \mathrm{F}^{r+s}\,P^{\mathrm{ch}}(k+m+1) \quad \text{for} \quad X \in \mathrm{F}^r\,P^{\mathrm{ch}}(k+1),\quad Y \in \mathrm{F}^s\,P^{\mathrm{ch}}(m+1).$$

To this end, we want to show that the left-hand side of (6.20) vanishes for all $h \in \mathrm{F}^{r+s-1}\,\mathscr{O}_{k+m+1}^{\star T}$. By linearity, we can suppose that $h = fg$ as in (6.19) and the number of divisors of h is $\leq r+s-1$. Since the divisors of f and g are disjoint, the number of divisors of h is the sum of the number of divisors of f and g. Hence, $f \in \mathrm{F}^{r-1}\,\mathscr{O}_{k+1}^{\star T}$ or $g \in \mathrm{F}^{s-1}\,\mathscr{O}_{k+m+1}^{\star T}$. Then we apply formula (6.20) to compute $Y \circ_1 X$. In the first case, we have $X(f) = 0$. In the second case, applying some derivatives and setting $z_1 = \cdots = z_k = z_0$ in g, we will obtain an element of $\mathrm{F}^{s-1}\,\mathscr{O}_{m+1}^{\star T}$, which is annihilated by Y. □

As a consequence of Proposition 8.1, the associated graded spaces

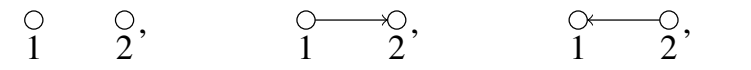
$$\mathrm{gr}^r\,P^{\mathrm{ch}}(n) = \mathrm{F}^r\,P^{\mathrm{ch}}(n)/\,\mathrm{F}^{r+1}\,P^{\mathrm{ch}}(n) \tag{8.5}$$

form a graded operad (see the end of Sect. 3.1).

8.2. n-graphs

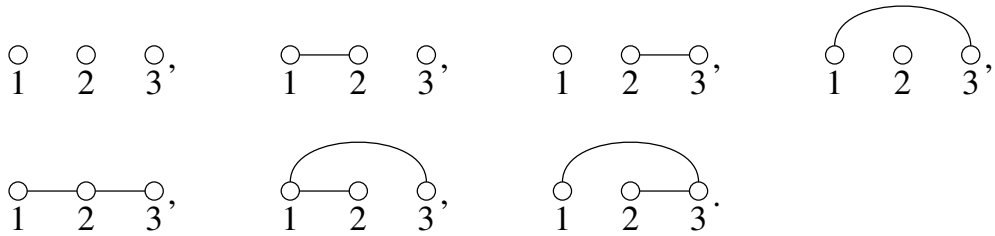
For $n \geq 1$, we define an *n-graph* as a graph Γ with the set of vertices $\{1, \dots, n\}$ and an arbitrary collection of oriented edges, denoted $E(\Gamma)$. We denote by $\mathscr{G}(n)$ the collection of all n-graphs without tadpoles, and by $\mathscr{G}_0(n)$ the collection of all acyclic n-graphs, i.e., n-graphs that have no cycles (including tadpoles and multiple edges).

For example, the set $\mathscr{G}_0(1)$ consists of the graph with a single vertex labeled 1 and no edges, the set $\mathscr{G}_0(2)$ consists of three graphs:



$$E(\Gamma) = \emptyset, \qquad E(\Gamma) = \{1 \longrightarrow 2\}, \quad E(\Gamma) = \{2 \longrightarrow 1\} \tag{8.6}$$

and $\mathscr{G}_0(3)$ consists of the following graphs, with arbitrary orientation of all edges:



(8.7)

By convention, we also let $\mathscr{G}_0(0) = \mathscr{G}(0)$ be the set consisting of a single element (the empty graph, with 0 vertices).

An *oriented cycle* C of an n-graph $\Gamma \in \mathscr{G}(n)$ is, by definition, a collection of edges of Γ forming a closed sequence (possibly with self intersections):

$$C = \{i_1 \longrightarrow i_2,\, i_2 \longrightarrow i_3, \dots,\, i_{s-1} \longrightarrow i_s,\, i_s \longrightarrow i_1\} \subset E(\Gamma). \tag{8.8}$$

8.3. The maps X^Γ

For an oriented graph $\Gamma \in \mathscr{G}(n)$, we define

$$p_\Gamma = p_\Gamma(z_1, \dots, z_n) = \prod_{(i \to j) \in E(\Gamma)} z_{ij}^{-1} \in \mathrm{F}^r\, \mathscr{O}_n^{\star T}, \quad z_{ij} = z_i - z_j, \tag{8.9}$$

where the product is over all edges of Γ and r is the number of edges. Note that if we change the orientation of a single edge of Γ, then p_Γ will change sign. For any graph G with a set of vertices labeled by an index set I, we introduce the notation

$$\lambda_G = \sum_{i \in I} \lambda_i, \quad \partial_{z_G} = \sum_{i \in I} \partial_{z_i}, \quad \partial_G = \sum_{i \in I} \partial_i, \tag{8.10}$$

where $\partial_i = 1 \otimes \cdots \otimes \partial \otimes \cdots \otimes 1$ denotes the action of ∂ on the i-th factor in a tensor product $V^{\otimes n}$. Note that, by translation covariance, we have $\partial_{z_G} p_G = 0$.

Lemma 8.2. *For an acyclic graph $\Gamma \in \mathscr{G}_0(n)$, we have*

$$\mathbb{F}[\partial_{z_1}, \dots, \partial_{z_n}] p_\Gamma = \mathbb{F}[z_{ij}^{-1} \,|\, (i \to j) \in E(\Gamma)] p_\Gamma .$$

Proof. Clearly, applying derivatives to the function p_Γ, we get an element of the space $\mathbb{F}[z_{ij}^{-1} \,|\, (i \to j) \in E(\Gamma)] p_\Gamma$. Hence, we only need to show the opposite inclusion, i.e., that for arbitrary exponents $m_{ij} \geq 1$, we have

$$\prod_{(i \to j) \in E(\Gamma)} z_{ij}^{-m_{ij}} \in \mathbb{F}[\partial_{z_1}, \dots, \partial_{z_n}] p_\Gamma . \tag{8.11}$$

Assuming, by induction, that (8.11) holds, we show how to apply derivatives in order to increase arbitrarily the exponents of the function $\prod_{(i \to j) \in E(\Gamma)} z_{ij}^{-m_{ij}}$. Fix an edge $e = (\alpha \to \beta)$ of the graph Γ, and let $\Gamma \setminus e$ be the graph obtained by deleting the edge e from Γ. Since by assumption Γ is acyclic, the connected components Γ_α and Γ_β of α and β in $\Gamma \setminus e$ are disjoint. Then it is easy to check that

$$\begin{aligned} -\frac{1}{m_{\alpha\beta}} \partial_{z_{\Gamma_\alpha}} \prod_{(i \to j) \in E(\Gamma)} z_{ij}^{-m_{ij}} &= \frac{1}{m_{\alpha\beta}} \partial_{z_{\Gamma_\beta}} \prod_{(i \to j) \in E(\Gamma)} z_{ij}^{-m_{ij}} \\ &= z_{\alpha\beta}^{-1} \prod_{(i \to j) \in E(\Gamma)} z_{ij}^{-m_{ij}} . \end{aligned}$$

The claim follows. □

Lemma 8.3. *The space $\mathrm{F}^r \mathscr{O}_n^{\star T}$ is generated as a $\mathscr{D}_n^T$-module by the functions p_Γ, with $\Gamma \in \mathscr{G}_0(n)$ acyclic graphs with at most r edges.*

Proof. Clearly, every function in $\mathrm{F}^r \mathscr{O}_n^{\star T}$ can be written as a linear combination of functions of the form

$$f \, z_{i_1 j_1}^{-m_{i_1 j_1}} \cdots z_{i_r j_r}^{-m_{i_r j_r}} , \tag{8.12}$$

with $m_{i_\ell j_\ell} \geq 0$ and f polynomial. We need to show that the function (8.12) can be obtained starting from some p_Γ and acting with $\mathscr{D}_n^T$. Let $\Gamma \in \mathscr{G}(n)$ be the graph with edges $(i_1 \to j_1), \dots, (i_r \to j_r)$. By a computation similar to (8.2) (*cf.* (8.15) below), if the graph is not acyclic, then the function (8.12) lies in $\mathrm{F}^{r-1} \mathscr{O}_n^{\star T}$ and the claim holds by induction. For an acyclic graph Γ, as an immediate consequence of Lemma 8.2, we have that the function (8.12) is generated by p_Γ. □

By restriction, for every $X \in P^{\mathrm{ch}}(n)$, we have maps

$$\begin{aligned} X^{\Gamma}_{\lambda_1,\dots,\lambda_n} &: V^{\otimes n} \longrightarrow V[\lambda_1,\dots,\lambda_n]/\langle \partial + \lambda_1 + \cdots + \lambda_n \rangle, \\ v_1 \otimes \cdots \otimes v_n &\longmapsto X^{z_1,\dots,z_n}_{\lambda_1,\dots,\lambda_n}(v_1,\dots,v_n; p_\Gamma). \end{aligned} \tag{8.13}$$

By (8.4), if $X \in \mathrm{F}^r P^{\mathrm{ch}}(n)$, we have $X^\Gamma = 0$ for graphs Γ with fewer than r edges. Furthermore, relations among the p_Γ's lead to the following relations for the maps X^Γ.

Lemma 8.4. *Let $\Gamma \in \mathscr{G}(n)$ be a graph with r edges containing an oriented cycle $C \subset E(\Gamma)$. Then we have the following* cycle relations:

(a) $X^\Gamma = 0$ for all $X \in \mathrm{F}^r P^{\mathrm{ch}}(n)$;

(b) $\sum_{e\in C} Y^{\Gamma\setminus e} = 0$ for all $Y \in \mathrm{F}^{r-1} P^{\mathrm{ch}}(n)$, where $\Gamma\setminus e$ is the graph obtained from Γ by removing the edge e.

Proof. After relabeling the vertices, we can assume that

$$C = \{1 \longrightarrow 2,\ 2 \longrightarrow 3,\dots,\ s-1 \longrightarrow s,\ s \longrightarrow 1\}.$$

Then p_Γ has a factor

$$p_C = \frac{1}{z_{12}z_{23}\cdots z_{s-1,s}z_{s1}}.$$

Since C has s edges, we expect $p_C \in \mathrm{F}^s \mathscr{O}_n^{\star T}$; however, we claim that in fact $p_C \in \mathrm{F}^{s-1} \mathscr{O}_n^{\star T}$. Indeed, using the relation

$$\sum_{e\in C} z_e = z_{12} + z_{23} + \cdots + z_{s-1,s} + z_{s1} = 0, \tag{8.14}$$

we have as in (8.2),

$$\begin{aligned} &-p_C \\ &= \frac{-z_{12}}{z_{12}^2 z_{23}\cdots z_{s-1,s}z_{s1}} \\ &= \frac{z_{23}}{z_{12}^2 z_{23}\cdots z_{s-1,s}z_{s1}} + \cdots + \frac{z_{s-1,s}}{z_{12}^2 z_{23}\cdots z_{s-1,s}z_{s1}} + \frac{z_{s1}}{z_{12}^2 z_{23}\cdots z_{s-1,s}z_{s1}}. \end{aligned} \tag{8.15}$$

In particular, $p_\Gamma \in \mathrm{F}^{r-1} \mathscr{O}_n^{\star T}$, which implies claim (a). Claim (b) follows from the equation

$$0 = \sum_{e\in C} z_e p_\Gamma = \sum_{e\in C} p_{\Gamma\setminus e}.$$

□

Example 8.5. Let $Y \in \mathrm{F}^{r-1} P^{\mathrm{ch}}(n)$ and $\Gamma' \in \mathscr{G}_0(n)$ be a graph with $(r-1)$ edges. For a fixed edge $e = (i \to j)$ of Γ', we denote by Γ the graph obtained by adding the opposite edge $e' = (j \to i)$. Then Γ has a 2-cycle $C = \{e, e'\}$, and $\Gamma'' = \Gamma \setminus e$ is obtained from $\Gamma' = \Gamma \setminus e'$ by reversing the orientation of e. In this case, (10.5) implies $Y^{\Gamma'} = -Y^{\Gamma''}$.

We will derive additional relations from the sesquilinearity conditions (6.12) for $X \in \mathrm{F}^r P^{\mathrm{ch}}(n)$.

Lemma 8.6. *Let $X \in \mathrm{F}^r P^{\mathrm{ch}}(n)$ and $\Gamma \in \mathscr{G}(n)$ be a graph with r edges. Denote the connected components of Γ by Γ_α. Then we have the following* sesquilinearity relations:

(a) $(\partial_{\lambda_i} - \partial_{\lambda_j}) X^\Gamma_{\lambda_1,\dots,\lambda_n} = 0$ *for any* $(i \to j) \in E(\Gamma)$, *which means that* X^Γ *is a polynomial of the sums* λ_{Γ_α};

(b) $X^\Gamma_{\lambda_1,\dots,\lambda_n}(\partial_{\Gamma_\alpha}(v_1 \otimes \cdots \otimes v_n)) = -\lambda_{\Gamma_\alpha} X^\Gamma_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n)$.

Proof. If $(i \to j)$ is an edge of a graph Γ with r edges, then $z_{ij} p_\Gamma \in \mathrm{F}^{r-1} \mathscr{O}_n^{\star T}$. Hence,

$$X^{z_1,\dots,z_n}_{\lambda_1,\dots,\lambda_n}(v_1,\dots,v_n; z_{ij} p_\Gamma) = 0.$$

Claim (a) then follows from the sesquilinearity condition (6.12). Next, let us prove claim (b). Since Γ is a disjoint union of the Γ_α's, the function p_Γ is the product of the corresponding p_{Γ_α}'s. By the translation covariance of p_{Γ_α}, we have $\partial_{z_{\Gamma_\alpha}} p_{\Gamma_\alpha} = 0$, and hence $\partial_{z_{\Gamma_\alpha}} p_\Gamma = 0$. Claim (b) then follows again from (6.12). □

8.4. Compositions of the maps X^Γ

Now we will investigate how the maps (8.13) compose. For $X \in P^{\mathrm{ch}}(k+1)$ and $Y \in P^{\mathrm{ch}}(m+1)$, their $\circ_1$-product $Y \circ_1 X \in P^{\mathrm{ch}}(k+m+1)$ is given by (6.20). We want to find $(Y \circ_1 X)^\Gamma$, where $\Gamma \in \mathscr{G}(k+m+1)$ is a graph whose vertices are labeled by $0, 1, \dots, k+m$. In order to apply (6.20) for $h = p_\Gamma$, according to (6.19), we factor

$$p_\Gamma = f(z_0,\dots,z_k) g(z_0,\dots,z_{k+m}), \quad f = p_{\Gamma'}, \quad g = p_{\Gamma''}. \tag{8.16}$$

Here Γ' is the subgraph of Γ with vertices $0, 1, \dots, k$ and all edges from Γ among these vertices; Γ'' is the subgraph of Γ that includes all edges of Γ not in Γ'. The factorization (8.16) holds because $E(\Gamma)$ is the disjoint union of $E(\Gamma')$ and $E(\Gamma'')$.

Setting $z_1 = \cdots = z_k = z_0$ in $p_{\Gamma''}$ corresponds to contracting the vertices $0, 1, \dots, k$ to a single vertex labeled 0. We let $\bar{\Gamma}''$ be the graph with vertices

labeled $0, k+1, \dots, k+m$ and edges obtained from the edges of Γ'' by replacing any vertex $0 \le i \le k$ with 0, keeping the same orientation. Then

$$p_{\Gamma''}|_{z_1=\dots=z_k=z_0} = p_{\bar{\Gamma}''}. \tag{8.17}$$

Finally, introduce graphs G_i $(0 \le i \le k)$ as follows. Take the connected component of the vertex i in Γ'' and remove from it the vertex i and all edges connected to i. Then G_i is the resulting subgraph of Γ''. Note that, by construction, the vertices of G_i form a subset of $\{k+1, \dots, k+m\}$. For another description of the graphs Γ', $\bar{\Gamma}''$ and G_i, see Examples 9.1 and 9.5 below.

Proposition 8.7. *With the above notation, suppose that the graph $\bar{\Gamma}''$ is acyclic. Then*

$$\begin{aligned}
&(Y \circ_1 X)^{\Gamma}_{\lambda_0,\lambda_1,\dots,\lambda_{k+m}}(v_0, v_1, \dots, v_{k+m}) \\
&= Y^{\bar{\Gamma}''}_{\lambda_{\Gamma'},\lambda_{k+1},\dots,\lambda_{k+m}} \\
&\quad (X^{\Gamma'}_{\lambda_0+\lambda_{G_0}+\partial_{G_0},\dots,\lambda_k+\lambda_{G_k}+\partial_{G_k}}(v_0, \dots, v_k), v_{k+1}, \dots, v_{k+m})
\end{aligned} \tag{8.18}$$

for $X \in P^{\mathrm{ch}}(k+1)$ and $Y \in P^{\mathrm{ch}}(m+1)$. Here we also use the notation (8.10) with ∂_i representing the action of ∂ on v_i.

Assume, in addition, that $X \in \mathrm{F}^r P^{\mathrm{ch}}(k+1)$, $Y \in \mathrm{F}^s P^{\mathrm{ch}}(m+1)$, Γ' has r edges and $\bar{\Gamma}''$ has s edges. Then equation (8.18) holds without the assumption that $\bar{\Gamma}''$ is acyclic.

Proof. In order to apply (6.20), we need to compute $\partial_{z_i} p_{\Gamma''}$. After possibly changing a sign, we will assume that all edges of Γ'' are oriented as $(i \to j)$ with $i < j$. The assumption that $\bar{\Gamma}''$ has no cycles implies that G_i and G_l are disconnected for $0 \le i, l \le k$. Let E_i be the set of all edges of Γ'' starting from the vertex i. Then we can write

$$E(\Gamma'') = \bigsqcup_{i=0}^{k} (E_i \sqcup E(G_i)) \sqcup F$$

for some subset F of edges among vertices $k+1, \dots, k+m$. Thus

$$p_{\Gamma''} = p_F \prod_{i=0}^{k} p_{E_i} p_{G_i},$$

where $p_F = \prod_{e \in F} z_e^{-1}$, and similarly for p_{E_i}.

For every edge $(i \to j) \in E_i$, we have $-\partial_{z_i} z_{ij}^{-1} = \partial_{z_j} z_{ij}^{-1}$. Hence

$$-\partial_{z_i} p_{E_i} = \sum_{(i \to j) \in E_i} \partial_{z_j} p_{E_i} = \partial_{z_{G_i}} p_{E_i}.$$

Using that $\partial_{z_{G_i}} p_{G_i} = 0$ and $\partial_{z_{G_i}} p_F = 0$, this implies

$$-\partial_{z_i} p_{\Gamma''} = \partial_{z_{G_i}} p_{\Gamma''}.$$

The statement then follows from (6.20), by applying equation (8.17) and the sesqui-linearity (6.12), after observing that $\lambda_{\Gamma'} = \lambda'_0 = \lambda_0 + \lambda_1 + \cdots + \lambda_k$.

For the last assertion of the proposition, if $\bar{\Gamma}''$ has a cycle, then by Lemma 8.4 (a) the right-hand side of (8.18) vanishes. Hence, we need to check that, in this case, the left-hand side of (8.18) vanishes as well. This follows from formula (6.20) and the fact that, after differentiating $p_{\Gamma''}$ and setting $z_1 = \cdots = z_k = z_0$, the resulting function is in $\mathrm{F}^{r-1}\, \mathscr{O}^{\star T}_{m+1}$. □

We can summarize all the previous results as follows:

Corollary 8.8. *(a) For every $X \in \mathrm{F}^r\, P^{\mathrm{ch}}(k+1)$ and every graph $\Gamma \in \mathscr{G}(k+1)$ with at most r edges, the map*

$$X^\Gamma \colon V^{\otimes(k+1)} \longrightarrow V[\lambda_0, \ldots, \lambda_k]/\langle \partial + \lambda_0 + \cdots + \lambda_k \rangle$$

defined by (8.13), satisfies the cycle relations (a) and (b) from Lemma 8.4 and the sesquilinearity relations (a) and (b) from Lemma 8.6.

(b) For $X \in \mathrm{F}^r\, P^{\mathrm{ch}}(k+1)$, $Y \in \mathrm{F}^s\, P^{\mathrm{ch}}(m+1)$, and for $\Gamma \in \mathscr{G}(k+m+1)$ such that Γ' has at most r edges and $\bar{\Gamma}''$ has at most s edges, equation (8.18) holds.

(c) If $X \in \mathrm{F}^r\, P^{\mathrm{ch}}(k+1)$ is such that $X^\Gamma = 0$ for all graphs $\Gamma \in \mathscr{G}(k+1)$ with r edges, then $X \in \mathrm{F}^{r+1}\, P^{\mathrm{ch}}(k+1)$.

Hence, we have an induced injective map defined on the associated graded space $\mathrm{gr}^r\, P^{\mathrm{ch}}(k+1)$, such that

$$\bar{X} \longmapsto \tilde{X} = \{X^\Gamma \mid \Gamma \in \mathscr{G}(k+1) \ \text{ with } \ r \ \text{ edges}\}.$$

Proof. Claim (a) is given by Lemmas 8.4, 8.6. Claim (b) is given by Proposition 8.7. Claim (c) follows from Lemma 8.3 and the sesquilinearity conditions. □

Using this corollary, in Sect. 10 below, we will provide a more detailed description of the associated graded operad $\mathrm{gr}\, P^{\mathrm{ch}}$.

8.5. *Refinement of the filtration on P^{ch}*

We refine the filtration of the chiral operad P^{ch} introduced in Sect. 8.1 as follows. Let V be a vector superspace with an increasing filtration

$$\mathrm{F}^{-1} V = \{0\} \subset \mathrm{F}^0 V \subset \mathrm{F}^1 V \subset \mathrm{F}^2 V \subset \cdots \subset V. \tag{8.19}$$

This induces an increasing filtration on the tensor products

$$\mathrm{F}^s(V^{\otimes(k+1)} \otimes \mathscr{O}^{\star T}_{k+1}) = \sum_{r_0+r_1+\cdots+r_{k+1}=s} \mathrm{F}^{r_0} V \otimes \cdots \otimes \mathrm{F}^{r_k} V \otimes \mathrm{F}^{r_{k+1}} \mathscr{O}^{\star T}_{k+1},$$

if $s \geq 0$, and $\mathrm{F}^s = 0$ if $s < 0$. For example, for $k = 1$, we have

$$\mathrm{F}^s(V^{\otimes 2} \otimes \mathscr{O}_2^{\star T}) = (\mathrm{F}^s(V^{\otimes 2}) \otimes \mathscr{O}_2^T) + (\mathrm{F}^{s-1}(V^{\otimes 2}) \otimes \mathscr{O}_2^{\star T}). \tag{8.20}$$

The corresponding refined filtered space $\mathrm{F}^r\, P^{\mathrm{ch}}(k+1)$ is defined as the set of elements $X \in P^{\mathrm{ch}}(k+1)$ such that

$$X(\mathrm{F}^s(V^{\otimes(k+1)} \otimes \mathscr{O}_{k+1}^{\star T})) \subset (\mathrm{F}^{s-r}\, V)[\lambda_0, \dots, \lambda_k]/\langle \partial + \lambda_0 + \cdots + \lambda_k \rangle, \tag{8.21}$$

for every s. This is a decreasing filtration, possibly infinite in both directions.

Proposition 8.9. *With the above refined filtration, $P^{\mathrm{ch}}(V)$ is a filtered operad (cf. (3.10)). Hence, we have the corresponding Lie superalgebra filtration $\mathrm{F}^r\, W^{\mathrm{ch}}(V)$ of $W^{\mathrm{ch}}(V)$.*

Proof. The proof of the first statement is the same as for Proposition 8.1. The last assertion follows from Theorem 3.4 (c). □

Recall that a *filtered* vertex algebra is a vertex algebra V with an increasing filtration (8.19) such that

$$:(\mathrm{F}^p\, V)(\mathrm{F}^q\, V): \subset \mathrm{F}^{p+q}\, V \quad \text{and} \quad [\mathrm{F}^p\, V\, {}_\lambda\, \mathrm{F}^q\, V] \subset \mathrm{F}^{p+q-1}\, V[\lambda], \tag{8.22}$$

for all p, q.

Theorem 8.10. *Let V be a filtered vector superspace. Under the correspondence from Theorem 6.12, the structures of filtered non-unital vertex algebra on V are in bijection with the odd elements $X \in \mathrm{F}^1\, W_1^{\mathrm{ch}}(\Pi V)$ satisfying $X \square X = 0$.*

Proof. If V is a filtered vertex algebra, then, due to (8.22), the corresponding X satisfies

$$X_{\lambda_0,\lambda_1}^{z_0,z_1}(\mathrm{F}^p\, V \otimes \mathrm{F}^q\, V \otimes 1) = [\mathrm{F}^p\, V_{\lambda_0}\, \mathrm{F}^q\, V] \subset \mathrm{F}^{p+q-1}\, V[\lambda_0]$$

and

$$X_{\lambda_0,\lambda_1}^{z_0,z_1}\left(\mathrm{F}^p\, V \otimes \mathrm{F}^q\, V \otimes \frac{1}{z_{10}}\right) = :(\mathrm{F}^p\, V)(\mathrm{F}^q\, V): \subset \mathrm{F}^{p+q}\, V.$$

By (8.20), this means that $X \in \mathrm{F}^1\, W_1^{\mathrm{ch}}(\Pi V)$. □

9. The cooperad of n-graphs

9.1. Cocomposition of n-graphs

As in Sect. 8.2, let $\mathscr{G}(n)$ be the collection of all n-graphs which have no tadpoles, and $\mathscr{G}_0(n)$ be the collection of all acyclic n-graphs. Fix an n-tuple $(m_1, \dots, m_n)$ of positive integers, and let $M_1, \dots, M_n$ as in (2.9). We define the *cocomposition map*

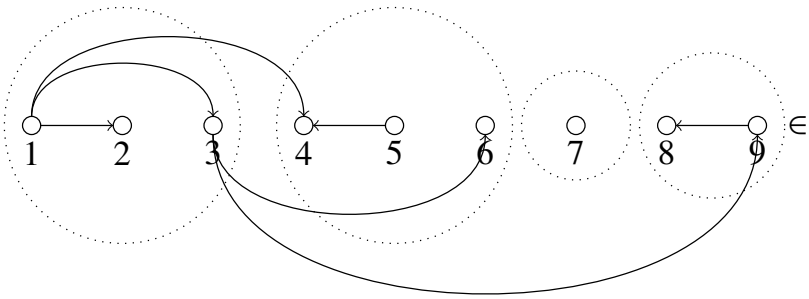
$$\Delta^{m_1 \cdots m_n} : \mathscr{G}(M_n) \longrightarrow \mathscr{G}(n) \times \mathscr{G}(m_1) \times \cdots \times \mathscr{G}(m_n), \tag{9.1}$$

denoted

$$\Gamma \longmapsto \Delta_0^{m_1 \cdots m_n}(\Gamma), \Delta_1^{m_1 \cdots m_n}(\Gamma), \dots, \Delta_n^{m_1 \cdots m_n}(\Gamma), \tag{9.2}$$

as follows. $\Delta_1^{m_1 \cdots m_n}(\Gamma)$ is the subgraph of Γ associated to the vertices $\{1, \dots, M_1\}$, $\Delta_2^{m_1 \cdots m_n}(\Gamma)$ is the subgraph of Γ associated to the vertices $\{M_1 + 1, \dots, M_2\}$ (which we relabel $\{1, \dots, m_2\}$), and so on up to $\Delta_n^{m_1 \cdots m_n}(\Gamma)$, which is the subgraph of Γ associated to the last m_n vertices $\{M_{n-1} + 1, \dots, M_n\}$ (which we relabel $\{1, \dots, m_n\}$), and finally $\Delta_0^{m_1 \cdots m_n}(\Gamma)$ is the graph obtained by collapsing the first m_1 vertices of Γ (and all edges among them) into a single vertex (which we label 1), the second m_2 vertices of Γ into a single vertex (which we label 2), and so on up to the last m_n vertices of Γ into a single vertex (which we label n).

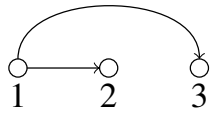
For example, consider the list of integers $(3, 3, 1, 2)$, and the 9-graph

$\Gamma =$ 

$\in \mathscr{G}_0(9).$

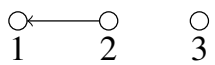
(9.3)

Then, the cocomposition $\Delta^{3312}(\Gamma) \in \mathscr{G}(4) \times \mathscr{G}_0(3) \times \mathscr{G}_0(3) \times \mathscr{G}_0(1) \times \mathscr{G}_0(2)$ consists of the following graphs: the subgraph of Γ associated to the first three vertices, is

$\Delta_1^{3312}(\Gamma) =$ 

$\in \mathscr{G}_0(3),$

the subgraph Γ associated to the second three vertices (and relabeling the vertices), is

$\Delta_2^{3312}(\Gamma) =$ 

$\in \mathscr{G}_0(3),$

the subgraph associated to the seventh vertex is just $\Delta_3^{3312}(\Gamma) = \overset{\circ}{1} \in \mathscr{G}_0(1)$, the subgraph of Γ associated to the last two vertices is

$$\Delta_4^{3312}(\Gamma) = \quad \underset{1}{\circ} \longleftarrow \underset{2}{\circ} \in \mathscr{G}_0(2),$$

and finally, collapsing all these subgraphs into single vertices, we get

$$\Delta_0^{3312}(\Gamma) = \quad \underset{1}{\circ} \rightleftarrows \underset{2}{\circ} \quad \underset{3}{\circ} \quad \underset{4}{\circ} \in \mathscr{G}(4).$$

Note that if Γ is acyclic, then all the subgraphs $\Delta_i^{m_1 \cdots m_n}(\Gamma)$, for $i = 1, \dots, n$, are acyclic as well, while, in general, this is not the case for $\Delta_0^{m_1 \cdots m_n}(\Gamma)$.

Example 9.1. A special case is when $m_1 = k+1$ and $m_2 = \cdots = m_n = 1$. With the notation of Sect. 8.4, we have in this case $\Delta_1^{(k+1)1\cdots 1}(\Gamma) = \Gamma'$, $\Delta_2^{(k+1)1\cdots 1}(\Gamma) = \cdots = \Delta_n^{(k+1)1\cdots 1}(\Gamma) = \circ$, and $\Delta_0^{(k+1)1\cdots 1}(\Gamma) = \bar{\Gamma}''$.

Lemma 9.2. *For every $m_1, \dots, m_n$, there is a natural bijective correspondence*

$$\Delta \colon E(\Gamma) \xrightarrow{\sim} E(\Delta_0^{m_1 \cdots m_n}(\Gamma)) \sqcup E(\Delta_1^{m_1 \cdots m_n}(\Gamma)) \sqcup \cdots \sqcup E(\Delta_n^{m_1 \cdots m_n}(\Gamma)). \tag{9.4}$$

Proof. An edge $e \in E(\Gamma)$ has either both tail and head contained in one of the subsets $\{M_{i-1}+1, \dots, M_i\}$, for some $i = 1, \dots, n$, in which case it corresponds to an edge of $\Delta_i^{m_1 \cdots m_n}(\Gamma)$, or it does not, in which case it corresponds to an edge of $\Delta_0^{m_1 \cdots m_n}(\Gamma)$. □

It follows from Lemma 9.2 that the cooperad of graphs $\mathscr{G}$ is graded by the number of edges.

Lemma 9.3. *Let $C \subset E(\Gamma)$ be an oriented cycle of an n-graph $\Gamma \in \mathscr{G}(n)$. Then,*

(a) either $\Delta(C) \subset E(\Delta_i^{m_1 \cdots m_n}(\Gamma))$, in which case $\Delta(C)$ is an oriented cycle of $\Delta_i^{m_1 \cdots m_n}(\Gamma) \in \mathscr{G}(m_i)$;
(b) or, $\Delta(C) \cap E(\Delta_0^{m_1 \cdots m_n}(\Gamma))$ is an oriented cycle of $\Delta_0^{m_1 \cdots m_n}(\Gamma) \in \mathscr{G}(n)$.

Proof. Obvious. □

Let, as above, $m_1, \dots, m_n$ be positive integers, and let $\Gamma \in \mathscr{G}(M_n)$. We now introduce an important notion, which will be essential in Sect. 10.

Definition 9.4. *Let $k \in \{1,\dots,M_n\}$ and $j \in \{1,\dots,n\}$. We say that j is* externally connected *to k (via the graph Γ and its cocomposition $\Delta^{m_1\cdots m_n}(\Gamma)$) if there is an unoriented path (without repeating edges) of $\Delta_0^{m_1\cdots m_n}(\Gamma)$ joining j to i, where $i \in \{1,\dots,n\}$ is such that $k \in \{M_{i-1}+1,\dots,M_i\}$, and the edge out of i is the image, via the map Δ in (9.4), of an edge which has its head or tail in k. We denote by*

$$\mathscr{E}(k) = \mathscr{E}(\Gamma, m_1, \dots, m_n; k) \subset \{1,\dots,n\},$$

the set of all $j \in \{1,\dots,n\}$ which are externally connected to k. Moreover, given a set of variables $x_1,\dots,x_n$, we denote

$$X(k) = X(\Gamma, m_1, \dots, m_n; k) = \sum_{j\in\mathscr{E}(k)} x_j\,. \tag{9.5}$$

For example, for the graph in (9.3), we have

$$\begin{aligned} &X(1) = x_1 + x_2 + x_4,\ X(2) = 0,\ X(3) = x_1 + x_2 + x_4,\\ &X(4) = x_1 + x_2 + x_4,\ X(5) = 0,\ X(6) = x_1 + x_2 + x_4,\ X(7) = 0,\\ &X(8) = 0,\ X(9) = x_1 + x_2. \end{aligned}$$

Note that, if $k \in \{M_{i-1}+1,\dots,M_i\}$, then $i \notin \mathscr{E}(k)$ unless $\Delta_0^{m_1\cdots m_n}(\Gamma)$ is not acyclic.

Example 9.5. In the setting of Example 9.1, let $m_1 = k+1$ and $m_2 = \cdots = m_n = 1$. Assuming that $\bar\Gamma''$ is acyclic, for every $\ell = 0,\dots,k$, the set $\mathscr{E}(\ell)$ coincides with the set of vertices of the graph G_ℓ defined in Sect. 8.4.

9.2. *Coassociativity of the cocomposition map of n-graphs*

The collection of sets $\mathscr{G}(n)$, $n \geq 0$, together with the cocomposition maps (9.1), defines a *cooperad* [LV12], or, equivalently, the dual $\mathscr{G}^*$ is naturally an operad.

We will not give a formal definition of what a cooperad is (since we will never use it), but we will prove here the main conditions: coassociativity, in Proposition 9.6 below, and coequivariance with respect to the action of the symmetric group, in the next Sect. 9.3, see Proposition 9.7.

Fix a list $m_1,\dots,m_n$ of n positive integers, denote $M_i = \sum_{j=1}^i m_j$, $i = 0,\dots,n$, as in (2.9), then fix a list $\ell_1,\dots,\ell_{M_n}$ of M_n positive integers, and denote $L_j = \sum_{k=1}^j \ell_k$, $j = 0,\dots,M_n$, as in (2.14). Given a graph $\Gamma \in \mathscr{G}_0(L_{M_n})$, we can apply to it the cocomposition $\Delta^{\ell_1\cdots\ell_{M_n}}$, to get

$$\begin{aligned} &\Delta_0^{\ell_1\cdots\ell_{M_n}}(\Gamma) \in \mathscr{G}(M_n),\\ &\Delta_1^{\ell_1\cdots\ell_{M_n}}(\Gamma) \in \mathscr{G}(\ell_1),\ \dots,\ \Delta_{M_n}^{\ell_1\cdots\ell_{M_n}}(\Gamma) \in \mathscr{G}(\ell_{M_n}), \end{aligned}$$

and, to the first graph above, we can further apply the cocomposition map $\Delta^{m_1\cdots m_n}$ (9.1), to get

$$\Delta_0^{m_1\cdots m_n}(\Delta_0^{\ell_1\cdots\ell_{M_n}}(\Gamma)) \in \mathscr{G}(n),$$
$$\Delta_1^{m_1\cdots m_n}(\Delta_0^{\ell_1\cdots\ell_{M_n}}(\Gamma)) \in \mathscr{G}(m_1), \dots, \Delta_n^{m_1\cdots m_n}(\Delta_0^{\ell_1\cdots\ell_{M_n}}(\Gamma)) \in \mathscr{G}(m_n).$$

Alternatively, we can consider the n integers (summing to L_{M_n})

$$K_1 := L_{M_1} = \sum_{j=1}^{M_1} \ell_j,\ K_2 := L_{M_2} - L_{M_1} = \sum_{j=M_1+1}^{M_2} \ell_j, \dots,$$
$$K_n := L_{M_n} - L_{M_{n-1}} = \sum_{j=M_{n-1}+1}^{M_n} \ell_j,$$

we can apply the corresponding cocomposition map $\Delta^{K_1\cdots K_n}$ to Γ, to get

$$\Delta_0^{K_1\cdots K_n}(\Gamma) \in \mathscr{G}(n),$$
$$\Delta_1^{K_1\cdots K_n}(\Gamma) \in \mathscr{G}(K_1), \dots, \Delta_n^{K_1\cdots K_n}(\Gamma) \in \mathscr{G}(K_n),$$

and, to each of the graph in the second line, we can apply the corresponding cocomposition map $\Delta^{\ell_{M_{i-1}+1}\cdots\ell_{M_i}}$, $i = 1, \dots, n$ to get

$$\Delta_0^{\ell_{M_{i-1}+1}\cdots\ell_{M_i}}(\Delta_i^{K_1\cdots K_n}(\Gamma)) \in \mathscr{G}(m_i),$$
$$\Delta_1^{\ell_{M_{i-1}+1}\cdots\ell_{M_i}}(\Delta_i^{K_1\cdots K_n}(\Gamma)) \in \mathscr{G}(\ell_{M_{i-1}+1}), \dots,$$
$$\Delta_{m_i}^{\ell_{M_{i-1}+1}\cdots\ell_{M_i}}(\Delta_i^{K_1\cdots K_n}(\Gamma)) \in \mathscr{G}(\ell_{M_i}).$$

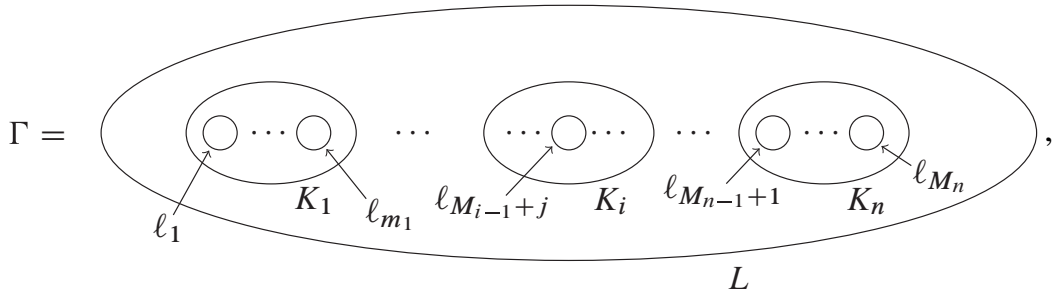
Proposition 9.6. *The cocomposition maps* (9.1) *of graphs satisfy the following coassociativity conditions:*

(i) $\Delta_0^{m_1\cdots m_n}(\Delta_0^{\ell_1\cdots\ell_{M_n}}(\Gamma)) = \Delta_0^{K_1\cdots K_n}(\Gamma)$ *in* $\mathscr{G}(n)$;

(ii) $\Delta_i^{m_1\cdots m_n}(\Delta_0^{\ell_1\cdots\ell_{M_n}}(\Gamma)) = \Delta_0^{\ell_{M_{i-1}+1}\cdots\ell_{M_i}}(\Delta_i^{K_1\cdots K_n}(\Gamma))$ *in* $\mathscr{G}(m_i)$, *for every* $i = 1, \dots, n$;

(iii) $\Delta_{M_{i-1}+j}^{\ell_1\cdots\ell_{M_n}}(\Gamma) = \Delta_j^{\ell_{M_{i-1}+1}\cdots\ell_{M_i}}(\Delta_i^{K_1\cdots K_n}(\Gamma))$ *in* $\mathscr{G}(\ell_{ij})$, *for every* $i = 1, \dots, n$ *and* $j = 1, \dots, m_i$.

Proof. All claims become obvious if they are explained "pictorially". Consider an arbitrary graph, which we can depict as follows:



where each of the intermediate ovals surround subgraphs of $K_1, \dots, K_n$ vertices respectively, and, inside the i-th oval, the inner circles surround subgraphs of $\ell_{M_{i-1}+1}, \dots, \ell_{M_i}$ vertices respectively.

In condition (i), the graph $\Delta_0^{K_1 \cdots K_n}(\Gamma)$ in the right-hand side is obtained starting from Γ and collapsing all intermediate subgraphs (= intermediate ovals) to single vertices. On the other hand, the graph $\Delta_0^{m_1 \cdots m_n}(\Delta_0^{\ell_1 \cdots \ell_{M_n}}(\Gamma))$ in the left-hand side is obtained by first collapsing all the inner subgraphs (= inner circles) to single vertices and then, in the resulting graph, by further collapsing the intermediate subgraphs (= intermediate ovals) to single vertices. The result is obviously the same.

In condition (ii), the graph $\Delta_i^{m_1 \cdots m_n}(\Delta_0^{\ell_1 \cdots \ell_{M_n}}(\Gamma))$ in the left-hand side is obtained starting from Γ by collapsing all inner subgraphs (= inner circles) to single vertices, and then, in the resulting graph, by taking the i-th intermediate subgraph. On the other hand, the graph $\Delta_0^{\ell_{M_{i-1}+1} \cdots \ell_{M_i}}(\Delta_i^{K_1 \cdots K_n}(\Gamma))$ in the right-hand side is obtained by first taking the i-th intermediate subgraph (= intermediate oval) of Γ, and then, inside it, by collapsing all inner subgraphs (=inner circles) to single vertices. The result is obviously the same.

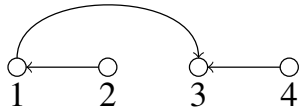
Finally, in condition (iii), the graph $\Delta_{M_{i-1}+j}^{\ell_1 \cdots \ell_{M_n}}(\Gamma)$ in the left-hand side is obtained by looking at the $(M_{i-1} + j)$-th inner subgraph (= inner circle) of Γ, which is the j-th circle inside the i-th intermediated oval, while the graph $\Delta_j^{\ell_{M_{i-1}+1} \cdots \ell_{M_i}}(\Delta_i^{K_1 \cdots K_n}(\Gamma))$ in the right-hand side is obtained by first taking the i-th intermediate subgraph (= intermediate oval) of Γ, and then, inside it, by taking the j-th inner subgraph (= inner circle). The result is obviously the same. □

9.3. Coequivariance of the cocomposition map of n-graphs

For every $n \geq 1$, there is a natural (left) action of the symmetric group S_n on the set $\mathscr{G}_0(n)$ of acyclic n-graphs, and on the set $\mathscr{G}(n)$ of all n-graphs. It is defined as follows: given the n-graph Γ and the permutation $\sigma \in S_n$, we define $\sigma(\Gamma)$ to be the same graph as Γ, but with the vertex which was labeled 1 relabeled as $\sigma(1)$, and so on up to the vertex which was labeled n, which is relabeled as

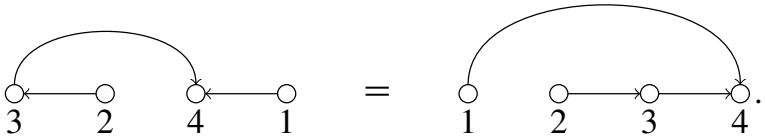
$\sigma(n)$. For example, if $\Gamma \in \mathscr{G}_0(4)$ is the following 4-graph:

$$\Gamma = \quad \in \mathscr{G}(4),$$



and $\sigma \in S_4$ is the permutation $\sigma = (1\ 3\ 4)$ (in the standard cycle decomposition), then

$$\sigma(\Gamma) = \quad = \quad .$$



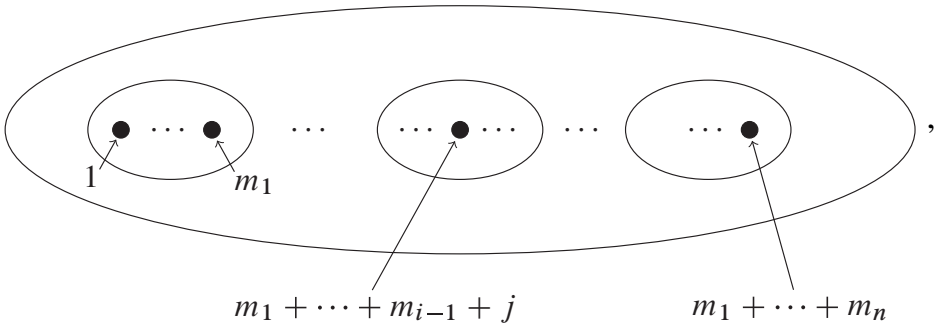
Proposition 9.7. *For every positive integers $n, m_1, \dots, m_n$, every permutations $\sigma \in S_n$, $\tau_1 \in S_{m_1}$, ..., $\tau_n \in S_{m_n}$, and every graph $\Gamma \in \mathscr{G}_0(m_1 + \cdots + m_n)$, we have*

$$\begin{aligned} &\Delta^{m_{\sigma^{-1}(1)} \cdots m_{\sigma^{-1}(n)}}((\sigma(\tau_1, \dots, \tau_n))(\Gamma)) \\ &= (\sigma(\Delta_0^{m_1 \cdots m_n}(\Gamma)), \tau_{\sigma^{-1}(1)}(\Delta_{\sigma^{-1}(1)}^{m_1 \cdots m_n}(\Gamma)), \dots, \tau_{\sigma^{-1}(n)}(\Delta_{\sigma^{-1}(n)}^{m_1 \cdots m_n}(\Gamma))), \end{aligned} \tag{9.6}$$

where the composition of permutations $\sigma(\tau_1, \dots, \tau_n)$ is defined by (2.10).

Proof. Also for this proposition we provide a "pictorial" proof. Consider an arbitrary acyclic $(m_1 + \cdots + m_n)$-graph Γ, which we depict as:

$$\Gamma = \quad ,$$

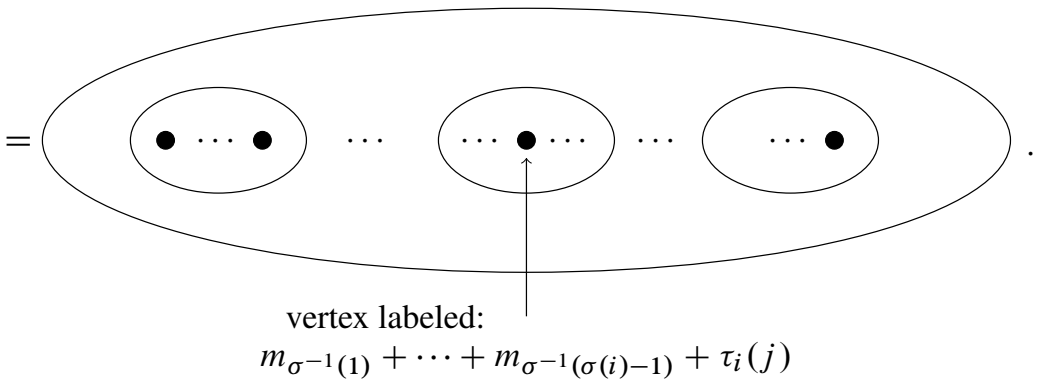


(9.7)

where we represented only the vertices (not the edges), labeled from 1 to $m_1 + \cdots + m_n$, grouped (by the inner ovals) in groups of $m_1, \dots, m_n$ vertices. Hence, as indicated, the vertex in the i-th oval ($i = 1, \dots, n$), in the j-th position within that oval ($j = 1, \dots, m_i$) is labeled $m_1 + \cdots + m_{i-1} + j$.

When we apply the permutation $\sigma(\tau_1, \dots, \tau_n) \in S_{m_1 + \cdots + m_n}$ to the graph Γ, we get, by the way the symmetric group acts on $\mathscr{G}_0(m_1 + \cdots + m_n)$, the exact same picture, but with the vertices labeled according to the action of the permutation $\sigma(\tau_1, \dots, \tau_n)$, given by formula (2.12). Hence, we have

$(\sigma(\tau_1,\dots,\tau_n))(\Gamma)$



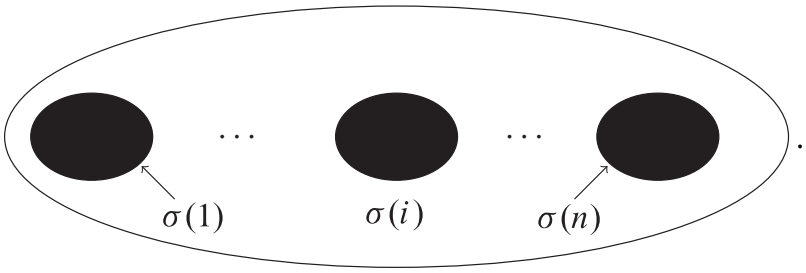
Then, to get the picture of $(\sigma(\tau_1,\dots,\tau_n))(\Gamma)$, with the vertices in the correct order, we should rearrange the vertices of picture (9.8) by moving the vertex labeled by 1 (which, in the picture (9.8), is in the $i = \sigma^{-1}(1)$-th oval, in $\tau^{-1}_{\sigma^{-1}(1)}(1)$-th position) in first position, the vertex labeled 2 in second position, and so on. Hence, in this rearrangement, the i-th oval of picture (9.7) will be moved to position $\sigma(i)$, and, within that oval, the j-th vertex will be moved to position $\tau_i(j)$.

Note that, while, in picture (9.7) the ovals contain, in the order they are depicted, $m_1,\dots,m_n$ vertices respectively, in the rearranged graph $(\sigma(\tau_1,\dots,\tau_n))(\Gamma)$, where the vertex labeled 1 come first, the vertex labeled 2 comes second, and so on, the vertices will be grouped in ovals containing $m_{\sigma^{-1}(1)},\dots,m_{\sigma^{-1}(n)}$ vertices respectively. Hence, we should apply the cocomposition map $\Delta^{m_{\sigma^{-1}(1)}\cdots m_{\sigma^{-1}(n)}}$ to it.

According to the definition, the graph

$$\Delta_0^{m_{\sigma^{-1}(1)}\cdots m_{\sigma^{-1}(n)}}((\sigma(\tau_1,\dots,\tau_n))(\Gamma))$$

is obtained by collapsing all the ovals in picture (9.8) to single vertices:



(9.8)

Obviously, this is the same graph as

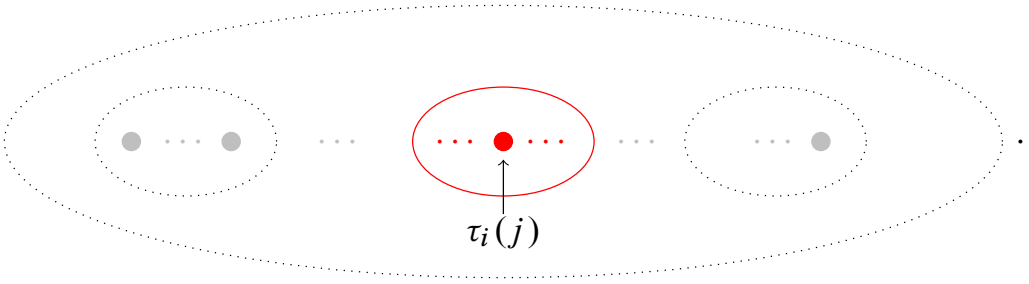
$$\sigma(\Delta^{m_1\cdots m_n}(\Gamma)),$$

where we first collapse all the inner ovals of Γ in picture (9.7) to single vertices, and then we apply the permutation $\sigma \in S_n$, i.e., we relabel the vertices according to σ.

Next, according to the definition, the graph

$$\Delta_{\sigma(i)}^{m_{\sigma^{-1}(1)}\cdots m_{\sigma^{-1}(n)}}((\sigma(\tau_1,\dots,\tau_n))(\Gamma))$$

is the subgraph corresponding to the $\sigma(i)$-th oval of the graph $(\sigma(\tau_1,\dots,\tau_n))(\Gamma)$ (rearranged), i.e., the i-th oval of picture (9.8):



Obviously, this is the same as the graph

$$\tau_i(\Delta_i^{m_1\cdots m_n}(\Gamma)),$$

where we first take the subgraph of Γ corresponding to the i-th oval of picture (9.7), and then we apply the permutation $\tau_i \in S_{m_i}$, i.e., we relabel the vertices according to τ_i. □

10. The operad governing Poisson vertex superalgebras

10.1. Definition of a Poisson vertex superalgebra

Recall that a *Poisson vertex superalgebra* (abbreviated PVA) is a commutative associative superalgebra V endowed with an even derivation ∂ and a Lie conformal superalgebra λ-bracket $\{\cdot\,{}_\lambda\,\cdot\}$ satisfying the left Leibniz rule:

$$\{a_\lambda bc\} = \{a_\lambda b\}c + (-1)^{p(b)p(c)}\{a_\lambda c\}b. \tag{10.1}$$

10.2. Definition of the operad $P^{\rm cl}$

Let $V = V_{\bar 0} \oplus V_{\bar 1}$ be a vector superspace endowed with an even endomorphism $\partial \in \operatorname{End} V$. The operad $P^{\rm cl}$ is the collection of superspaces $P^{\rm cl}(n)$ defined as follows. As a vector superspace, $P^{\rm cl}(n)$ is the space of all maps

$$f : \mathscr{G}(n) \times V^{\otimes n} \longrightarrow V[\lambda_1,\dots,\lambda_n]/\langle\partial + \lambda_1 + \cdots + \lambda_n\rangle, \tag{10.2}$$

which are linear in the second factor, mapping the n-graph $\Gamma \in \mathscr{G}(n)$ and the monomial $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ to the polynomial

$$f_{\lambda_1,\dots,\lambda_n}^{\Gamma}(v_1 \otimes \cdots \otimes v_n), \tag{10.3}$$

satisfying the *cycle relations* and the *sesquilinearity conditions* described below. The cycle relations state that

$$f^\Gamma = 0 \ \text{ unless } \ \Gamma \in \mathscr{G}_0(n), \tag{10.4}$$

and if $C \subset E(\Gamma)$ is an oriented cycle of Γ, then

$$\sum_{e \in C} f^{\Gamma \setminus e} = 0, \tag{10.5}$$

where $\Gamma \setminus e$ is the graph obtained from Γ by removing the edge e. Note that these are the same relations as in Lemma 8.4. Condition (10.5) follows from (10.4) unless Γ contains a unique oriented cycle. In the special case of oriented cycles of length 2, the cycle relation (10.5) means that changing orientation of a single edge of the n-graph $\Gamma \in \mathscr{G}(n)$ amounts to a change of sign of f^Γ.

To write the *sesquilinearity conditions*, let $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_s$ be the decomposition of Γ as disjoint union of its connected components, and let $I_1, \dots, I_s \subset \{1, \dots, n\}$ be the sets of vertices associated to these connected components. For example, for the graph Γ in (9.3), we have $\Gamma = \Gamma_1 \sqcup \Gamma_2$, with $I_1 = \{1, 2, 3, 4, 5, 6, 8, 9\}$ and $I_2 = \{7\}$. Then for every $\alpha = 1, \dots, s$, we have two sesquilinearity conditions. The first one states

$$\frac{\partial}{\partial \lambda_i} f^\Gamma_{\lambda_1, \dots, \lambda_n}(v_1 \otimes \cdots \otimes v_n) \ \text{ is the same for all } \ i \in I_\alpha. \tag{10.6}$$

In other words, the polynomial $f^\Gamma_{\lambda_1, \dots, \lambda_n}(v_1 \otimes \cdots \otimes v_n)$ is a function of the variables $\lambda_{\Gamma_\alpha} = \sum_{i \in I_\alpha} \lambda_i$, $\alpha = 1, \dots, s$ (*cf.* (8.10)), and not of the variables $\lambda_1, \dots, \lambda_n$ separately. The second sesquilinearity condition is, again in the notation (8.10),

$$f^\Gamma_{\lambda_1, \dots, \lambda_n}(\partial_{\Gamma_\alpha}(v_1 \otimes \cdots \otimes v_n)) = -\lambda_{\Gamma_\alpha} f^\Gamma_{\lambda_1, \dots, \lambda_n}(v_1 \otimes \cdots \otimes v_n). \tag{10.7}$$

These are the same relations as in Lemma 8.6.

Remark 10.1. Since Γ is a disjoint union of its connected components Γ_α, the second sesquilinearity condition (10.7) implies

$$f^\Gamma_{\lambda_1, \dots, \lambda_n}(\partial_\Gamma v) = -\sum_{i=1}^n \lambda_i \, f^\Gamma_{\lambda_1, \dots, \lambda_n}(v) = \partial(f^\Gamma_{\lambda_1, \dots, \lambda_n}(v)), \quad v \in V^{\otimes n} \tag{10.8}$$

(*cf.* Remark 6.5).

The space $P^{\mathrm{cl}}(n)$ decomposes as a direct sum

$$P^{\mathrm{cl}}(n) = \bigoplus_{r \geq 0} \mathrm{gr}^r \, P^{\mathrm{cl}}(n), \tag{10.9}$$

where $\mathrm{gr}^r \, P^{\mathrm{cl}}(n)$ is the subspace of all maps (10.2) vanishing on graphs Γ with number of edges not equal to r.

Remark 10.2. Let $V = \bigoplus_r \mathrm{gr}^r V$ be a graded vector space, and consider the induced grading of the tensor powers $V^{\otimes k}$. Then the classical operad $P^{\mathrm{cl}}(V)$ has a refined grading defined as follows: $f \in \mathrm{gr}^r P^{\mathrm{cl}}(k)(V)$ if, for every graph $\Gamma \in \mathscr{G}(k)$ with s edges, we have

$$f^{\Gamma}_{\lambda_1,\dots,\lambda_k}(\mathrm{gr}^t V^{\otimes k}) \subset (\mathrm{gr}^{s+t-r} V)[\lambda_1,\dots,\lambda_k]/\langle \partial + \lambda_1 + \cdots + \lambda_k \rangle.$$

The grading (10.9) corresponds to the special case when $V = \mathrm{gr}^0 V$.

The $\mathbb{Z}/2\mathbb{Z}$-grading of the superspace $P^{\mathrm{cl}}(n)$ is induced by that of the vector superspace V (as before, the variables λ_i are even and commute). We also have a natural right action of the symmetric group S_n on $P^{\mathrm{cl}}(n)$ by (parity preserving) linear maps, defined by the following formula ($f \in P^{\mathrm{cl}}(n)$, $\Gamma \in \mathscr{G}(n)$, $v_1,\dots,v_n \in V$):

$$(f^{\sigma})^{\Gamma}_{\lambda_1,\dots,\lambda_n}(v_1 \otimes \cdots \otimes v_n) = f^{\sigma(\Gamma)}_{\sigma(\lambda_1,\dots,\lambda_n)}(\sigma(v_1 \otimes \cdots \otimes v_n)), \tag{10.10}$$

where $\sigma(\lambda_1,\dots,\lambda_n)$ is defined by (2.8), $\sigma(v_1 \otimes \cdots \otimes v_n)$ is defined by (2.2), and $\sigma(\Gamma)$ is defined in Sect. 9.3.

Next, we define the composition maps of the operad P^{cl}. Let $f \in P^{\mathrm{cl}}(n)$ and $g_1 \in P^{\mathrm{cl}}(m_1),\dots,g_n \in P^{\mathrm{cl}}(m_n)$. Let M_i, $i = 0,\dots,n$, and Λ_i, $i = 1,\dots,n$, be as in (5.7). Let $\Gamma \in \mathscr{G}(M_n)$ and consider its cocomposition $\Delta^{m_1\cdots m_n}(\Gamma)$ defined in Sect. 9.1. We let

$$(f(g_1,\dots,g_n))^{\Gamma} : V^{\otimes M_n} \longrightarrow V[\lambda_1,\dots,\lambda_{M_n}]/\langle \partial + \lambda_1 + \cdots + \lambda_{M_n} \rangle$$

be defined by the following formula:

$$\begin{aligned}
&(f(g_1,\dots,g_n))^{\Gamma}_{\lambda_1,\dots,\lambda_{M_n}}(v_1 \otimes \cdots \otimes v_{M_n}) \\
&= f^{\Delta_0^{m_1\cdots m_n}(\Gamma)}_{\Lambda_1,\dots,\Lambda_n}(((|_{x_1=\Lambda_1+\partial}(g_1)^{\Delta_1^{m_1\cdots m_n}(\Gamma)}_{\lambda_1+X(1),\dots,\lambda_{M_1}+X(M_1)}) \otimes \cdots \\
&\quad \otimes (|_{x_n=\Lambda_n+\partial}(g_n)^{\Delta_n^{m_1\cdots m_n}(\Gamma)}_{\lambda_{M_{n-1}+1}+X(M_{n-1}+1),\dots,\lambda_{M_n}+X(M_n)}))(v_1 \otimes \cdots \otimes v_{M_n})).
\end{aligned} \tag{10.11}$$

In formula (10.11) we are using the following notation. Given the graphs $\Gamma_1 \in \mathscr{G}(m_1)$, $\dots$, $\Gamma_n \in \mathscr{G}(m_n)$, we let, recalling (5.8),

$$\begin{aligned}
&((g_1)^{\Gamma_1}_{\lambda_1,\dots,\lambda_{M_1}} \otimes \cdots \otimes (g_n)^{\Gamma_n}_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}})(v_1 \otimes \cdots \otimes v_{M_n}) \\
&:= \pm (g_1)^{\Gamma_1}_{\lambda_1,\dots,\lambda_{M_1}}(v_1 \otimes \cdots \otimes v_{M_1}) \otimes \cdots \otimes \\
&\qquad (g_n)^{\Gamma_n}_{\lambda_{M_{n-1}+1},\dots,\lambda_{M_n}}(v_{M_{n-1}+1} \otimes \cdots \otimes v_{M_n}),
\end{aligned} \tag{10.12}$$

with $\pm$ the same as (5.9). We are using the notation (9.5) for the variables $X(1),\dots,X(M_n)$ appearing in (10.11). Finally, for polynomials $P(\lambda) = \sum_m p_m \lambda^m$

and $Q(\mu) = \sum_n q_n \mu^n$ with coefficients in V, we denote

$$(|_{x=\partial} P(\lambda + y)) \otimes (|_{y=\partial} Q(\mu + x)) = \sum_{m,n} ((\mu + \partial)^n p_m) \otimes ((\lambda + \partial)^m q_n), \tag{10.13}$$

and by $\partial g_\lambda(w_1 \otimes \cdots \otimes w_m)$ we mean $\partial(g_\lambda(w_1 \otimes \cdots \otimes w_m))$.

Remark 10.3. In view of Examples 9.1 and 9.5, in the special case $m_1 = k + 1, m_2 = \cdots = m_n = 1$ and letting $n = k + m + 1$, formula (10.11) reduces to (8.18).

Lemma 10.4. *With the above notation, the right-hand side of* (10.11) *is a well-defined element of* $V[\lambda_1, \ldots, \lambda_{M_n}]/\langle \partial + \lambda_1 + \cdots + \lambda_{M_n} \rangle$, *for every* $f \in P^{\mathrm{cl}}(n)$ *and* $g_1 \in P^{\mathrm{cl}}(m_1), \ldots, g_n \in P^{\mathrm{cl}}(m_n)$.

Proof. First observe that, if $\Delta_0^{m_1 \cdots m_n}(\Gamma)$ is not acyclic, then the right-hand side of (10.11) is 0, since by assumption f satisfies (10.4). On the other hand, if $\Delta_0^{m_1 \cdots m_n}(\Gamma)$ is acyclic, then by the observation at the end of Sect. 9.1, the variable x_i does not appear in $X(k)$ when $k \in \{M_{i-1} + 1, \ldots, M_i\}$. This makes the right-hand side of (10.11) a well-defined polynomial for given polynomials

$$(g_i)^{\Gamma_i}_{\lambda_{M_{i-1}+1}, \ldots, \lambda_{M_i}}(v_{M_{i-1}+1} \otimes \cdots \otimes v_{M_i}). \tag{10.14}$$

However, (10.14) are only determined up to adding elements of

$$\langle \partial + \lambda_{M_{i-1}+1} + \cdots + \lambda_{M_i} \rangle = \langle \Lambda_i + \partial \rangle,$$

and we need to check that the right-hand side of (10.11) will remain the same after that. Fix $1 \le i \le n$, and replace in (10.11) the polynomial (10.14) with

$$(\Lambda_i + \partial)(h_i)^{\Gamma_i}_{\lambda_{M_{i-1}+1}, \ldots, \lambda_{M_i}}(v_{M_{i-1}+1} \otimes \cdots \otimes v_{M_i})$$

for some map

$$h_i : \mathcal{G}(m_i) \times V^{\otimes m_i} \longrightarrow V[\lambda_{M_{i-1}+1}, \ldots, \lambda_{M_i}].$$

Let us introduce the shorthand notation

$$\tilde{g}_i^G = (|_{x_i = \Lambda_i + \partial}(g_i)^G_{\lambda_{M_{i-1}+1} + X(M_{i-1}+1), \ldots, \lambda_{M_i} + X(M_i)}), \tag{10.15}$$

for an arbitrary graph G. Then in (10.11), we need to replace $\tilde{g}_i^{\Delta_i^{m_1 \cdots m_n}(\Gamma)}$ with

$$|_{x_i = \Lambda_i + \partial}(\Lambda_i + \partial + X(M_{i-1} + 1) + \cdots + X(M_i)) \tilde{h}_i^{\Delta_i^{m_1 \cdots m_n}(\Gamma)}. \tag{10.16}$$

It follows from the definition (9.5) of $X(k)$, that

$$X(M_{i-1} + 1) + \cdots + X(M_i) = \sum_j x_j,$$

where the sum is over all $1 \leq j \leq n$ such that j is connected by an unoriented path with i in the graph $\Delta_0^{m_1 \cdots m_n}(\Gamma)$. Together with x_i, this gives the sum of all x_j where j is a vertex of the connected component Γ_i^0 of i in $\Delta_0^{m_1 \cdots m_n}(\Gamma)$. Then after setting all $x_j = \Lambda_j + \partial$, we obtain

$$\sum_{j \in \Gamma_i^0} (\Lambda_j + \partial_j) = \Lambda_{\Gamma_i^0} + \partial_{\Gamma_i^0},$$

using again the notation (8.10). Then after applying $f_{\Lambda_1,\dots,\Lambda_n}^{\Delta_0^{m_1 \cdots m_n}(\Gamma)}$ we get 0, because f satisfies the second sesquilinearity condition (10.7). □

Lemma 10.5. *For every $f \in P^{\mathrm{cl}}(n)$ and $g_1 \in P^{\mathrm{cl}}(m_1), \dots, g_n \in P^{\mathrm{cl}}(m_n)$, the composition $f(g_1, \dots, g_n)$, defined by (10.11), is an element of $P^{\mathrm{cl}}(M_n)$.*

Proof. We need to check that $f(g_1, \dots, g_n)$ satisfies the cycle relations (10.4), (10.5) and the sesquilinearity conditions (10.6), (10.7). Observe that if $\Gamma \in \mathscr{G}(M_n)$ contains a cycle, then one of the graphs $\Delta_i^{m_1 \cdots m_n}(\Gamma)$, $i = 0, 1, \dots, n$, must contain a cycle as well. Evaluating f for $i = 0$ or g_i for $1 \leq i \leq n$, we obtain 0, because f and g_i satisfy (10.4). Therefore, $f(g_1, \dots, g_n)$ satisfies the first cycle relation (10.4).

To prove the second cycle relation (10.5), consider an oriented cycle $C \subset E(\Gamma)$ of $\Gamma \in \mathscr{G}(n)$. Recalling (10.11), we need to show that

$$\sum_{e \in C} f_{\Lambda_1,\dots,\Lambda_n}^{\Delta_0^{m_1 \cdots m_n}(\Gamma \setminus e)} (\tilde{g}_1^{\Delta_i^{m_1 \cdots m_n}(\Gamma \setminus e)} \otimes \cdots \otimes \tilde{g}_n^{\Delta_n^{m_1 \cdots m_n}(\Gamma \setminus e)}) = 0, \tag{10.17}$$

where we use the notation (10.15). Given an edge $e \in C$, consider its image $\Delta(e)$ under the map (9.4). Clearly, we have

$$\Delta_i^{m_1 \cdots m_n}(\Gamma \setminus e) = \begin{cases} \Delta_i^{m_1 \cdots m_n}(\Gamma) \setminus \Delta(e), & \text{if } \Delta(e) \in E(\Delta_i^{m_1 \cdots m_n}(\Gamma)), \\ \Delta_i^{m_1 \cdots m_n}(\Gamma), & \text{otherwise.} \end{cases}$$

Hence, by Lemma 9.2, the left-hand side of (10.17) is equal to

$$\begin{aligned} &\sum_{e' \in \Delta(C) \cap E(\Delta_0^{m_1 \cdots m_n}(\Gamma))} f_{\Lambda_1,\dots,\Lambda_n}^{\Delta_0^{m_1 \cdots m_n}(\Gamma) \setminus e'} (\tilde{g}_1^{\Delta_i^{m_1 \cdots m_n}(\Gamma \setminus e)} \otimes \cdots \otimes \tilde{g}_n^{\Delta_n^{m_1 \cdots m_n}(\Gamma \setminus e)}) \\ &+ \sum_{i=1}^{n} \sum_{e' \in \Delta(C) \cap E(\Delta_i^{m_1 \cdots m_n}(\Gamma))} f_{\Lambda_1,\dots,\Lambda_n}^{\Delta_0^{m_1 \cdots m_n}(\Gamma)} (\tilde{g}_1^{\Delta_1^{m_1 \cdots m_n}(\Gamma)} \otimes \cdots \otimes \\ &\qquad \tilde{g}_i^{\Delta_i^{m_1 \cdots m_n}(\Gamma) \setminus e'} \otimes \cdots \otimes \tilde{g}_n^{\Delta_n^{m_1 \cdots m_n}(\Gamma)}). \end{aligned} \tag{10.18}$$

If $\Delta(C) \subset E(\Delta_i^{m_1\cdots m_n}(\Gamma))$, then (10.18) reduces to

$$\sum_{e'\in\Delta(C)} f_{\Lambda_1,\dots,\Lambda_n}^{\Delta_0^{m_1\cdots m_n}(\Gamma)}(\tilde{g}_1^{\Delta_1^{m_1\cdots m_n}(\Gamma)}\otimes\cdots\otimes\tilde{g}_i^{\Delta_i^{m_1\cdots m_n}(\Gamma)\setminus e'}\otimes\cdots\otimes\tilde{g}_n^{\Delta_n^{m_1\cdots m_n}(\Gamma)}). \tag{10.19}$$

In this case, by Lemma 9.3 (a), $\Delta(C)$ is an oriented cycle of $\Delta_i^{m_1\cdots m_n}(\Gamma)$, hence (10.19) vanishes by the second cycle condition (10.5) for g_i. On the other hand, if $\Delta(C)$ is not contained in $E(\Delta_i^{m_1\cdots m_n}(\Gamma))$ for any $i = 1,\dots,n$, then by Lemma 9.3, $\Delta(C) \cap E(\Delta_0^{m_1\cdots m_n}(\Gamma))$ is an oriented cycle of $\Delta_0^{m_1\cdots m_n}(\Gamma)$. In this case, the first sum of (10.18) vanishes since f satisfies (10.5). Moreover, each term in the second sum of (10.18) vanishes as well, since $\Delta_0^{m_1\cdots m_n}(\Gamma)$ is not acyclic and f satisfies (10.4). We conclude that $f(g_1,\dots,g_n)$ satisfies the second cycle condition (10.5) as claimed.

Next, we will prove that $f(g_1,\dots,g_n)$ satisfies the first sesquilinearity relation (10.6). Let $(h \to k)$ be an edge in the graph Γ. We need to prove that the right-hand side of (10.11) is a polynomial of $(\lambda_h + \lambda_k)$ and not of λ_h and λ_k separately. First, suppose that for some $i = 1,\dots,n$, we have

$$h,k \in \{M_{i-1}+1,\dots,M_i\}, \quad \text{i.e.,} \quad h = M_{i-1}+r, \ \ k = M_{i-1}+q,$$

for some $r,q \in \{1,\dots,m_i\}$. In this case, λ_h and λ_k are both summands of Λ_i; hence $f_{\Lambda_1,\dots,\Lambda_n}$ has the required property (of being polynomial of $(\lambda_h+\lambda_k)$ and not of λ_h and λ_k separately). The image of $(h \to k)$ under the map (9.4) is an edge $(r \to q)$ in $\Delta_i^{m_1\cdots m_n}(\Gamma)$. Thus, $(g_i)_{\lambda_{M_{i-1}+1},\dots,\lambda_{M_i}}^{\Delta_i^{m_1\cdots m_n}(\Gamma)}$ is a polynomial of $(\lambda_h+\lambda_k)$ by the first sesquilinearity property of g_i.

Now suppose that

$$h \in \{M_{i-1}+1,\dots,M_i\}, \quad k \in \{M_{j-1}+1,\dots,M_j\},$$

for different $i,j \in \{1,\dots,n\}$. In this case, $(i \to j)$ is an edge in the graph $\Delta_0^{m_1\cdots m_n}(\Gamma)$. Therefore, $f_{\Lambda_1,\dots,\Lambda_n}^{\Delta_0^{m_1\cdots m_n}(\Gamma)}$ is a polynomial of $(\Lambda_i+\Lambda_j)$, and hence of $(\lambda_h+\lambda_k)$. Furthermore, by the assumption (10.6) on g_t $(t = 1,\dots,n)$ and by the definition (9.5) of the variables $X(1),\dots,X(M_n)$, all the λ_l's of the same connected component of λ_h and λ_k appear as summed in the polynomial $\tilde{g}_t^{\Delta_t^{m_1\cdots m_n}(\Gamma)}$. We conclude that (10.6) holds for $f(g_1,\dots,g_n)$, as claimed.

Finally, we will show that $f(g_1,\dots,g_n)$ satisfies the second sesquilinearity relation (10.7). Let G be one of the connected components of Γ, and consider the image $G_i = \Delta_i^{m_1\cdots m_n}(G)$ $(0 \le i \le n)$ of G under the map (9.4). Note that if G_0 contains a cycle, then $\Delta_0^{m_1\cdots m_n}(\Gamma)$ does, which implies $(f(g_1,\dots,g_n))^\Gamma = 0$. Hence, we can suppose that G_0 is acyclic. Then it is easy to see that all G_i $(0 \le i \le n)$ are connected. Furthermore, the set of vertices of G is the disjoint

union of the sets of vertices of G_i $(1 \le i \le n)$. Thus, using again the notation (8.10), we have

$$\lambda_G + \partial_G = \sum_{i=1}^{n} (\lambda_{G_i} + \partial_{G_i}),$$

and to prove (10.7) for $f(g_1, \dots, g_n)$, it is enough to show that

$$(f(g_1, \dots, g_n))^{\Gamma}_{\lambda_1, \dots, \lambda_{M_n}}((\lambda_{G_i} + \partial_{G_i})v) = 0, \quad v \in V^{\otimes M_n}, \quad 1 \le i \le n. \tag{10.20}$$

Since g_i itself satisfies (10.7), we have from (10.11):

$$f^{\Delta_0^{m_1 \cdots m_n}(\Gamma)}_{\Lambda_1, \dots, \Lambda_n}(\tilde{g}_1^{\Delta_i^{m_1 \cdots m_n}(\Gamma)} \otimes \cdots \otimes \tilde{g}_n^{\Delta_n^{m_1 \cdots m_n}(\Gamma)})\Big(\Big(\lambda_{G_i} + \partial_{G_i} + \sum_{k \in G_i} X(k)\Big)v\Big)$$

$$= 0.$$

As in the proof of Lemma 10.4 (*cf.* (10.16)), we see from the definition (9.5) of $X(k)$, that

$$\sum_{k \in G_i} X(k) = \sum_{j \in G_0 \setminus \{i\}} x_j.$$

After setting all $x_j = \Lambda_j + \partial$, we can add x_i to the above sum, because g_i is defined only up to adding elements of $\langle \Lambda_i + \partial \rangle$. We obtain

$$\sum_{j \in G_0} x_j|_{x_j = \Lambda_j + \partial} = \Lambda_{G_0} + \partial_{G_0}.$$

After applying $f^{\Delta_0^{m_1 \cdots m_n}(\Gamma)}_{\Lambda_1, \dots, \Lambda_n}$ to this we get 0, since f satisfies (10.7). This proves (10.20) and finishes the proof of the lemma. □

Theorem 10.6. *The vector superspaces $P^{\mathrm{cl}}(n)$, $n \ge 0$, together with the actions of the symmetric groups S_n given by* (10.10) *and the composition maps defined by* (10.11)*, form an operad, which is graded by* (10.9).

Proof. First, let us check that $f^\sigma \in P^{\mathrm{cl}}(n)$ for every $f \in P^{\mathrm{cl}}(n)$ and $\sigma \in S_n$. The cycle relations (10.4) and (10.5) for f^σ are obvious, using the fact that if $C \subset E(\Gamma)$ is an oriented cycle of Γ, then $\sigma(C)$ (obtained by applying σ to the tails and heads of all edges in C) is an oriented cycle of $\sigma(\Gamma)$. Next, if $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_s$ is a disjoint union of connected components, then $\sigma(\Gamma)$ is a disjoint union of connected components $\sigma(\Gamma_1) \sqcup \cdots \sqcup \sigma(\Gamma_s)$. From here, it is easy to derive the sesquilinearity conditions (10.6) and (10.7) for f^σ. Thus, $f^\sigma \in P^{\mathrm{cl}}(n)$.

We have already shown in Lemma 10.5 that the composition $f(g_1, \dots, g_n) \in P^{\mathrm{cl}}(M_n)$ for $f \in P^{\mathrm{cl}}(n)$ and $g_i \in P^{\mathrm{cl}}(m_i)$. It is clear by construction that the action of the symmetric group (10.10) and the composition maps (10.11) are

parity preserving linear maps $P^{\mathrm{cl}}(n) \to P^{\mathrm{cl}}(n)$ and $P^{\mathrm{cl}}(n) \otimes P^{\mathrm{cl}}(m_1) \otimes \cdots \otimes P^{\mathrm{cl}}(m_n) \to P^{\mathrm{cl}}(M_n)$, respectively. The unity axioms (3.3) are obvious, where the unit $1 \in P^{\mathrm{cl}}(1)$ is the identity operator

$$1^{\bullet}_{\lambda}(v) = v + \langle \partial + \lambda \rangle \in V[\lambda]/\langle \partial + \lambda \rangle \cong V,$$

and $\bullet$ represents the graph with one vertex. The fact that P^{cl} is a graded operad follows from Lemma 9.2 and the definition (10.10), (10.11) of the operad structure. To finish the proof of the theorem, we need to verify the associativity (3.2) of the composition and the equivariance (3.4) of the symmetric group action.

To prove the associativity axiom, given $f \in P^{\mathrm{cl}}(n)$, $g_i \in P^{\mathrm{cl}}(m_i)$ and $h_{ij} \in P^{\mathrm{cl}}(\ell_{ij})$, we need to show that $\varphi = \psi$, where

$$\begin{aligned}
\varphi &= f(g_1(h_{11}, \dots, h_{1m_1}), \dots, g_n(h_{n1}, \dots, h_{nm_n})), \\
\psi &= (f(g_1, \dots, g_n))(h_{11}, \dots, h_{1m_1}, \dots, h_{n1}, \dots, h_{nm_n}).
\end{aligned}$$

Let us introduce the lexicographically ordered index sets

$$\begin{aligned}
\mathscr{J} &= \{(ij) \,|\, 1 \le i \le n,\ 1 \le j \le m_i\}, \\
\mathscr{K} &= \{(ijk) \,|\, 1 \le i \le n,\ 1 \le j \le m_i,\ 1 \le k \le \ell_{ij}\},
\end{aligned}$$

and the notation

$$\begin{aligned}
&\Lambda_i = \sum_{j=1}^{m_i} \sum_{k=1}^{\ell_{ij}} \lambda_{ijk}, \quad \Lambda_{ij} = \sum_{k=1}^{\ell_{ij}} \lambda_{ijk}, \quad 1 \le i \le n, \ (ij) \in \mathscr{J}, \\
&L_i = \sum_{j=1}^{m_i} \ell_{ij}, \qquad\qquad L = \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \ell_{ij}.
\end{aligned}$$

Then for any graph $\Gamma \in \mathscr{G}_0(L)$ and vectors $v_{ijk} \in V$, we find from the definition of composition (10.11):

$$\begin{aligned}
&\varphi^{\Gamma}_{(\lambda_{ijk})_{(ijk)\in\mathscr{K}}}\Big(\bigotimes_{(ijk)\in\mathscr{K}} v_{ijk} \Big) \\
&= f^{\Delta_0^{L_1 \cdots L_n}(\Gamma)}_{\Lambda_1,\dots,\Lambda_n}\Big(\bigotimes_{i=1}^{n} (g_i)^{\Delta_0^{\ell_{i1}\cdots\ell_{im_i}}(\Delta_i^{L_1\cdots L_n}(\Gamma))}_{\Lambda_{i1},\dots,\Lambda_{im_i}} \\
&\quad \Big(\bigotimes_{j=1}^{m_i} (h_{ij})^{\Delta_j^{\ell_{i1}\cdots\ell_{im_i}}(\Delta_i^{L_1\cdots L_n}(\Gamma))}_{\lambda_{ij1},\dots,\lambda_{ij\ell_{ij}}} \Big(\bigotimes_{k=1}^{\ell_{ij}} v_{ijk} \Big) \Big) \Big), \\
&\psi^{\Gamma}_{(\lambda_{ijk})_{(ijk)\in\mathscr{K}}}\Big(\bigotimes_{(ijk)\in\mathscr{K}} v_{ijk} \Big)
\end{aligned}$$

$$= f^{\Delta_0^{m_1\cdots m_n}(\Delta_0^{\ell_{11}\cdots\ell_{nm_n}}(\Gamma))}_{\Lambda_1,\dots,\Lambda_n}\Big(\bigotimes_{i=1}^{n}(g_i)^{\Delta_i^{m_1\cdots m_n}(\Delta_0^{\ell_{11}\cdots\ell_{nm_n}}(\Gamma))}_{\Lambda_{i1},\dots,\Lambda_{im_i}}$$
$$\Big(\bigotimes_{j=1}^{m_i}(h_{ij})^{\Delta^{\ell_{11}\cdots\ell_{nm_n}}_{m_1+\cdots+m_{i-1}+j}(\Gamma)}_{\lambda_{ij1},\dots,\lambda_{ij\ell_{ij}}}\Big(\bigotimes_{k=1}^{\ell_{ij}}v_{ijk}\Big)\Big)\Big).$$

The above right-hand sides are equal by Proposition 9.6, thus proving the associativity axiom (3.2).

Now we will prove the supersymmetric equivariance (3.4). Let $f\in P^{\mathrm{cl}}(n)$, $g_i\in P^{\mathrm{cl}}(m_i)$ as before, and $\sigma\in S_n$, $\tau_i\in S_{m_i}$ for $1\le i\le n$. Then for a graph $\Gamma\in\mathscr{G}_0(m_1+\cdots+m_n)$ and vectors $v_{ij}\in V$, we compute:

$$(f^\sigma(g_1^{\tau_1},\dots,g_n^{\tau_n}))^\Gamma_{\lambda_{11},\dots,\lambda_{nm_n}}\Big(\bigotimes_{(ij)\in\mathscr{J}}v_{ij}\Big)$$
$$=(f^\sigma)^{\Delta_0^{m_1\cdots m_n}(\Gamma)}_{\Lambda_1,\dots,\Lambda_n}\Big(\bigotimes_{i=1}^{n}(g_i^{\tau_i})^{\Delta_i^{m_1\cdots m_n}(\Gamma)}_{\lambda_{i1},\dots,\lambda_{im_i}}\Big(\bigotimes_{j=1}^{m_i}v_{ij}\Big)\Big)$$
$$=f^{\sigma(\Delta_0^{m_1\cdots m_n}(\Gamma))}_{\Lambda_{\sigma^{-1}(1)},\dots,\Lambda_{\sigma^{-1}(n)}}\Big(\sigma\Big(\bigotimes_{i=1}^{n}(g_i)^{\tau_i(\Delta_i^{m_1\cdots m_n}(\Gamma))}_{\lambda_{i\tau_i^{-1}(1)},\dots,\lambda_{i\tau_i^{-1}(m_i)}}\Big(\tau_i\Big(\bigotimes_{j=1}^{m_i}v_{ij}\Big)\Big)\Big)\Big)$$
$$=\epsilon_g(\sigma)\epsilon_{\otimes_{i=1}^n\tau_i(\otimes_{j=1}^{m_1}v_{ij})}(\sigma)\,f^{\sigma(\Delta_0^{m_1\cdots m_n}(\Gamma))}_{\Lambda_{\sigma^{-1}(1)},\dots,\Lambda_{\sigma^{-1}(n)}}$$
$$\Big(\bigotimes_{i=1}^{n}(g_{\sigma^{-1}(i)})^{\tau_{\sigma^{-1}(i)}(\Delta_{\sigma^{-1}(i)}^{m_1\cdots m_n}(\Gamma))}_{\lambda_{\sigma^{-1}(i)\tau^{-1}_{\sigma^{-1}(i)}(1)},\dots,\lambda_{\sigma^{-1}(i)\tau^{-1}_{\sigma^{-1}(i)}(m_{\sigma^{-1}(i)})}}$$
$$\Big(\tau_{\sigma^{-1}(i)}\Big(\bigotimes_{j=1}^{m_{\sigma^{-1}(i)}}v_{\sigma^{-1}(i)j}\Big)\Big)\Big)$$
$$=\epsilon_g(\sigma)\epsilon_{\otimes_{i=1}^n\tau_i(\otimes_{j=1}^{m_1}v_{ij})}(\sigma)\,f^{\Delta_0^{m_{\sigma^{-1}(1)}\cdots m_{\sigma^{-1}(n)}}((\sigma(\tau_1,\dots,\tau_n))(\Gamma))}_{\Lambda_{\sigma^{-1}(1)},\dots,\Lambda_{\sigma^{-1}(n)}}$$
$$\Big(\bigotimes_{i=1}^{n}(g_{\sigma^{-1}(i)})^{\Delta_i^{m_{\sigma^{-1}(1)}\cdots m_{\sigma^{-1}(n)}}((\sigma(\tau_1,\dots,\tau_n))(\Gamma))}_{\lambda_{\sigma^{-1}(i)\tau^{-1}_{\sigma^{-1}(i)}(1)},\dots,\lambda_{\sigma^{-1}(i)\tau^{-1}_{\sigma^{-1}(i)}(m_{\sigma^{-1}(i)})}}$$
$$\Big(\tau_{\sigma^{-1}(i)}\Big(\bigotimes_{j=1}^{m_{\sigma^{-1}(i)}}v_{\sigma^{-1}(i)j}\Big)\Big)\Big)$$
$$=\epsilon_g(\sigma)(f(g_{\sigma^{-1}(1)},\dots,g_{\sigma^{-1}(n)}))^{(\sigma(\tau_1,\dots,\tau_n))(\Gamma)}_{\lambda_{\sigma^{-1}(1)\tau^{-1}_{\sigma^{-1}(1)}(1)},\dots,\lambda_{\sigma^{-1}(n)\tau^{-1}_{\sigma^{-1}(n)}(m_{\sigma^{-1}(n)})}}$$
$$\Big((\sigma(\tau_1,\dots,\tau_n))\Big(\bigotimes_{(ij)\in\mathscr{J}}v_{ij}\Big)\Big)$$

$$= \epsilon_g(\sigma)((f(g_{\sigma^{-1}(1)},\dots,g_{\sigma^{-1}(n)}))^{\sigma(\tau_1,\dots,\tau_n)})^{\Gamma}_{\lambda_{11},\dots,\lambda_{nm_n}}\Big(\bigotimes_{(ij)\in\mathscr{J}} v_{ij}\Big).$$

For the first equality above, we used the definition (10.11) of the composition; for the second equality, the definition (10.10) of the action of the symmetric group on $P^{\mathrm{cl}}(n)$; for the third equality, the definition (2.2)–(2.3) of the action of S_n on a tensor product of n vector superspaces; for the fourth equality, Proposition 9.7; for the fifth equality, we used again (10.11), (2.2) and the definition of the composition of permutations; and for the last equality, we used again (10.10).

This completes the proof of the theorem. □

10.3. Poisson vertex algebras and the operad P^{cl}

As in Sect. 7.4, given the vector superspace V, with parity p, and the even endomorphism $\partial \in \mathrm{End}(V)$, we denote by ΠV the same vector space with reversed parity $\bar{p} = 1 - p$. Consider the corresponding operad $P^{\mathrm{cl}}(\Pi V)$ from Sect. 10.2 and the associated $\mathbb{Z}$-graded Lie superalgebra $W^{\mathrm{cl}}(\Pi V) := W(P^{\mathrm{cl}}(\Pi V))$ given by Theorem 3.4.

Theorem 10.7. *We have a bijective correspondence between the odd elements $X \in W_1^{\mathrm{cl}}(\Pi V)$ such that $X \square X = 0$ and the Poisson vertex superalgebra structures on V, defined as follows. The commutative associative product and the λ-bracket of the Poisson vertex superalgebra V corresponding to X are given by*

$$ab = (-1)^{p(a)} X^{\bullet\rightarrow\bullet}(a \otimes b), \quad [a_\lambda b] = (-1)^{p(a)} X^{\bullet\,\bullet}_{\lambda,-\lambda-\partial}(a \otimes b). \quad (10.21)$$

Proof. Note that, by the first sesquilinearity condition (10.6), the polynomial $X^{\bullet\rightarrow\bullet}_{\lambda_1,\lambda_2}$ depends only on $\lambda_1 + \lambda_2 \equiv -\partial$. Hence, it is independent of λ_1, λ_2. For this reason, in the first equation of (10.21) we omitted the subscripts λ_1, λ_2.

First, we check that the symmetry of X translates to the commutativity of the product ab and the skew-symmetry of the λ-bracket $[a_\lambda b]$. We have

$$\begin{aligned} X^{\bullet\rightarrow\bullet}(v_1 \otimes v_2) &= (X^{(12)})^{\bullet\rightarrow\bullet}(v_1 \otimes v_2) = (-1)^{\bar{p}(v_1)\bar{p}(v_2)} X^{\bullet\leftarrow\bullet}(v_2 \otimes v_1) \\ &= (-1)^{p(v_1)+p(v_2)+p(v_1)p(v_2)} X^{\bullet\rightarrow\bullet}(v_2 \otimes v_1), \end{aligned}$$

which, by the first equation in (10.21), is equivalent to the symmetry condition of the product: $v_1 v_2 = (-1)^{p(v_1)p(v_2)} v_2 v_1$. Similarly, evaluating the identity $X = X^{(12)}$ on the disconnected graph $\bullet\ \bullet$, we get

$$\begin{aligned} X^{\bullet\,\bullet}_{\lambda,-\lambda-\partial}(v_1 \otimes v_2) &= (X^{(12)})^{\bullet\,\bullet}_{\lambda,-\lambda-\partial}(v_1 \otimes v_2) \\ &= (-1)^{\bar{p}(v_1)\bar{p}(v_2)} X^{\bullet\,\bullet}_{-\lambda-\partial,\lambda}(v_2 \otimes v_1), \end{aligned}$$

which, by the second equation in (10.21), is equivalent to the skew-symmetry condition (5.3) of the λ-bracket.

Next, we need to prove that the condition $X\square X = 0$ translates to three conditions: the associativity of the product ab, the Jacobi identity (5.3) for the λ-bracket $[a_\lambda b]$, and the Leibniz rule (10.1). Recall that, by (3.14),

$$X\square X = X \circ_1 X + X \circ_2 X + (X \circ_2 X)^{(12)}.$$

Since, by construction, $X\square X$ is invariant by the action of the symmetric group, to impose the condition $X\square X = 0$ is the same as to impose $(X\square X)^\Gamma = 0$ for each of the three graphs:

(10.22)

Evaluating all three summands of $X\square X$ on the disconnected graph $\bullet\quad\bullet\quad\bullet$, we get, by the definition (10.11) of the composition maps,

$$\begin{aligned}
&(X \circ_1 X)^{\bullet\bullet\bullet}_{\lambda_1,\lambda_2,\lambda_3}(v_1 \otimes v_2 \otimes v_3)\\
&= X^{\bullet\bullet}_{\lambda_1+\lambda_2,\lambda_3}(X^{\bullet\bullet}_{\lambda_1,\lambda_2}(v_1 \otimes v_2) \otimes v_3)\\
&= (-1)^{p(v_2)}[[v_{1\lambda_1} v_2]_{\lambda_1+\lambda_2} v_3],\\
&(X \circ_2 X)^{\bullet\bullet\bullet}_{\lambda_1,\lambda_2,\lambda_3}(v_1 \otimes v_2 \otimes v_3)\\
&= (-1)^{\bar p(v_1)} X^{\bullet\bullet}_{\lambda_1,\lambda_2+\lambda_3}(v_1 \otimes X^{\bullet\bullet}_{\lambda_2,\lambda_3}(v_2 \otimes v_3))\\
&= (-1)^{1+p(v_2)}[v_{1\lambda_1}[v_{2\lambda_2} v_3]],\\
&((X \circ_2 X)^{(12)})^{\bullet\bullet\bullet}_{\lambda_1,\lambda_2,\lambda_3}(v_1 \otimes v_2 \otimes v_3)\\
&= (-1)^{\bar p(v_1)\bar p(v_2)}(X \circ_2 X)^{\bullet\bullet\bullet}_{\lambda_2,\lambda_1,\lambda_3}(v_2 \otimes v_1 \otimes v_3)\\
&= (-1)^{p(v_2)+p(v_1)p(v_2)}[v_{2\lambda_2}[v_{1\lambda_1} v_3]].
\end{aligned}$$

Hence, the condition $(X\square X)^{\bullet\bullet\bullet} = 0$ is equivalent to the Jacobi identity (5.3) for the λ-bracket.

Evaluating all three summands of $X\square X$ on the second graph in (10.22), we get, by the definition (10.11) of the composition maps,

$$\begin{aligned}
&(X \circ_1 X)^{\bullet\bullet\to\bullet}_{\lambda_1,\lambda_2,\lambda_3}(v_1 \otimes v_2 \otimes v_3)\\
&= X^{\bullet\to\bullet}(X^{\bullet\bullet}_{\lambda_1,\lambda_2+x_3}(v_1 \otimes v_2) \otimes (|_{x_3=\lambda_3+\partial} v_3))\\
&= (-1)^{p(v_2)}[v_{1\lambda_1} v_2] v_3,\\
&(X \circ_2 X)^{\bullet\bullet\to\bullet}_{\lambda_1,\lambda_2,\lambda_3}(v_1 \otimes v_2 \otimes v_3)\\
&= (-1)^{\bar p(v_1)} X^{\bullet\bullet}_{\lambda_1,\lambda_2+\lambda_3}(v_1 \otimes X^{\bullet\to\bullet}(v_2 \otimes v_3))\\
&= (-1)^{1+p(v_2)}[v_{1\lambda_1} v_2 v_3],
\end{aligned}$$

$$
\begin{aligned}
&((X\circ_2 X)^{(12)})^{\bullet\ \bullet\longrightarrow\bullet}_{\lambda_1,\lambda_2,\lambda_3}(v_1\otimes v_2\otimes v_3)\\
&=(-1)^{\bar p(v_1)\bar p(v_2)}(X\circ_2 X)^{\overset{\frown}{\bullet\ \bullet}\ \bullet}_{\lambda_2,\lambda_1,\lambda_3}(v_2\otimes v_1\otimes v_3)\\
&=(-1)^{\bar p(v_2)+\bar p(v_1)\bar p(v_2)}X^{\bullet\longrightarrow\bullet}((|_{x_2=\lambda_2+\partial}v_2)\otimes X^{\bullet\ \bullet}_{\lambda_1,\lambda_3+x_2}(v_1\otimes v_3))\\
&=(-1)^{p(v_2)+p(v_1)p(v_2)}v_2[v_{1\lambda_1}v_3].
\end{aligned}
$$

Hence, the condition $(X\square X)^{\bullet\ \longrightarrow\bullet}=0$ is equivalent to the Leibniz rule (10.1).

Finally, we evaluate all three summands of $X\square X$ on the third graph in (10.22). We get

$$
\begin{aligned}
&(X\circ_1 X)^{\bullet\longrightarrow\bullet\longrightarrow\bullet}_{\lambda_1,\lambda_2,\lambda_3}(v_1\otimes v_2\otimes v_3)=X^{\bullet\longrightarrow\bullet}(X^{\bullet\longrightarrow\bullet}(v_1\otimes v_2)\otimes v_3)\\
&=(-1)^{p(v_2)}(v_1v_2)v_3,\\
&(X\circ_2 X)^{\bullet\longrightarrow\bullet\longrightarrow\bullet}_{\lambda_1,\lambda_2,\lambda_3}(v_1\otimes v_2\otimes v_3)=(-1)^{\bar p(v_1)}X^{\bullet\longrightarrow\bullet}(v_1\otimes X^{\bullet\longrightarrow\bullet}(v_2\otimes v_3))\\
&=(-1)^{1+p(v_2)}v_1(v_2v_3),\\
&((X\circ_2 X)^{(12)})^{\bullet\longrightarrow\bullet\longrightarrow\bullet}_{\lambda_1,\lambda_2,\lambda_3}(v_1\otimes v_2\otimes v_3)\\
&=(-1)^{\bar p(v_1)\bar p(v_2)}(X\circ_2 X)^{\overset{\frown}{\bullet\ \bullet}\ \bullet}_{\lambda_2,\lambda_1,\lambda_3}(v_2\otimes v_1\otimes v_3)\\
&=(-1)^{\bar p(v_2)+\bar p(v_1)\bar p(v_2)}X^{\overset{\frown}{\bullet\ \bullet}}(\cdots)=0.
\end{aligned}
$$

The last equality holds because X evaluated on a graph containing a cycle is, by definition, zero. Hence, the condition $(X\square X)^{\bullet\longrightarrow\bullet\longrightarrow\bullet}=0$ is equivalent to the associativity of the product. This completes the proof. □

In view of Theorem 10.7, we have the definition of the corresponding cohomology complex.

Definition 10.8. *Let V be a Poisson vertex superalgebra. The corresponding* PVA cohomology complex *is defined as*

$$(W^{\mathrm{cl}}(\Pi V),\operatorname{ad}X),$$

where $X\in W_1(\Pi V)_{\bar 1}$ is given by (10.21).

Remark 10.9. Let $V=\bigoplus_r \operatorname{gr}^r V$ be a graded vector superspace. Recall from Remark 10.2 that we have the corresponding grading of the operad $P^{\mathrm{cl}}(\Pi V)$, and hence of the Lie superalgebra $W^{\mathrm{cl}}(\Pi V)$. It is easy to check, as for Theorem 8.10, that under the correspondence from Theorem 10.7, the structures of graded Poisson vertex algebra on V are in bijection with the odd elements $X\in \operatorname{gr}^1 W_1^{\mathrm{cl}}(\Pi V)$ satisfying $X\square X=0$. (Recall that V is a graded Poisson vertex algebra if $(\operatorname{gr}^p V)(\operatorname{gr}^q V)\subset \operatorname{gr}^{p+q}V$ and $[\operatorname{gr}^p V_\lambda \operatorname{gr}^q V]\subset \operatorname{gr}^{p+q-1}V[\lambda]$.)

10.4. Relation between $\operatorname{gr} P^{\mathrm{ch}}$ *and* P^{cl}

Recall from Corollary 8.8 that for every $\bar{X} \in \operatorname{gr}^r P^{\mathrm{ch}}(k+1)$ (with a representative $X \in \mathrm{F}^r P^{\mathrm{ch}}(k+1)$) and every graph $\Gamma \in \mathscr{G}(k+1)$ with r edges, we have the map

$$X^\Gamma (= X(p_\Gamma)) \colon V^{\otimes(k+1)} \longrightarrow V[\lambda_0, \dots, \lambda_k]/\langle \partial + \lambda_0 + \cdots + \lambda_k \rangle.$$

Theorem 10.10. *For every vector superspace V with an $\mathbb{F}[\partial]$-module structure, there is a canonical injective morphism of graded operads*

$$\operatorname{gr} P^{\mathrm{ch}}(V) \hookrightarrow P^{\mathrm{cl}}(V), \tag{10.23}$$

mapping $\bar{X} \in \operatorname{gr}^r P^{\mathrm{ch}}(k+1)$ to:

$$\{X^\Gamma \mid \Gamma \in \mathscr{G}(k+1) \ \text{ with } \ r \ \text{ edges}\} \in \operatorname{gr}^r P^{\mathrm{cl}}(k+1). \tag{10.24}$$

This morphism is a bijection $\operatorname{gr} P^{\mathrm{ch}}(k+1) \xrightarrow{\sim} P^{\mathrm{cl}}(k+1)$ *for* $k = -1, 0, 1$.

Proof. As a consequence of Corollary 8.8 (a)–(b) and the definition of the graded operad P^{cl}, the map (10.24) is a well-defined morphism of operads, and by Corollary 8.8 (c) this morphism is injective. Surjectivity of the morphism for $k = -1, 0$ is immediate. Let us prove it for $k = 1$. Recall that $\operatorname{gr} P^{\mathrm{cl}}(2) = \operatorname{gr}^0 P^{\mathrm{cl}}(2) \oplus \operatorname{gr}^1 P^{\mathrm{cl}}(2)$. By definition, $\operatorname{gr}^0 P^{\mathrm{cl}}(2)$ consists of maps

$$X^{\bullet\bullet} : V^{\otimes 2} \longrightarrow V[\lambda],$$

satisfying the sesquilinearity conditions

$$\begin{aligned} X_\lambda^{\bullet\bullet}((\partial v_0) \otimes v_1) &= -\lambda X_\lambda^{\bullet\bullet}(v_0 \otimes v_1), \\ X_\lambda^{\bullet\bullet}(v_0 \otimes (\partial v_1)) &= (\lambda + \partial) X_\lambda^{\bullet\bullet}(v_0 \otimes v_1). \end{aligned}$$

A preimage $\tilde{X} \in F^0 P^{\mathrm{ch}}(2) = P^{\mathrm{ch}}(2)$ can be constructed by letting

$$\begin{aligned} &\tilde{X}(v_0, v_1; z_{01}^n) \\ &= \begin{cases} \Big(-\dfrac{\partial}{\partial\lambda}\Big)^n X_\lambda^{\bullet\bullet}(v_0 \otimes v_1) \ \text{ if } \ n \geq 0, \\ (-1)^m \displaystyle\int_0^\lambda d\mu_1 \cdots \int_0^{\mu_{m-1}} d\mu_m \, X_{\mu_m}^{\bullet\bullet}(v_0 \otimes v_1) \ \text{ if } \ n = -m \leq -1. \end{cases} \end{aligned}$$

It is not hard to check that, indeed, $\tilde{X}$ is a well-defined element of $P^{\mathrm{ch}}(2)$ and the image of its coset $[\tilde{X}] \in \operatorname{gr}^0 P^{\mathrm{ch}}(2) = F^0 P^{\mathrm{ch}}(2)/F^1 P^{\mathrm{ch}}(2)$ via the morphism (10.23) coincides with $X^{\bullet\bullet}$. Next, $\operatorname{gr}^1 P^{\mathrm{cl}}(2)$ consists of $\mathbb{F}[\partial]$-module homomorphisms

$$X^{\bullet\to\bullet} : V^{\otimes 2} \longrightarrow V.$$

A preimage $\tilde{X} \in F^1 P^{\mathrm{ch}}(2)$ can be constructed by letting

$$\tilde{X}(v_0, v_1; z_{01}^n) = \begin{cases} 0 \ \text{ if } \ n \geq 0, \\ \dfrac{(-1)^m}{m!} X^{\bullet\to\bullet}((\partial + \lambda)^m v_0 \otimes v_1) \ \text{ if } \ n = -m-1 \leq -1. \end{cases}$$

Again, it is not hard to check that $\tilde{X}$ is a well-defined element of $F^1 P^{\mathrm{ch}}(2) = \mathrm{gr}^1 P^{\mathrm{ch}}(2)$ and its image via the morphism (10.23) coincides with $X^{\bullet\to\bullet}$. □

Example 10.11. Let V be a non-unital vertex algebra. By Theorem 6.12, the vertex algebra structure of V corresponds to an odd element $X \in W_1^{\mathrm{ch}}(\Pi V) = P^{\mathrm{ch}}(2)(\Pi V)$ such that $X \square X = 0$. The filtration (8.4) of $P^{\mathrm{ch}}(2)$ is

$$\mathrm{F}^0 P^{\mathrm{ch}}(2) = P^{\mathrm{ch}}(2), \quad \mathrm{F}^1 P^{\mathrm{ch}}(2) = \{f \mid f(\mathscr{O}_2^T) = 0\}, \quad \mathrm{F}^2 P^{\mathrm{ch}}(2) = 0.$$

Hence, the image X_0 of $\mathrm{gr}^0 X$ in $P^{\mathrm{cl}}(2)$ via the map defined in Theorem 10.10, is the element

$$X_0{}^{\bullet\ \bullet} = X(1), \quad X_0{}^{\bullet\to\bullet} = 0.$$

Thus we obtain a PVA structure on V, where the λ-bracket is the same as for the vertex algebra V, and the commutative associative product is zero.

Recall that if V is a filtered vector space, then $P^{\mathrm{ch}}(V)$ is a filtered operad with respect to the refined filtration introduced in Sect. 8.5, and $P^{\mathrm{cl}}(\mathrm{gr}\, V)$ is a graded operad with respect to the refined grading introduced in Remark 10.2. Then Theorem 10.10 still holds:

Theorem 10.12. *We have a canonical injective morphism of graded operads from*

$$\mathrm{gr}\, P^{\mathrm{ch}}(V) \hookrightarrow P^{\mathrm{cl}}(\mathrm{gr}\, V). \tag{10.25}$$

Explicitly, $\bar{f} \in \mathrm{gr}^r P_k^{\mathrm{ch}}(V)$, with a representative $f \in \mathrm{F}^r P^{\mathrm{ch}}(V)$, is mapped to the element $\tilde{f} \in \mathrm{gr}^r P_k^{\mathrm{cl}}(\mathrm{gr}\, V)$ defined as follows. If $\Gamma \in \mathscr{G}(k)$ has s edges and $\bar{v}_1 \otimes \cdots \otimes \bar{v}_k \in \mathrm{gr}^t(V^{\otimes k}) = \bigoplus_{r_1+\cdots+r_k=t} \mathrm{gr}^{r_1} V \otimes \cdots \otimes \mathrm{gr}^{r_k} V$, we let

$$\tilde{f}^{\Gamma}_{\lambda_1,\dots,\lambda_k}(\bar{v}_1 \otimes \cdots \otimes \bar{v}_k) = f^{z_1,\dots,z_k}_{\lambda_1,\dots,\lambda_k}(v_1, \dots, v_k; p_\Gamma) + \mathrm{F}^{s+t-r-1} V \tag{10.26}$$

in $(\mathrm{gr}^{s+t-r} V)[\lambda_1, \dots, \lambda_k]/\langle \partial + \lambda_1 + \cdots + \lambda_k \rangle$.

Proof. Straightforward. □

Let V be a filtered vertex algebra and let $\mathrm{gr}\, V$ be the associated graded Poisson vertex algebra. By Theorem 8.10, the vertex algebra structure of V corresponds to an odd element $X \in \mathrm{F}^1 W_1^{\mathrm{ch}}(\Pi V) = \mathrm{F}^1 P^{\mathrm{ch}}(2)(\Pi V)$ such that $X \square X = 0$. Moreover, by Remark 10.9, the Poisson vertex algebra structure of $\mathrm{gr}\, V$ corresponds to an odd element $\tilde{X} \in \mathrm{gr}^1 W_1^{\mathrm{cl}}(\Pi\, \mathrm{gr}\, V) = \mathrm{gr}^1 P^{\mathrm{cl}}(2)(\Pi\, \mathrm{gr}\, V)$ such that $\tilde{X} \square \tilde{X} = 0$.

Theorem 10.13. *The image of* $\bar{X} \in \mathrm{gr}^1\, W_1^{\mathrm{ch}}(\Pi V)$ *via the morphism defined by Theorem 10.12 is* $\tilde{X}$.

Proof. The proof follows by construction. □

Obviously, the morphism of operads defined in Theorem 10.12 induces a Lie superalgebra injective homomorphism

$$\mathrm{gr}\, W^{\mathrm{ch}}(\Pi V) \hookrightarrow W^{\mathrm{cl}}(\mathrm{gr}\, \Pi V). \tag{10.27}$$

Moreover, by Theorem 10.13, $\bar{X} = \mathrm{gr}\, X$, where $X \in W_1^{\mathrm{ch}}(\Pi V)$ is associated to the vertex algebra structure of V, is mapped by the homomorphism (10.27) to $\tilde{X} \in W_1^{\mathrm{cl}}(\mathrm{gr}\, \Pi V)$, associated to the PVA structure of $\mathrm{gr}\, V$. Summarizing, we have:

Theorem 10.14. *Let* V *be a filtered vertex algebra and let* $\mathrm{gr}\, V$ *be the associated graded Poisson vertex algebra. Denote by* $X \in \mathrm{F}^1\, W_1^{\mathrm{ch}}(\Pi V)$ *the element corresponding to the vertex algebra structure of* V *by* (6.31) *(cf. Theorem 8.10), and denote by* $\tilde{X} \in \mathrm{gr}^1\, W_1^{\mathrm{cl}}(\mathrm{gr}\, \Pi V)$ *the element corresponding to the Poisson vertex algebra structure of* $\mathrm{gr}\, V$ *by* (10.21) *(cf. Remark 10.9).*

(a) There is a canonical injective homomorphism of graded Lie superalgebras

$$\mathrm{gr}\, W^{\mathrm{ch}}(\Pi V) \hookrightarrow W^{\mathrm{cl}}(\mathrm{gr}\, \Pi V), \tag{10.28}$$

mapping $\bar{X} \in \mathrm{gr}^1\, W^{\mathrm{ch}}(\Pi V)$ *to* $\tilde{X} \in \mathrm{gr}^1\, W_1^{\mathrm{cl}}(\mathrm{gr}\, \Pi V)$.

(b) Hence, we have an injective morphism of complexes

$$(\mathrm{gr}\, W^{\mathrm{ch}}(\Pi V), d_{\bar{X}} = \mathrm{gr}\, \mathrm{ad}\, X) \longrightarrow (W^{\mathrm{cl}}(\mathrm{gr}\, \Pi V), d_{\tilde{X}} = \mathrm{ad}\, \tilde{X}). \tag{10.29}$$

(c) As a consequence, we have the corresponding Lie superalgebra homomorphism of cohomologies:

$$H(\mathrm{gr}\, W^{\mathrm{ch}}(\Pi V), d_{\bar{X}}) \longrightarrow H(W^{\mathrm{cl}}(\Pi V), d_{\tilde{X}}). \tag{10.30}$$

Remark 10.15. It is interesting to understand whether the morphism (10.25) is in fact an isomorphism. In the recent paper [BDSHK18], we prove this under the assumption that the filtration of V is induced from a grading such that $V \cong \mathrm{gr}\, V$ as $\mathbb{F}[\partial]$-modules. In this case, (10.28) and (10.30) are Lie superalgebra isomorphisms and, since the cohomology of a complex is majorized by the cohomology of the associated graded complex, we get the inequalities

$$\dim H^k(W^{\mathrm{ch}}(\Pi V), d_X) \leq \dim H^k(W^{\mathrm{cl}}(\mathrm{gr}\, \Pi V), d_{\tilde{X}}) \tag{10.31}$$

for every $k \geq 0$.

10.5. A finite analog of the operad P^{cl}

For a vector superspace V, we can define a finite analog P^{fn} of the operad P^{cl} introduced in Sect. 10.2 as follows (*cf.* [Mar96]). We let $P^{\mathrm{fn}}(n)$ be the space of all maps

$$f : \mathscr{G}(n) \times V^{\otimes n} \longrightarrow V, \tag{10.32}$$

which are linear in the second factor, mapping the n-graph $\Gamma \in \mathscr{G}(n)$ and the monomial $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ to the vector $f^\Gamma(v_1 \otimes \cdots \otimes v_n)$, satisfying the cycle relations (10.4) and (10.5). The action of the symmetric groups S_n is given by

$$(f^\sigma)^\Gamma(v_1 \otimes \cdots \otimes v_n) = f^{\sigma(\Gamma)}(\sigma(v_1 \otimes \cdots \otimes v_n)), \tag{10.33}$$

where $\sigma(v_1 \otimes \cdots \otimes v_n)$ is defined by (2.2), and $\sigma(\Gamma)$ is defined in Sect. 9.3. As for the composition maps, using the cocomposition maps on graphs defined in (9.2), we let

$$(f(g_1, \dots, g_n))^\Gamma = f^{\Delta_0^{m_1 \cdots m_n}(\Gamma)}(g_1{}^{\Delta_1^{m_1 \cdots m_n}(\Gamma)} \otimes \cdots \otimes g_n{}^{\Delta_n^{m_1 \cdots m_n}(\Gamma)}), \tag{10.34}$$

for $f \in P^{\mathrm{fn}}(n)$, $g_1 \in P^{\mathrm{fn}}(m_1)$, $\dots$, $g_n \in P^{\mathrm{fn}}(m_n)$, and $\Gamma \in \mathscr{G}(M_n)$.

The same proof as for Theorem 10.7 leads to:

Theorem 10.16. *We have a bijective correspondence between the odd elements $X \in W_1^{\mathrm{fn}}(\Pi V)$ such that $X \square X = 0$ and the Poisson superalgebra structures on V, given by*

$$ab = (-1)^{p(a)} X^{\bullet\to\bullet}(a \otimes b), \quad \{a, b\} = (-1)^{p(a)} X^{\bullet\ \bullet}(a \otimes b). \tag{10.35}$$

11. The variational Poisson cohomology and the PVA cohomology

11.1. The Lie superalgebra $W^{\partial,\mathrm{as}}(\Pi V)$

In this section, we review the construction of the cohomology complex associated to a Poisson vertex algebra introduced in [DSK13]. Let V be a commutative associative superalgebra with an even derivation ∂. As usual, we denote by p the parity of V and by ΠV the space V with reversed parity $\bar{p}$. For $k \geq -1$, we let $W_k^{\partial,\mathrm{as}}(\Pi V)$ be the subspace of $W_k^{\partial}(\Pi V)$ (*cf.* Sect. 5.3) consisting of all linear maps

$$\begin{aligned} f : V^{\otimes n} &\longrightarrow \mathbb{F}_-[\lambda_1, \dots, \lambda_n] \otimes_{\mathbb{F}[\partial]} V, \\ v_1 \otimes \cdots \otimes v_n &\longmapsto f_{\lambda_1, \dots, \lambda_n}(v_1 \otimes \cdots \otimes v_n), \end{aligned}$$

satisfying the sesquilinearity conditions (5.5) and the following Leibniz rules:

$$\begin{aligned} &f_{\lambda_1,\dots,\lambda_n}(v_1,\dots,u_i\,w_i,\dots,v_n)\\ &=(-1)^{p(w_i)(s_{i+1,k}+k-i)}\,f_{\lambda_1,\dots,\lambda_i+\partial,\dots,\lambda_n}(v_1,\dots,u_i,\dots,v_n)_{\to}w_i\\ &\quad+(-1)^{p(u_i)(p(w_i)+s_{i+1,k}+k-i)}\,f_{\lambda_1,\dots,\lambda_i+\partial,\dots,\lambda_n}(v_1,\dots,w_i,\dots,v_n)_{\to}u_i\,, \end{aligned} \tag{11.1}$$

where the arrow means that ∂ is moved to the right and s_{ij} is as in (7.7).

Proposition 11.1 ([DSK13, Proposition 5.1–5.2]). *The space*

$$W^{\partial,\mathrm{as}}(\Pi V)=\bigoplus_{k\geq -1}W_k^{\partial,\mathrm{as}}(\Pi V)$$

is a subalgebra of the Lie superalgebra $W^{\partial}(\Pi V)$. Moreover, there is a bijective correspondence between the odd elements $\bar{X}\in W_1^{\partial,\mathrm{as}}(\Pi V)$ such that $[\bar{X},\bar{X}]=0$ and the Poisson vertex algebra λ-brackets on V, given by

$$[a_\lambda b]=(-1)^{p(a)}\bar{X}_{\lambda,-\lambda-\partial}(a\otimes b). \tag{11.2}$$

As a consequence, given a Poisson vertex algebra λ-bracket on V, we have the corresponding cohomology complex $(W^{\partial,\mathrm{as}}(\Pi V),d_{\bar{X}})$ with differential $d_{\bar{X}}=\operatorname{ad}\bar{X}$.

11.2. Relation between $W^{\mathrm{cl}}(\Pi V)$ and $W^{\partial,\mathrm{as}}(\Pi V)$

Let V be a Poisson vertex algebra. Recall that, by Theorem 10.7, associated to the PVA structure of V there is an odd element $X\in W_1^{\mathrm{cl}}(\Pi V)$ such that $[X,X]=2X\Box X=0$, and we thus have the corresponding cohomology complex

$$(W^{\mathrm{cl}}(\Pi V),\operatorname{ad}X). \tag{11.3}$$

Moreover, by Proposition 11.1, we also have an odd element $\bar{X}\in W_1^{\partial,\mathrm{as}}(\Pi V)$ such that $[\bar{X},\bar{X}]=0$, and we thus have the corresponding cohomology complex

$$(W^{\partial,\mathrm{as}}(\Pi V),\operatorname{ad}\bar{X}). \tag{11.4}$$

By (10.21) and (11.2), we have

$$\bar{X}=X^{\bullet\bullet}. \tag{11.5}$$

It is natural to ask what is the relation between the two cohomology theories (11.3) and (11.4).

Recall that the operad $P^{\mathrm{cl}}(\Pi V)$, hence the Lie superalgebra $W^{\mathrm{cl}}(\Pi V)$, has a grading gr^r defined in (10.9): an element $f\in\mathrm{gr}^r\,W_k^{\mathrm{cl}}(\Pi V)$ vanishes on all

graphs Γ with $(k+1)$ vertices and number of edges not equal to r. Hence, every $f \in W_k^{\mathrm{cl}}(\Pi V)$ decomposes as a finite sum

$$f = \sum_{r \geq 0} f_r, \tag{11.6}$$

where ${f_r}^\Gamma = {f}^\Gamma$ if Γ has r edges, and ${f_r}^\Gamma = 0$ otherwise. In particular, the element X decomposes as

$$X = X_0 + X_1,$$

and the condition $[X, X] = 0$ is equivalent to

$$[X_0, X_0] = [X_1, X_1] = [X_0, X_1] = 0.$$

Hence, we have two anticommuting differentials $d_{X_0} = \operatorname{ad} X_0$ and $d_{X_1} = \operatorname{ad} X_1$ on $W^{\mathrm{cl}}(\Pi V)$, which are homogeneous of degree 0 and 1 respectively.

Lemma 11.2. *We have a natural Lie algebra isomorphism*

$$W^\partial(\Pi V) \xrightarrow{\sim} \operatorname{gr}^0 W^{\mathrm{cl}}(\Pi V), \tag{11.7}$$

mapping $\bar{f} \in W^\partial(\Pi V)$ *to the element* $f \in \operatorname{gr}^0 W^{\mathrm{cl}}(\Pi V)$ *such that*

$$f^{\bullet \cdots \bullet} = \bar{f} \quad \textit{and} \quad f^\Gamma = 0 \ \textit{if} \ |E(\Gamma)| \neq \emptyset.$$

Proof. It follows from the definitions, that we have an isomorphism between the operads $\mathscr{C}\!\mathit{hom}$ and $\operatorname{gr}^0 P^{\mathrm{cl}}$. The statement of the lemma is an obvious consequence of this fact. □

Lemma 11.3. *Let* $\bar{f} \in W_k^\partial(\Pi V)$ *and let* f_0 *be its image in* $\operatorname{gr}^0 W_k^{\mathrm{cl}}(\Pi V)$ *via the isomorphism* (11.7)*. We have:*

(a) $d_X f_0 = 0 \iff d_{X_0} f_0 = d_{X_1} f_0 = 0$;
(b) $d_{X_0} f_0 = 0 \iff d_{\bar{X}} \bar{f} = 0$;
(c) $d_{X_1} f_0 = 0 \iff \bar{f} \in W_k^{\partial,\mathrm{as}}(\Pi V)$.

Hence,

$$f_0 \in \operatorname{Ker}(d_X) \iff \bar{f} \in \operatorname{Ker}(d_{\bar{X}}|_{W^{\partial,\mathrm{as}}(\Pi V)}).$$

Proof. Claim (a) is obvious, by looking at the various degrees separately. Claim (b) follows from Lemma 11.2. Let us prove claim (c). Note that $d_{X_1} f_0 = [X_1, f_0] \in \operatorname{gr}^1 W_{k+1}(\Pi V)$. Hence, to impose the condition $[X_1, f_0] = 0$ it is enough to evaluate it on graphs with 1 edge, and, by symmetry, on the graph

$$\Gamma = \bullet \quad \bullet \ \cdots \ \bullet \quad \bullet \longrightarrow \bullet.$$

By definition, $[X_1, f_0] = X_1 \square f_0 - (-1)^{\bar{p}(f_0)} f_0 \square X_1$, and we will compute the two summands separately. By (3.13) and (10.10), we have

$$(X_1 \square f_0)^\Gamma_{\lambda_1, \ldots, \lambda_{k+2}}(v_1 \otimes \cdots \otimes v_{k+2})$$

$$= \sum_{\sigma\in S_{k+1,1}} ((X_1 \circ_1 f_0)^{\sigma^{-1}})^{\Gamma}_{\lambda_1,\dots,\lambda_{k+2}}(v_1\otimes\cdots\otimes v_{k+2})$$
$$= \sum_{\sigma\in S_{k+1,1}} (X_1 \circ_1 f_0)^{\sigma^{-1}(\Gamma)}_{\sigma^{-1}(\lambda_1,\dots,\lambda_{k+2})}(\sigma^{-1}(v_1\otimes\cdots\otimes v_{k+2})).$$

Observe that, since f_0 has zero degree, $(X_1 \circ_1 f_0)^{\sigma^{-1}(\Gamma)} = 0$ if the subgraph obtained from $\sigma^{-1}(\Gamma)$ by deleting the vertex labeled $(k+2)$ has an edge. This leaves only two shuffles in the above sum: $\sigma =$ the identity and $\sigma =$ the transposition of $(k+1)$ and $(k+2)$. In the latter case, $\sigma^{-1}(\Gamma)$ is the same as Γ with reversed orientation of the edge, which leads to a minus sign. Hence, by (10.11) and (10.21), we get

$$\begin{aligned}
&(X_1\square f_0)^{\Gamma}_{\lambda_1,\dots,\lambda_{k+2}}(v_1\otimes\cdots\otimes v_{k+2})\\
&= (X_1\circ_1 f_0)^{\Gamma}_{\lambda_1,\dots,\lambda_{k+2}}(v_1\otimes\cdots\otimes v_{k+2})\\
&\quad - (-1)^{\bar p(v_{k+1})\bar p(v_{k+2})}(X_1\circ_1 f_0)^{\Gamma}_{\lambda_1,\dots,\lambda_{k+2},\lambda_{k+1}}(v_1\otimes\cdots\otimes v_{k+2}\otimes v_{k+1})\\
&= X_1^{\bullet\to\bullet}(\bar f_{\lambda_1,\dots,\lambda_{k+1}+x_{k+2}}(v_1\otimes\cdots\otimes v_{k+1})\otimes(|_{x_{k+2}=\lambda_{k+2}+\partial v_{k+2}}))\\
&\quad - (-1)^{\bar p(v_{k+1})\bar p(v_{k+2})}X_1^{\bullet\to\bullet}(\bar f_{\lambda_1,\dots,\lambda_k,\lambda_{k+2}+x_{k+1}}(v_1\otimes\cdots\otimes v_k\otimes v_{k+2})\\
&\quad\otimes(|_{x_{k+1}=\lambda_{k+1}+\partial v_{k+1}}))\\
&= (-1)^{1+\bar p(\bar f)+\bar p(v_1)+\cdots+\bar p(v_{k+1})}(\bar f_{\lambda_1,\dots,\lambda_{k+1}+\lambda_{k+2}+\partial}(v_1\otimes\cdots\otimes v_{k+1})\to v_{k+2}\\
&\quad + (-1)^{p(v_{k+1})p(v_{k+2})}\bar f_{\lambda_1,\dots,\lambda_{k+1}+\lambda_{k+2}+\partial}(v_1\otimes\cdots\otimes v_{k+2})\to v_{k+1}).
\end{aligned}$$

As for the second summand in the bracket $[X_1, f_0]$, we have, by (3.13) and (10.10),

$$(f_0\square X_1)^{\Gamma}_{\lambda_1,\dots,\lambda_{k+2}}(v_1\otimes\cdots\otimes v_{k+2})$$
$$= \sum_{\sigma\in S_{2,k}} ((f_0\circ_1 X_1)^{\sigma^{-1}})^{\Gamma}_{\lambda_1,\dots,\lambda_{k+2}}(v_1\otimes\cdots\otimes v_{k+2})$$
$$= \sum_{\sigma\in S_{2,k}} (f_0\circ_1 X_1)^{\sigma^{-1}(\Gamma)}_{\sigma^{-1}(\lambda_1,\dots,\lambda_{k+2})}(\sigma^{-1}(v_1\otimes\cdots\otimes v_{k+2})).$$

Since f_0 has zero degree, $(f_0\circ_1 X_1)^{\sigma^{-1}(\Gamma)} = 0$ unless the only edge of the graph $\sigma^{-1}(\Gamma)$ connect the vertices labeled 1 and 2. This happens for only one shuffle, given by

$$\sigma(1) = k+1,\ \sigma(2) = k+2,\ \sigma(i) = i-2 \quad \text{for } i = 3,\dots,k+2.$$

Hence, by (10.11) and (10.21), we have

$$(f_0\square X_1)^{\Gamma}_{\lambda_1,\dots,\lambda_{k+2}}(v_1\otimes\cdots\otimes v_{k+2})$$

$$
\begin{aligned}
&= \bar{\epsilon}_\sigma(v)\, \bar{f}_{\lambda_{k+1}+\lambda_{k+2},\lambda_1,\dots,\lambda_k}(X^{\bullet\to\bullet}(v_{k+1}\otimes v_{k+2})\otimes v_1\otimes\cdots\otimes v_k)\\
&= (-1)^{p(v_{k+1})}\bar{\epsilon}_\sigma(v)\, \bar{f}_{\lambda_{k+1}+\lambda_{k+2},\lambda_1,\dots,\lambda_k}(v_{k+1}v_{k+2}\otimes v_1\otimes\cdots\otimes v_k)\\
&= (-1)^{1+\bar{p}(v_1)+\cdots+\bar{p}(v_{k+1})}\, \bar{f}_{\lambda_1,\dots,\lambda_k,\lambda_{k+1}+\lambda_{k+2}}(v_1\otimes\cdots\otimes v_k\otimes v_{k+1}v_{k+2}),
\end{aligned}
$$

where

$$
\bar{\epsilon}_\sigma(v) = (-1)^{(\bar{p}(v_{k+1})+\bar{p}(v_{k+2}))\sum_{i=1}^k \bar{p}(v_i)}.
$$

In the last equality we used the symmetry condition on $\bar{f}\in W^\partial(\Pi V)$, and the fact that

$$
\bar{p}(v_{k+1}v_{k+2}) = 1+\bar{p}(v_{k+1})+\bar{p}(v_{k+2}).
$$

Combining the above results, we conclude that the condition $[X_1, f_0]=0$ is equivalent to the equation

$$
\begin{aligned}
&\bar{f}_{\lambda_1,\dots,\lambda_{k+1}+\lambda_{k+2}+\partial}(v_1\otimes\cdots\otimes v_{k+1})_\to v_{k+2}\\
&+(-1)^{p(v_{k+1})p(v_{k+2})}\bar{f}_{\lambda_1,\dots,\lambda_{k+1}+\lambda_{k+2}+\partial}(v_1\otimes\cdots\otimes v_{k+2})_\to v_{k+1}\\
&=\bar{f}_{\lambda_1,\dots,\lambda_k,\lambda_{k+1}+\lambda_{k+2}}(v_1\otimes\cdots\otimes v_k\otimes v_{k+1}v_{k+2}),
\end{aligned}
$$

i.e., $\bar{f}$ satisfies the Leibniz rule (11.1). This proves claim (c). The last assertion of the lemma is an obvious consequence of the previous claims. □

Theorem 11.4. *We have a canonical injective homomorphism of Lie superalgebras*

$$
H(W^{\partial,\mathrm{as}}(\Pi V), d_{\bar{X}}) \hookrightarrow H(W^{\mathrm{cl}}(\Pi V), d_X) \tag{11.8}
$$

induced by the map (11.7).

Proof. By Lemmas 11.2 and 11.3, the map (11.7) restricts to a Lie superalgebra isomorphism

$$
\mathrm{Ker}(d_{\bar{X}}|_{W^{\partial,\mathrm{as}}(\Pi V)}) \xrightarrow{\sim} \mathrm{Ker}(d_X)\cap \mathrm{gr}^0\, W^{\mathrm{cl}}(\Pi V). \tag{11.9}
$$

Note that, by degree considerations, we have

$$
d_X(W^{\mathrm{cl}}(\Pi V))\cap \mathrm{gr}^0\, W^{\mathrm{cl}}(\Pi V) = \{[X_0, g_0]\,|\, g_0\in \mathrm{gr}^0\, W^{\mathrm{cl}}(\Pi V)\,,\ [X_1,g_0]=0\}.
$$

It follows that, under the isomorphism (11.9), $d_{\bar{X}}(W^{\partial,\mathrm{as}}(\Pi V))$ maps bijectively to $d_X(W^{\mathrm{cl}}(\Pi V))\cap \mathrm{gr}^0\, W^{\mathrm{cl}}(\Pi V)$. Hence, (11.9) induces an isomorphism

$$
H(W^{\partial,\mathrm{as}}(\Pi V), d_{\bar{X}}) \xrightarrow{\sim} \frac{\mathrm{Ker}(d_X)\cap \mathrm{gr}^0\, W^{\mathrm{cl}}(\Pi V)}{d_X(W^{\mathrm{cl}}(\Pi V))\cap \mathrm{gr}^0\, W^{\mathrm{cl}}(\Pi V)}. \tag{11.10}
$$

The claim follows since the RHS of (11.10) is a subalgebra of $H(W^{\mathrm{cl}}(\Pi V), d_X)$. □

Remark 11.5. The map (11.8) is an isomorphism for the 0-th and 1-st cohomologies. Therefore, by Remark 10.15 we have the following inequality

$$\dim H^k(W^{\mathrm{ch}}(\Pi V), d_X) \leq \dim H^k(W^{\partial,\mathrm{as}}(\mathrm{gr}\,\Pi V), d_{\tilde{X}}) \tag{11.11}$$

for $k = 0, 1$ provided that V and $\mathrm{gr}\,V$ are isomorphic as $\mathbb{F}[\partial]$-modules. In [BDSHKV19], we prove that (11.8) is an isomorphism, provided that, as a differential algebra, V is an algebra of differential polynomials in finitely many variables.

11.3. Application to the free boson

Let $\mathscr{F}$ be a differential field with the derivation ∂. Consider the Lie conformal algebra of N free bosons

$$R = \mathscr{F}[\partial]u_1 \oplus \cdots \oplus \mathscr{F}[\partial]u_N \oplus \mathscr{F}K,$$

with the λ-brackets on the generators $u_1, \dots, u_N$ given by

$$[u_{i\,\lambda} u_j] = \lambda \delta_{ij} K, \quad i, j = 1, \dots, N,$$

where K is central and $\partial K = 0$. Its universal enveloping vertex algebra is

$$\widetilde{B} = \mathscr{F}[K, u_i^{(n)} \,|\, i = 1, \dots, N,\ n \in \mathbb{Z}_+], \quad \partial u_i^{(n)} = u_i^{(n+1)},$$

with the increasing filtration defined by letting the degrees of $u_i^{(n)}$ and K equal 1. The associated graded of the vertex algebra $\widetilde{B}$ is the Poisson vertex algebra

$$\widetilde{\mathscr{B}} := \mathrm{gr}\,\widetilde{B} = \mathscr{F}[K, u_i^{(n)} \,|\, i = 1, \dots, N,\ n \in \mathbb{Z}_+], \quad \partial u_i^{(n)} = u_i^{(n+1)},$$
$$\partial K = 0,$$

with the λ-bracket on generators given by $\{u_{i\,\lambda} u_j\} = \lambda \delta_{ij} K$ for $i, j = 1, \dots, N$, where again K is central. By (11.11) we have

$$\dim H^k(\widetilde{B}) \leq \dim H^k(\widetilde{\mathscr{B}}), \tag{11.12}$$

for $k = 0, 1$, where on the left we have the cohomology of the vertex algebra $\widetilde{B}$ while on the right we have the variational Poisson cohomology of the PVA $\widetilde{\mathscr{B}}$. In fact, due to Remarks 10.15 and 11.5, the inequality (11.12) holds for all $k \geq 0$.

We are interested in the quotients $B = \widetilde{B}/\langle K - 1\rangle$ and $\mathscr{B} = \widetilde{\mathscr{B}}/\langle K - 1\rangle$ by the ideals generated by $(K - 1)$. Then B is the vertex algebra of N free bosons

$$B = \mathscr{F}[u_i^{(n)} \,|\, i = 1, \dots, N,\ n \in \mathbb{Z}_+],$$

while $\mathscr{B}$ is the Poisson vertex algebra

$$\mathscr{B} = \mathscr{F}[u_i^{(n)} \,|\, i = 1, \dots, N,\, n \in \mathbb{Z}_+],$$

with the λ-bracket on generators given by $\{u_{i\,\lambda} u_j\} = \lambda \delta_{ij}$ for $i, j = 1, \dots, N$. It is not hard to relate the cohomologies of B and $\mathscr{B}$ to those of $\widetilde{B}$ and $\widetilde{\mathscr{B}}$, respectively, and to show that

$$\dim H^k(B) \leq \dim H^k(\mathscr{B}), \quad k \geq 0 \tag{11.13}$$

(see [BDSK19] for details).

It was proved in [DSK12] and [DSK13], respectively, that $\dim H^k(\mathscr{B}) = \binom{N+1}{k+1}$ if $\mathscr{F} = \mathbb{F}$ with $\partial \mathbb{F} = 0$, and $\dim H^k(\mathscr{B}) = \binom{N}{k+1}$ if $\mathscr{F}$ is linearly closed. The representatives of cohomology classes were explicitly computed. Using those results, it is easy to find representatives of a basis of the space of Casimirs for $\mathscr{B}$, and of the space of derivations of $\mathscr{B}$ modulo inner derivations. For $\mathscr{F} = \mathbb{F}$, representatives of a basis of $H^0(\mathscr{B}) \subset \mathscr{B}/\partial\mathscr{B}$ are the Casimir elements

$$1, u_1, \dots, u_N, \tag{11.14}$$

and representatives of a basis of $H^1(\mathscr{B}) = \mathrm{Der}(\mathscr{B})/\mathrm{Inder}(\mathscr{B})$ are the following derivations,

$$\begin{aligned} &\frac{\partial}{\partial u_i},\ i = 1, \dots, N, \quad \text{and} \\ &D_{ij} = \sum_{n \in \mathbb{Z}_+} \left(u_i^{(n)} \frac{\partial}{\partial u_j^{(n)}} - u_j^{(n)} \frac{\partial}{\partial u_i^{(n)}} \right),\ 1 \leq i < j \leq N. \end{aligned} \tag{11.15}$$

If the field $\mathscr{F}$ is linearly closed, it contains x such that $\partial x = 1$, hence we have $1 = \partial x \equiv 0$ in $\mathscr{B}/\partial\mathscr{B}$, and $\frac{\partial}{\partial u_i} = \{x u_{i\,\lambda} \cdot\}|_{\lambda=0}$, $i = 1, \dots, N$, are inner derivations, while the remaining elements in (11.14) and (11.15) are linearly independent representatives.

Note that, in the case when $\mathscr{F} = \mathbb{F}$, the elements (11.14) are Casimirs of B, linearly independent over $\mathbb{F}$. Hence, $\dim(\mathrm{Cas}(B)) \geq N + 1$. On the other hand, by Theorem 7.6 and the inequality (11.13) the opposite inequality holds. It follows that

$$\dim(\mathrm{Cas}(B)) = N + 1,$$

and the elements (11.14) form a basis of $\mathrm{Cas}(B)$.

Next, still in the case $\mathscr{F} = \mathbb{F}$, derivations (11.15) are actually derivations of the Lie conformal algebra R. Hence, they uniquely extend to derivations of its universal enveloping vertex algebra B, and it is easy to see that they are linearly independent modulo inner derivations of B. Hence, $\dim(\mathrm{Der}(B)/\mathrm{Inder}(B)) \geq$

$\frac{1}{2}N(N+1)$. On the other hand, by Theorem 7.6 and the inequality (11.13), the opposite inclusion holds. It follows that

$$\dim(\mathrm{Der}(B)/\mathrm{Inder}(B)) = \binom{N+1}{2},$$

and the derivations (11.15) are representatives of a basis of $\mathrm{Der}(B)/\mathrm{Inder}(B)$.

Similarly, in the case when $\mathscr{F}$ is linearly closed, we obtain

$$\dim(\mathrm{Cas}(B)) = N \quad \text{and} \quad \dim(\mathrm{Der}(B)/\mathrm{Inder}(B)) = \binom{N}{2},$$

with the same representatives as for $\mathscr{B}$, described above.

A. Relation to chiral algebras

In [BD04] the authors introduced an algebro-geometric rendition of the theory of vertex algebras, which they called *chiral algebras*. In this section we outline the relation of the above results with their definitions.

A.1. Chiral operations

Consider a smooth algebraic curve X over $\mathbb{F}$. For any right $\mathscr{D}_X$-module $\mathscr{A}$, Beilinson and Drinfeld construct an operad $\mathscr{P}^{\mathrm{ch}}_{\mathscr{A}}$ whose $(k+1)$-ary operations are

$$\mathscr{P}^{\mathrm{ch}}_{\mathscr{A}}(k+1) = \mathrm{Hom}_{\mathscr{D}_{X^{k+1}}\text{-mod}}(j_* j^* \mathscr{A}^{\boxtimes(k+1)}, \Delta_* \mathscr{A}),$$

where j is the inclusion of the open complement of the diagonal divisor on X^{k+1} (union of hypersurfaces $x^i = x^j$ for $i \neq j$), and $\Delta\colon \mathbb{A}^1 \to \mathbb{A}^{k+1}$ is the diagonal embedding $x \mapsto (x, \dots, x)$. A *non-unital chiral algebra* on X is by definition a morphism of operads

$$\mathscr{Lie} \longrightarrow \mathscr{P}^{\mathrm{ch}}_{\mathscr{A}}.$$

In particular it is defined by a binary operation

$$\mathscr{P}^{\mathrm{ch}}_{\mathscr{A}}(2) \ni \mu\colon j_* j^* \mathscr{A} \boxtimes \mathscr{A} \longrightarrow \Delta_* \mathscr{A}, \tag{A.1}$$

satisfying skew-symmetry and Jacobi identity.

The dualizing sheaf ω_X of X carries a canonical chiral algebra structure given by the residue map. For this we define μ as the cokernel of the inclusion

$$0 \longrightarrow \omega_X \boxtimes \omega_X \longrightarrow j_* j^* \omega_X \boxtimes \omega_X \xrightarrow{\mu} \Delta_* \omega_X \longrightarrow 0.$$

Skewsymmetry follows from the isomorphism $\omega_X \boxtimes \omega_X \simeq \omega_{X^2}$, which is skew-equivariant for the action of $\mathbb{Z}/2\mathbb{Z}$ by permutation of the two factors. The Jacobi identity is a little subtler to prove and is a consequence of the Cousin resolution of ω_{X^3} with respect to the diagonal stratification (see [BD04] or [FBZ04]). A non-unital chiral algebra $\mathscr{A}$ is called *unital* or simply a *chiral algebra* if there is a morphism $\omega_X \to \mathscr{A}$ of $\mathscr{D}_X$-modules such that the restriction of the multiplication $\mu_{\mathscr{A}}$ of $\mathscr{A}$ to $j_* j^* \omega_X \boxtimes \mathscr{A} \to j_* j^* \mathscr{A} \boxtimes A$ coincides with the cokernel of the sequence

$$0 \longrightarrow \omega_X \boxtimes \mathscr{A} \longrightarrow j_* j^* \omega_X \boxtimes \mathscr{A} \longrightarrow \Delta_* \mathscr{A} \longrightarrow 0. \tag{A.2}$$

A.2. $\mathscr{D}$-modules on the line

In the particular case when $X = \mathbb{A}^1$ is the affine line over $\mathbb{F}$, any $\mathscr{D}_X$-module $\mathscr{A}$ is determined by the $\Gamma(\mathbb{A}^1, \mathscr{D}_{\mathbb{A}^1})$-module $A := \Gamma(\mathbb{A}^1, \mathscr{A})$ of global sections. The same is true for the $\mathscr{D}_{X^{k+1}}$-modules

$$j_* j^* \mathscr{A}^{\boxtimes(k+1)} \quad \text{and} \quad \Delta_* \mathscr{A}.$$

Let $\mathscr{D}_{k+1}$ be the algebra of regular differential operators on $(k+1)$ variables $z_0, \dots, z_k$ as in Sect. 6.2, and let I be the left ideal generated by $\{z_0 - z_i\}_{i=1}^k$. Let $\mathscr{O}_{k+1} = \mathbb{F}[z_0, \dots, z_k]$ and recall the algebra $\mathscr{O}^\star_{k+1}$ of functions defined in Sect. 6.2. It is naturally an $\mathscr{O}_{k+1}$-module, as is $\mathscr{D}_{k+1}$. Notice that $A^{\otimes(k+1)}$ is naturally a $\mathscr{D}_{k+1}$-module. We have

$$\Gamma(\mathbb{A}^{k+1}, j_* j^* \mathscr{A}^{\boxtimes(k+1)}) = \mathscr{O}^\star_{k+1} \otimes_{\mathscr{O}_{k+1}} A^{\otimes(k+1)},$$

and the $\mathscr{D}_{k+1}$-module structure is by the action on the right factor.

Consider $I \backslash \mathscr{D}_{k+1}$, which is a $(\mathscr{D}_1 - \mathscr{D}_{k+1})$ bimodule as follows. The action of $\mathscr{D}_{k+1}$ is by multiplication on the right. The action of $\mathscr{D}_1 = \mathbb{F}[z][\partial_z]$ on the left is defined by letting z act as multiplication on the left by z_0 and ∂_z act as multiplication on the left by $\sum_{i=0}^k \partial_{z_i}$. We have

$$\Gamma(\mathbb{A}^{k+1}, \Delta_* \mathscr{A}) = A \otimes_{\mathscr{D}_1} (I \backslash \mathscr{D}_{k+1}), \tag{A.3}$$

with its natural right $\mathscr{D}_{k+1}$-module structure by right multiplication on the right factor. We have

$$\mathscr{P}^{\mathrm{ch}}_{\mathscr{A}}(k+1) = \mathrm{Hom}_{\mathscr{D}_{k+1}}(\mathscr{O}^\star_{k+1} \otimes_{\mathscr{O}_{k+1}} A^{\otimes(k+1)}, A \otimes_{\mathscr{D}_1} (I \backslash \mathscr{D}_{k+1})). \tag{A.4}$$

A.3. Equivariant $\mathscr{D}$-modules

Let X be a smooth scheme, G an algebraic group acting on X, and $\mathscr{F}$ a quasi-coherent sheaf of $\mathscr{O}_X$-modules. Denote by $a\colon G\times X\to X$ the G-action and by $\pi_2\colon G\times X\to X$ the projection to the second factor. We say $\mathscr{F}$ is *G-equivariant* if there exists an isomorphism of $\mathscr{O}_{G\times X}$-modules

$$\alpha\colon a^*\mathscr{F}\longrightarrow \pi_2^*\mathscr{F} \tag{A.5}$$

such that:

(1) the diagram

$$\begin{array}{ccc} (1_G\times a)^*\,\pi_2^*\mathscr{F} & \longrightarrow & \pi_3^*\mathscr{F} \\ \uparrow & & \uparrow \\ (1_G\times a)^*\,a^*\mathscr{F} & = & (m\times 1_X)^*\,a^*\mathscr{F} \end{array} \tag{A.6}$$

commutes in the category of $\mathscr{O}_{G\times G\times X}$-modules[1];

(2) the pullback

$$(e\times 1_X)^*\alpha\colon \mathscr{F}\longrightarrow\mathscr{F},$$

where $e\in G$ is the unit, is the identity map.

A $\mathscr{D}_X$-module $\mathscr{F}$ is called *strongly equivariant* if a given α as in (A.5) is an isomorphism of $\mathscr{D}_{G\times X}$-modules and the diagram (A.6) is in the category of $\mathscr{D}_{G\times G\times X}$-modules. The module $\mathscr{F}$ is said to be *weakly equivariant* if α is an isomorphism of $\mathscr{O}_G\otimes\mathscr{D}_X$-modules.

A.4. Equivariant $\mathscr{D}$-modules on the line

Consider the affine line $\mathbb{A}^1$ over a field $\mathbb{F}$, with its natural action of the additive group $\mathbb{G}_a$ by translations. Let $\mathscr{F}$ be a translation equivariant $\mathscr{O}_{\mathbb{A}^1}$-module. Let $0\in\mathbb{A}^1$ be the origin. The functor 0^* of taking the fiber at 0 defines an equivalence of categories between translation equivariant quasi-coherent sheaves on the line and vector spaces. The inverse functor associates to the vector space V the sheaf associated to the $\mathbb{F}[x]$-module $V[x]$ and the action of $t\in\mathbb{G}_a$ is given by $v(x)\mapsto v(x+t)$. The isomorphism α as in (A.5) is given by

$$\alpha\colon V[t,x]\longrightarrow V[t,x],\quad v(t,x)\longmapsto v(t,x+t). \tag{A.7}$$

Notice that $V[x]$ has a canonical right $\mathscr{D}_1$-module structure with ∂_x acting by $-d/dx$. Similarly, $V[t,x]$ has right action of $\mathscr{D}_2=\Gamma(\mathbb{G}_a\times\mathbb{A}^1,\mathscr{D}_{\mathbb{G}_a\times\mathbb{A}^1})$. The map (A.7) is a morphism of $\mathscr{D}_2$-modules. In fact, we have an equivalence of categories between strongly equivariant $\mathscr{D}$-modules on $\mathbb{A}^1$ and vector spaces.

[1] Here $\pi_3: G\times G\times X\to X$ is the projection map.

Now let $\partial \in \mathrm{End}(V)$. As in Sect. 6.3, we have a $\mathscr{D}_1$-module structure on $V[x]$ defined by letting x act as multiplication by x and ∂_x act by $\partial - d/dx$. The map (A.7) no longer commutes with the action of ∂_t; hence, it defines a weakly equivariant $\mathscr{D}$-module structure on the sheaf associated to the $\mathscr{D}_1$-module $V[x]$. In other words, differentiating the $\mathbb{G}_a$ action on $V[x]$, we obtain that ∂_x acts by $-d/dx$, which does not coincide with the action obtained from the $\mathscr{D}_1$-module structure. The assignment $(V, \partial) \mapsto V[x]$ defines an equivalence of categories between weakly equivariant $\mathscr{D}$-modules on $\mathbb{A}^1$ and pairs (V, ∂) of a vector space and an endomorphism.

A.5. Equivariant operads

Let $\mathscr{P}$ be a symmetric operad and G be a group. We say that $\mathscr{P}$ is *G-equivariant* if every space $\mathscr{P}(n)$ carries an action of G and the composition maps (3.1) are morphisms of G-modules. In particular, this implies that the action of G commutes with the symmetric group action on each $\mathscr{P}(n)$. It is clear that the spaces of invariants $\mathscr{P}(n)^G$ (respectively, coinvariants $\mathscr{P}(n)_G$) form a suboperad (respectively, quotient operad) of $\mathscr{P}$.

A.6. Equivariant chiral operations

Consider a weakly equivariant $\mathscr{D}$-module $\mathscr{A}$ on the line corresponding to the pair (V, ∂). The $\mathscr{D}_{k+1}$-module (A.3) is in this case given by

$$V \otimes_{\mathbb{F}[\partial]} \mathbb{F}[x][\partial_0, \dots, \partial_k], \tag{A.8}$$

where we view $\mathbb{F}[x][\partial_0, \dots, \partial_k]$ as a $(\mathbb{F}[\partial] - \mathscr{D}_{k+1})$-bimodule as follows. The left action of ∂ is given by $\sum_{i=0}^{k} \partial_i$. The right action of ∂_i is by multiplication by ∂_i, and the right action of z_i is given by

$$f(x, \partial_0, \dots, \partial_k) \cdot z^i = x f(x, \partial_0, \dots, \partial_k) + \frac{\partial}{\partial_{\partial^i}} f(x, \partial_0, \dots, \partial_k). \tag{A.9}$$

In this case, (A.4) reads

$$\mathscr{P}^{\mathrm{ch}}_{\mathscr{A}}(k+1) = \mathrm{Hom}_{\mathscr{D}_{k+1}}(\mathscr{O}^{\star}_{k+1} \otimes V^{\otimes(k+1)}, V \otimes_{\mathbb{F}[\partial]} \mathbb{F}[x][\partial_0, \dots, \partial_k]).$$

The group $\mathbb{G}_a$ acts on these operations as follows. Given $t \in \mathbb{G}_a$ and $\varphi \in \mathscr{P}^{\mathrm{ch}}_A(k+1)$, we obtain a new operation φ^t by letting

$$\begin{aligned} &\varphi^t(f(z_0, \dots, z_k) \otimes v_0 \otimes \cdots \otimes v_k) \\ &:= \varphi(f(z_0 - t, \dots, z_k - t) \otimes v_0 \otimes \cdots \otimes v_k)|_{x=x+t}. \end{aligned} \tag{A.10}$$

The set of translation invariant operations $\mathscr{P}_{\mathscr{A}}^{T,\mathrm{ch}} \subset \mathscr{P}_{\mathscr{A}}^{\mathrm{ch}}$ defines a suboperad. A weakly translation equivariant $\mathscr{D}$-module $\mathscr{A}$ on $\mathbb{A}^1$ is called a non-unital *weakly translation equivariant* chiral algebra if the multiplication (A.1) is translation invariant. For instance, the unital chiral algebra $\omega_{\mathbb{A}^1}$ is weakly translation equivariant. A unital chiral algebra is called *translation equivariant* if the morphism $\omega_{\mathbb{A}^1} \to \mathscr{A}$ is equivariant for the $\mathbb{G}_a$-action.

Lemma A.1. *Let V be an $\mathbb{F}[\partial]$-module, and $\mathscr{A}$ be its associated weakly equivariant $\mathscr{D}$-module on $\mathbb{A}^1$. Let P^{ch} be the operad from Proposition 6.7 associated to V. Then we have an isomorphism of operads $\mathscr{P}_{\mathscr{A}}^{T,\mathrm{ch}} \simeq P^{\mathrm{ch}}$.*

Proof. Recall the algebra of translation invariant differential operators $\mathscr{D}_{k+1}^T$ of Sect. 6.2. The action of $\mathbb{G}_a$ on $\mathbb{A}^1$ induces an action on $\Delta_* \mathscr{A}$ and consequently on its global sections (A.8), which is given simply by $x \mapsto x + t$. The space of invariant sections is a $\mathscr{D}_{k+1}^T$-module isomorphic to (6.7). Indeed, we have an isomorphism

$$\begin{aligned} V \otimes_{\mathbb{F}[\partial]} \mathbb{F}[\partial_0, \dots, \partial_k] &\xrightarrow{\sim} V[\lambda_0, \dots, \lambda_k]/\langle \partial + \lambda_0 + \dots + \lambda_k \rangle, \\ v \otimes f(\partial_0, \dots, \partial_k) &\longmapsto f(-\lambda_0, \dots, -\lambda_k)v, \end{aligned} \tag{A.11}$$

which is compatible with the action of $\mathscr{D}_1^T = \mathbb{F}[\partial]$. Similarly, the space of $\mathbb{G}_a$-invariant sections of $\mathscr{O}_{k+1}^{\star} \otimes V^{\otimes(k+1)}$ is given by $\mathscr{O}_{k+1}^{\star T} \otimes V^{\otimes(k+1)}$ and is a $\mathscr{D}_{k+1}^T$-module as in Sect. 6.3.

For $\varphi \in \mathscr{P}_{\mathscr{A}}^{T,\mathrm{ch}}(k+1)$, restricting φ to $\mathscr{O}_{k+1}^{\star T} \otimes V^{\otimes(k+1)}$, we see that by (A.10)

$$f \otimes v_0 \otimes \dots \otimes v_k$$

does not depend on x; therefore, by (A.11) it defines a vector in

$$V[\lambda_0, \dots, \lambda_k]/\langle \partial + \lambda_0 + \dots + \lambda_k \rangle.$$

Hence, φ defines an element of $P^{\mathrm{ch}}(k+1)$.

Conversely, given X as in (6.11) satisfying the sesquilinearity conditions (6.12), we extend X to a morphism $\varphi \in \mathscr{P}_{\mathscr{A}}^{T,\mathrm{ch}}(k+1)$ as follows. By a Taylor expansion, we can express any function $f(z_0, \dots, z_k) \in \mathscr{O}_{k+1}^{\star}$ as a finite sum

$$\sum g_i(z_0, z_1, \dots, z_k) z_0^{n_i},$$

for some $g_i \in \mathscr{O}_{k+1}^{\star T}$ and some nonnegative integers n_i. We define

$$\varphi(f \otimes v_0 \otimes \dots \otimes v_k) := \sum x^{n_i} X(g_i \otimes v_0 \otimes \dots \otimes v_k),$$

where we identified $X(g_i \otimes v_0 \otimes \dots \otimes v_k)$ with a translation invariant vector in (A.8) by (A.11). It is clear that φ is translation invariant and it is a morphism of $\mathscr{D}_{k+1}$-modules. □

Corollary A.2 ([BD04, 0.15]). *There is an equivalence of categories between weakly translation equivariant chiral algebras on $\mathbb{A}^1$ and vertex algebras.*

Proof. We first prove the analogous statement for non-unital algebras. The non-unital weakly translation equivariant chiral algebras are given by morphisms of operads $\mathscr{Lie} \to \mathscr{P}_{\mathscr{A}}^{T,\mathrm{ch}}$ for a weakly equivariant $\mathscr{D}$-module $\mathscr{A}$. By Lemma A.1, we have an $\mathbb{F}[\partial]$-module V and a morphism of operads $\mathscr{Lie} \to P^{\mathrm{ch}}$. In a similar way as in Remark 4.3, these correspond to an odd element $X \in W_1^{\mathrm{ch}}(\Pi V)$ satisfying $X \square X = 0$. The result then follows from Theorem 6.12. Under this equivalence, the unit vertex algebra $\mathbb{F}$ corresponds to the chiral algebra $\omega_{\mathbb{A}^1}$.

Consider now a translation equivariant unital chiral algebra V on $\mathbb{A}^1$. Then the morphism $\omega_{\mathbb{A}^1} \to \mathscr{A}$ corresponds to a morphism of vertex algebras $\mathbb{F} \to V$. The image of $1 \in \mathbb{F}$ is the vacuum vector $|0\rangle$ of V. Indeed, since $\omega_{\mathbb{A}^1} \to \mathscr{A}$ is a morphism of $\mathscr{D}$-modules, we have $\partial|0\rangle = 0$. If $X \in W_1^{\mathrm{ch}}(\Pi V)$ is the corresponding operation, it follows from (A.2) that

$$X\left(|0\rangle \otimes v \otimes \frac{1}{z_1 - z_0}\right) = v, \quad v \in V,$$

from where the vacuum axioms follow. □

A.7. Lie conformal operad

In addition to the operad $P_{\mathscr{A}}^{\mathrm{ch}}$ associated to any $\mathscr{D}_X$-module $\mathscr{A}$, Beilinson and Drinfeld define an operad $P_{\mathscr{A}}^{*}$ by letting

$$\mathscr{P}_{\mathscr{A}}^{*}(k+1) = \mathrm{Hom}_{\mathscr{D}_{X^{k+1}}\text{-mod}}(\mathscr{A}^{\boxtimes(k+1)}, \Delta_* \mathscr{A}).$$

In the case of $X = \mathbb{A}^1$ and $\mathscr{A}$ a weakly equivariant $\mathscr{D}$-module, we let $\mathscr{P}_{\mathscr{A}}^{T,*}$ be the suboperad of $\mathbb{G}_a$-invariant operations. We have in the same way as Lemma A.1 the following:

Lemma A.3. *Let V be an $\mathbb{F}[\partial]$-module, and $\mathscr{A}$ be its associated weakly equivariant $\mathscr{D}$-module on $\mathbb{A}^1$. Let $\mathscr{Chom}$ be the operad from Sect. 5.2 associated to V. Then we have an isomorphism of operads $\mathscr{P}_{\mathscr{A}}^{T,*} \simeq \mathscr{Chom}$.*

A.8. Classical operations

For any smooth algebraic curve X over $\mathbb{F}$ and any right $\mathscr{D}_X$-module $\mathscr{A}$ on it, in [BD04, 1.4.27] the authors define an operad of *classical operations* $\mathscr{P}_{\mathscr{A}}^{c}$ as follows. Let $\mathscr{Lie}$ be the Lie operad, that is, $\mathscr{Lie}(k+1)$ is the vector space with a basis consisting of all formal symbols

$$[x_{\sigma(0)}, [x_{\sigma(1)}, \cdots [x_{\sigma(k-1)}, x_{\sigma(k)}] \cdots]], \quad \sigma \in S_{k+1}, \quad \sigma(0) = 0.$$

The composition in $\mathscr{Lie}$ is defined by replacing the corresponding variables and expanding using the Jacobi and skew-symmetry identities. For each $k \geq 0$ and each $(m_0, \dots, m_k)$-shuffle σ as in Sect. 2.5, we let

$$\mathscr{Lie}_\sigma := \mathscr{Lie}(m_0) \otimes \cdots \otimes \mathscr{Lie}(m_k),$$

and

$$\mathscr{P}^*_{\mathscr{A}}(\sigma) = \mathrm{Hom}_{\mathscr{D}_{X^{k+1}}\text{-mod}}(\mathscr{A}^{\otimes m_0} \boxtimes \cdots \boxtimes \mathscr{A}^{\otimes m_k}, \Delta_* \mathscr{A}),$$

where $\Delta\colon X \hookrightarrow X^{k+1}$ is the small diagonal embedding. Finally, put

$$\mathscr{P}^c_{\mathscr{A}}(n+1) = \bigoplus_{k=0}^{n} \bigoplus_{\sigma} \mathscr{P}^*_{\mathscr{A}}(\sigma) \otimes \mathscr{Lie}_\sigma,$$

where the inner sum is over $(m_0, \dots, m_k)$-shuffles σ such that $\sum_{i=0}^k m_i = n+1$. The composition in $\mathscr{P}^c_{\mathscr{A}}$ is defined as the tensor product of the compositions in the operad $\mathscr{P}^*_{\mathscr{A}}$ defined in A.7 and the compositions in the $\mathscr{Lie}$ operad (see [BD04, 1.4.27] for details). The operad $\mathscr{P}^c_{\mathscr{A}}$ defined this way is graded, with the grading given by k in the above sum.

Remark A.4. In [BD04] the authors work with unordered sets and equivalence relations on these sets, namely, instead of defining the n-ary operations $\mathscr{P}(n)$ for an operad, they define the I-ary operations $\mathscr{P}(I)$ for any finite nonempty set I. Similarly, composition is defined for any equivalence relation S in I instead of a shuffle σ. For the equivalence of these two approaches, see [GK94].

In the case when $X = \mathbb{A}^1$ and $\mathscr{A}$ is a weakly $\mathbb{G}_a$-equivariant $\mathscr{D}_X$-module, we consider the translation equivariant suboperad $\mathscr{P}^{T,c}_{\mathscr{A}}$, and in the same way as in A.6, we have the following:

Theorem A.5. *Let V be an $\mathbb{F}[\partial]$-module, and $\mathscr{A}$ be its associated weakly equivariant $\mathscr{D}$-module on $\mathbb{A}^1$. Let P^{cl} be the operad from Theorem 10.6 associated to V. Then we have an isomorphism of graded operads $\mathscr{P}^{T,c}_{\mathscr{A}} \simeq P^{\mathrm{cl}}$.*

Sketch of the proof. The proof relies on a theorem by Chapoton and Livernet [CL01], which states that the operad of pre-Lie algebras[2] is isomorphic to the operad of rooted trees. Using this theorem, one can associate to any n-ary operation in the operad $\mathscr{Lie}$ a connected graph $\Gamma \in \mathscr{G}(n)$ as in Sect. 8.2. More generally, given an $(m_0, \dots, m_k)$-shuffle σ and

$$\tau_0 \otimes \cdots \otimes \tau_k \in \mathscr{Lie}_\sigma,$$

we obtain a graph $\Gamma \in \mathscr{G}(\sum m_i)$ with $(k+1)$ connected components, the i-th component of which being a connected graph in $\mathscr{G}(m_i)$. By Lemma A.3, to

[2] That is algebras satisfying an even version of (3.15).

any element of $\mathscr{P}_{\mathscr{A}}^{T,*}(\sigma)$ we associate an operation f^{Γ} satisfying the sesquilinearity conditions (10.6), (10.7). The operation f^{Γ} satisfies in addition the cycle relations (10.4), (10.5), since the graph Γ comes from an operation in $\mathscr{L}ie$ and therefore satisfies skew-symmetry (as opposed to the general graph that by Chapoton–Livernet's theorem defines only a pre-Lie algebra operation). This defines an isomorphism of graded vector spaces $\mathscr{P}_{\mathscr{A}}^{T,c}(n) \simeq P^{\mathrm{cl}}(n)$ for all n. One readily checks that this isomorphism is compatible with compositions in both operads. □

The operad $\mathscr{P}_{\mathscr{A}}^{\mathrm{ch}}$ carries a natural filtration given by the diagonal stratification of X^n for each n. It gives rise to the associated graded operad $\mathrm{gr}\,\mathscr{P}_{\mathscr{A}}^{\mathrm{ch}}$. In [BD04, 3.2.5] the authors produce a canonical embedding of graded operads

$$\mathrm{gr}\,\mathscr{P}_{\mathscr{A}}^{\mathrm{ch}} \hookrightarrow \mathscr{P}_{\mathscr{A}}^{c},$$

and claim that it is an isomorphism if $\mathscr{A}$ is a projective $\mathscr{D}_X$-module. In the case of $X = \mathbb{A}^1$ and considering the translation invariant suboperads, this embedding is the geometric counterpart to Theorem 10.10.

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